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THE GAUSS IMAGE OF FLAT SURFACES IN R^4

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Let M be a surface of zero Gaussian curvature in $\boldsymbol{R}^4$ which has flat normal connection. Let $G_{2,4}$ denote the Grassmann manifold consisting of oriented 2-dimensional linear subspaces of $\boldsymbol{R}^4$. The Gauss map $G: M \to G_{2,4}$ is defined by assigning each point of M to the tangent plane of M at the point.

In this paper we study the structure of the image of M by the Gauss map. In [4], C. Thas showed that the Gauss image of a surface of zero Gaussian curvature in $\boldsymbol{R}^4$ which has flat normal connection is flat. Theorem 1 gives a further information on the structure of the Gauss image. Namely, under the identification $G_{2,4}=S^2\left(\frac{1}{\sqrt{2}}\right)\times S^2\left(\frac{1}{\sqrt{2}}\right)$, the Gauss image of M is the Riemannian product of two curves, one in the first factor of $S^2\times S^2$ and one in the second factor. We compute the geodesic curvatures of those curves and show that if those curves are totally geodesic, then M is the Riemannian product of two plane curves.

In §2, we give some local formulas for principal curvatures and show that if certain functions defined from principal curvatures vanish everywhere, then the surface is the Riemannian product of two plane curves. In §3, we look at $G_{2,4}$ and give some basic formulas. In §4, we prove our theorems for the geometry of the Gauss image of M.

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1. Preliminaries.

Let M be a connected n-dimensional C^∞ Riemannian manifold and let $\psi: M\to \boldsymbol{R}^N$ be an isometric immersion of M into an N-dimensional Euclidean space $\boldsymbol{R}^N$. Let D and $\bar{D}$ denote the covariant differentiations of M and $\boldsymbol{R}^N$ respectively. Let X, Y be tangent vector fields on M. Then

$$\bar{D}_X Y = D_X Y + B(X, Y) \tag{1.1}$$

where $B(X, Y)$ is the normal component of $\bar{D}_X Y$.

Let ξ be a normal vector field on M. We write

$$\bar{D}_X \xi = -A_\xi X + D_X^\perp \xi \tag{1.2}$$

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where $A_\xi X$ and $D_X^\perp \xi$ are the tangential and normal components of $\bar{D}_X \xi$. Then we have

$$\langle A_\xi X, Y\rangle = \langle B(X, Y), \xi\rangle \tag{1.3}$$

where $\langle\,,\rangle$ denotes the inner product of $\boldsymbol{R}^N$. The linear transformation A_ξ on the tangent bundle TM is called the *shape operator* of M with respect to ξ. Since A_ξ is symmetric, i. e.

$$\langle A_\xi X, Y\rangle = \langle X, A_\xi Y\rangle, \tag{1.4}$$

all eigenvalues of A_ξ are real. An eigenvalue of A_ξ is called a *principal curvature* with respect to ξ. An eigenvector of A_ξ is called a *principal vector* with respect to ξ.

Let R and $R^\perp$ be the curvature tensors associated with D and $D^\perp$ respectively, i. e.

$$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{1.5}$$

$$R^\perp(X, Y)\xi = D_X^\perp D_Y^\perp \xi - D_Y^\perp D_X^\perp \xi - D_{[X,Y]}^\perp \xi \tag{1.6}$$

where X, Y, Z are tangent to M and ξ is normal to M.

Then for any tangent vector fields X, Y, Z, W and normal vector fields ξ, η, we have the following equations:

$$\langle R(X, Y)Z, W\rangle = -\langle B(X, Z), B(Y, W)\rangle + \langle B(Y, Z), B(X, W)\rangle \tag{1.7}$$

(Gauss equation)

$$\langle R^\perp(X, Y)\xi, \eta\rangle = \langle (A_\xi A_\eta - A_\eta A_\xi)X, Y\rangle \tag{1.8}$$ (Ricci equation)

$$(D_X B)(Y, Z) - (D_Y B)(X, Z) = 0 \tag{1.9}$$ (Codazzi equation)

In the last formula, the covariant derivative of B is defined by

$$(D_X B)(Y, Z) = D_X B(Y, Z) - B(D_X Y, Z) - B(Y, D_X Z). \tag{1.10}$$

The normal connection $D^\perp$ is said to be *flat* if $R^\perp = 0$. (1.8) implies that $D^\perp$ is flat at $p \in M$ if and only if

$$A_\xi A_\eta = A_\eta A_\xi \tag{1.11}$$

for any two normal vectors ξ and η at p. Thus if $D^\perp$ is flat at $p \in M$, there exists an orthonormal base $e_1, \cdots, e_n$ of $T_p M$ such that each e_i $(i=1, \cdots, n)$ is a principal vector with respect to any normal vector at p.

2. Local Formulas for Flat Surfaces in $\boldsymbol{R}^4$.

Let M be a surface in $\boldsymbol{R}^4$ which has zero Gaussian curvature and flat normal connection.

In this section we derive some local formulas for M.

Since the normal connection of M is flat, there exists an orthonormal local frame (e_1, e_2) of TM such that e_1 and e_2 are common principal vectors for all normal vectors at each point. For each normal vector ξ we set $h_{ij}(\xi)=\langle A_\xi e_i, e_j\rangle$ $(i, j=1, 2)$. Since e_1 and e_2 are principal vectors for any ξ, we have $h_{12}(\xi)=h_{21}(\xi)=0$.

