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UNICITY THEOREMS FOR THE GAUSS MAPS OF COMPLETE MINIMAL SURFACES, II

Dedicated to Professor Mitsuru Nakai on his 60th birthday

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§ 1. Introduction.

The Gauss map of complete minimal surfaces in $\boldsymbol{R}^m$ have many properties which have analogies to value-distribution-theoretic properties of holomorphic curves in the complex projective space. The author gave some of them in his papers [5]~[8]. Moreover, in [9] he obtained the following analogy of Nevanlinna's unicity theorem ([11]):

THEOREM. *Let M and $\tilde{M}$ be two nonflat minimal surfaces immersed in $\boldsymbol{R}^3$ and let $G: M \to S^2$ and $\tilde{G}: \tilde{M} \to S^2$ be the Gauss maps of M and $\tilde{M}$ respectively. Suppose that there is a conformal diffeomorphism Φ between M and $\tilde{M}$. If M or $\tilde{M}$ is complete and there are seven distinct directions $n_1, \cdots, n_7 \in S^2$ such that $G^{-1}(n_j) = (\tilde{G} \cdot \Phi)^{-1}(n_j)$ $(1 \leqq j \leqq 7)$, then $G \equiv \tilde{G} \cdot \Phi$.*

He gave also more precise results for the case where both of M and $\tilde{M}$ are complete and have finite total curvature. The purpose of this paper is to give some generalizations of these results to minimal surfaces in $\boldsymbol{R}^m$ for the case $m > 3$.

As is well-known, the set of all oriented 2-planes in $\boldsymbol{R}^m$ containing the origin is canonically identified with the quadric

$$Q_{m-2}(\boldsymbol{C}) := \{(w_1 : \cdots : w_m) ; w_1^2 + \cdots + w_m^2 = 0\}$$

in $P^{m-1}(\boldsymbol{C})$. For a minimal surface $x := (x_1, \cdots, x_m) : M \to \boldsymbol{R}^m$ immersed in $\boldsymbol{R}^m$ the Gauss map G of M is defined as the map of M into $Q_{m-2}(\boldsymbol{C})$ which maps each $p \in M$ to the point in $Q_{m-2}(\boldsymbol{C})$ corresponding to the oriented tangent plane of M at p. We may regard M as a Riemann surface with a conformal metric and G as a holomorphic map of M into $P^{m-1}(\boldsymbol{C})$.

As in the case $m=3$, we consider two nonflat minimal surfaces

$$x := (x_1, \cdots, x_m) : M \to \boldsymbol{R}^m, \qquad \tilde{x} := (\tilde{x}_1, \cdots, \tilde{x}_m) : \tilde{M} \to \boldsymbol{R}^m$$

and their Gauss maps $G : M \to P^N(\boldsymbol{C})$ and $\tilde{G} : \tilde{M} \to P^N(\boldsymbol{C})$, where $N := m-1$. Sup-

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pose that there is a conformal diffeomorphism Φ between M and $\tilde{M}$. Then the Gauss map of the minimal surface $\tilde{x}\cdot\Phi : M\to \boldsymbol{R}^m$ is given by $\tilde{G}\cdot\Phi$. Consider the holomorphic maps $f:=G : M\to P^N(\boldsymbol{C})$, $g:=\tilde{G}\cdot\Phi : M\to P^N(\boldsymbol{C})$ and assume that they satisfy the following:

ASSUMPTION 1.1. *There exist hyperplanes* $H_1, \cdots, H_q$ *in* $P^N(\boldsymbol{C})$ *located in general position such that*

(i) $f^{-1}(H_j)=g^{-1}(H_j)\neq M$ *for every* j,

(ii) $f=g$ *on* $\bigcup_{j=1}^q f^{-1}(H_j)-K$ *for a compact subset* K *of* M.

The main result is stated as follows:

THEOREM 1.2. *Under the above assumption, we have necessarily* $f\equiv g$

(A) *if* $q>m^2+m(m-1)/2$ *for the case where* M *is complete and has infinite total curvature or*

(B) *if* $q\geqq m^2+m(m-1)/2$ *for the case where* $K=\emptyset$ *and* M *and* $\tilde{M}$ *are both complete and have finite total curvature.*

We shall give some estimates for divisors on an open Riemann surface in §2 and construct a pseudo-metric with strictly negative curvature associated with two holomorphic maps f and g into $P^N(\boldsymbol{C})$ satisfying Assumption 1.1 in §3. After these preparations, in §§4∼5 we shall prove some unicity theorems for holomorphic maps into $P^N(\boldsymbol{C})$ defined on an open Riemann surfaces with complete conformal metrics. Theorem 1.2 will be proved in §6.

§2. Some estimates for divisors.

Let M be a Riemann surface. In this paper, a divisor ν on M means a map $\nu : M\to\boldsymbol{R}$ whose support $\mathrm{Supp}(\nu):=\overline{\{z\,;\,\nu(z)\neq 0\}}$ has no accumulation point in M. We say that a complex-valued function u on M has mild singularities if it can be written as

$$|u(z)|=|z-a|^{\sigma}u^*(z)\prod_i\left|\frac{1}{\log|g_i(z)v_i(z)|}\right|^{\tau_i} \tag{2.1}$$

on a neighborhood of each $a\in M$ with a real number σ, finitely many nonnegative numbers τ_i, positive C^∞ function u^*, v_i and nonzero holomorphic functions g_i, where z is a holomorphic local coordinate around a. For a function u with mild singularities, we define the divisor ν_u of u by

$$\nu_u(a):=\text{the number } \sigma \text{ for the representation (2.1)}$$

for each $a\in M$.

Let $f : M\to \boldsymbol{P}^n(\boldsymbol{C})$ be a nondegenerate holomorphic map. For $a\in M$ taking an open neighborhood D of a contained in the domain of a holomorphic local coordinate, we choose a reduced representation $f:=(f_0:\cdots:f_n)$ on D, where

f_i's are holomorphic functions on D without common zero. Consider the holomorphic map

$$F_k := F^{(0)} \wedge F^{(1)} \wedge \cdots \wedge F^{(k)} : D \to \bigwedge^{k+1} C^{n+1} \tag{2.2}$$

for $0 \leqq k \leqq n$, where $F^{(0)} \equiv F := (f_0, \cdots, f_n)$ and $F^{(l)} := (f_0^{(l)}, \cdots, f_n^{(l)})$ for each $l = 0, 1, \cdots$. The norm of F_k is given by

$$|F_k| := \Big(\sum_{0 \leqq i_0 < \cdots < i_k \leqq n} |W(f_{i_0}, \cdots, f_{i_k})|^2 \Big)^{1/2},$$

where $W(f_{i_0}, \cdots, f_{i_k})$ denotes the Wronskian of $f_{i_0}, \cdots, f_{i_k}$. Set $\nu_k := \nu_{|F_k|}$ for $0 \leqq k \leqq n$. The divisor ν_n is nothing but the divisor of $W(f_0, f_1, \cdots, f_n)$. These are globally well-defined on M. Because, for another reduced representation $f := (\tilde{f}_0 : \cdots : \tilde{f}_n)$ we can write $(\tilde{f}_0, \cdots, \tilde{f}_n) = hF$ with a nowhere vanishing holomorphic function h and F_k is multiplied by h^{k+1}, and for another holomorphic local coordinate ζ, F_k is multiplied by $(dz/d\zeta)^{k(k+1)/2}$.

We now take a hyperplane H with $f(M) \subsetneqq H$ given by

$$H : \bar{c}_0 w_0 + \cdots + \bar{c}_n w_n = 0 .$$

For each reduced representation $f := (f_0 : \cdots : f_n)$ we set

$$F \equiv F(H) := \bar{c}_0 f_0 + \cdots + \bar{c}_n f_n \tag{2.3}$$

and define the pull-back of the divisor H via f by $\nu(f, H) := \nu_F$, which is well-defined on M.

