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ON THE CONSTRUCTION OF LINEARLY INDEPENDENT VECTORS WITH VARIABLE COMPONENTS

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§ 1. Introduction.

We use the same notations as in a previous paper [1]. Let J be a closed interval $[\gamma, \delta]=\{t \mid \gamma \leqq t \leqq \delta,\ t \in \mathbf{R}\}$. Let $C^{\mu}(J, \mathbf{C})$ denote the totality of complex-valued functions defined and of class C^{μ} on J ($\mu=0, 1, \cdots, \infty$). Hereafter we fix some μ.

For the sake of brevity, we denote $C^{\mu}(J, \mathbf{C})$ by $K(J)$, and $K(J)^{n}$ by $M(J)$:

$$M(J)=\{\boldsymbol{f}(t)=\mathrm{col}(f_{1}(t), f_{2}(t), \cdots, f_{n}(t)) \mid f_{j}(t) \in K(J),\ j=1, 2, \cdots, n\}.$$

Let $X(t)$ be an $n \times h$ matrix whose components all belong to $K(J)$:

$$X(t)=\begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1h}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2h}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nh}(t) \end{pmatrix}, \tag{1.1}$$

where h is an integer such that $1 \leqq h \leqq n-1$, and suppose that a condition

$$\operatorname{rank} X(t)=h \tag{1.2}$$

is satisfied on J.

The first purpose of this paper is to prove the following theorem:

THEOREM 1. *Let $X(t)$ be the $n \times h$ matrix given above and satisfying the condition* (1.2) *on J. Then there exists a vector $\boldsymbol{y}(t) \in M(J)$ such that*

$$\begin{cases} \operatorname{rank} \boldsymbol{y}(t)=1 \quad \text{on } J, \\ \operatorname{rank}(X(t), \boldsymbol{y}(t))=h+1 \quad \text{on } J. \end{cases} \tag{1.3}$$

As a corollary of Theorem 1, we obtain immediately the following theorem:

THEOREM 2. *Let $X(t)$ be the $n \times h$ matrix given above and satisfying the condition* (1.2) *on J. Then there exists an $n \times (n-h)$ matrix $Y(t)$ whose components*

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all belong to $K(J)$:

$$Y(t)=\begin{pmatrix} y_{1,h+1}(t) & y_{1,h+2}(t) & \cdots & y_{1n}(t) \\ y_{2,h+1}(t) & y_{2,h+2}(t) & \cdots & y_{2n}(t) \\ \vdots & \vdots & & \vdots \\ y_{n,h+1}(t) & y_{n,h+2}(t) & \cdots & y_{nn}(t) \end{pmatrix} \tag{1.4}$$

such that

$$\begin{cases} \operatorname{rank} Y(t)=n-h & \textit{on } J, \\ \operatorname{rank}(X(t), Y(t))=n & \textit{on } J. \end{cases} \tag{1.5}$$

Now, let I be a closed interval $[\alpha, \beta]=\{t \mid \alpha \leqq t \leqq \beta,\ t \in \mathbf{R}\}$ and let $B(t)$ be a square matrix of degree n whose components all belong to $K(I)$:

$$B(t)=\begin{pmatrix} b_{11}(t) & b_{12}(t) & \cdots & b_{1n}(t) \\ b_{21}(t) & b_{22}(t) & \cdots & b_{2n}(t) \\ \vdots & \vdots & & \vdots \\ b_{n1}(t) & b_{n2}(t) & \cdots & b_{nn}(t) \end{pmatrix}. \tag{1.6}$$

We assume that for a positive integer s: $2 \leqq s \leqq n-1$, a condition

$$\operatorname{rank} B(t)=n-s(=r) \tag{1.7}$$

is satisfied on I, and consider a linear equation

$$B(t)\boldsymbol{f}(t)=\boldsymbol{o} \quad \text{on } I; \quad \boldsymbol{f}(t) \in M(I). \tag{1.8}$$

We denote the totality of solutions of (1.8) by $W(I)$:

$$W(I)=\{\boldsymbol{f}(t) \in M(I) \mid B(t)\boldsymbol{f}(t)=\boldsymbol{o} \text{ on } I\}.$$

Then, we know that there exist s vectors $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_s(t)$ belonging to $W(I)$, such that

$$\operatorname{rank}(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_s(t))=s \quad \text{on } I.$$

For the proof of this fact, see, for example, the proof of Theorem in the previous paper [1].

The second purpose of this paper is to prove the following theorem:

THEOREM 3. *Let* $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_{s'}(t)$ *be* s' *prescribed vectors belonging to* $W(I)$ *and satisfying a condition*

$$\operatorname{rank}(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_{s'}(t))=s' \quad \textit{on } I, \tag{1.9}$$

where s' *is a positive integer such that* $1 \leqq s' < s$.