LEMMA 2.1. *For each point p of M there exists an orthonormal basis (e_3, e_4) of $T_p^\perp M$, the normal space of M at p, which satisfies*

$$h_{22}(e_3)=0, \qquad h_{11}(e_4)=0. \tag{2.1}$$

Proof. Let $(\tilde{e}_3, \tilde{e}_4)$ be any orthonormal basis of $T_p^\perp M$. Since the Gaussian curvature is zero, it follows from (1.7) that

$$\det A_{\tilde{e}_3}+\det A_{\tilde{e}_4}=0, \tag{2.2}$$

or equivalently,

$$h_{11}(\tilde{e}_3)h_{22}(\tilde{e}_3)+h_{11}(\tilde{e}_4)h_{22}(\tilde{e}_4)=0. \tag{2.3}$$

Set

$$\begin{aligned} e_3&=\cos\theta\,\tilde{e}_3+\sin\theta\,\tilde{e}_4 \\ e_4&=-\sin\theta\,\tilde{e}_3+\sin\theta\,\tilde{e}_4. \end{aligned} \tag{2.4}$$

Then

$$h_{22}(e_3)=\cos\theta\, h_{22}(\tilde{e}_3)+\sin\theta\, h_{22}(\tilde{e}_4) \tag{2.5}$$

$$h_{11}(e_4)=-\sin\theta\, h_{11}(\tilde{e}_3)+\cos\theta\, h_{11}(\tilde{e}_4). \tag{2.6}$$

(2.3) implies that the system of equations (2.1) has a non-trivial solution for θ.

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Let $\lambda=h_{11}(e_3)$ and $\mu=h_{22}(e_4)$. Let $M_1=\{p\in M: A_\xi\neq 0 \text{ for all } \xi\in T_p^\perp M\}$. Then $p\in M_1$ if and only if λ and μ satisfy $\lambda\mu\neq 0$ at p. In [3], Reckziegel proved that the principal curvatures and principal vector fields with respect to a continuous normal vector field ξ are C^∞ on $U\subset M$ if no point in U is umbilical with respect to A_ξ. By this theorem, we see that e_1 and e_2 define C^∞ vector fields on M_1, and λ and μ define C^∞ functions on M_1. We continue to denote these vector fields and functions by e_1, e_2 and λ, μ. We define a 1-form ω_{AB} $(A, B=1, \cdots, 4)$ on M_1 by

$$\omega_{AB}(X)=\langle \bar{D}_X e_A, e_B\rangle \tag{2.7}$$

for $X\in TM$. Setting $X=e_1$ and $Y=Z=e_2$ in (1.9), we obtain

$$\bar{D}_{e_1}(\mu e_4)+\mu\omega_{12}(e_2)e_4+\lambda\omega_{21}(e_2)e_3=0 \tag{2.8}$$

and this gives

$$e_1\mu-\mu\omega_{21}(e_2)=0 \tag{2.9}$$

and

$$\lambda\omega_{21}(e_2)-\mu\omega_{34}(e_1)=0. \tag{2.10}$$

Similarly, if we set $X=e_2$ and $Y=Z=e_1$ in (1.9), we obtain

$$e_2\lambda-\lambda\omega_{12}(e_1)=0 \tag{2.11}$$

and

$$\mu\omega_{12}(e_1)+\lambda\omega_{34}(e_2)=0. \tag{2.12}$$

Since $[e_1, e_2]=\bar{D}_{e_1}e_2-\bar{D}_{e_2}e_1=\omega_{21}(e_1)e_1+\omega_{12}(e_2)e_2$, we have

$$\langle R(e_1, e_2)e_2, e_1\rangle=e_1(\omega_{21}(e_2))+e_2(\omega_{12}(e_1))-(\omega_{12}(e_1))^2-(\omega_{21}(e_2))^2 \tag{2.13}$$

Combining (2.9) and (2.11) with (2.13), we have on M_1

$$\begin{aligned}\langle R(e_1, e_2)e_2, e_1\rangle&=e_1\left(\frac{e_1\mu}{\mu}\right)+e_2\left(\frac{e_2\lambda}{\lambda}\right)-\left(\frac{e_2\lambda}{\lambda}\right)^2-\left(\frac{e_1\mu}{\mu}\right)^2\\&=\frac{e_1e_1\mu}{\mu}-\frac{2(e_1\mu)^2}{\mu^2}+\frac{e_2e_2\lambda}{\lambda}-\frac{2(e_2\lambda)^2}{\lambda^2}\\&=-\mu e_1e_1\frac{1}{\mu}-\lambda e_2e_2\frac{1}{\lambda}\\&=-\lambda\mu\left(\frac{1}{\lambda}e_1e_1\frac{1}{\mu}+\frac{1}{\mu}e_2e_2\frac{1}{\lambda}\right).\end{aligned} \tag{2.14}$$

Since the Gaussian curvature is zero, (2.14) gives

$$\frac{1}{\lambda}e_1e_1\frac{1}{\mu}+\frac{1}{\mu}e_2e_2\frac{1}{\lambda}=0 \tag{2.15}$$

on M_1. The normal curvature on M_1 is given by

$$\begin{aligned}&\langle R^{\perp}(e_1, e_2)e_3, e_4\rangle\\&=e_1(\omega_{34}(e_2))-e_2(\omega_{34}(e_1))+\omega_{12}(e_1)\omega_{34}(e_1)-\omega_{21}(e_2)\omega_{34}(e_2)\\&=e_1\left(-\frac{\mu e_2\lambda}{\lambda^2}\right)-e_2\left(\frac{\lambda e_1\mu}{\mu^2}\right)+\left(\frac{e_2\lambda}{\lambda}\right)\left(\frac{\lambda e_1\mu}{\mu^2}\right)-\left(\frac{e_1\mu}{\mu}\right)\left(-\frac{\mu e_2\lambda}{\lambda^2}\right)\\&=-\frac{\mu e_1e_2\lambda}{\lambda^2}+\frac{2\mu(e_1\lambda)(e_2\lambda)}{\lambda^3}-\frac{\lambda e_2e_1\mu}{\mu^2}+\frac{2\lambda(e_1\mu)(e_2\mu)}{\mu^3}\\&=\mu e_1e_2\frac{1}{\lambda}+\lambda e_2e_1\frac{1}{\mu}\\&=\lambda\mu\left(\frac{1}{\lambda}e_1e_2\frac{1}{\lambda}+\frac{1}{\mu}e_2e_1\frac{1}{\mu}\right).\end{aligned} \tag{2.16}$$