We next consider q hyperplanes $H_1, \cdots, H_q$ in $P^n(C)$ given by

$$H_j : \langle w, A_j \rangle \equiv \bar{c}_{j0} w_0 + \cdots + \bar{c}_{jn} w_n = 0 \qquad (1 \leqq j \leqq q),$$

where $A_j := (c_{j0}, \cdots, c_{jn}) \in C^{n+1} - \{0\}$. For $R \subseteqq Q := \{1, 2, \cdots, q\}$ we denote by $d(R)$ the dimension of the vector subspace of C^{n+1} generated by $\{A_j ; j \in R\}$. Following [3], we say that $H_1, \cdots, H_q$ are in N-subgeneral position if $d(R) = n+1$ for all $R \subseteqq Q$ with $\# R \geqq N+1$, where $\# A$ means the number of elements of a set A. In particular case $N = n$, these are said to be in general position. In [12] E. I. Noachka gave the following theorem:

THEOREM 2.4. *For given hyperplanes $H_1, H_2, \cdots, H_q$ in $P^n(C)$ located in N-subgeneral position, there are some rational numbers $\omega(1), \cdots, \omega(q)$ and θ satisfying the following conditions;*

(i) $0 < \omega(j) \leqq \theta \leqq 1$ $(1 \leqq j \leqq q)$,

(ii) $\sum_{j=1}^{q} \omega(j) = n+1+\theta(q-2N+n-1)$,

(iii) $\dfrac{n+1}{2N-n+1} \leqq \theta \leqq \dfrac{n+1}{N+1}$,

(iv) *if $R \subset Q$ and $0 < \# R \leqq N+1$, then* $\sum_{j \in R} \omega(j) \leqq d(R)$.

For the proof, see [3] or [10, §2.4].

We call constants $\omega(j)$ $(1 \leqq j \leqq q)$ and θ with the properties of Theorem 2.4 Nochka weights and a Nochka constant for $H_1, \cdots, H_q$ respectively.

Related to Nochka weights, we have the following:

PROPOSITION 2.5. *Let $H_1, \cdots, H_q$ be hyperplanes in $P^n(\boldsymbol{C})$ located in N-subgeneral position and let $\omega(1), \cdots, \omega(q)$ be Nochka weights for them, where $q > 2N-n+1$. For each $R \subseteqq Q := \{1, 2, \cdots, q\}$ with $0 < \# R \leqq N+1$ and real constants $E_1, \cdots, E_q$ with $E_j \geqq 1$, there are some $R' \subseteqq R$ such that $\# R' = d(R') = d(R)$ and*

$$\prod_{j \in R} E_j^{\omega(j)} \leqq \prod_{j \in R'} E_j .$$

For the proof, see [3] or [10, Proposition 2.4.15].

For later use, we shall give the following estimate for divisors.

PROPOSITION 2.6. *Let f be a nondegenerate holomorphic map of a domain in $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$ with a reduced representation $f = (f_0 : \cdots : f_n)$ and let $H_1, \cdots, H_q$ be hyperplanes in N-subgeneral position with Nochka weights $\omega(1), \cdots, \omega(q)$ respectively. Then,*

$$\sum_{j=1}^{q} \omega(j) \nu(f, H_j) \leqq \nu_n + \frac{n(n+1)}{2} .$$

Proof. For an arbitrary $a \in M$ set $m_j := \nu(f, H_j)(a)$ and $S := \{j ; m_j > 0\}$. Then $\# S \leqq N$. For, otherwise, f_i $(0 \leqq i \leqq n)$ are represented as linear combinations of $\{F(H_j) ; j \in S\}$ and so $f_0, \cdots, f_n$ have a common zero at a. We choose a set R with $S \subseteqq R \subseteqq \{1, \cdots, q\}$ and $\# R = N+1$. Then we see $d(R) = n+1$ by the assumption. Set $E_j := e^{m_j}$. By Proposition 2.5 there exists some $j_0, j_1, \cdots, j_n$ in R such that $H_{j_0}, \cdots, H_{j_n}$ are linearly independent and $\prod_{j \in R} E_j^{\omega(j)} \leqq \prod_{l=0}^{n} E_{j_l}$, so that

$$\sum_{j \in R} \omega(j) m_j \leqq \sum_{l=0}^{n} m_{j_l} . \tag{2.7}$$

Set $\psi_l := F(H_{j_l})$ and define $\varphi := \dfrac{W(\psi_0, \psi_1, \cdots, \psi_n)}{\psi_0 \psi_1 \cdots \psi_n}$. Since $f_0, f_1, \cdots, f_n$ are represented as linear combinations of $\psi_0, \cdots, \psi_n$, $W(f_0, f_1, \cdots, f_n)$ is a nonzero constant multiple of $W(\psi_0, \psi_1, \cdots, \psi_n)$. This implies that

$$\nu_\varphi(a) = \nu_n(a) - \sum_{l=0}^{n} m_{j_l} . \tag{2.8}$$

On the other hand, the meromorphic function φ is expanded as

$$\varphi=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \dfrac{\psi_0'}{\psi_0} & \dfrac{\psi_1'}{\psi_1} & \cdots & \dfrac{\psi_n'}{\psi_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\psi_0^{(n)}}{\psi_0} & \dfrac{\psi_1^{(n)}}{\psi_1} & \cdots & \dfrac{\psi_n^{(n)}}{\psi_n} \end{vmatrix} = \sum_{\binom{0\cdots n}{i_0\cdots i_n}} \operatorname{sgn}\begin{pmatrix} 0 & \cdots & n \\ i_0 & \cdots & i_n \end{pmatrix} \frac{\psi_{i_1}'}{\psi_{i_1}} \frac{\psi_{i_2}''}{\psi_{i_2}} \cdots \frac{\psi_{i_n}^{(n)}}{\psi_{i_n}}.$$

Since the order of the pole of each $\psi_j^{(l)}/\psi_j$ is at most l, each term in the above expansion of φ has a pole of order at most $n(n+1)/2$ at a. Therefore, we have $\nu_\varphi(a)\geqq -n(n+1)/2$. By the use of (2.7) and (2.8) we conclude

$$\sum_{j=1}^{q} \omega(j)\nu(f, H_j)(a) \leqq \sum_{l=0}^{n} m_{j_l} = \nu_n - \nu_\varphi(a) \leqq \nu_n + \frac{n(n+1)}{2}.$$

The proof of Proposition 2.6 is completed.

§ 3. Pseudo-metrics with strictly negative curvature.

Let M be an open Riemann surface and ds^2 a pseudo-metric on M, namely, a metric on M with isolated singularities which is locally written as $ds^2=\lambda^2|dz|^2$ in terms of a nonnegative real-valued function λ with mild singularities and a holomorphic local coordinate z. We define the divisor of ds^2 by $\nu_{ds}:=\nu_\lambda$ for each local expression $ds^2=\lambda^2|dz|^2$, which is globally well-defined on M. We say that ds^2 is a continuous pseudo-metric if $\nu_{ds}\geqq 0$ everywhere.

DEFINITION 3.1. We define the Ricci form of ds^2 by

$$\operatorname{Ric}[ds^2] := -dd^c[\log \lambda^2]$$

for each local expression $\nu_{ds}:=\nu_\lambda$, where we mean by $[u]$ the current associated with a locally integrable function u and by d and $d^c(:=(\sqrt{-1}/4\pi)(\bar{\partial}-\partial))$ the differential operators for the currents.

