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ON AN ESTIMATE FOR $\int_0^\infty m(t, E(-z, q))t^{-1-\beta}dt$

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In this paper we shall give a lower estimate for

$$\int_0^\infty \frac{m(t,\ E(-z,\ q))}{t^{1+\beta}}dt,$$

where $E(z, q)$ is the Weierstrass primary factor of genus q, β a constant satisfying $q<\beta<q+1$ and $m(t, f)$ the Nevanlinna proximity function. Our result is the following

THEOREM.
$$\int_0^\infty \frac{m(t,\ E(-z,\ q))}{t^{1+\beta}}dt>\frac{1}{\beta^2\kappa(\beta)},$$
where $\kappa(\beta)$ *is the constant defined by*

$$\begin{cases} \dfrac{|\sin \pi\beta|}{q+|\sin \pi\beta|} & (q<\beta<q+1/2), \\ \dfrac{|\sin \pi\beta|}{q+1} & (q+1/2\leqq\beta<q+1). \end{cases}$$

In the above estimation equality does not occur. In order to show this inequality part we need a rough tracing of the level curve

$$\log|1+te^{i\theta}|+\sum_{j=1}^{q}(-1)^j j^{-1}t^j \cos j\theta=0.$$

However we do not need its precise analysis.

Proof. Let us consider

$$I_F=\int_0^\infty \frac{1}{\pi}\int_F \log|E(-te^{i\theta},\ q)|d\theta\frac{dt}{t^{1+\beta}},$$

where F is a measurable subset of $[0, \pi]$. Evidently

$$I_F\leqq\int_0^\infty \frac{m(t,\ E(-z,\ q))}{t^{1+\beta}}dt$$

for any F. Further it is known [1] that it is possible to change the order of

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integration. It is known that

$$\frac{1}{\pi}\int_0^\infty \frac{\log|E(-te^{i\theta}, q)|}{t^{1+\beta}}dt = \frac{\cos\theta\beta}{\beta\sin\pi\beta}.$$

Hence

$$I_F = \int_F \frac{\cos\theta\beta}{\beta\sin\pi\beta}d\theta.$$

If q is even, that is, $\sin\pi\beta>0$, then

$$\frac{1}{\beta}\int_F \cos\theta\beta\, d\theta \leqq \frac{1}{\beta}\int_0^\pi (\cos\theta\beta)^+ d\theta$$

$$= \begin{cases} (q+\sin\beta\pi)/\beta^2 & (q<\beta<q+1/2), \\ (q+1)/\beta^2 & (q+1/2\leqq\beta<q+1). \end{cases}$$

Here equality occurs by a suitable choice of F. If q is odd, that is, $\sin\pi\beta<0$, then

$$\frac{1}{\beta}\int_F \cos\theta\beta\, d\theta \geqq \frac{1}{\beta}\int_0^\pi (\cos\theta\beta)^- d\theta$$

$$= \begin{cases} -(q+|\sin\pi\beta|)/\beta^2 & (q<\beta<q+1/2), \\ -(q+1)/\beta^2 & (q+1/2\leqq\beta<q+1). \end{cases}$$

Again equality occurs by a suitable choice of F in this case. Denoting this special F by F_0

$$I_{F_0} = 1/\kappa(\beta)\beta^2.$$

Hence

$$\int_0^\infty \frac{m(t, E(-z, q))}{t^{1+\beta}}dt \geqq \frac{1}{\kappa(\beta)\beta^2}.$$

At the origin the rays defined by $\cos(q+1)\theta=0$ are tangents of branches of the level curve indicated already. Around the point at infinity the rays defined by $\cos q\theta=0$ are asymptotics of branches. There is a loop around -1, which starts from the origin and ends at the origin and lies in the sector defined by

$$\frac{2q+1}{2(q+1)}\pi < \theta < \frac{2q+3}{2(q+1)}\pi.$$

Hence

$$m(t, E(-z, q)) > \frac{1}{\pi}\int_{F_0} \log|E(-te^{i\theta}, q)|\, d\theta$$

for any t. Hence we have the desired inequality part.

BIBLIOGRAPHY

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