Since the normal connection is flat, (2.16) gives

$$\frac{1}{\lambda}e_1e_2\frac{1}{\lambda}+\frac{1}{\mu}e_2e_1\frac{1}{\mu}=0. \tag{2.17}$$

LEMMA 2.2. *On M_1 we have*

$$\left[\frac{1}{\lambda}e_1, \frac{1}{\mu}e_2\right]=0. \tag{2.18}$$

Proof.

$$\begin{aligned}\left[\frac{1}{\lambda}e_1, \frac{1}{\mu}e_2\right]&=D_{(1/\lambda)e_1}\frac{1}{\mu}e_2-D_{(1/\mu)e_2}\frac{1}{\lambda}e_1\\ &=-\frac{e_1\mu}{\lambda\mu^2}e_2+\frac{1}{\lambda\mu}\omega_{21}(e_1)+\frac{e_2\lambda}{\lambda^2\mu}e_1-\frac{1}{\lambda\mu}\omega_{12}(e_2)e_2\\ &=\frac{1}{\lambda^2\mu}(e_2\lambda-\lambda\omega_{12}(e_1))e_1-\frac{1}{\lambda\mu^2}(e_1\mu-\mu\omega_{21}(e_2))e_2\\ &=0 \qquad \text{(by (2.9) and (2.11))}.\end{aligned}$$

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By Lemma 2.2, there exists a local coordinate system (u_1, u_2) on M_1 such that

$$\frac{\partial}{\partial u_1}=\frac{1}{\lambda}e_1, \qquad \frac{\partial}{\partial u_2}=\frac{1}{\mu}e_2. \tag{2.19}$$

Now we define functions f and g by

$$f=e_1\frac{1}{\mu}, \qquad g=e_2\frac{1}{\lambda}. \tag{2.20}$$

Then (2.15) and (2.17) can be rewritten as

$$\frac{\partial f}{\partial u_1}+\frac{\partial g}{\partial u_2}=0 \tag{2.21}$$

and

$$\frac{\partial f}{\partial u_2}+\frac{\partial g}{\partial u_1}=0, \tag{2.22}$$

respectively.

If we set $v_1=u_1-u_2$ and $v_2=u_1+u_2$, (2.21) and (2.22) imply that there exist functions $\phi_1(v_1)$ and $\phi_2(v_2)$ such that

$$f(v_1, v_2)=\phi_1(v_1)+\phi_2(v_2) \tag{2.23}$$

$$g(v_1, v_2)=\phi_1(v_1)-\phi_2(v_2). \tag{2.24}$$

In § 4 we will give a geometric interpretation of ϕ_1 and ϕ_2 in terms of the Gauss map of M.

Note that $f=C_1$ and $g=C_2$ (C_1 and C_2 are constant) are trivial solutions for the system of differential equations (2.21) and (2.22). We now show that these solutions characterize the "standard" torus in $\boldsymbol{R}^4$.

PROPOSITION 2.1. *Let M be a closed surface in $\boldsymbol{R}^4$ which has zero Gaussian curvature and flat normal connection. Suppose $M_1=M$. Then f is constant if and only if M is the product of two closed plane curves.*

Proof. If M is a product space, it is easy to see that $f\equiv 0$ and $g\equiv 0$.

Suppose that $f\equiv C_1$ on M. Then (2.21) and (2.22) imply that g is also constant on M. We set $g\equiv C_2$. Since $M_1=M$, λ and μ are globally defined C^∞ functions. Since M is compact, λ and μ must have critical points p_1 and p_2 respectively. Then $f(p_2)=0$ and $g(p_1)=0$, which implies that $f\equiv 0$ and $g\equiv 0$ on M. Thus we have $e_1\mu\equiv 0$ and $e_2\lambda\equiv 0$. By (2.9), (2.10), (2.11) and (2.12), we obtain

$$\omega_{12}(e_1)\equiv 0, \quad \omega_{21}(e_2)\equiv 0, \quad \omega_{34}(e_1)\equiv 0, \quad \omega_{34}(e_2)\equiv 0, \tag{2.25}$$

Hence

$$\bar{D}_{e_1}e_2\equiv 0, \quad \bar{D}_{e_2}e_1\equiv 0, \quad \bar{D}_{e_1}e_4\equiv 0, \quad \bar{D}_{e_2}e_3\equiv 0. \tag{2.26}$$

Take $p_0\in M$ and let $\Gamma_i(s_i)$ $(i=1,2)$ be the maximal integral curve of e_i with initial point p_0, i.e. $\dfrac{d\Gamma_i}{ds_i}(s_i)=e_i(\Gamma_i(s_i))$ and $\Gamma_i(0)=p_0$. We choose a Cartesian coordinate system (x_1, x_2, x_3, x_4) on $\boldsymbol{R}^4$ such that

$$x_A(p_0)=0 \quad \text{and} \quad \frac{\partial}{\partial x_A}(p_0)=e_A(p_0) \quad \text{for } A=1, 2, 3, 4. \tag{2.27}$$