Let f, g be distinct nonconstant holomorphic maps of M into $P^N(\boldsymbol{C})$ and consider a nonzero function $\chi(w, \tilde{w})$ on $\boldsymbol{C}^{N+1}\times\boldsymbol{C}^{N+1}$ having the following:

PROPERTY 3.2. *It is bilinear with respect to the variables w and $\tilde{w}$, and $\chi(w, w)=0$ for all $w\in\boldsymbol{C}^{N+1}$.*

Suppose that they satisfy the following:

ASSUMPTION 3.3. *There exist hyperplanes $H_1, \cdots, H_q$ in $P^N(\boldsymbol{C})$ located in general positien such that*

(i) *$\chi(f, g)\not\equiv 0$ and $f^{-1}(H_j)=g^{-1}(H_j)\neq M$ for every j,*

(ii) *$\chi(f, g)=0$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)-K$ for a compact subset K of M.*

Take the smallest projective linear subspaces P_f and P_g of $P^N(\boldsymbol{C})$ which include the images $f(M)$ and $g(M)$ respectively. Set $n_f := \dim P_f$ and $n_g := \dim P_g$. Then the maps $f: M \to P_f$ and $g: M \to P_g$ are nondegenerate. We regard hyperplanes $H_1 \cap P_f, \cdots, H_q \cap P_f$ as hyperplanes in P_f, which are located in N-subgeneral position. We take Nochka weights $\omega_f(1), \cdots, \omega_f(q)$ and a Nochka constant θ_f for these hyperplanes. Similarly, we take $\omega_g(1), \cdots, \omega_g(q)$ and θ_g for the hyperplanes $H_1 \cap P_g, \cdots, H_q \cap P_g$ with respect to the map g. Choose reduced representations $f := (f_0 : \cdots : f_{n_f})$, $g := (g_0 : \cdots : g_{n_g})$ on M in terms of some homogeneous coordinates on P_f and P_g respectively. Set $F := (f_0, \cdots, f_{n_f})$ and $G := (g_0, \cdots, g_{n_g})$ and consider the maps F_k and G_k defined as (2.2) for f and g. Taking unit vectors $A_j \in \boldsymbol{C}^{n_f+1}$, $B_j \in \boldsymbol{C}^{n_g+1}$ with

$$H_j \cap P_f : \langle w, A_j \rangle = 0 \ (w \in P_f), \qquad H_j \cap P_g : \langle \tilde{w}, B_j \rangle = 0 \ (\tilde{w} \in P_g),$$

we define $F(H_j) := \langle F, A_j \rangle$, $G(H_j) := \langle G, B_j \rangle$ and the contact functions $\varphi_k^f(H_j) := |F_k \vee A_j|^2 / |F_k|^2$ for $0 \leqq k \leqq n_f$ and $\varphi_k^g(H_j) := |G_k \vee A_j|^2 / |G_k|^2$ for $0 \leqq k \leqq n_g$, where $X \vee Y$ denotes the interior product of vectors X and Y (c.f., [8, §3]).

Set $\sigma_n := n(n+1)/2$, $\tau_n := \sum_{k=1}^n \sigma_k$ and

$$\gamma_f := \theta_f(q - 2N + n_f - 1), \qquad \gamma_g := \theta_g(q - 2N + n_g - 1). \tag{3.4}$$

Suppose that

$$\frac{\sigma_{n_f}}{\gamma_f} + \frac{\sigma_{n_g}}{\gamma_g} < 1. \tag{3.5}$$

Choose positive numbers $\varepsilon_1, \varepsilon_2$ such that $\gamma_f > \varepsilon_1 \tau_{n_f}$, $\gamma_g > \varepsilon_2 \tau_{n_g}$ and

$$\frac{\sigma_{n_f} + \varepsilon_1 \sigma_{n_f+1}}{\gamma_f - \varepsilon_1 \tau_{n_f}} + \frac{\sigma_{n_g} + \varepsilon_2 \sigma_{n_g+1}}{\gamma_g - \varepsilon_2 \tau_{n_g}} < 1 \tag{3.6}$$

and define the functions

$$\begin{aligned} \eta_f &:= \left(\frac{|F|^{\gamma_f - \varepsilon_1 \sigma_{n_f+1}} |F_{n_f}| \prod_{k=0}^{n_f} |F_k|^{\varepsilon_1}}{\prod_{j=1}^q (|F(H_j)| \prod_{k=0}^{n_f-1} \log(\delta / \varphi_k^f(H_j)))^{\omega_f(j)}} \right)^{1/(\sigma_{n_f} + \varepsilon_1 \tau_{n_f})}, \\ \eta_g &:= \left(\frac{|G|^{\gamma_g - \varepsilon_2 \sigma_{n_g+1}} |G_{n_g}| \prod_{k=0}^{n_g} |G_k|^{\varepsilon_2}}{\prod_{j=1}^q (|G(H_j)| \prod_{k=0}^{n_g-1} \log(\delta / \varphi_k^g(H_j)))^{\omega_g(j)}} \right)^{1/(\sigma_{n_g} + \varepsilon_2 \tau_{n_g})}, \end{aligned} \tag{3.7}$$

where δ is a sufficiently large positive constant which is specified later. As is easily seen, $\eta_f^2 |dz|^2$ and $\eta_g^2 |dz|^2$ are globally well-defined on M. Take p_1 and p_2 with

$$p_1 + p_2 = 1, \qquad p_1 \geqq \frac{\sigma_{n_f} + \varepsilon_1 \tau_{n_f}}{\gamma_f - \varepsilon_1 \sigma_{n_f+1}}, \qquad p_2 \geqq \frac{\sigma_{n_g} + \varepsilon_2 \tau_{n_g}}{\gamma_g - \varepsilon_2 \sigma_{n_g+1}}$$

and define

$$d\tau^2 := \frac{|\chi(F, G)|^2}{|F|^2 |G|^2} \eta_f^{2p_1} \eta_g^{2p_2} |dz|^2. \tag{3.8}$$

DEFINITION 3.9. We say that a continuous pseudo-metric ds^2 has strictly negative curvature on M if there is a positive constant C such that

$$-\mathrm{Ric}\,[ds^2] \geqq C\Omega_{ds^2},$$

where Ω_{ds^2} denotes the area form for ds^2, namely, $\Omega_{ds^2} := \lambda^2(\sqrt{-1}/2)dz\wedge d\bar{z}$ for each local expression $ds^2=\lambda^2|dz|^2$.

PROPOSITION 3.10. *For a sufficiently large δ, the pseudo-metric $d\tau^2$ given by* (3.8) *is continuous and has strictly negative curvature on $M-K$.*

Proof. We first show that $\nu_{d\tau}(z)\geqq 0$ for all $z\in M$. This is obvious if $z\notin \bigcup_{j=1}^q f^{-1}(H_j)(=\bigcup_{j=1}^q g^{-1}(H_j))$. Otherwise, since

$$\nu_{n_f}-\sum_{j=1}^q \omega_f(j)\nu(f, H_j)\geqq -\sigma_{n_f}, \qquad \nu_{n_g}-\sum_{j=1}^q \omega_g(j)\nu(g, H_j)\geqq -\sigma_{n_g}$$

by Proposition 2.6, we obtain

$$\begin{aligned}\nu_{d\tau}&\geqq \nu_\chi+p_1\Big(\nu_{n_f}-\sum_{j=1}^q \omega_f(j)\nu(f, H_j)\Big)\frac{1}{\sigma_{n_f}+\varepsilon_1\tau_{n_f}}\\ &\quad+p_2\Big(\nu_{n_g}-\sum_{j=1}^q \omega_g(j)\nu(g, H_j)\Big)\frac{1}{\sigma_{n_g}+\varepsilon_2\tau_{n_g}}\\ &\geqq \nu_\chi-\frac{p_1\sigma_{n_f}}{\sigma_{n_f}+\varepsilon_1\tau_{n_f}}-\frac{p_2\sigma_{n_g}}{\sigma_{n_g}+\varepsilon_2\tau_{n_g}}\\ &\geqq \nu_\chi-1\,.\end{aligned}$$

On the other hand, by the assumption we have $\chi=0$ on $\bigcup_{j=1}^q f^{-1}(H_j)$ and hence $\nu_\chi\geqq 1$ there. This concludes that $d\tau^2$ is continuous on M.