Then there exist $(s-s')$ *vectors* $\boldsymbol{y}_{s'+1}(t), \boldsymbol{y}_{s'+2}(t), \cdots, \boldsymbol{y}_s(t)$ *belonging to* $W(I)$ *and satisfying conditions*

$$(1.10)\qquad \begin{cases} \operatorname{rank}(\boldsymbol{y}_{s'+1}(t), \cdots, \boldsymbol{y}_s(t))=s-s' & on\ I, \\ \operatorname{rank}(\boldsymbol{x}_1(t), \cdots, \boldsymbol{x}_{s'}(t), \boldsymbol{y}_{s'+1}(t), \cdots, \boldsymbol{y}_s(t))=s & on\ I. \end{cases}$$

In general, we denote a minor of degree r of the matrix $B(t)$ which is given by (1.6), by

$$B\begin{pmatrix} j_1 & j_2 \cdots & j_r \\ k_1 & k_2 \cdots & k_r \end{pmatrix} = \begin{vmatrix} b_{j_1k_1}(t) & b_{j_1k_2}(t) \cdots & b_{j_1k_r}(t) \\ b_{j_2k_1}(t) & b_{j_2k_2}(t) \cdots & b_{j_2k_r}(t) \\ \vdots & \vdots & \vdots \\ b_{j_rk_1}(t) & b_{j_rk_2}(t) \cdots & b_{j_rk_r}(t) \end{vmatrix}$$

$$\begin{pmatrix} 1 \leqq j_1 < j_2 < \cdots < j_r \leqq n \\ 1 \leqq k_1 < k_2 < \cdots < k_r \leqq n \end{pmatrix},$$

and then, a minor of degree h of the $n \times h$ matrix $X(t)$ which is given by (1.1), is especially denoted by

$$X\begin{pmatrix} k_1 & k_2 \cdots & k_h \\ 1 & 2 \cdots & h \end{pmatrix} = \begin{vmatrix} x_{k_11}(t) & x_{k_12}(t) \cdots & x_{k_1h}(t) \\ x_{k_21}(t) & x_{k_22}(t) \cdots & x_{k_2h}(t) \\ \vdots & \vdots & \vdots \\ x_{k_h1}(t) & x_{k_h2}(t) \cdots & x_{k_hh}(t) \end{vmatrix}$$

$$(1 \leqq k_1 < k_2 < \cdots < k_h \leqq n).$$

In § 2, we shall give two lemmas which will be used for the proof of Theorem 1, and in § 3, we shall prove Theorem 1.

In § 4, we shall give a summary of the matters which are necessary for the proof of Theorem 3, and in §§ 5-6, we shall prove Theorem 3.

§ 2. Lemmas.

LEMMA 1. *Let* J_0 *be a closed interval* $[\gamma_0, \delta_0]=\{t \mid \gamma_0 \leqq t \leqq \delta_0,\ t \in \mathbf{R}\}$ *and let* $\varphi_\tau(t)$ $(\tau=1, 2, \cdots, \tau_0)$ *be a finite number of real-valued continuous functions defined on* J_0. *Then there exists a closed interval* $J^*=[\gamma^*, \delta^*]$ *contained in* J_0, *such that each of* $\varphi_\tau(t)$ $(\tau=1, 2, \cdots, \tau_0)$ *is one-signed or identically equal to zero on* J^* *respectively.*

Proof. Put

$$E_+^{(1)}=\{t \in J_0 \mid \varphi_1(t)>0\}, \quad E_-^{(1)}=\{t \in J_0 \mid \varphi_1(t)<0\},$$

$$E_0^{(1)}=\{t \in J_0 \mid \varphi_1(t)=0\}.$$

Then, $E_+^{(1)}$, $E_-^{(1)}$ and $E_0^{(1)}$ are disjoint with each other and $E_+^{(1)} \cup E_-^{(1)} \cup E_0^{(1)} = J_0$. $E_+^{(1)}$ and $E_-^{(1)}$ are relatively open on J_0. Therefore, if $E_+^{(1)} \neq \emptyset$ or $E_-^{(1)} \neq \emptyset$, we can find a closed interval $J_1^*=[\gamma_1^*, \delta_1^*] \subset J_0$ such that $\varphi_1(t)>0$ or $\varphi_1(t)<0$ on J_1^*. If $E_+^{(1)}=\emptyset$ and $E_-^{(1)}=\emptyset$, we see $\phi_1(t) \equiv 0$ on $J_1^*=J_0$.

By repeating the process just described, for the interval J_1^* and the functions

$\varphi_\tau(t)$ $(\tau=2, 3, \cdots, \tau_0)$ successively, we obtain the desired interval $J^*=[\gamma^*, \delta^*]$.