Then (2.26) implies that e_2 and e_4 are constant along Γ_1, that is,

$$\begin{aligned} e_2(\Gamma_1(s_1))&=e_2(p_0)=(0, 1, 0, 0) \\ e_4(\Gamma_1(s_1))&=e_4(p_0)=(0, 0, 0, 1) \end{aligned} \quad \text{for any } s_1. \tag{2.28}$$

Hence $\Gamma_1(s_1)$ is orthogonal to $(0, 1, 0, 0)$ and $(0, 0, 0, 1)$ at each s_1. Thus Γ_1 lies in the x_1-x_3 plane. Similarly, we see that Γ_2 lies in the x_2-x_4 plane. We write Γ_1 and Γ_2 as

$$\begin{aligned} \Gamma_1(s_1)&=(\Gamma_{11}(s_1), 0, \Gamma_{13}(s_1), 0) \\ \Gamma_2(s_2)&=(0, \Gamma_{22}(s_2), 0, \Gamma_{24}(s_2)). \end{aligned} \tag{2.29}$$

Let $\alpha_s(t)$ be the integral curve of e_2 with initial point $\Gamma_1(s)$, i.e.

$$\begin{aligned} \alpha_s'(t)&=e_2(\alpha_s(t)) \\ \alpha_s(0)&=\Gamma_1(s). \end{aligned} \tag{2.30}$$

We write $\alpha_s(t)$ as

$$\alpha_s(t)=\alpha(s, t)=(\alpha_1(s, t), \alpha_2(s, t), \alpha_3(s, t), \alpha_4(s, t)). \tag{2.31}$$

Then it follows from (2.30) that

(2.32) $$e_2(\alpha_s(t))=\frac{\partial\alpha}{\partial t}=\sum_{A=1}^{4}\frac{\partial\alpha_A}{\partial t}\frac{\partial}{\partial x_A}.$$

We claim that

(2.33) $$e_1(\alpha_s(t))=\frac{\partial\alpha}{\partial s}=\sum_{A=1}^{4}\frac{\partial\alpha_A}{\partial s}\frac{\partial}{\partial x_A}.$$

To prove this, we write

(2.34) $$e_1(\alpha_s(t))=a(s,\ t)\frac{\partial\alpha}{\partial s}+b(s,\ t)\frac{\partial\alpha}{\partial t}.$$

This is possible since $\frac{\partial\alpha}{\partial s}$ and $\frac{\partial\alpha}{\partial t}$ span the tangent plane of M at each point. Since $\bar{D}_{e_1}e_2=\bar{D}_{e_2}e_1=0$ by (2.26), at each point of M we have

(2.35) $$[e_1,\ e_2]=\bar{D}_{e_1}e_2-\bar{D}_{e_2}e_1=0.$$

It follows from (2.32), (2.34) and (2.35) that

(2.36) $$[e_1(\alpha_s(t)),\ e_2(\alpha_s(t))]=-\frac{\partial a}{\partial t}\frac{\partial\alpha}{\partial s}-\frac{\partial b}{\partial t}\frac{\partial\alpha}{\partial t}=0.$$

Thus we have

(2.37) $$\frac{\partial a}{\partial t}=0,\quad \frac{\partial b}{\partial t}=0\quad \text{for all } s \text{ and } t.$$

Hence $a(s,\ t)=a(s,\ 0)$ and $b(s,\ t)=b(s,\ 0)$, but since $\alpha_s(0)=\Gamma_1(s)$, $a(s,\ 0)=1$ and $b(s,\ 0)=0$. Therefore $a(s,\ t)=1$ and $b(s,\ t)=0$ for all s and t. This completes the proof for (2.33).

Since $\bar{D}_{e_1}e_2=0$, we have

(2.38) $$\frac{\partial^2\alpha}{\partial s\partial t}=0$$

or

(2.39) $$\frac{\partial^2\alpha_A}{\partial s\partial t}=0\qquad (A=1,\ 2,\ 3,\ 4).$$

(2.39) implies that $\alpha_A(s,\ t)$ can be written as

(2.40) $$\alpha_A(s,\ t)=\alpha_{A1}(s)+\alpha_{A2}(t).$$

Since $\alpha(s,\ 0)=\Gamma_1(s)$ and $\alpha(0,\ t)=\Gamma_2(t)$, it follows from (2.29) and (2.40) that

(2.41) $$\alpha(s,\ t)=(\Gamma_{11}(s),\ \Gamma_{22}(t),\ \Gamma_{13}(s),\ \Gamma_{24}(t)).$$

$\Gamma_1(s)$ and $\Gamma_2(t)$ are smooth curves on $M=M_1$ which are defined for $-\infty<s<+\infty$ and $-\infty<t<+\infty$. (Note that s and t are the arc length parameters of Γ_1 and Γ_2 respectively.) By (2.41) we have

(2.42) $$\Gamma_1\times\Gamma_2=\{\alpha(s,\ t):\ -\infty<s,\ t<+\infty\}.$$

From the definition of $\alpha(s, t)$, (2.42) implies that $\Gamma_1 \times \Gamma_2$ is an open subset of M. Since $\Gamma_1 \times \Gamma_2$ is closed and M is connected, $M=\Gamma_1 \times \Gamma_2$. Since M is compact, $\Gamma_1 = M \cap \boldsymbol{R}^2$ is a closed plane curve. (Similarly Γ_2 is a closed plane curve.)