To complete the proof, we choose a sufficiently large δ so that

$$\begin{aligned}&dd^c\log\eta_f^2\geqq\frac{\gamma_f-\varepsilon_1\sigma_{n_f+1}}{\sigma_{n_f}+\varepsilon_1\tau_{n_f}}\Omega_f+C_1\eta_f^2dd^c|z|^2,\\ (3.11)\qquad &dd^c\log\eta_g^2\geqq\frac{\gamma_g-\varepsilon_2\sigma_{n_g+1}}{\sigma_{n_g}+\varepsilon_2\tau_{n_g}}\Omega_g+C_1\eta_g^2dd^c|z|^2\end{aligned}$$

on $M-(K\cup\bigcup_{j=1}^q f^{-1}(H_j))$ for some positive constants C_1 by the same arguments as in [8, pp. 31$\sim$32], where $\Omega_f := dd^c\log|F|^2$ and $\Omega_g := dd^c\log|G|^2$. We then have

$$\begin{aligned}-\mathrm{Ric}\,[d\tau^2]&=dd^c\log|\chi|^2-\Omega_f-\Omega_g+p_1dd^c\log\eta_f^2+p_2dd^c\log\eta_g^2\\ &\geqq C_1(p_1\eta_f^2+p_2\eta_g^2)dd^c|z|^2\\ &\quad+\Big(p_1\frac{\gamma_f-\varepsilon_1\sigma_{n_f+1}}{\sigma_{n_f}+\varepsilon_1\tau_{n_f}}-1\Big)\Omega_f+\Big(p_2\frac{\gamma_g-\varepsilon_2\sigma_{n_g+1}}{\sigma_{n_g}+\varepsilon_2\tau_{n_g}}-1\Big)\Omega_g\\ &\geqq C_1\eta_f^{2p_1}\eta_g^{2p_2}dd^c|z|^2\,.\end{aligned}$$

Since we have $|\chi|\leqq C_2|F||G|$ for some constant $C_2>0$, we can conclude that $d\tau^2$ has strictly negative curvature. The proof of Proposition 3.10 is completed.

§4. Unicity theorems for holomorphic curves.

Let M be an open Riemann surface and ds^2 a complete conformal metric on M. We now recall the following definition and result given in [8].

DEFINITION 4.1. Let Ω_1 and Ω_2 be C^∞ differentiable (1, 1)-currents on M. For some $c>0$, by $\Omega_1 \prec_c \Omega_2$ we mean that there exist some divisor ν and a bounded continuous nonnegative function k with mild singularities on M such that $\nu\geqq c$ everywhere on Supp(ν) and

$$\Omega_1+[\nu]=\Omega_2+dd^c[\log k^2],$$

where $[\nu]$ denotes the current associated with ν. The notation $\Omega_1\prec\Omega_2$ means that $\Omega_1\prec_c\Omega_2$ for some $c>0$.

THEOREM 4.2. *Let M be an open Riemann surface with a complete conformal metric ds^2. If there exists a continuous pseudo-metric $d\tau^2$ on M whose curvature is strictly negative on $M-K$ for some compact set K such that, for some constant p with $0<p<1$,*

$$-\mathrm{Ric}\,[ds^2]\prec_{1-p} p(-\mathrm{Ric}\,[d\tau^2])$$

on $M-K$, then M is of finite type, namely, biholomorphic with a compact Riemann surface with finitely many points removed.

For the proof, see [8, pp. 24~27].

To state the main result of this section, we give the following:

DEFINITION 4.3. Let $f, g: M\to P^N(\boldsymbol{C})$ be nonconstant holomorphic maps. We say that they satisfy the condition $(C)_{\rho_1,\rho_2}$ for some $\rho_1, \rho_2>0$ if

$$-\mathrm{Ric}\,[ds^2]\prec\rho_1\Omega_f+\rho_2\Omega_g$$

on $M-K$ for some compact set K.

THEOREM 4.4. *Let M be an open Riemann surface with a complete conformal metric ds^2 and f, g nonconstant holomorphic maps of M into $P^N(\boldsymbol{C})$. Suppose that, for some $\rho_1, \rho_2>0$,*

(i) *M is not of finite type,*

(ii) *f and g satisfy the condition $(C)_{\rho_1,\rho_2}$,*

(iii) *$\gamma_f, \gamma_g>0$ for the numbers γ_f, γ_g defined by* (3.4) *and Assumption* 3.3 *holds for some χ with Property* 3.2.

Then, it holds that

$$(\rho_1+1)\frac{\sigma_{n_f}}{\gamma_f}+(\rho_2+1)\frac{\sigma_{n_g}}{\gamma_g}\geqq 1 .$$

Proof. The proof is given by reduction to absurdity. Suppose that the conclusion does not hold. We then have the stituation considered in §3. We use the previous notations without permission. Set

$$A:=\frac{\gamma_f-\varepsilon_1\sigma_{n_f+1}}{\sigma_{n_f}+\varepsilon_1\tau_{n_f}},\qquad B:=\frac{\gamma_g-\varepsilon_2\sigma_{n_g+1}}{\sigma_{n_g}+\varepsilon_2\tau_{n_g}}, \tag{4.5}$$

where ε_1 and ε_2 are chosen so that $\gamma_f>\varepsilon_1\sigma_{nf+1}>0$, $\gamma_g>\varepsilon_2\rho_{n_g+1}>0$ and

$$\frac{\rho_1+1}{A}+\frac{\rho_2+1}{B}<1 . \tag{4.6}$$

Set

$$p_1:=\frac{B\rho_1+\rho_2-\rho_1}{A\rho_2+B\rho_1},\qquad p_2:=\frac{A\rho_2+\rho_1-\rho_2}{A\rho_2+B\rho_1}.$$

Then, we see easily

$$p_1A-1>0,\qquad p_2B-1>0,\qquad p_1+p_2=1 \tag{4.7}$$

and

$$\frac{p_1A-1}{\rho_1}=\frac{p_2B-1}{\rho_2}=\frac{AB-A-B}{A\rho_2+B\rho_1}>1 . \tag{4.8}$$

Using these constants ε_1, ε_2 and a sufficiently large δ, we define the functions η_f and η_g by (3.7) and pseudo-metric $d\tau^2$ by (3.8), which is continuous and has strictly negative on $M-K$ according to Proposition 3.10.

