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CHARACTERIZATIONS OF BLOCH FUNCTIONS ON THE UNIT BALL OF C^n

Dedicated to my father on his 60th birthday

BY Lou ZENGJIAN

Abstract

We give some new characterizations of Bloch functions on the unit ball in C^n. This extends a theorem of S. B. Lee.

1. Introduction.

The properties and characterizations of Bloch function on the unit ball have been studied in [1, 2, 3, 4]. In this paper we give some new characterizations of Bloch functions, i. e. we give several equivalent conditions for a function to be a Bloch function.

Before we state our main theorem, we fix some notations and definitions used in this paper.

Let C^n denote the n-dimensional vector space. Let B_n denote the open unit ball in C^n with boundary dB_n and let a denote the rotation-invariant positive measure on ∂B_n for which $\sigma(\partial B_n)=1$. Let U^n denote the unit polydisk in C^n, $A(U^n)$ denote the space of all functions which holomorphic in U^n and continue on $\bar{U}^n$.

Throughout the paper, all the functions we consider are supposed to be holomorphic in B_n.

For a function / holomorphic in B_n, let $(R^\beta f)(z)=\sum_{\alpha\geq 0}|\alpha|^\beta a_\alpha z^\alpha$ denote the radial derivative of $f(z)=\sum_{\alpha\geq 0}a_\alpha z^\alpha$ and $(D^\beta f)(z)=\sum_{\alpha\geq 0}(|\alpha|+1)^\beta a_\alpha z^\alpha$ the fractional derivative of $f(\beta>0)$. For $0<p<\infty$, we set

$$M_p(r, f)=\left(1/V\int_{\partial B_n}|f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p}$$

where V is the Euclidean volume of ∂B_n.

DEFINITION 1.1. A holomorphic function $f: B_n\to C$ is said to be in $H^p(B_n)$

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$(0<p<\infty)$ if

$$\|f\|_{H^p(B_n)}=\lim_{r\to 1} M_p(r, f)<\infty .$$

DEFINITION 1.2. A function / is said to be in $G^p(B_n)$ $(0<p<\infty)$ if

$$\|f\|_{G^p(B_n)}=\left(\int_0^1 M_1(r, D^1 f)^p dr\right)^{1/p}<\infty .$$

DEFINITION 1.3. Let $f:\Omega\to C^n$ be an analytic function on the bounded homogeneous domain Ω in C^n. For $z\in\Omega$, we set

$$Q_f(z)=\sup\left\{\frac{|\langle(\nabla f)(z), \bar{w}\rangle|}{H_s(w, \bar{w})^{1/2}} : 0\neq w\in C^n\right\}$$

where $(\nabla f)(z)=(\partial f/\partial z_1(z),\cdots, \partial f/\partial z_n(z))$ and $H_s(w, \bar{w})$ denotes the Bergman metric on Ω and $\langle u, \bar{v}\rangle$ means $\sum_{i=1}^n u_i v_i$. A holomorphic function on Ω is called a Bloch function if

$$\sup\{Q_f(z): z\in\Omega\}<\infty .$$

The space $B(\Omega)$ of Bloch functions on Ω forms a Banach space with the Bloch norm ([1])

$$\|f\|_{B(\Omega)}=|f(0)|+\sup\{Q_f(z): z\in\Omega\} .$$

Let $\Omega=U^n$. Then $f\in B(U^n)$ if and only if ([1])

$$\sup\left\{\left|\frac{\partial f}{\partial z_j}(z)\right|(1-|z_j|^2): z\in U^n\right\}<\infty , \qquad 1\leqq j\leqq n . \tag{1}$$

Let $\Omega=B_n$. Then $f\in B(B_n)$ if and only if ([1])

$$\sup\{|(\nabla f)(z)|(1-|z|): z\in B_n\}<\infty . \tag{2}$$

The main result of this paper is the following

Theorem. Let g be a holomorphic function defined on B_n. Then the following conditions are all equivalent.