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3. Geometry of $G_{2,4}$.

Let $G_{2,4}$ denote the Grassmann manifold consisting of oriented 2-dimensional linear subspaces of $\boldsymbol{R}^4$. We now recall some basic facts for $G_{2,4}$. (See [2] for more details.)

Let $P_0 \in G_{2,4}$ and let $\{\mathring{e}_1, \cdots, \mathring{e}_4\}$ be a fixed orthonormal basis of $\boldsymbol{R}^4$ such that

$$P_0 = \mathring{e}_1 \wedge \mathring{e}_2 . \tag{3.1}$$

Let P be an element of $G_{2,4}$ in a neighborhood of P_0. Let $\{e_1, \cdots, e_4\}$ be an orthonormal frame of $\boldsymbol{R}^4$ such that

$$P = e_1 \wedge e_2 . \tag{3.2}$$

We write $e_1 = \sum_{A=1}^{4} a_{1A} \mathring{e}_A$ and $e_2 = \sum_{A=1}^{4} a_{2A} \mathring{e}_A$, where a_{1A} and a_{2A} satisfy

$$\sum_A (a_{1A})^2 = 1 \tag{3.3}$$

$$\sum_A (a_{2A})^2 = 1 \tag{3.4}$$

$$\sum_A a_{1A} a_{2A} = 0 . \tag{3.5}$$

We assign to P an element $z=(z_1, z_2, z_3, z_4)$ of $\boldsymbol{C}^4$, where

$$z_A = a_{1A} + i a_{2A} \qquad (A=1, \cdots, 4) . \tag{3.6}$$

Since λz corresponds to the same plane P for all $\lambda \in \boldsymbol{C} - \{0\}$, the correspondence $\varphi : P \mapsto z$ can be regarded as a map from $G_{2,4}$ into $P_3\boldsymbol{C}$. Moreover, (3.3)-(3.5) imply that

$$\sum_A z_A^2 \equiv 0 . \tag{3.7}$$

Thus φ defines a map from $G_{2,4}$ into Q, where Q is the quadric in $P_3\boldsymbol{C}$ which is defined by the equation (3.7). φ is a bijection, and moreover, φ becomes an isometry when we equip $G_{2,4}$ with the standard invariant metric and Q with the metric induced from the Fubini-Study metric on $P_3\boldsymbol{C}$. Let F be a map from Q into $\boldsymbol{C} \times \boldsymbol{C}$ which is defined by

$$F(z_1, z_2, z_3, z_4) = (w_1, w_2) , \tag{3.8}$$

where

$$w_1=\frac{z_3+iz_4}{z_1-iz_2}, \quad w_2=\frac{-z_3+iz_4}{z_1-iz_2}. \tag{3.9}$$

If we regard F as a map from Q into $P_1\boldsymbol{C}\times P_1\boldsymbol{C}$, then F is a bijection. Moreover when we equip $P_1\boldsymbol{C}\times P_1\boldsymbol{C}$ with the metric

$$ds^2=\frac{2|dw_1|^2}{(1+|w_1|^2)^2}+\frac{2|dw_2|^2}{(1+|w_2|^2)^2}, \tag{3.10}$$

F becomes an isometry. Since $P_1\boldsymbol{C}\times P_1\boldsymbol{C}$ is isometric to $S^2\left(\frac{1}{\sqrt{2}}\right)\times S^2\left(\frac{1}{\sqrt{2}}\right)$ under the metric (3.10), $F\circ\varphi$ defines an isometry between $G_{2,4}$ and $S^2\left(\frac{1}{\sqrt{2}}\right)\times S^2\left(\frac{1}{\sqrt{2}}\right)$.

Let $P(t)$ be a curve in $G_{2,4}$ with $P(0)=P_0$. Let $e_1(t)=\sum_A a_{1A}(t)\hat{e}_A$ and $e_2(t)=\sum_A a_{2A}(t)\hat{e}_A$ be orthonormal vectors in $\boldsymbol{R}^4$ which span $P(t)$. Set $z_A(t)=a_{1A}(t)+ia_{2A}(t)$. We identify $P(t)$ with a curve $(z_1(t), z_2(t), z_3(t), z_4(t))$ on Q. Since $z_1\neq 0$ in a neighborhood of P_0, (z_1, z_2, z_3, z_4) and $\left(1, \frac{z_2}{z_1}, \frac{z_3}{z_1}, \frac{z_4}{z_1}\right)$ represent the same element of $G_{2,4}$. Hence

$$P'(t)=\left(0, \left(\frac{z_2}{z_1}\right)', \left(\frac{z_3}{z_1}\right)', \left(\frac{z_4}{z_1}\right)'\right). \tag{3.11}$$

It follows from (3.11) that

$$P'(0)=(0,\ iz_1'(0)-z_2'(0),\ z_3'(0),\ z_4'(0)). \tag{3.12}$$

Since $\sum_A z_A(t)^2\equiv 0$, we have $iz_1'(0)-z_2'(0)=0$. Hence

$$P'(0)=(0,\ 0,\ z_3'(0),\ z_4'(0)). \tag{3.13}$$

As in (2.7), we set $\omega_{AB}=\langle\bar{D}e_A, e_B\rangle$ $(A, B=1, \cdots, 4)$. Since $da_{i\alpha}=\omega_{i\alpha}$ $(i=1,2, \alpha=3,4)$, (3.13) implies that $\{\omega_{i\alpha}: i=1,2, \alpha=3,4\}$ form a co-frame of $T_{P_0}G_{2,4}$. We define another co-frame $\{\theta_{(i\alpha)}\}$ by