Now, we represent each H_j as

$$H_j:\bar{c}_{j0}w_0+\cdots+\bar{c}_{jN}w_N=0 .$$

As in [8, p. 32], taking some holomorphic local coordinate z, for each j, k $(1\leqq j\leqq q,\ 0\leqq k\leqq n_f-1)$ we choose $i_1,\cdots,i_k$ with $0\leqq i_1<\cdots<i_k\leqq N$ such that

$$\psi_{jk}^z:=\sum_{l\neq i_1,\cdots,i_k}\bar{c}_{jl}W(f_l,f_{i_1},\cdots,f_{i_k})\not\equiv 0 ,$$

where we set $\psi_{j0}^z:=F(H_j)$. We then have $|\psi_{jk}^z|^2/|F_k|^2\leqq\varphi_k^f(H_j)$ and, by the theorem of identity, $\psi_{jk}^\zeta\not\equiv 0$ for every holomorphic local coordinate ζ. Set

$$k_f:=\left(\frac{\prod_{1\leqq j\leqq q,\,0\leqq k\leqq n_f-1}|\psi_{jk}^z|^{\varepsilon_1/q}\log^{\omega_f(j)}(\delta/\varphi_k^f(H_j))}{\prod_{0\leqq k\leqq n_f-1}|F_k|^{\varepsilon_1}}\right)^{1/(\sigma_{n_f}+\varepsilon_1\tau_{n_f})},$$

which is a well-defined function with mild singularities on $M-K$. Since

$$\frac{|\psi_{jk}^z|^{\varepsilon_1/q}\log^{\omega_f(j)}(\delta/\varphi_k^f(H_j))}{|F_k|^{\varepsilon_1/q}}\leqq\sup_{0<x\leqq 1}x^{\varepsilon_1/q}\log^{\omega_f(j)}\left(\frac{\delta}{x^2}\right)<+\infty ,$$

k_f is bounded. Set

$$\phi_1 := \left(\frac{|F_{n_f}| \prod_{j,k} |\psi_{jk}^z|^{\varepsilon_1/q}}{\prod_{j=1}^q |F(H_j)|^{\omega_f(j)}}\right)^{1/(\sigma_{n_f}+\varepsilon_1 \tau_{n_f})}$$

on the domain of each holomorphic local coordinate z. Then, we have $\nu_{\eta_f} \leqq \nu_{\phi_1}$, and $\eta_f k_f = |F|^A \phi_1$, so that

$$dd^c \log (\eta_f k_f)^2 = A\Omega_f + dd^c \log |\phi_1|^2 .$$

Similarly, we can choose a bounded continuous nonnegative function k_g with mild singularities and a locally defined nonzero meromorphic function ϕ_2 satisfying the condition $\nu_{\eta_g} \leqq \nu_{\phi_2}$ such that

$$dd^c \log (\eta_g k_g)^2 = B\Omega_g + dd^c \log |\phi_2|^2 .$$

On the other hand, we have $dd^c \log |\phi_i|^2 = [\nu_{\phi_i}]$ $(i=1, 2)$ by Poincaré-Lelong formula. Therefore, we obtain

$$\begin{aligned} -\mathrm{Ric}\,[d\tau^2] &+ p_1 dd^c \log k_f^2 + p_2 dd^c \log k_g^2 \\ &= [\nu_\chi] - \Omega_f - \Omega_g + p_1 dd^c \log (\eta_f k_f)^2 + p_2 dd^c \log (\eta_g k_g)^2 \\ &= (p_1 A - 1)\Omega_f + (p_2 B - 1)\Omega_g + [\nu_0] . \end{aligned}$$

where $\nu_0 := \nu_\chi + p_1 \nu_{\phi_1} + p_2 \nu_{\phi_2}$. By (4.7) and (4.8) this yields that

$$-\mathrm{Ric}\,[d\tau^2] + dd^2 \log k_f^{2p_1} k_g^{2p_2} = \frac{AB-A-B}{A\rho_2 + B\rho_1}(\rho_1 \Omega_f + \rho_2 \Omega_g) + [\nu_0] . \tag{4.9}$$

We have also the inequality $\nu_0 \geqq \nu_\chi + p_1 \nu_{n_f} + p_2 \nu_{\eta_g} = \nu_{d\tau} \geqq 0$ on $M-K$ and, moreover, there is a positive constant c_0 with $\nu_0 \geqq c_0$ on $\mathrm{Supp}\,(\nu_0)$. Set

$$p := \frac{A\rho_2 + B\rho_1}{AB-A-B}.$$

Then, $0 < p < 1$ by (4.8) and the identity (4.9) can be rewritten as

$$\rho_1 \Omega_f + \rho_2 \Omega_g \prec_{pc_0} p(-\mathrm{Ric}\,[d\tau^2]) .$$

The assumption (ii) yields that

$$-\mathrm{Ric}\,[ds^2] \prec \rho_1 \Omega_f + \rho_2 \Omega_g \prec p(-\mathrm{Ric}\,[d\tau^2]) .$$

We may write this

$$-\mathrm{Ric}\,[ds^2] \prec_{1-p} p(-\mathrm{Ric}\,[d\tau^2]) ,$$

because we can choose ε_1 and ε_2 so that $1-p$ is sufficiently small. This contradicts Theorem 4.2. The proof of Theorem 4.4 is completed.

§ 5. Holomorphic maps defined on a Riemann surface of finite type.

In this section, we give a unicity theorem similar to the previous section for holomorphic maps into $P^N(\boldsymbol{C})$ defined on an open Riemann surface M of finite type which has a complete conformal metric ds^2.

DEFINITION 5.1. Let $f, g: M \to P^N(\boldsymbol{C})$ be nonconstant holomorphic maps. We say that they satisfy the condition $(C)'_{\rho_1, \rho_2}$ for $\rho_1, \rho_2 > 0$ if there is a bounded continuous nonnegative function k with mild singularities such that

$$-\mathrm{Ric}\,[ds^2] \leqq \rho_1 \Omega_f + \rho_2 \Omega_g + dd^c \log k^2 \tag{5.2}$$

on M.

THEOREM 5.3. *In the same situation as in Theorem* 4.4, *suppose that, for some* $\rho_1, \rho_2 > 0$,

(i) *M is of finite type,*

(ii) *f and g satisfy the condition $(C)'_{\rho_1, \rho_2}$,*

(iii) *$\gamma_f, \gamma_g > 0$ and Assumption* 3.3 *holds for $K := \emptyset$ and some χ with Property* 3.2.

Then, it holds that

$$(\rho_1+1)\frac{\sigma_{n_f}}{\gamma_f} + (\rho_2+1)\frac{\sigma_{n_g}}{\gamma_g} \geqq 1.$$

Proof. Assume that the conclusion does not hold. We use the same notations as in the proof of Theorem 4.4, where $K := \emptyset$. Using the same constants p_1, p_2 and functions η_f, η_g as in the proof of Theorem 4.4, we construct a continuous pseudo-metric $d\tau^2$ on M, which has strictly negative curvature on M. Here, we note that the universal covering surface of M is biholomorphic with the unit disc. For, there is no continuous pseudo-metric with strictly negative curvature on a Riemann surface whose universal covering surface is biholomorphic with $\boldsymbol{C}$. By the generalized Schwarz' lemma ([1, pp. 12∼14]), there exists a positive constant C_0 such that

$$d\tau^2 \leqq C_0 d\sigma_M^2,$$

where $d\sigma_M^2$ denotes the Poincaré metric on M. By the assumption, M is biholomorphic with a compact Riemann surface $\overline{M}$ with finitely many points a_l's removed. For each a_l we take a neighborhood U_l of a_l which is biholomorphic with $\Delta^* := \{z; 0 < |z| < 1\}$, where $z(a_l) = 0$. The Poincaré metric on the domain Δ^* is given by

$$d\sigma_{\Delta^*} := \frac{4|dz|^2}{|z|^2 \log^2 |z|^2}.$$

By the use of the distance decreasing property of Poincaré metric we have

$$d\tau^2 \leqq C_l \frac{|dz|^2}{|z|^2 \log^2 |z|^2}$$

for some $C_l > 0$. This implies that, for a neighborhood U_l^* of a_l which is relatively compact in U_l,

$$\int_{U_l^*} \Omega_{d\tau^2} < +\infty .$$

Since $\bar{M}$ is compact, we have

$$\int_M \Omega_{d\tau^2} \leqq \int_{\bar{M} - \cup_l U_l^*} \Omega_{d\tau^2} + \sum_l \int_{U_l^*} \Omega_{d\tau^2} < +\infty . \tag{5.4}$$