© The function g is a Bloch function, *i.e.* $g\in B(B_n)$

® $f*g\in A(U^n)$ for all $f\in G^1(B_n)$

© $f*g\in B(U^n)$ for all $f\in G^1(B_n)$

® $f*g\in B(U^n)$ for all $f\in H^1(B_n)$

where $(f*g)(z)=\sum_{\alpha\geqq 0} a_\alpha b_\alpha \omega_\alpha z^\alpha$ is the Hadamard product of $f(z)-\sum_{\alpha\geqq 0} a_\alpha z^\alpha$ and $g(z)=\sum_{\alpha\geqq 0} b_\alpha z^\alpha$, and

$$\omega_\alpha=\int_{\partial B_n} |\zeta^\alpha|^2 d\sigma(\zeta)=\frac{(n-1)!\,\alpha!}{(n-1+|\alpha|)!}$$

Now let X and Y be two holomorphic function spaces. We let (X, Y) denote the collection of all multipliers from X to Y. That is, (X, Y) is the set of all holomorphic function g such that for every $f\in X$, $f*g\in Y$.

From Theorem we have

COROLLARY. $B(B_n)=(G^1(B_n),A(U^n))=(G^1(B_n),\ B(U^n))$
$=(H^1(B_n),\ B(U^n))$

This Corollary extends a theorem of S. B. Lee ([7, Theorem 3.6]).

2. Proof of Theorem.

At first, it is easy to see that the defining condition (2) of Bloch functions is equivalent to either of the following two conditions

$$\sup\{(1-|z|)|(R^1f)(z)|:\ z\in B_n\}<\infty$$

or

$$\text{srp}\{(1-|z|)|(D^1f)(z)|:\ z\in B_n\}<\infty \tag{3}$$

We prove the theorem proving the implications ①→②→③→①, ①- ›④ and ④→③. From [5, Theorem 1(b)] we have $G^1(B_n)\subset H^1(B_n)$. Moreover the proper inclusion $A(U^n)\subset B(U^n)$ are well known. So the implications ②→③ and ④→③ are obvious.

Proof of ①→②. Suppose $g(z)=\sum_{\alpha\geq 0}x_\alpha z^\alpha\in B(B_n)$ and let $f(z)=\sum_{\alpha\geq 0}a_\alpha z^\alpha \in G^1(B_n),\ z\in U^n$, we have

$$\int_{\partial B_n} D^1f(\rho\zeta)D^1g(\rho z\zeta)d\sigma(\zeta)\sum_{n=0}(n+1)^2\rho^{2n}\sum_{\alpha\geq 0}a_\alpha x_\alpha\omega_\alpha z^\alpha$$

where $z\bar{\zeta}=(z_1\bar{\zeta}_1,\ \cdots,\ z_n\bar{\zeta}_n)$ Since $\int_0^1\rho^{2n+1}\log(1/\rho)d\rho=1/4(n+1)^2$, we have

$$4\int_0^1\rho\log(1/\rho)\int_{\partial B_n} D^1f(\rho\zeta)D^1g(\rho z\bar{\zeta})d\sigma(\zeta)d\rho\sum_{\alpha\geq 0} a_\alpha x_\alpha\omega_\alpha z^\alpha=(f*g)(z)$$

By the inequality $\rho\log(1/\rho)\leqq 1-\rho,\ 0<\rho\leqq 1$, we have

$$|(f*g)(z)|\leqq 4\int_0^1\int_{\partial B_n}(1-\rho)|D^1f(\rho\zeta)||D^1g(\rho z\bar{\zeta})|d\sigma(\zeta)d\rho$$

$$\leqq 4\|g\|_{B(B_n)}\int_0^1 M_1(\rho,\ D^1f)d\rho$$

$$\leqq 4\|g\|_{B(B_n)}\|f\|_{G^1(B_n)}$$

For $f\in G^1(B_n)$, $z\in\bar{U}^n$, set $f_z(\zeta)=f(z\zeta)$. Then, since the correspondence $\bar{U}^n\ni z\to f_z\in G^1(B_n)$ is continuous, we have

$$|f*g(z_1)-f*g(z_2)|=|(f_{z_1}-f_{z_2})*g(e)|$$

$$\leqq C\|f_{z_1}-f_{z_2}\|_{G^1(B_n)}\|g\|_{B(B_n)}$$

$$\longrightarrow 0 \qquad \text{as} \quad |z_1-z_2|\longrightarrow 0$$

Hence $f*g\in A(U^n)$ for all $f\in G^1(B_n)$.

