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BOUNDED ANALYTIC FUNCTIONS AND METRICS OF CONSTANT CURVATURE ON RIEMANN SURFACES

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1. Introduction.

Let $\boldsymbol{B}(\Omega)$ be the set of bounded analytic functions $f: \Omega \to \Delta$, where Ω is a simply connected hyperbolic Riemann surface and Δ is the unit disc. Let $\boldsymbol{B}_0(\Omega)$ be the set of locally schlicht functions belonging to $\boldsymbol{B}(\Omega)$. The Poincaré metric λ_Ω of the surface Ω has constant curvature $\equiv -4$. Explicitly, we have $\lambda_\Delta(z) = |dz|/(1-|z|^2)$. It is well known that the pull-back $f^*\lambda_\Delta(z) = |f'(z)||dz|/(1-|f(z)|^2)$ of λ_Δ via $f \in \boldsymbol{B}_0(\Omega)$ (resp. $f \in B(\Omega)$) is a metric of constant curvature $\equiv -4$ regular (resp. with isolated singularities) on Ω. The main result of this paper asserts that the converse of the above relation holds. For simplicity, we denote by $M(X)$ the set of C^∞ conformal metrics of constant curvature $\equiv -4$ on a Riemann surface X.

THEOREM 1. *For all $\lambda \in M(\Omega)$, there exists an $f \in \boldsymbol{B}_0(\Omega)$ such that $\lambda = f^*\lambda_\Delta$. Moreover, $\lambda = g^*\lambda_\Delta$ for $g \in \boldsymbol{B}_0(\Omega)$ if and only if g is of the form $g = \gamma \circ f$ where $\gamma \in$ Möb(Δ), the set of Möbius transformations leaving Δ fixed.*

THEOREM 2. *Let E be an arbitrary closed discrete subset of Ω. If $\lambda \in M(\Omega \setminus E)$ has non-positive integral indices at every point in E, then there exists an $f \in \boldsymbol{B}(\Omega)$ such that $\lambda = f^*\lambda_\Delta$ on $\Omega \setminus E$. Moreover, $\lambda = g^*\lambda_\Delta$ on $\Omega \setminus E$ if and only if g is of the form $g = \gamma \circ f$ with $\gamma \in$ Möb(Δ).*

Theorems 1 and 2 show that the set of metrics of constant negative curvature is in a one-to-one correspondence with the set of bounded analytic functions in Ω modulo Möb(Δ). Theorem 2 is an improvement of Theorem 29.1 in Heins [2] and many of the results concerning metrics of constant curvature in [2] are easy consequence of Theorem 1. Also, Theorem 1 allows us to define the monodromy homomorphism χ of a metric of constant curvature. Theorem 6 answers the question when the image of the homomorphism χ acts discontinuously on Δ. In the last section, we prove a theorem which shows that a theorem in [5] is false.

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2. Proof of theorem 1.

The following Lemma is useful.

LEMMA 3. *Let $f: N \to \Omega$ be holomorphic where N is a connected neighborhood of a simply connected hyperbolic Riemann surface Ω. Assume that, for $z \in N$, f satisfies the equality $f^*\lambda_\Omega(z) = \lambda_\Omega(z)$. Then f is a restriction of a conformal automorphism of Ω.*

Proof. By conformal invariance, we may assume that Ω is the unit disc. But in this case Lemma 3 is well known. See [2, p. 39]. ■

LEMMA 4. *Let X and Y be surfaces of constant curvature κ. Then the surfaces X and Y are locally isometric.*

Proof. This is well known. See [3, p. 169]. ■

Proof of theorem 1. By conformal invariance, we may assume $\Omega = \Delta$. Applying Lemma 4 to the unit discs equipped with the metric $\lambda \in M(\Delta)$ and the Poincaré metric λ_Δ, we have an open covering $\{U_\alpha\}_{\alpha \in \Delta}$ and a set of conformal (or anti-conformal) mappings $f_\alpha: U_\alpha \to \Delta$ $(\alpha \in \Delta)$ with $\lambda | U_\alpha = f_\alpha^* \lambda_\Delta$. By taking $\bar{f}_\alpha$, if f_α is anti-conformal, we may assume that each f_α is conformal. Lemma 3 implies that there exists a set $\{\gamma_{\alpha\beta}\} \in \text{Möb}(\Delta)$ such that $f_\alpha(z) = \gamma_{\alpha\beta} \circ f_\beta(z)$ for $z \in U_\alpha \cap U_\beta$, $\alpha, \beta \in \Delta$. Thus $f_\alpha(z)$ can be analytically continued along all paths contained in Δ. Since Δ is simply connected, the monodromy theorem implies that there exists an $f \in \boldsymbol{B}_0(\Omega)$ satisfying $\lambda = f^*\lambda_\Delta$. The latter half of Theorem 1 is clear from Lemma 3. This completes the proof. ■