Now, for any value $t_0\in\mathbf{R}$, we put

$$e_+(t\,;t_0)=\begin{cases} 0 & \text{for } t\leqq t_0,\\ \exp\left\{-\dfrac{1}{(t-t_0)^2}\right\} & \text{for } t>t_0\,; \end{cases}$$

$$e_-(t\,;t_0)=\begin{cases} \exp\left\{-\dfrac{1}{(t-t_0)^2}\right\} & \text{for } t<t_0,\\ 0 & \text{for } t\geqq t_0, \end{cases}$$

and for any values t_1, $t_2\in\mathbf{R}$ such that $t_1<t_2$, we put

$$e(t\,;t_1, t_2)=\begin{cases} 0 & \text{for } t\leqq t_1,\\ \exp\left\{-\dfrac{1}{(t-t_1)^2}-\dfrac{1}{(t-t_2)^2}\right\} & \text{for } t_1<t<t_2,\\ 0 & \text{for } t\geqq t_2. \end{cases}$$

Then we see that the functions $e_+(t\,;t_0)$, $e_-(t\,;t_0)$ and $e(t\,;t_1, t_2)$ belong to $C^\infty(\mathbf{R}, \mathbf{R})$.

Next, let $J_1=(\gamma_1, \delta_1)$ and $J_2=(\gamma_2, \delta_2)$ be open intervals on $\mathbf{R}$ such that $\gamma_1<\gamma_2<\delta_1<\delta_2$.

Furthermore let $\theta(t)$ and $\omega(t)$ be functions belonging to $K(\bar{J}_1)$ and to $K(\bar{J}_2)$ respectively, such that each of $\theta_{(r)}(t)$ $(=\mathrm{Re}\,\theta(t))$, $\theta_{(i)}(t)$ $(=\mathrm{Im}\,\theta(t))$, $\omega_{(r)}(t)$ $(=\mathrm{Re}\,\omega(t))$ and $\omega_{(i)}(t)$ $(=\mathrm{Im}\,\omega(t))$ is one-signed or identically equal to zero on $\bar{J}_1\cap\bar{J}_2=[\gamma_2, \delta_1]$.

Under these circumstances, we shall prove the following lemma:

LEMMA 2. *Let*

$$f(t)=(c_1+id_1)e(t\,;\gamma_1, \delta_1)\,;\quad i=\sqrt{-1},$$

where c_1 *and* d_1 *are real non-zero constants. Then, there exist two real non-zero constants* c_2 *and* d_2 *such that a function*

$$g(t)=(c_2+id_2)e(t\,;\gamma_2, \delta_2)$$

satisfies conditions

$$g(t)-\omega(t)f(t)\neq 0 \quad \text{on } J_2=(\gamma_2, \delta_2), \tag{2.1}$$

and

$$f(t)-\theta(t)g(t)\neq 0 \quad \text{on } J_1=(\gamma_1, \delta_1). \tag{2.2}$$

Proof. We, at the beginning, take note of the fact that the functions $\theta_{(r)}(t)$, $\theta_{(i)}(t)$, $\omega_{(r)}(t)$ and $\omega_{(i)}(t)$ are continuous and bounded on the interval

$\bar{J}_1 \cap \bar{J}_2 = [\gamma_2, \delta_1]$.

Let us put

$$f_{(r)}(t) = \text{Re}\, f(t), \quad f_{(i)}(t) = \text{Im}\, f(t), \quad g_{(r)}(t) = \text{Re}\, g(t), \quad g_{(i)}(t) = \text{Im}\, g(t).$$

We show first that by choosing either c_2 or d_2 suitably, we can make the function $g(t)$ satisfy the condition (2.1).

Since $\omega(t)f(t) \equiv 0$ on the interval $[\delta_1, \delta_2)$, we have only to determine the non-zero constants c_2 and d_2, such that the condition (2.1) is satisfied on the interval $J_1 \cap J_2 = (\gamma_2, \delta_1)$ instead of J_2.

For the determination of the constants c_2 and d_2, we shall distinguish four cases, according to the values of $\omega_{(r)}(t)$ and $\omega_{(i)}(t)$ on $\bar{J}_1 \cap \bar{J}_2$:

Case I-(i) $\omega(t) \equiv 0$ on $\bar{J}_1 \cap \bar{J}_2$,
Case I-(ii) $\omega_{(r)}(t) \neq 0$ and $\omega_{(i)}(t) \equiv 0$ on $\bar{J}_1 \cap \bar{J}_2$,
Case I-(iii) $\omega_{(r)}(t) \equiv 0$ and $\omega_{(i)}(t) \neq 0$ on $\bar{J}_1 \cap \bar{J}_2$,
Case I-(iv) $\omega_{(r)}(t) \neq 0$ and $\omega_{(i)}(t) \neq 0$ on $\bar{J}_1 \cap \bar{J}_2$.

In Case I-(i), the condition (2.1) is satisfied for all non-zero values of c_2 and d_2, because we have $\omega(t)f(t) \equiv 0$ on $\bar{J}_2$.