$$\theta_{(13)}=\omega_{23}, \quad \theta_{(14)}=\omega_{24}, \quad \theta_{(23)}=-\omega_{13}, \quad \theta_{(24)}=-\omega_{14} \tag{3.14}$$

and let $\{\hat{e}_i\wedge\hat{e}_\alpha\}$ denote the dual frame of $\{\theta_{(i\alpha)}\}$. In a similar way, if an orthonormal frame $\{e_A\}$ is given at $P\in G_{2,4}$, $\{e_i\wedge e_\alpha\}$ is defined as a frame for $T_PG_{2,4}$. We note that, under these definitions, the following formal expression makes real sense now:

$$\begin{aligned} d(e_1\wedge e_2)&=de_1\wedge e_2+e_1\wedge de_2 \\ &=-\omega_{13}e_2\wedge e_3-\omega_{14}e_2\wedge e_4+\omega_{23}e_1\wedge e_3+\omega_{24}e_1\wedge e_4. \end{aligned} \tag{3.15}$$

Using (3.10), we see that the metric on $G_{2,4}$ is given by $\sum_{i,\alpha}(\omega_{i\alpha})^2$. This implies that $\{e_i\wedge e_\alpha\}$ forms an orthonormal frame at each point of $G_{2,4}$.

Let $\{\theta_{(i\alpha)(j\beta)}\}$ be the connection forms associated with $\{\theta_{(i\alpha)}\}$. Then the structure equation is written as

$$d\theta_{(i\alpha)}=\sum_{j,\beta}\theta_{(j\beta)}\wedge\theta_{(j\beta)(i\alpha)}. \tag{3.16}$$

Let i' and α' denote the complements of i and α in $\{1,2\}$ and $\{3,4\}$ respectively.

LEMMA 3.1. $\theta_{(i'\alpha)(i\alpha)}=\omega_{i'i}$

$\theta_{(i\alpha')(i\alpha)}=\omega_{\alpha'\alpha}$

$\theta_{(i'\alpha')(i\alpha)}=0.$

Proof. We write (3.14) as

$$\theta_{(i\alpha)}=\varepsilon_i\omega_{i'\alpha}, \tag{3.17}$$

where $\varepsilon_i=(-1)^{i+1}$. Since $\omega_{i\alpha}$ satisfies the structure equation

$$d\omega_{i\alpha}=\sum_A\omega_{iA}\wedge\omega_{A\alpha}, \tag{3.18}$$

we have

$$\begin{aligned} d\theta_{(i\alpha)}&=\varepsilon_i d\omega_{i'\alpha} \\ &=\varepsilon_i(\sum_A\omega_{i'A}\wedge\omega_{A\alpha}) \\ &=\varepsilon_i(\omega_{i'i}\wedge\omega_{i\alpha}+\omega_{i'\alpha'}\wedge\omega_{\alpha'\alpha}) \\ &=\theta_{(i'\alpha)}\wedge\omega_{i'i}+\theta_{(i\alpha')}\wedge\omega_{\alpha'\alpha}. \end{aligned} \tag{3.19}$$

The lemma follows from (3.16) and (3.19). Q. E. D.

As we see above, $G_{2,4}$ can be identified with $S^2\left(\frac{1}{\sqrt{2}}\right)\times S^2\left(\frac{1}{\sqrt{2}}\right)$. For brevity, we write S_1 and S_2 for those two factors. Let V_1 and V_2 be the 2-dimensional linear subspaces of $T_PG_{2,4}$ which are tangent to S_1 and S_2 respectively.

LEMMA 3.2. *V_1 is spanned by $e_1\wedge e_3-e_2\wedge e_4$ and $e_1\wedge e_4+e_2\wedge e_3$, and V_2 is spanned by $e_1\wedge e_3+e_2\wedge e_4$ and $e_1\wedge e_4-e_2\wedge e_3$.*

Proof. It suffices to prove the lemma for P_0. Let $\tilde{X}\in T_{P_0}G_{2,4}$. $\tilde{X}\in V_1$ if and only if

$$dw_2(\tilde{X})=0. \tag{3.20}$$

Using (3.9), we see that (3.20) is equivalent to

$$dz_3(\tilde{X})=idz_4(\tilde{X}). \tag{3.21}$$

Since $dz_\alpha=\omega_{1\alpha}+i\omega_{2\alpha}$, (3.21) implies that

$$\omega_{13}(\tilde{X})=-\omega_{24}(\tilde{X}), \qquad \omega_{23}(\tilde{X})=\omega_{14}(\tilde{X}). \tag{3.22}$$

Thus

$$\begin{aligned}\tilde{X}&=-\omega_{13}(\tilde{X})e_2\wedge e_3-\omega_{14}(\tilde{X})e_2\wedge e_4+\omega_{23}(\tilde{X})e_1\wedge e_3+\omega_{24}(\tilde{X})e_1\wedge e_4\\ &=\omega_{24}(\tilde{X})(e_2\wedge e_3+e_1\wedge e_4)+\omega_{23}(\tilde{X})(e_1\wedge e_3-e_2\wedge e_4).\end{aligned} \tag{3.23}$$

Similarly, any vector $\tilde{Y}$ in V_2 can be written as a linear combination of $e_1\wedge e_3+e_2\wedge e_4$ and $e_2\wedge e_3-e_1\wedge e_4$. Q. E. D.