3. Monodromy homomorphisms inducd by metrics.

Let X be a hyperbolic Riemann surface. The Poincaré metric λ_X on X is defined by requiring the identity

$$\lambda_\Delta = \pi^* \lambda_X$$

where $\pi: \Delta \to X$ is a holomorphic universal covering. Conversely, we note that if $\lambda \in M(X)$ satisfies the identity $\lambda_\Delta = \pi^*\lambda$ with $\pi: \Delta \to X$ holomorphic, then λ is the Poincaré metric because we have $\lambda = \lambda_X$ from the inequality

$$\lambda_\Delta = \pi^*\lambda \leqq \pi^*\lambda_X \leqq \lambda_\Delta .$$

This observation leads us to the following definition. Let E be a (possibly empty) closed discrete subset of X. We call $\lambda \in M(X \setminus E)$ a *branched Poincaré metric* on X with singularity on E if λ satisfies

$$\lambda_\Delta = \pi^*\lambda \quad \text{on} \quad \pi^{-1}(X \setminus E) \tag{3.1}$$

for some holomorphic map $\pi: \Delta \to X$.

LEMMA 5. *Let λ be a branched Poincaré metric on X with singularity on E satisfying $\lambda_\Delta = \pi^* \lambda$. Then $\pi: \Delta \to X$ is a normal branched covering whose branch points are contained in E. (Here, we use the word "normal branched covering" to mean that the restriction $\pi | \pi^{-1}(X \setminus E)$ is a normal regular covering.* [1, p. 38])

Conversely, every holomorphic normal branched covering $\pi: \Delta \to X$ yields a branched Poincaré metric λ on X with $\lambda_\Delta = \pi^* \lambda$.

Proof. First, assume that λ is a branched Poincaré metric. Let G be the group $\{\gamma \in \text{Möb}(\Delta) | \pi \circ \gamma = \pi \text{ on } \Delta\}$. It is clear that G is discontinuous on Δ since π is non-constant. We claim that $\pi(x) = \pi(y)$ for $x, y \in \Delta$ if and only if there exists a $\gamma \in G$ with $y = \gamma(x)$. Observe that by (3.1) the orders of the derivative π' at x and y are the same. Hence we can solve, at least locally, the equation $\pi \circ \gamma = \pi$ for γ where γ is holomorphic near x and satisfies $y = \gamma(x)$. Thus,

$$\lambda_\Delta(z) = \pi^* \lambda(z) = (\pi \circ \gamma)^* \lambda(z) = \gamma^* \pi^* \lambda(z) = \gamma^* \lambda_\Delta(z)$$

for z in a neighborhood of x. Then it follows from Lemma 3 that γ is a restriction of a Möbius transformation fixing Δ. By analytic continuation we conclude that $\gamma \in G$, proving the claim.

We next show that $\pi(\Delta)$ contains $X \setminus E$. Otherwise, it follows from the theory of S-K metrics due to Heins [2] that the upper envelope η of the Perron family of metrics on $\pi(\Delta) \setminus E$ generated by $\lambda | \pi(\Delta) \setminus E$ and the Poincaré metric on $\pi(\Delta)$ belongs to the set $M(\pi(\Delta) \setminus E)$. η satisfies the conditions $\eta > \lambda$ and $\pi^* \eta \in M(\Delta)$, so that we have $\lambda_\Delta < \pi^* \eta$. This contradicts the maximality of the Poincaré metric λ_Δ in $M(\Delta)$. Thus $\pi(\Delta) \setminus E = X \setminus E$. Since the map π is locally schlicht except on $\pi^{-1}(E)$, it is seen that the set of elliptic fixed points of G is contained in $\pi^{-1}(E)$. We conclude that $X \setminus E$ is conformally equivalent to the quotient Riemann surface $(\Delta \setminus \pi^{-1}(E))/G$ and that the map π may be identified with the natural projection $\Delta \to \Delta/G$. Hence π is a normal branched covering with possible branch points or punctures in E.