We now take a nowhere zero holomorphic form ω on M. Since $dd^c \log \eta_f^2 \geqq A\Omega_f$ and $dd^c \log \eta_g^2 \geqq B\Omega_g$ by (3.11), we can find subharmonic functions v_1 and v_2 such that

$$\eta_f^2 |dz|^2 = e^{v_1} |F|^{2A} |\omega|^2, \quad \eta_g^2 |dz|^2 = e^{v_2} |G|^{2B} |\omega|^2 .$$

Set $v_3 := \log |\chi|^2 + p_1 v_1 + p_2 v_2$ and

$$v := v_3 + (p_1 A - 1 - \rho_1) \log |F|^2 + (p_2 B - 1 - \rho_2) \log |G|^2 ,$$

which is subharmonic by (4.8). We then have

$$d\tau^2 = e^{v_3} |F|^{2(p_1 A - 1)} |G|^{2(p_2 B - 1)} |\omega|^2 = e^v |F|^{2\rho_1} |G|^{2\rho_2} |\omega|^2 . \tag{5.5}$$

Now, take the functions $\lambda > 0$ with $ds^2 = \lambda^2 |\omega|^2$ and k as in Definition 5.1. By (5.2) there is a subharmonic function $w (\not\equiv -\infty)$ such that

$$e^w \lambda^2 = k^2 |F|^{2\rho_1} |G|^{2\rho_2} .$$

Combining this with (5.5), we obtain

$$e^{v+w} ds^2 \leqq C_0 d\tau^2$$

on M for some positive constant C_0. Here, we can apply the result of S. T. Yau in [15] to see

$$\int_M e^{v+w} \Omega_{ds^2} = +\infty ,$$

because M is complete with respect to the metric ds^2 and $v+w$ is subharmonic. This contradicts the assertion (5.4). The proof of Theorem 5.3 is completed.

Related to Theorem 5.3, we shall prove more precise unicity theorem of holomorphic curve for a particular case. To state this, we give the following:

DEFINITION 5.6. Let f be a nonconstant holomorphic map of an open Riemann surface M into $P^N(C)$. We say that f has an essential singularity at an isolated end of M if we can take a not relatively compact connected open subset D of M, called a neighborhood of the end, such that there is a biholomorphic map Φ of $\{z \in C; 0<|z|<2\}$ onto an open neighborhood of $\bar{D}$ satisfying the condition that $\Phi(\partial D)=\{z; |z|=1\}$, $\Phi(D)=\Delta^* := \{z; 0<|z|<1\}$ and the map $f \cdot \Phi: \Delta^* \to P^N(C)$ has an essential singularity at the origin.

THEOREM 5.7. *Let M be an open Riemann surface and f, g nonconstant holomorphic maps of M into $P^N(C)$ at least one of which has an essential singularity at an isolated end of M. Suppose that $\gamma_f, \gamma_g > 0$ and Assumption* 3.3 *holds for some χ with Property* 3.2 *on some neighborhood of the end. Then*

$$\frac{\sigma_{n_f}}{\gamma_f} + \frac{\sigma_{n_g}}{\gamma_g} \geqq 1 .$$

For the proof, we recall the second main theorem in the classical value distribution theory of holomorphic curves. Let f be a nonconstant holomorphic map of an open neighborhood of $\Delta_{s,+\infty} := \{z; s \leqq |z| < +\infty\}$ into $P^N(C)$. The order function of f is defined by

$$T_f(r) := \int_s^r \frac{dt}{t} \int_{s \leqq |z| < t} \Omega_f ,$$

which can be rewritten as

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log |F(re^{i\theta})| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log |F(se^{i\theta})| d\theta + O(\log r) \tag{5.8}$$

with $F := (f_0, \cdots, f_n)$ for a reduced representation $f := (f_0 : \cdots : f_n)$ ([10, Corollary 3.1.12]). For a divisor ν on an open neighborhood of $\Delta_{s,+\infty}$ the counting function of ν is defined by

$$N(r, \nu) := \int_s^r \Big(\sum_{s \leqq |z| \leqq t} \nu(z) \Big) \frac{dt}{t} .$$

Let H be a hyperplane with $f(\Delta_{s,\infty}) \nsubseteq H$. By definition, the counting function of H for f is given by

$$N_f(r, H) := N(r, \nu(f, H)) .$$

According to Jensen's formula, we have

$$N_f(r, H) = \frac{1}{2\pi} \int_0^{2\pi} \log |F(H)(re^{i\theta})| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log |F(H)(se^{i\theta})| d\theta + O(\log r),$$

where $F(H)$ is the function defined by (2.3) ([10, Corollary 3.1.8]).

The second main theorem in value distribution theory is stated as follows:

THEOREM 5.9. *Let $f: \Delta_{s,+\infty} \to P^N(C)$ be a nonconstant holomorphic map and $H_1, \cdots, H_q$ hyperplanes in $P^N(C)$ located in general position such that $f(\Delta_{s,+\infty}) \nsubseteq \bigcup_j H_j$. Consider the least projective linear subspace P_f which includes the image* *of f and set $n_f := \dim P_f$. Take Nochka weights $\omega_f(1), \cdots, \omega_f(q)$ and a Nochka constant θ_f for f considered as a map into P_f. Then,*

$$\gamma_f T_f(r) \leqq \sum_{j=1}^{q} \omega_f(j) N_f(r, H_j) - N(r, \nu_{n_f}) + O(\log(rT_f(r)) \tag{5.10}$$

for all r not including in a set E with $\int_E (1/r)dr < \infty$, where $\gamma_f = \theta_f(q-2N+n_f-1)$.

This is given in [12]. The details of the proof are described in [2] (cf., [10, Theorem 3.2.12]).

Proof of Theorem 5.7. Changing notation if necessary, we may assume that f has an essential singularity at an isolated end of M. Then there is a not relatively compact open subset D of M and a map Φ satisfying the conditions stated in Definition 5.6. By the identity theorem we have only to prove that $f \cdot \Phi \equiv g \cdot \Phi$ on Δ^*. On the other hand, Δ^* is biholomorphically equivalent to $\Delta_{1,+\infty}$. Therefore, there is no harm in assuming that $M = \Delta_{1,+\infty}$ and f has an essential singularity at ∞. By Theorem 5.9 we have (5.10) and, similarly,

$$\gamma_g T_g(r) \leqq \sum_{j=1}^{q} \omega_g(j) N_g(r, H_j) - N(r, \nu_{n_g}) + O(\log(rT_g(r))$$

outside a set E with $\int_E (1/r)dr < \infty$. On the other hand, by Proposition 2.6 we get

$$\sum_{j=1}^{q} \omega_f(j) N_f(r, H_j) - N(r, \nu_{n_f}) \leqq \sigma_{n_f} N(r, \min(\tilde{\nu}_f, 1)),$$

$$\sum_{j=1}^{q} \omega_g(j) N_g(r, H_j) - N(r, \nu_{n_g}) \leqq \sigma_{n_g} N(r, \min(\tilde{\nu}_g, 1)),$$

where $\tilde{\nu}_f := \sum_j \nu(f, H_j)$ and $\tilde{\nu}_g := \sum_j \nu(g, H_j)$. By Assumption 3.3 we have easily

$$N(r, \min(\tilde{\nu}_f, 1)) \leqq N(r, \chi(f, g)), \qquad N(r, \min(\tilde{\nu}_g, 1)) \leqq N(r, \chi(f, g)).$$

By (5.8) and the bilinearity of χ,

$$N(r, \nu_{\chi(f,g)}) \leqq T_f(r) + T_g(r) + O(\log r).$$

These imply that

$$T_f(r) + T_g(r) \leqq \left(\frac{\sigma_{n_f}}{\gamma_f} + \frac{\sigma_{n_g}}{\gamma_g}\right)(T_f(r) + T_g(r)) + O(\log(rT_f(r)T_g(r))). \tag{5.11}$$

On the other hand, since f has an essential singularity at ∞, it holds that

$$\lim_{r \to \infty} \frac{\log r}{T_f(r) + T_g(r)} = 0$$

([10, Proposition 3.3.3]). Dividing both sides of (5.11) by $T_f(r)+T_g(r)$ and tending r to ∞, we can conclude that

$$\frac{\sigma_{n_f}}{\gamma_f}+\frac{\sigma_{n_g}}{\gamma_g}\geqq 1\,.$$

This gives Theorem 5.7.