In Case I-(ii), since

$$\text{Re}\, \omega(t)f(t) = \omega_{(r)}(t)f_{(r)}(t) = c_1\omega_{(r)}(t)e(t\,;\gamma_1, \delta_1) \quad \text{on } \bar{J}_1 \cap \bar{J}_2\,;$$

$$\text{Im}\, \omega(t)f(t) = \omega_{(r)}(t)f_{(i)}(t) = d_1\omega_{(r)}(t)e(t\,;\gamma_1, \delta_1) \quad \text{on } \bar{J}_1 \cap \bar{J}_2,$$

the condition (2.1) is satisfied, if we choose either the constant c_2 with the opposite sign to $c_1\omega_{(r)}(t)$ on $\bar{J}_1 \cap \bar{J}_2$, or the constant d_2 with the opposite sign to $d_1\omega_{(r)}(t)$ on $\bar{J}_1 \cap \bar{J}_2$.

In Case I-(iii), since

$$\text{Re}\, \omega(t)f(t) = -\omega_{(i)}(t)f_{(i)}(t) = -d_1\omega_{(i)}(t)e(t\,;\gamma_1, \delta_1) \quad \text{on } \bar{J}_1 \cap \bar{J}_2\,;$$

$$\text{Im}\, \omega(t)f(t) = \omega_{(i)}(t)f_{(r)}(t) = c_1\omega_{(i)}(t)e(t\,;\gamma_1, \delta_1) \quad \text{on } \bar{J}_1 \cap \bar{J}_2,$$

the condition (2.1) is satisfied, if we choose either the constant c_2 with the opposite sign to $-d_1\omega_{(i)}(t)$ on $\bar{J}_1 \cap \bar{J}_2$, or the constant d_2 with the opposite sign to $c_1\omega_{(i)}(t)$ on $\bar{J}_1 \cap \bar{J}_2$.

In Case I-(iv), we have

$$\begin{aligned} \text{Re}\, \omega(t)f(t) &= \omega_{(r)}(t)f_{(r)}(t) - \omega_{(i)}(t)f_{(i)}(t) \\ &= \{c_1\omega_{(r)}(t) - d_1\omega_{(i)}(t)\}\, e(t\,;\gamma_1, \delta_1) \quad \text{on } \bar{J}_1 \cap \bar{J}_2\,; \end{aligned}$$

$$\begin{aligned} \text{Im}\, \omega(t)f(t) &= \omega_{(r)}(t)f_{(i)}(t) + \omega_{(i)}(t)f_{(r)}(t) \\ &= \{d_1\omega_{(r)}(t) + c_1\omega_{(i)}(t)\}\, e(t\,;\gamma_1, \delta_1) \quad \text{on } \bar{J}_1 \cap \bar{J}_2. \end{aligned}$$

Since

$$\{c_1\omega_{(r)}(t)\}\cdot\{-d_1\omega_{(i)}(t)\}=-c_1d_1\omega_{(r)}(t)\omega_{(i)}(t)\neq 0 \quad \text{on } \bar{J}_1\cap\bar{J}_2\,;$$

$$\{d_1\omega_{(r)}(t)\}\cdot\{c_1\omega_{(i)}(t)\}=c_1d_1\omega_r(t)\omega_{(i)}(t)\neq 0 \quad \text{on } \bar{J}_1\cap\bar{J}_2,$$

one of these two products has the positive sign. Therefore the two factors $c_1\omega_{(r)}(t)$ and $-d_1\omega_{(i)}(t)$, or $d_1\omega_{(r)}(t)$ and $c_1\omega_{(i)}(t)$ in the above product which has the positive sign, have the same sign as each other on $\bar{J}_1\cap\bar{J}_2$. Hence, one of $\operatorname{Re}\omega(t)f(t)$ and $\operatorname{Im}\omega(t)f(t)$ has the definite sign on $\bar{J}_1\cap\bar{J}_2$.

If $\operatorname{Re}\omega(t)f(t)$ has the definite sign on $\bar{J}_1\cap\bar{J}_2$, then we choose the constant c_2 with the opposite sign to $\operatorname{Re}\omega(t)f(t)$ on $\bar{J}_1\cap\bar{J}_2$. If $\operatorname{Im}\omega(t)f(t)$ has the definite sign on $\bar{J}_1\cap\bar{J}_2$, then we choose the constant d_2 with the opposite sign to $\operatorname{Im}\omega(t)f(t)$ on $\bar{J}_1\cap\bar{J}_2$.

The procedure stated above, means that by choosing suitably one of the constants c_2 and d_2 in all cases, we can make the condition (2.1) be satisfied.

Next, under the circumstances that the condition (2.1) has been satisfied by determining suitably one of the constants c_2 and d_2, we shall show that we can choose the other of them so that the condition (2.2) is satisfied.

Since $f(t)\neq 0$ on the interval $J_1=(\gamma_1, \delta_1)$ and $\theta(t)g(t)\equiv 0$ on the interval $(\gamma_1, \gamma_2]$ for all non-zero values of c_2 and d_2, we have only to determine the non-zero constants c_2 and d_2, so that the condition (2, 2) is satisfied on the interval $J_1\cap J_2=(\gamma_2, \delta_1)$ instead of J_1.