4. Gauss Image of Flat Surfaces in $\boldsymbol{R}^4$.

Let M be a surface in $\boldsymbol{R}^4$ and let $\{e_1, \cdots, e_4\}$ be an orthonormal frame field defined in an open set $U\subset M$ such that e_1 and e_2 are tangent to M at each point of U. We define the Gauss map $G: M\rightarrow G_{2,4}$ by

$$G(p)=e_1(p)\wedge e_2(p) \tag{4.1}$$

for each $p\in M$. Let $dG: TM\rightarrow TG_{2,4}$ denote the differential of G. Then it follows from (3.15) that

$$dG(X)=-\omega_{13}(X)e_2\wedge e_3-\omega_{14}(X)e_2\wedge e_4+\omega_{23}(X)e_1\wedge e_3+\omega_{24}(X)e_1\wedge e_4 \tag{4.2}$$

for each $X\in T_pM$.

LEMMA 4.1. *The Gauss map G is singular at p if and only if* $\det A_\xi=0$ *for any normal vector ξ at p.*

Proof. Suppose G is singular at p and $dG(ae_1+be_2)=0$. Then, by (4.2), we have

$$\begin{aligned}&ah_{11}(e_3)+bh_{12}(e_3)=0\\ &ah_{11}(e_4)+bh_{12}(e_4)=0\\ &ah_{12}(e_3)+bh_{22}(e_3)=0\\ &ah_{12}(e_4)+bh_{22}(e_4)=0.\end{aligned} \tag{4.3}$$

(4.3) implies that $\det A_{e_3}=\det A_{e_4}=0$. Since e_3 (or e_4) can be chosen arbitrarily in $T_p^\perp M$, it follows that $\det A_\xi=0$ for any $\xi\in T_p^\perp M$. Conversely, if $\det A_{e_3}=\det A_{e_4}=0$, (4.3) has a non-trivial solution for a and b, which implies that G is singular at p. Q. E. D.

Now we prove the main theorem of this paper.

THEOREM 1. *Let M be a surface in $\boldsymbol{R}^4$ which has zero Gaussian curvature and flat normal connection. Let $p\in M$. Suppose that the Gauss map $G: M\rightarrow G_{2,4}=S_1\times S_2$ is non-degenerate at p. Then there exists a neighborhood U of p and* *curves* $\gamma_1 \subset S_1$ *and* $\gamma_2 \subset S_2$ *such that* $G(U)$ *is the Riemannian product of* γ_1 *and* γ_2.

Proof. We write the shape operators in the following matrix forms:

$$A_{e_3}=\begin{pmatrix} h_{11}(e_3) & h_{12}(e_3) \\ h_{21}(e_3) & h_{22}(e_3) \end{pmatrix}, \qquad A_{e_4}=\begin{pmatrix} h_{11}(e_4) & h_{12}(e_4) \\ h_{21}(e_4) & h_{22}(e_4) \end{pmatrix}. \tag{4.4}$$

By Lemma 2.1, we can find an orthonormal frame $\{e_1. \cdots, e_4\}$ such that the shape operators are written as

$$A_{e_3}=\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_{e_4}=\begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}. \tag{4.5}$$

Since G is non-degenerate at p, it follows from Lemma 4.1 that $\lambda\mu \neq 0$ in a neighborhood U' of p. Then, by Reckziegel's theorem ([3]), each e_A $(A=1, \cdots, 4)$ is a C^∞ vector field and λ and μ are C^∞ functions on U'. Using (4.2), we have

$$dG(e_1)=-\lambda e_2 \wedge e_3 \tag{4.6}$$

$$dG(e_2)=\mu e_1 \wedge e_4. \tag{4.7}$$

Let $X_1=\frac{1}{\lambda}e_1-\frac{1}{\mu}e_2$ and $X_2=\frac{1}{\lambda}e_1+\frac{1}{\mu}e_2$. Then it follows from (4.6) and (4.7) that

$$dG(X_1)=-e_2 \wedge e_3 - e_1 \wedge e_4 \tag{4.8}$$

$$dG(X_2)=-e_2 \wedge e_3 + e_1 \wedge e_4. \tag{4.9}$$

From Lemma 3.2, we see that $dG(X_1)$ and $dG(X_2)$ are tangent to S_1 and S_2 respectively.

Let (v_1, v_2) be the coordinate system on M which is introduced in §2. Then we have

$$\begin{aligned} \frac{\partial}{\partial v_1} &= \frac{1}{\lambda}e_1-\frac{1}{\mu}e_2=X_1 \\ \frac{\partial}{\partial v_2} &= \frac{1}{\lambda}e_1+\frac{1}{\mu}e_2=X_2. \end{aligned} \tag{4.10}$$

$G(M)$ is parameterized by (v_1, v_2) through G and locally $G(M)$ can be expressed as $\{(f_1(v_1, v_2), f_2(v_1, v_2))\}$, where f_1 and f_2 are differentiable maps from an open set of $\boldsymbol{R}^2$ into S_1 and S_2 respectively. Since $dG\left(\frac{\partial}{\partial v_1}\right)=dG(X_1)$ is tangent to S_1, we have

$$df_2\left(\frac{\partial}{\partial v_1}\right)\equiv 0. \tag{4.11}$$

Thus there exists a differentiable map $\gamma_2(v_2)$ from an open set of $\boldsymbol{R}$ into S_2

such that

$$f_2(v_1, v_2)=\gamma_2(v_2). \tag{4.12}$$

Similarly, $dG\left(\frac{\partial}{\partial v_2}\right)$ is tangent to S_2 and we have

$$df_1\left(\frac{\partial}{\partial v_2}\right)\equiv 0, \tag{4.13}$$

and hence there exists a differentiable map $\gamma_1(v_1)$ from an open set of $\boldsymbol{R}$ into S_1 such that

$$f_1(v_1, v_2)=\gamma_1(v_1). \tag{4.14}$$

This completes the proof of the theorem. Q. E. D.