The second statement of the Lemma is proved similarly as in the case of regular coverings, and we omit its proof. ∎

Now assume that $\pi: \Delta \to X$ is a holomorphic universal covering. Since $\pi^* \lambda \in M(\Delta)$ whenever $\lambda \in M(X)$, Theorem 1 guarantees the existence of an $f \in \boldsymbol{B}_0(\Delta)$ such that $\pi^* \lambda = f^* \lambda_\Delta$. Noting that $\pi^* \lambda$ is Γ-invariant where Γ is the covering group for the covering π, we conclude from Lemma 3 that there exists a homomorphism $\chi: \Gamma \to \text{Möb}(\Delta)$ such that

$$f \circ \gamma = \chi(\gamma) \circ f \qquad \text{for all} \quad \gamma \in \Gamma. \tag{3.2}$$

We shall call the homomorphism χ the *monodromy homomorphism* induced by the metric $\lambda \in M(X)$. Observe that, for fixed π, χ is uniquely determined up to an inner automorphism of Möb(Δ). It is natural to ask when is the image $\chi(\Gamma)$ discontinuous. The answer is given by the following:

THEOREM 6. *Let χ: $\Gamma\rightarrow$Möb(Δ) be the monodromy homomorphism induced by the metric $\lambda\in M(X)$. Then $\chi(\Gamma)$ acts discontinuously on Δ if and only if there exists a Riemann surface Y and a branched Poincaré metric η on Y such that $\lambda=F^{*}\eta$ for some F: $X\rightarrow Y$ holomorphic.*

Proof. Fix an $f\in \boldsymbol{B}_{0}(\Delta)$ such that $\pi^{*}\lambda=f^{*}\lambda_{\Delta}$ as above. First, assume that the group $\chi(\Gamma)$ is discontinuous on Δ. Let Y be the quotient Riemann surface $\Delta/\chi(\Gamma)$. Then by (3.2) f induces a holomorphic map F: $X\rightarrow Y$ such that $F\circ\pi=\pi_{1}\circ f$ where π_{1}: $\Delta\rightarrow Y$ denotes the natural projection. Let η be the branched Poincaré metric on Y determined by $\lambda_{\Delta}=\pi_{1}^{*}\eta$. Now we have

$$\pi^{*}\lambda=f^{*}\lambda_{\Delta}=f^{*}(\pi_{1}^{*}\eta)=(F\circ\pi)^{*}\eta=\pi^{*}(F^{*}\eta),$$

concluding that $\lambda=F^{*}\eta$, as desired.

Conversely, assume that $\lambda=F^{*}\eta$ where F: $X\rightarrow Y$ is holomorphic and η is a branched Poincaré metric on Y. By Lemma 5 there exists a holomorphic normal branched covering π_{1}: $\Delta\rightarrow Y$ with $\lambda_{\Delta}=\pi_{1}^{*}\eta$. Let $E\subset Y$ be the set of singularities of η. Since both metrics $F^{*}\eta$ and $\pi_{1}^{*}\eta$ are regular, we find that $n(p, F)=n(q, \pi_{1})$ for every pair (p, q) such that $F(p)=\pi_{1}(q)$ where $n(p, F)$ denotes the multiplicity of F at p. This observation allows us to conclude that there exists a holomorphic map g: $\Delta\rightarrow\Delta$ satisfying the condition $F\circ\pi=\pi_{1}\circ g$. Thus we have

$$\pi^{*}\lambda=\pi^{*}(F^{*}\eta)=(\pi_{1}\circ g)^{*}\eta=g^{*}\pi_{1}^{*}\eta=g^{*}\lambda_{\Delta},$$

so that $f^{*}\lambda_{\Delta}=g^{*}\lambda_{\Delta}$. Lemma 3 shows that there exists a $\gamma\in$Möb(Δ) with $f=\gamma\circ g$. Hence we conclude that $\chi(\Gamma)\subset\gamma\circ\Gamma_{1}\circ\gamma^{-1}$ where Γ_{1} denotes the covering group of the branched covering π_{1}. This completes the proof of Theorem 6 since Γ_{1} is discontinuous. ■

4. Behavior of the metric at isolated singularities.

As an application of Theorem 1 we study the behavior of a metric λ of constant curvature arround a puncture. The problem being local, we may assume without loss of generality that $\lambda\in M(\Delta\setminus\{0\})$. Let U be the upper half-plane. Since $(e^{iz})^{*}\lambda\in M(U)$, by using Theorem 1 we obtain an $f\in \boldsymbol{B}_{0}(U)$ such that $(e^{iz})^{*}\lambda=f^{*}\lambda_{\Delta}$. Let χ be the monodromy homomorphism induced by λ and set $\gamma=\chi(\tau)$ where $\tau(z)=z+2\pi$ is a generator for the covering group of e^{iz}: $U\rightarrow\Delta\setminus\{0\}$. Hence we have that $f\circ\tau=\gamma\circ f$.