§6. Unicity theorems for the Gauss maps of complete minimal surfaces.

In this section, we shall give some unicity theorems for the Gauss map of complete minimal surfaces in $\boldsymbol{R}^m$ by applying the results in the previous sections.

Consider two nonflat minimal surfaces

$$x:=(x_1,\cdots,x_m):M\to\boldsymbol{R}^m,\qquad \tilde{x}:=(\tilde{x}_1,\cdots,\tilde{x}_m):\tilde{M}\to\boldsymbol{R}^m$$

and let G and $\tilde{G}$ be the Gauss maps of M and $\tilde{M}$ respectively. Assume that there is a conformal diffeomorphism Φ of M onto $\tilde{M}$. We may regard M and $\tilde{M}$ as Riemann surfaces and Φ as a biholomorphic isomorphism between M and $\tilde{M}$. Consider the maps

$$f:=G:M\to P^N(\boldsymbol{C}),\qquad g:=\tilde{G}\cdot\Phi:M\to P^N(\boldsymbol{C})\,,$$

where $N:=m-1$, and holomorphic forms $\omega_i:=\partial x_i$ on M and $\tilde{\omega}_i:=\partial\tilde{x}_i$ on $\tilde{M}$. As is well-known, the Gauss maps of M and $\tilde{M}$ are represented as

$$G=(\omega_1:\cdots:\omega_m),\qquad \tilde{G}=(\tilde{\omega}_1:\cdots:\tilde{\omega}_m)\,, \tag{6.1}$$

and the induced metrics of M and $\tilde{M}$ are given by

$$ds^2=2(|\omega_1|^2+\cdots+|\omega_m|^2),\qquad d\tilde{s}^2=2(|\tilde{\omega}_1|^2+\cdots+|\tilde{\omega}_m|^2) \tag{6.2}$$

respectively (e. g., [13]). We can easily obtain the following:

$$-\mathrm{Ric}\,[ds^2]=-\mathrm{Ric}\,[\Phi^*(d\tilde{s}^2)]=\Omega_f=\Omega_g\,. \tag{6.3}$$

Therefore, f and g satisfy the condition $(C)_{1/2,1/2}$, where we may take $K=\emptyset$ in Definition 4.3 and hence the condition $(C)'_{1/2,1/2}$ is also satisfied.

THEOREM 6.4. *In the above situation, assume that $x:M\to\boldsymbol{R}^m$ is complete and has infinite total curvature and that f and g satisfy Assumption* 1.1. *If $q>m^2+m(m-1)/2$, then $f\equiv g$.*

For a paricular case where f has an essential singularity at an isolated end, the same conclusion holds under the only assumption $q>m^2$.

Proof. Assume that $f\not\equiv g$ under the assumption of the first part of Theorem 6.4. For reduced representations $f=(f_1:\cdots:f_m)$ and $g=(g_1:\cdots:g_m)$, the function

$$\chi((w_1, \cdots, w_m); (\tilde{w}_1, \cdots, \tilde{w}_m)) := w_i \tilde{w}_j - w_j \tilde{w}_i$$

satisfies Assumption 3.3 for some distinct indices i, j. Setting $\rho_1 = \rho_2 := 1/2$, we apply Theorem 4.4 to see

$$\frac{n_f(n_f+1)}{2\theta_f(q-N+n_f-1)} + \frac{n_g(n_g+1)}{2\theta_g(q-N+n_g-1)} \geqq \frac{2}{3}.$$

On the other hand, by Theorem 2.4 (iii) we see

$$\frac{n_f(n_f+1)}{\theta_f(q-2N+n_f-1)} \leqq \frac{n_f(2N-n_f+1)}{q-2N+n_f-1}$$

and, since $1 \leqq n_f \leqq N$ and $q > (N+1)^2$, we have

$$\frac{N(N+1)}{q-N-1} - \frac{n_f(2N-n_f+1)}{q-2N+n_f-1}$$
$$= \frac{(N-n_f)((N-n_f+1)q-(N+1)(2N-n_f+1))}{(q-N-1)(q-2N+n_f-1)} \geqq 0.$$

These are also true if θ_f and n_f are replaced by θ_g and n_g respectively. Thus, we obtain

$$\frac{n_f(n_f+1)}{\theta_f(q-2N+n_f-1)} \leqq \frac{N(N+1)}{q-N-1}, \qquad \frac{n_g(n_g+1)}{\theta_g(q-2N+n_g-1)} \leqq \frac{N(N+1)}{q-N-1}. \tag{6.5}$$

and so

$$\frac{N(N+1)}{q-N-1} \geqq \frac{2}{3}.$$

This leads to an absurd conclusion

$$q \leqq \frac{3}{2} N(N+1) + N + 1 = m^2 + \frac{m(m-1)}{2}.$$

The first part of Theorem 6.4 is completely proved.

The proof of the latter half is also given by reduction to absurdity. Suppose that $f \not\equiv g$, and take some χ with Property 3.2 which satisfies Assumption 3.3. By assumption, f has an essential singularity at an isolated end and $q > (N+1)^2$. Then, since $\gamma_f, \gamma_g > 0$ for the constants γ_f and γ_g defined by (3.4), we can apply Theorem 5.7 to get

$$\frac{n_f(n_f+1)}{2\theta_f(q-N+n_f-1)} + \frac{n_g(n_g+1)}{2\theta_g(q-N+n_g-1)} \geqq 1. \tag{6.6}$$

The assertions (6.5) remains valid in this case too. By (6.6) we conclude

$$\frac{N(N+1)}{q-N-1} \geqq 1.$$

This contradicts the assumption for q. The proof of Theorem 6.4 is completed.

THEOREM 6.7. *In the same situation as in Theorem* 6.4, *suppose that* M, $\tilde{M}$ *are both complete and have finite total curvature. If* f *and* g *satisfy Assumption* 1.1 *for* $K=\emptyset$ *and* $q \geqq m^2+m(m-1)/2$, *then* $f \equiv g$.

Proof. Assume that $f \not\equiv g$ and take some χ with Property 3.2 which satisfies Assumption 3.3. For our purpose, we may replace $\tilde{x}: \tilde{M} \to \boldsymbol{R}^m$ by $\tilde{x} \cdot \Phi: M \to \boldsymbol{R}^m$ and so we may assume that $\tilde{M}=M$ and Φ is the identity map. By Chern-Osserman's theorem ([4]), we may set $M=\overline{M}-\{a_1, \cdots, a_K\}$ for a compact Riemann surface $\overline{M}$ and the forms $\omega_i := \partial x_i$, $\tilde{\omega}_i := \partial \tilde{x}_i$ $(1 \leqq i \leqq m)$ are meromorphically extended to $\overline{M}$. The induced metrics ds^2, $d\tilde{s}^2$ are also extended to $\overline{M}$ as pseudo-metrics. We consider the numbers n_f, n_g, $\omega_f(j)$'s, $\omega_g(j)$'s, θ_f, θ_g, γ_f and γ_g defined in § 3. The assertion (6.5) holds in this case too. The assumption implies that

$$\frac{\sigma_f}{\gamma_f} \leqq \frac{N(N+1)}{2(q-N-1)} \leqq \frac{1}{3}, \quad \frac{\sigma_g}{\gamma_g} \leqq \frac{N(N+1)}{2(q-N-1)} \leqq \frac{1}{3}.$$

Choose sufficiently small positive rational numbers ε_1 and ε_2 such that the constants A and B defined by (4.5) are both larger than $3-\varepsilon$ for an arbitrarily pre-assigned positive number ε. Settign $p_1=p_2=1/2$, we define the pseudo-metric $d\tau^2$ by (3.8), whose curvature is strictly negative on M.