For the accomplishment of our purpose, we shall distinguish four cases, according to the values of $\theta_{(r)}(t)$ and $\theta_{(i)}(t)$ on $\bar{J}_1\cap\bar{J}_2$:

Case II-(i) $\theta(t)\equiv 0$ on $\bar{J}_1\cap\bar{J}_2$,
Case II-(ii) $\theta_{(r)}(t)\neq 0$ and $\theta_{(i)}(t)\equiv 0$ on $\bar{J}_1\cap\bar{J}_2$,
Case II-(iii) $\theta_{(r)}(t)\equiv 0$ and $\theta_{(i)}(t)\neq 0$ on $\bar{J}_1\cap\bar{J}_2$,
Case II-(iv) $\theta_{(r)}(t)\neq 0$ and $\theta_{(i)}(t)\neq 0$ on $\bar{J}_1\cap\bar{J}_2$.

In Case II-(i), we have $\theta(t)g(t)\equiv 0$ on $\bar{J}_1\cap\bar{J}_2$ for all non-zero values of c_2 and d_2, and further $f(t)\neq 0$ on J_1. Hence the condition (2.2) is satisfied for all non-zero values of c_2 and d_2.

In Case II-(ii), we have

$$\operatorname{Re}\theta(t)g(t)=\theta_{(r)}(t)g_{(r)}(t)=c_2\theta_{(r)}(t)e(t\,;\gamma_2, \delta_2) \quad \text{on } \bar{J}_1\cap\bar{J}_2\,;$$

$$\operatorname{Im}\theta(t)g(t)=\theta_{(r)}(t)g_{(i)}(t)=d_2\theta_{(r)}(t)e(t\,;\gamma_2, \delta_2) \quad \text{on } \bar{J}_1\cap\bar{J}_2.$$

Although one of the constants c_2 and d_2 is already fixed in Cases I-(i)~I-(iv), if we choose the other of them so that either

" $c_2\theta_{(r)}(t)$ has the opposite sign to c_1 on $\bar{J}_1\cap\bar{J}_2$ ",

or

" $d_2\theta_{(r)}(t)$ has the opposite sign to d_1 on $\bar{J}_1\cap\bar{J}_2$ ",

then the condition (2.2) is satisfied.

In Case II-(iii), since

$$\mathrm{Re}\ \theta(t)g(t)=-\theta_{(i)}(t)g_{(i)}(t)=-d_2\theta_{(i)}(t)e(t\,;\gamma_2,\ \delta_2)\quad \text{on}\ \bar{J}_1\cap\bar{J}_2\,;$$

$$\mathrm{Im}\ \theta(t)g(t)=\theta_{(i)}(t)g_{(r)}(t)=c_2\theta_{(i)}(t)e(t\,;\gamma_2,\ \delta_2)\qquad \text{on}\ \bar{J}_1\cap\bar{J}_2,$$

we have only to determine one of the constants c_2 and d_2 so that either

" $-d_2\theta_{(i)}(t)$ has the opposite sign to c_1 on $\bar{J}_1\cap\bar{J}_2$",

or

" $c_2\theta_{(i)}(t)$ has the opposite sign to d_1 on $\bar{J}_1\cap\bar{J}_2$".

In Case II-(iv), we have

$$\mathrm{Re}\ \theta(t)g(t)=\theta_{(r)}(t)g_{(r)}(t)-\theta_{(i)}(t)g_{(i)}(t)$$
$$=\{c_2\theta_{(r)}(t)-d_2\theta_{(i)}(t)\}\,e(t\,;\gamma_2,\ \delta_2)\,;$$
$$\mathrm{Im}\ \theta(t)g(t)=\theta_{(r)}(t)g_{(i)}(t)+\theta_{(i)}(t)g_{(r)}(t)$$
$$=\{d_2\theta_{(r)}(t)+c_2\theta_{(i)}(t)\}\,e(t\,;\gamma_2,\ \delta_2)\,.$$

Although one of the constants c_2 and d_2 is already fixed in Cases I-(i)~I-(iv), we can choose the other of them so that either

" $c_2\theta_{(r)}(t)-d_2\theta_{(i)}(t)$ has the opposite sign to c_1 on $\bar{J}_1\cap\bar{J}_2$",

or

" $d_2\theta_{(r)}(t)+c_2\theta_{(i)}(t)$ has the opposite sign to d_1 on $\bar{J}_1\cap\bar{J}_2$".

By means of this choice, the condition (2.2) is satisfied.
Thus this lemma has been completely proved.

Remark 1. Replacing $f(t)=(c_1+id_1)e(t\,;\gamma_1,\ \delta_1)$ by

$$f(t)=(c_1+id_1)e_-(t\,;\delta_1)\,,$$

we obtain a result similar to Lemma 2.