LEMMA 7. *γ is not a hyperbolic transformation.*

Proof. Schwarz' lemma implies that f is hyperbolically distance-decreasing. Thus,

$$d_{\Delta}(f(z), \gamma(f(z)))=d_{\Delta}(f(z), f(\tau(z)))\leqq d_{U}(z, z+2\pi),$$

where $d_{\Delta}(\cdot, \cdot)$ and $d_U(\cdot, \cdot)$ denote the hyperbolic distance of Δ and U respectively. Letting $z=iy\to\infty(y\in\boldsymbol{R}^+)$, we have

$$\inf_{z\in\Delta} d_{\Delta}(z, \gamma(z))=0.$$

From this, it is easy to see that γ is not hyperbolic. ■

LEMMA 8. *The following estimates hold near the origin.*

(i) *If γ is parabolic, then*

$$\lambda(z)=\frac{1}{2|z|\ln(C_1/|z|)}+O(1),$$

with some constant $C_1>0$.

(ii) *If γ is elliptic, then there exist constants $C_2>0$, $C_3>0$ and an integer $k\geqq 0$ such that*

$$\lambda(z)=|z|^{\alpha+k-1}(C_2+O(z^{C_3})),$$

where $2\pi\alpha(0<\alpha<1)$ is the rotation angle of γ at a fixed point.

(iii) *If γ is the identity, then there exist a constant $C_4>0$ and an integer $k\geqq 0$ such that*

$$\lambda(z)=|z|^k(C_4+O(z)).$$

Proof. (i) Theorem 1 implies that there exists a locally schlicht function $f: U\to U$ such that $(e^{iz})^*\lambda=f^*\lambda_U$ and $f(z+2\pi)=f(z)+2\pi$ for all $z\in U$. Thus we have a Fourier series expansion

$$f(z)=z+\sum a_n e^{niz}.$$

Since $\mathrm{Im} f(z)$ is periodic, it can be regarded as a positive harmonic function on $\Delta\setminus\{0\}$. Hence, substituting $w=e^{iz}$ we have

$$\ln\frac{1}{|w|}+\mathrm{Im}\sum a_n w^n>0 \qquad \text{for} \quad 0<|w|<1.$$

This inequality easily implies that $a_n=0$ for all $n<0$. Therefore,

$$\lambda(w)|w|=\frac{|1+\sum_{n=0}^{\infty} nia_n w^n|}{2\ln(1/|w|)+2\,\mathrm{Im}\sum_{n=0}^{\infty} a_n w^n}=\frac{1}{2\ln(C_1/|w|)}+O(w) \quad (w\to 0)$$

with $C_1=e^{\mathrm{Im}\, a_0}$. This proves Case (i) of the Lemma.

(ii) As in Case (i) there exists a holomorphic function $f: U\to\Delta$ such that $(e^{iz})^*\lambda=f^*\lambda_{\Delta}$ and that $f(z+2\pi)=e^{2\pi\alpha i}f(z)$, $0<\alpha<1$. Expanding f in a Fourier series, we have

$$f(z)=e^{i\alpha z}\sum_{n=k}^{\infty} a_n e^{niz},$$

where $k(\geqq 0)$ is an integer and $a_k\neq 0$. A similar calculation as in Case (i) yields the desired estimate.

(iii) Since γ is the identity, $f: U \to \Delta$ is of the form

$$f(z)=\sum_{n=k}^{\infty} a_n e^{inz}, \qquad a_k \neq 0$$

with an integer $k \geqq 0$. This suffices to conclude the proof of Lemma 8. ■

We recall that the index v at an isolated singularity p of a metric of constant curvature λ is defined by

$$v=\lim_{z\to 0}\frac{\ln \lambda(z)}{\ln(1/|z|)}$$

where z is a coordinate centered at p(c.f. [2]). According to Lemma 8, this definition is legitimate and does not depend on the choice of the coordinate. We remark that the index v satisfies the inequality $v \leqq 1$ and that the equality occurs if and only if the Möbius transformation γ associated to the metric is parabolic. Also, observe that the index v is a non-positive integer if and only if the associated transformation γ is the identity.

In the proof of the last section we use the following corollary to Lemma 8.