Now, we choose a nonzero vector $(c_1, \cdots, c_m)$ such that

$$f(M) \cup g(M) \not\subseteq H_0 := \{(w_1 : \cdots : w_m);\ \bar{c}_1 w_1 + \cdots + \bar{c}_m w_m = 0\},$$

$$\omega := \bar{c}_1 \omega_1 + \cdots + \bar{c}_m \omega_m \not\equiv 0, \tag{6.8}$$

$$\nu_\omega(a_l) = \nu_{ds}(a_l)), \qquad \nu_\omega(a_l) = \nu_{d\tilde{s}}(a_l) \quad (1 \leqq l \leqq K).$$

Choose the functions ϕ_{jk} for f as in § 4 and $\tilde{\phi}_{jk}$ similarly for g and set

$$\eta_f^* := \left(\frac{|F(H_0)|^{\gamma_f - \varepsilon_1 \sigma_{n_f} + 1} |F_{n_f}| \prod_{j,k} |\phi_{jk}|^{\varepsilon_1/q}}{\prod_{j=1}^q |F(H_j)|^{\omega_f(j)}} \right)^{1/(\sigma_{n_f} + \varepsilon_1 \tau_{n_f})},$$

$$\eta_g^* := \left(\frac{|G(H_0)|^{\gamma_g - \varepsilon_2 \sigma_{n_g} + 1} |G_{n_g}| \prod_{j,k} |\tilde{\phi}_{jk}|^{\varepsilon_2/q}}{\prod_{j=1}^q |G(H_j)|^{\omega_g(j)}} \right)^{1/(\sigma_{n_g} + \varepsilon_2 \tau_{n_g})}.$$

We define a new pseudo-metric by

$$d\phi^2 := \frac{|\chi|^2}{|F(H_0)|^2 |G(H_0)|^2} \eta_f^* \eta_g^* |dz|^2.$$

Here, we may assume that all exponents appearing in the above are rational numbers. As is easily seen, $d\phi^2$ is a well-defined pseudo-metric on $\overline{M}$. On the other hand, for an arbitrary s-ple meromorphic form φ on M it holds that $\sum_{p \in \overline{M}} \nu_{d\varphi}(p) = s(2\gamma-2)$. For a sufficiently large integer s, we can find an s-ple meromorphic form $\varphi = \varphi_z (dz)^s$ such that $d\phi^2 = |\varphi_z|^{2/s} |dz|^2$ for each holomorphic local coordinate z. It follows that

$$\sum_{p\in\bar{M}}\nu_{d\phi}(p)=2\gamma-2\,, \tag{6.9}$$

where γ denotes the genus of $\bar{M}$. If we take a nonzero holomorphic function g with $\nu_g=\min\{\nu_{\omega_i};\ 1\leqq i\leqq m\}$ in a neighborhood of each $a\in\bar{M}$, we have

$$\nu_\omega=\nu_g+\nu_{\omega/g}=\nu_{ds}+\nu(f,\ H_0)$$

on $\bar{M}$. We define the degree of f by

$$d_f:=\sum_{z\in H_0}\nu(f,\ H_0)(z)\,,$$

which does not depend on the choice of the hyperplane H_0. We then have

$$2\gamma-2=\sum_{z\in\bar{M}}\nu_\omega(z)=\sum_{z\in\bar{M}}\nu_{ds}(z)+d_f=\sum_{l=1}^{K}\nu_{ds}(p_l)+d_f\,, \tag{6.10}$$

where we used the fact that $\nu_{ds}\equiv 0$ on M. Similarly, we get

$$2\gamma-2=\sum_{l=1}^{K}\nu_{d\tilde{s}}(a_l)+d_g\,. \tag{6.11}$$

Comparing the definition of $d\phi^2$ and $d\tau^2$, we have

$$\nu_{d\phi}=\nu_{d\tau}+\Big(\frac{A}{2}-1\Big)\nu_{F(H_0)}+\Big(\frac{B}{2}-1\Big)\nu_{G(H_0)}+\nu_0\,,$$

where

$$\nu_0:=\frac{\varepsilon_1}{2(\sigma_{n_f}+\varepsilon\tau_{n_f})}\Big(\frac{1}{q}\sum_{j,k}\nu_{\phi_{jk}^z}-\sum_{0\leq k\leq n_f}\nu_{|F_k|}\Big)$$
$$+\frac{\varepsilon_2}{2(\sigma_{n_g}+\varepsilon\tau_{n_g})}\Big(\frac{1}{q}\sum_{j,k}\nu_{\tilde{\phi}_{jk}^z}-\sum_{0\leq k\leq n_g}\nu_{|G_k|}\Big)\geqq 0\,.$$

Since $\nu_{d\tau}\geqq 0$ on M by Proposition 3.10, we have by the use of (6.9)

$$\begin{aligned}\sum_{l=1}^{K}\nu_{d\tau}(a_l)&=\sum_{z\in\bar{M}}\nu_{d\tau}(z)-\sum_{z\in M}\nu_{d\tau}(z)\\ &\leqq\sum_{z\in\bar{M}}\nu_{d\phi}(z)-\sum_{z\in\bar{M}}\Big(\Big(\frac{A}{2}-1\Big)\nu_{F(H_0)}(z)+\Big(\frac{B}{2}-1\Big)\nu_{G(H_0)}(z)\Big)\\ &=2\gamma-2-\Big(\frac{A}{2}-1\Big)d_f-\Big(\frac{B}{2}-1\Big)d_g\,.\end{aligned}$$

Since

$$2\gamma-2=\frac{1}{2}\sum_{l=1}^{K}(\nu_{ds}(a_l)+\nu_{d\tilde{s}}(a_l))+\frac{d_f+d_g}{2}$$

by (6.10) and (6.11), we obtain

$$\sum_{l=1}^{K}\nu_{d\tau}(a_l)\leqq\frac{1}{2}\Big(\sum_{l=1}^{K}\nu_{ds}(a_l)+\nu_{d\tilde{s}}(a_l)\Big)-\Big(\frac{A}{2}-\frac{3}{2}\Big)d_f-\Big(\frac{B}{2}-\frac{3}{2}\Big)d_g .$$

On the other hand, according to Chern-Osserman's theorem ([4, Lemma 2]),

$$\nu_{ds}(p_l)\leqq-2,\qquad \nu_{d\tilde{s}}(p_l)\leqq-2\quad(1\leqq l\leqq K). \tag{6.12}$$

This gives

$$\sum_{l=1}^{K}\nu_{d\tau}(a_l)\leqq-2K-\frac{1}{2}((A-3)d_f+(B-3)d_g).$$

Here, if we choose a sufficiently small positive ε_1, ε_2 and ε, then every term of the right hand side except the first may be assumed to be smaller than an arbitrarily small pre-assigned positive number. This implies that there is some l_0 with $\nu_{d\tau}(a_{l_0})<-1$. It follows that

$$\int_M d\Omega_{d\tau^2}=+\infty .$$

On the other hand, by the same argument as in the proof of Theorem 5.3 we can show $\int_M d\Omega_{d\tau^2}<\infty$ because (3.6) holds in this case too. This is a contradiction. We have Theorem 6.7.

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