Remark 2. Replacing $g(t)=(c_2+id_2)e(t\,;\gamma_2,\ \delta_2)$ by

$$g(t)=(c_2+id_2)e_+(t\,;\gamma_2)\,,$$

we obtain a result similar to Lemma 2.

§ 3. Proof of Theorem 1.

We can form, by assumption, a set $\{J_\iota\}_{\iota=1}^{\iota_0}$ of intervals possessing the following properties:

(i) $\bigcup_{\iota=1}^{\iota_0} J_\iota = J$;

(ii) $J_1=[\gamma_1, \delta_1)$, $J_{\iota_0}=(\gamma_{\iota_0}, \delta_{\iota_0}]$, $\gamma_1=\gamma$, $\delta_{\iota_0}=\delta$, $J_\iota=(\gamma_\iota, \delta_\iota)$ $(\iota=2, 3, \cdots, \iota_0-1)$;

(iii) $J_\iota \cap J_{\iota+1} \neq \emptyset$ $(\iota=1, 2, \cdots, \iota_0-1)$, $J_\iota \cap J_{\iota'}=\emptyset$ $(\iota+1<\iota', \iota=1, 2, \cdots, \iota_0-1)$, that is, $\gamma_1<\gamma_2<\delta_1<\cdots<\gamma_\iota<\delta_{\iota-1}<\gamma_{\iota+1}<\delta_\iota<\cdots<\delta_{\iota_0-2}<\gamma_{\iota_0}<\delta_{\iota_0-1}<\delta_{\iota_0}$ $(\iota=2, 3, \cdots, \iota_0-1)$;

(iv) For each J_ι, there exists a minor of degree h of $X(t)$ which does not vanish on $\bar{J}_\iota$.

We consider first the intervals J_1 and J_2, and choose two minors $X\begin{pmatrix} k_1 & k_2 \cdots k_h \\ 1 & 2 \cdots h \end{pmatrix}$ and $X\begin{pmatrix} m_1 & m_2 \cdots m_h \\ 1 & 2 \cdots h \end{pmatrix}$ of degree h of $X(t)$ such that a condition

$$X\begin{pmatrix} k_1 & k_2 \cdots k_h \\ 1 & 2 \cdots h \end{pmatrix} \neq 0 \tag{3.1}$$

is satisfied on $\bar{J}_1$ and a condition

$$X\begin{pmatrix} m_1 & m_2 \cdots m_h \\ 1 & 2 \cdots h \end{pmatrix} \neq 0 \tag{3.2}$$

is satisfied on $\bar{J}_2$.

We define an $(n-h)$-tuple $(k'_{h+1}, k'_{h+2}, \cdots, k'_n)$ for $1 \leqq k_1<k_2<\cdots<k_h \leqq n$ in such a way that $1 \leqq k'_{h+1}<k'_{h+2}<\cdots<k'_n \leqq n$ and $\{k_1, \cdots, k_h, k'_{h+1}, \cdots, k'_n\}=\{1, 2, \cdots, n\}$. That is, $k_1<k_2<\cdots<k_h$ and $k'_{h+1}<k'_{h+2}<\cdots<k'_n$ form a complete system of indices $\{1, 2, \cdots, n\}$. An $(n-h)$-tuple $(m'_{h+1}, m'_{h+2}, \cdots, m'_n)$ is also defined for $1 \leqq m_1<m_2<\cdots<m_h \leqq n$ in the same manner.

We put

$$\hat{\boldsymbol{x}}_{k_\rho}(t)=(x_{k_\rho 1}(t), x_{k_\rho 2}(t), \cdots, x_{k_\rho h}(t)) \quad (\rho=1, 2, \cdots, h),$$

$$\hat{\boldsymbol{x}}_{k'_\sigma}(t)=(x_{k'_\sigma 1}(t), x_{k'_\sigma 2}(t), \cdots, x_{k'_\sigma h}(t)) \quad (\sigma=h+1, h+2, \cdots, n),$$

$$\hat{\boldsymbol{x}}_{m_\rho}(t)=(x_{m_\rho 1}(t), x_{m_\rho 2}(t), \cdots, x_{m_\rho h}(t)) \quad (\rho=1, 2, \cdots, h),$$

$$\hat{\boldsymbol{x}}_{m'_\sigma}(t)=(x_{m'_\sigma 1}(t), x_{m'_\sigma 2}(t), \cdots, x_{m'_\sigma h}(t)) \quad (\sigma=h+1, h+2, \cdots, n).$$