COROLLARY 1. ([4], p. 73) *Let Ω be a hyperbolic plane region containing the origin. If λ is the Poincaré metric on $\Omega \setminus \{0\}$, then we have*

$$\lim_{z\to 0} \lambda(z)|z|\ln(1/|z|)=\frac{1}{2}.$$

Proof. We may assume without loss of generality that $\Delta \setminus \{0\} \subset \Omega \setminus \{0\}$. Let $\pi_1(=e^{iz}): U \to \Delta \setminus \{0\}$ and $\pi: U \to \Omega \setminus \{0\}$ be holomorphic universal coverings. Let $f: U \to U$ be a lift of the inclusion map $\Delta \setminus \{0\} \to \Omega \setminus \{0\}$ with respect to the coverings π_1 and π. From the identities $\pi_1 = \pi \circ f$ and $\pi^* \lambda = \lambda_\Delta$, we have $f^* \lambda_\Delta = \pi_1^* \lambda$. Since f is a lift, there exists a $\gamma \in \Gamma$ such that $f(z+2\pi)=\gamma \circ f(z)$, where Γ is a cover transformation group for the covering π. Since Γ does not contain elliptic transformations, Lemma 7 shows that γ is either a parabolic or the identity element. By Lemma 8 it suffices to show that γ is parabolic. Assume that γ is the identity. Then there exists a holomorphic function $h: \Delta \setminus \{0\} \to U$ such that

$$\pi \circ h = id \quad \text{on} \quad \Delta \setminus \{0\}. \tag{4.1}$$

By Riemann's removable singularity theorem, h is extended to a holomorphic function on Δ. This would contradicts the identity (4.1), since $0 \notin \pi(U)$. Hence γ is parabolic, as desired. ■

5. Proof of Theorem 2.

We consider the monodromy homomorphism $\chi: \Gamma \to \text{Möb}(\Delta)$ induced by $\lambda \in M(\Omega \setminus E)$ where Γ is the covering group for the holomorphic universal covering $\pi: \Delta \to \Omega \setminus E$. By assumption every index v is an integer $\leqq 0$. By similar reasoning as in the previous section concerning the classification of isolated singularities, we conclude that $\chi(\gamma)$ is the identity whenever $\gamma \in \Gamma$ is parabolic. Note that parabolic elements in Γ correspond to the punctures of the region $\Omega \setminus E$. Since Ω is simply connected, we see that Γ is generated by the parabolic elements, so that $\chi(\gamma)$ is the identity for all $\gamma \in \Gamma$. Hence F is invariant under the covering group Γ where F is a function in $\boldsymbol{B}_0(\Delta)$ such that $\pi^*\lambda = F^*\lambda_\Delta$. We claim that $f = F \circ \pi^{-1}: \Omega \setminus E \to \Delta$ is the desired function. It is easy to see that f is well defined and satisfies $\lambda = f^*\lambda_\Delta$. On the other hand, since f is bounded, f can be extended to a function in $\boldsymbol{B}(\Omega)$. This completes the proof. ∎

6. An inequality.

Let $d(z)$ denote the maximal Euclidean radius of the schlicht discs centered at $f(z)$ which is contained in the Riemannian image of a regular function f. In 1970 Pommerenke proved in [5] the following inequality.

(*) There exists a constant $\beta < 1$ such that

$$(1-|z|^2)|f'(z)| \leqq 2\beta d(z)\left(1+\ln \frac{1}{d(z)}\right), \quad z \in \Delta,$$

for every locally schlicht function $f: \Delta \to C$ with $\sup_{z\in\Delta} d(z) \leqq 1$.

Now we show, however, that such a constant $\beta < 1$ does not exist.

THEOREM 9. *There exists a universal covering f which does not satisfy the inequality* (*) *for every* $\beta < 1$.

Proof. Let E be the set $\{z \in \boldsymbol{C} \mid z = m + ni,\ m, n \in \boldsymbol{Z}\}$ and consider a holomorphic universal covering $f: \Delta \to \boldsymbol{C} \setminus E$. Then f is locally schlicht and satisfies $\sup_{z\in\Delta} d(z) \leqq 1$. Let $\lambda(z)|dz|$ be the Poincaré metric on the domain $\boldsymbol{C} \setminus E$. By definition, we have

$$\lambda(f(z))|f'(z)| = \frac{1}{1-|z|^2}.$$

It is clear that $d(z) = |f(z)|$ if $|f(z)| \leqq 1/2$. Thus, for $|f(z)|$ small,

$$\frac{(1-|z|^2)/f'(z)|}{2d(z)(1+\ln(1/d(z)))} = \frac{1}{2\lambda(w)|w|\ln(e/|w|)}$$

with $w = f(z)$. By Corollary 1, the right side of the above equation tends to 1 as $w = f(z) \to 0$. Hence the constant β must satisfy $\beta \geqq 1$ as desired. ∎

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