Now we compute the curvatures of γ_1 and γ_2. Let $G^*(TG_{2,4})$ denote the vector bundle on M induced from $TG_{2,4}$, the tangent bundle of $G_{2,4}$, by G. Let $\tilde{D}$ be the Riemannian connection on $G^*(TG_{2,4})$ associated with the metric $\tilde{g}=\sum_{i,\alpha}\omega_{i\alpha}^2$. It follows from Lemma 3.1 that

$$\tilde{D}(e_i\wedge e_\alpha)=\omega_{ii'}e_{i'}\wedge e_\alpha+\omega_{\alpha\alpha'}e_i\wedge e_{\alpha'}, \tag{4.15}$$

where i' and α' are the complements of i and α in $\{1,2\}$ and $\{3,4\}$, respectively. $\tilde{X}_1=\frac{1}{\sqrt{2}}dG(X_1)$ is a unit tangent vector of γ_1 and we have

$$\begin{aligned}\tilde{D}_{\tilde{X}_1}\tilde{X}_1&=\frac{1}{2}\tilde{D}_{X_1}(-e_2\wedge e_3-e_1\wedge e_4)\\ &=\frac{1}{2}(-\omega_{21}(X_1)e_1\wedge e_3-\omega_{34}(X_1)e_2\wedge e_4-\omega_{12}(X_1)e_2\wedge e_4-\omega_{43}(X_1)e_1\wedge e_3)\\ &=\frac{1}{2}\Big(-\frac{1}{\lambda}\omega_{21}(e_1)+\frac{1}{\mu}\omega_{21}(e_2)-\frac{1}{\lambda}\omega_{43}(e_1)+\frac{1}{\mu}\omega_{43}(e_2)\Big)e_1\wedge e_3\\ &\quad+\frac{1}{2}\Big(-\frac{1}{\lambda}\omega_{34}(e_1)+\frac{1}{\mu}\omega_{34}(e_2)-\frac{1}{\lambda}\omega_{12}(e_1)+\frac{1}{\mu}\omega_{12}(e_2)\Big)e_2\wedge e_4\\ &=2\Big(e_1\Big(\frac{1}{\mu}\Big)+e_2\Big(\frac{1}{\lambda}\Big)\Big)\tilde{\nu}_1,\end{aligned} \tag{4.16}$$

where $\tilde{\nu}_1=\frac{1}{\sqrt{2}}(e_2\wedge e_4-e_1\wedge e_3)$. $\tilde{\nu}_1$ is a unit vector which is normal to $G(M)$ and tangent to S_1. The last equality in (4.16) follows from the Codazzi equations (2.9)-(2.12). If we set $\tilde{X}_2=\frac{1}{\sqrt{2}}dG(X_2)$ and $\tilde{\nu}_2=\frac{1}{\sqrt{2}}(e_2\wedge e_4+e_1\wedge e_3)$, we obtain the following formula by a similar computation:

$$\tilde{D}_{\tilde{X}_2}\tilde{X}_2 = 2\left(e_1\left(\frac{1}{\mu}\right) - e_2\left(\frac{1}{\lambda}\right)\right)\tilde{\nu}_2. \tag{4.17}$$

Using (4.16) and (4.17), we can give a geometric interpretation of a formula obtained in § 2.

PROPOSITION 4.1. *Let $\phi_1(v_1)$ and $\phi_2(v_2)$ be the functions given in* (2.23) *and* (2.24). *Then the curvatures $\kappa_1(v_1)$ of $\gamma_1(v_2)$ and $\kappa_2(v_2)$ of $\gamma_2(v_2)$ are given by*

$$\begin{aligned} \kappa_1(v_1) &= 2\sqrt{2}\,\phi_1(v_1) \\ \kappa_2(v_2) &= 2\sqrt{2}\,\phi_2(v_2). \end{aligned} \tag{4.18}$$

In the following theorems, we give characterizations of surfaces which are product spaces in $\boldsymbol{R}^4$. Theorem 2 also follows from the work of Chen-Yamaguchi [1].

THEOREM 2. *Let M be a surface in $\boldsymbol{R}^4$ which has zero curvature and flat normal connection. If $G(M)$ is totally geodesic in $G_{2,4}$, M is locally the Riemannian product of two plane curves.*

Proof. If $G(M)$ is totally geodesic, the curvatures of γ_1 and γ_2 are identically zero. This implies that $e_1\left(\frac{1}{\mu}\right) \equiv 0$ and $e_2\left(\frac{1}{\lambda}\right) \equiv 0$ on M. Then, as we see in the proof for Proposition 2.1, M is locally the Riemannian product of two plane curves. Q. E. D.

THEOREM 3. *Let M be as in Theorem 2. In addition, suppose that M is compact. Then if the length of the mean curvature vector of $G(M) \subset G_{2,4}$ is constant, M is globally the Riemannian product of two closed plane curves.*

Proof. The mean curvature vector H of $G(M)$ in $G_{2,4}$ is given by $H = \kappa_1\tilde{\nu}_1 + \kappa_2\tilde{\nu}_2$. Hence, if $|H|^2 = \kappa_1(v_1)^2 + \kappa_2(v_2)^2$ is constant, both $\kappa_1(v_1)$ and $\kappa_2(v_2)$ must be constant. Then $e_1\left(\frac{1}{\mu}\right) = \frac{1}{\sqrt{2}}(\kappa_1(v_1) + \kappa_2(v_2))$ is constant and the theorem follows from Proposition 2.1. Q. E. D.

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