Proof of ③→①. For $g(z)=\sum_{\alpha\geqq 0}x_\alpha z^\alpha$ we define a linear operator T_g: $G^1(B_n)\to B(U^n)$ by $T_g(f)=f*g$. Then T_g is clearly closed, so T_g is a bounded linear operator from $G^1(B_n)$ to $B(U^n)$. Let

$$f(z)=\frac{(1-r^2)^n}{(1-r\langle z,\bar{\zeta}\rangle)^{2n}},\qquad 0\leqq r<1,\ \zeta\in\partial B_n$$

and

$$\Psi(z)=\sum_{\alpha\geqq 0}\frac{\Gamma(|\alpha|+2n)}{\Gamma(|\alpha|+n)}x_\alpha z^\alpha.$$

Then we have

$$f(z)=\sum_{\alpha\geqq 0}\frac{\Gamma(|\alpha|+2n)}{\Gamma(2n)\alpha!}\zeta^\alpha z^\alpha r^{|\alpha|}(1-r^2)^n,$$

and

$$f*g(re)=\frac{\Gamma(n)}{\Gamma(2n)}(1-r^2)^n\Psi(r^2\zeta),$$

where $e=(1,\cdots,1)$. Since $f\in G^1(B_n)$ and T_g is bounded, there is a constant C independent of r such that

$$\|f*g\|_{B(U^n)}\leqq C\|f\|_{G^1(B_n)}=O(1)$$

So from (1) we have

$$|(R^1\Psi)(r^2\zeta)|=O(1-r^2)^{-(n+1)} \tag{4}$$

Let

$$F_k(\zeta)=\sum_{|\alpha|=k}x_\alpha z^\alpha$$

From (4) we obtain

$$\left|\sum_{k=0}^{\infty}\frac{k\Gamma(k+2n)}{\Gamma(k+n)}F_k(\zeta)r^k\right|=O(1-r)^{-(n+1)} \tag{5}$$

By [6, Lemma 1] we have

$$\sum_{k=0}^{\infty}\frac{\Gamma(k+2n)}{\Gamma(k+n)}|F_k(\zeta)|r^k=O(1-r)^{-(n+1/2)} \tag{6}$$

So by the Stirling formula together with (5) and (6) we have

$$\begin{aligned}|(D^{n+1}g)(r\zeta)|&=\left|\sum_{k=0}^{\infty}(k+1)^{n+1}F_k(\zeta)r^k\right|\\&\leqq\left|\sum_{k=0}^{\infty}\left\{(k+1)^{n+1}\frac{\Gamma(k+n)}{k\Gamma(k+2n)}-1\right\}\frac{k\Gamma(k+2n)}{\Gamma(k+n)}F_k(\zeta)r^k\right|\\&\quad+\left|\sum_{k=0}^{\infty}\frac{k\Gamma(k+2n)}{\Gamma(k+n)}F_k(\zeta)r^k\right|\\&=O(1-r)^{-(n+1)}\end{aligned}$$

Hence

$$|D^1 g(r\zeta)| = O(1-r)^{-1}$$

From (3) we have $g \in B(B_n)$.

Proof of ①→④. Let $g \in B(B_n)$, Let / be in $H^1(B_n)$ and $z \in U^n$. Then

$$f * g(z) = \int_{\partial B_n} f(r\zeta) g(r^{-1} z \bar{\zeta}) d\sigma(\zeta),$$

where $\max\{|z_j|: 1 \leqq j \leqq n\} < r < 1$. For $\zeta \in \partial B_n$, by [1, Th. 3.4] $G(z) = g(z\bar{\zeta}) \in B(U^n)$, for $j = 1, \cdots, n$ and $z \in U^n$, we have

$$(1 - |z_j|^2) \left| \frac{\partial (f*g)}{\partial z_j}(z) \right| \leqq \int_{\partial B_n} |f(\zeta)| (1 - |z_j|^2) \left| \frac{\partial G}{\partial z_j}(z) \right| d\sigma(\zeta)$$
$$\leqq C \|f\|_{H^1(B_n)} \|G\|_{B(U^n)}$$

Hence we have $f * g \in B(U^n)$. This completes the proof of Theorem.

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