Then it follows from the conditions (3.1) and (3.2), that there exist functions $\theta_{\sigma\rho}(t)$ $(\rho=1, 2, \cdots, h; \sigma=h+1, h+2, \cdots, n)$ belonging to $K(\bar{J}_1)$ and functions $\omega_{\sigma\rho}(t)$ $(\rho=1, 2, \cdots, h; \sigma=h+1, h+2, \cdots, n)$ belonging to $K(\bar{J}_2)$, such that

$$\hat{\boldsymbol{x}}_{k'_\sigma}(t)=\sum_{\rho=1}^{h} \theta_{\sigma\rho}(t)\hat{\boldsymbol{x}}_{k_\rho}(t) \quad (\sigma=h+1, h+2, \cdots, n) \quad \text{on } \bar{J}_1, \tag{3.3}$$

and

$$\hat{\boldsymbol{x}}_{m'_\sigma}(t)=\sum_{\rho=1}^{h} \omega_{\sigma\rho}(t)\hat{\boldsymbol{x}}_{m_\rho}(t) \quad (\sigma=h+1, h+2, \cdots, n) \quad \text{on } \bar{J}_2. \tag{3.4}$$

The first step.

We determine a vector $\boldsymbol{y}(t)=\mathrm{col}(y_1(t), y_2(t), \cdots, y_n(t))$ on $\bar{J}_1$ in the following manner.

Concerning the component $y_{k'_{h+1}}(t)$, we put

$$y_{k'_{h+1}}(t)=(c_1+id_1)e_-(t;\delta_1);\quad i=\sqrt{-1},$$

where c_1 and d_1 are arbitrary real non-zero constants. As a matter of fact, it suffices for our present purpose that at least, any one of the constants c_1 and d_1 is not equal to zero. However, we take the constants c_1 and d_1 which are both non-zero for the sake of generality.

Concerning the other components of $\boldsymbol{y}(t)$, we put

$$y_{k_\rho}(t)\equiv 0 \text{ on } J\ (\rho=1, \cdots, h) \text{ and } y_{k'_\sigma}(t)\equiv 0 \text{ on } J\ (\sigma=h+2, \cdots, n).$$

Then, in virtue of the fact that $y_{k'_{h+1}}(t)\neq 0$ on J_1 and the condition (3.1) is satisfied on $\bar{J}_1$, we see

$$\mathrm{rank}\, \boldsymbol{y}(t)=1 \text{ on } J_1 \text{ and } \mathrm{rank}\,(X(t), \boldsymbol{y}(t))=h+1 \text{ on } J_1.$$

The second step.

We shall next construct a vector $\boldsymbol{y}(t)$, so that we have

$$\left\{\begin{array}{ll} \mathrm{rank}\, \boldsymbol{y}(t)=1 & \text{on } J_1\cup J_2; \\ \mathrm{rank}\,(X(t), \boldsymbol{y}(t))=h+1 & \text{on } J_1\cup J_2. \end{array}\right. \tag{3.5}$$

For the construction of $\boldsymbol{y}(t)$ on $J_1\cup J_2$, we shall distinguish three cases, according to the relation between the indices $(k_1, \cdots, k_h, k'_{h+1}, \cdots, k'_n)$ and $(m_1, \cdots, m_h, m'_{h+1}, \cdots, m'_n)$:

Case S-(i) There exists an index $\sigma(1)$ such that $h+1\leqq\sigma(1)\leqq n$, $m'_{\sigma(1)}=k'_{h+1}$.
Case S-(ii) There exist two indices $\rho(1)$ and $\sigma(2)$ such that

$$1\leqq\rho(1)\leqq h,\ m_{\rho(1)}=k'_{h+1} \text{ and } h+1\leqq\sigma(2)\leqq n,\ m'_{h+1}=k'_{\sigma(2)}.$$

Case S-(iii) There exist two indices $\rho(1)$ and $\rho(2)$ such that

$$1\leqq\rho(1)\leqq h,\ m_{\rho(1)}=k'_{h+1} \text{ and } 1\leqq\rho(2)\leqq h,\ m'_{h+1}=k_{\rho(2)}.$$

In Case S-(i), we modify the component $y_{m'_{\sigma(1)}}(t)$ $(\equiv y_{k'_{h+1}}(t))$ determined at the first step, in the following way:

$$y_{m'_{\sigma(1)}}(t)\ (\equiv y_{k'_{h+1}}(t))=(c_1+id_1)e_-(t;\delta_2),$$

and we leave the other components of $\boldsymbol{y}(t)$ as they are.

Then we have the condition (3.5).

In Cases S-(ii) and S-(iii), we must treat the function $\theta_{h+1,\rho(2)}(t)$ which appears in the relation (3.3), and the function $\omega_{h+1,\rho(1)}(t)$ which appears in the relation (3.4).

If we put

$$\theta_{h+1,\rho(2)}(t)=\varphi_1(t)+i\psi_1(t);\quad \varphi_1(t), \psi_1(t)\in C^\mu(\bar{J}_1;\mathbf{R}),$$
$$\omega_{h+1,\rho(1)}(t)=\varphi_2(t)+i\psi_2(t);\quad \varphi_2(t), \psi_2(t)\in C^\mu(\bar{J}_2;\mathbf{R}),$$

then, in virtue of Lemma 1, we can choose a closed subinterval $[\gamma_2^*, \delta_1^*]$ of the interval $[\gamma_2, \delta_1]$, such that each of the functions $\varphi_1(t)$, $\psi_1(t)$, $\varphi_2(t)$ and $\psi_2(t)$ is one-signed or identically equal to zero on the interval $[\gamma_2^*, \delta_1^*]$.

Replacing γ_2 and δ_1 by γ_2^* and δ_1^*, we can assume, without loss of generality, that each of the functions $\varphi_1(t)$, $\psi_1(t)$, $\varphi_2(t)$ and $\psi_2(t)$ is one-signed or identically equal to zero on the interval $[\gamma_2, \delta_1]$. On this occasion, we must modify additionally the functions $e_-(t;\delta_1)$ and $e(t;\gamma_2, \delta_2)$.

In Case S-(ii), we can determine, in virtue of Lemma 2, two real non-zero constants c_2 and d_2 so that the function

$$y_{m'_{h+1}}(t)=(c_2+id_2)e(t;\gamma_2, \delta_2)$$

satisfies a condition

$$y_{m'_{h+1}}(t)-\omega_{h+1,\rho(1)}(t)y_{k'_{h+1}}(t)\neq 0 \quad \text{on } J_2. \tag{3.6}$$

Concerning the other components of $\boldsymbol{y}(t)$, we put

$$y_{m_\rho}(t)\equiv 0 \quad \text{on } J-J_1 \ (\rho=1, 2, \cdots, h),$$

$$y_{m'_\sigma}(t)\equiv 0 \quad \text{on } J-J_1 \ (\sigma=h+2, \cdots, n).$$

Then we can verify that the condition (3.5) is satisfied, in the following way: By the same reasoning as in the first step, we first obtain

$$\operatorname{rank} \boldsymbol{y}(t)=1 \text{ on } J_1 \text{ and } \operatorname{rank}(X(t), \boldsymbol{y}(t))=h+1 \text{ on } J_1.$$

We next consider the vector $\boldsymbol{y}(t)$ and the matrix $(X(t), \boldsymbol{y}(t))$ on the interval J_2. We easily get rank $\boldsymbol{y}(t)=1$ on J_2, in virtue of the fact that $y_{m'_{h+1}}(t)\neq 0$ on J_2. Furthermore, making use of the relation (3.4) and the condition (3.6), and putting

$$\tilde{y}_{m'_{h+1}}(t)=y_{m'_{h+1}}(t)-\omega_{h+1,\rho(1)}(t)y_{k'_{h+1}}(t),$$

we can transform the matrix $(X(t), \boldsymbol{y}(t))$ on J_2, by means of elementary operations, in the following manner:

$$(X(t), \boldsymbol{y}(t)) \longrightarrow \begin{pmatrix} \hat{\boldsymbol{x}}_{m_1}(t) & 0 \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{m_{\rho(1)}}(t) & y_{m_{\rho(1)}}(t) \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{m_h}(t) & 0 \\ \hat{\boldsymbol{x}}_{m'_{h+1}}(t) & y_{m'_{h+1}}(t) \\ \vdots & 0 \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{m'_n}(t) & 0 \end{pmatrix}$$

$$\longrightarrow\begin{pmatrix} \hat{\boldsymbol{x}}_{m_1}(t) & 0 \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{m_{\rho(1)}}(t) & y_{m_{\rho(1)}}(t) \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{m_h}(t) & 0 \\ \boldsymbol{o} & \tilde{y}_{m'_{h+1}}(t) \\ \boldsymbol{o} & -\omega_{h+2,\rho(1)}(t)y_{m_{\rho(1)}}(t) \\ \vdots & \vdots \\ \boldsymbol{o} & -\omega_{n,\rho(1)}(t)y_{m_{\rho(1)}}(t) \end{pmatrix}\longrightarrow\begin{pmatrix} \hat{\boldsymbol{x}}_{m_1}(t) & 0 \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{m_{\rho(1)}}(t) & \vdots \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{m_h}(t) & 0 \\ \boldsymbol{o} & \tilde{y}_{m'_{h+1}}(t) \\ \boldsymbol{o} & 0 \\ \vdots & \vdots \\ \boldsymbol{o} & 0 \end{pmatrix}.$$

Therefore we obtain the condition (3.5), in virtue of the condition (3.6).

In Case S-(iii), we can determine, in virtue of Lemma 2, two real non-zero constants c_2 and d_2 so that the function

$$y_{m'_{h+1}}(t)=(c_2+id_2)e(t\,;\gamma_2,\delta_2)$$

satisfies a condition

$$(3.7)\qquad y_{k'_{h+1}}(t)-\theta_{h+1,\rho(2)}(t)y_{m'_{h+1}}(t)\neq 0 \quad \text{on } J_1$$

and the condition (3.6).

Further we define the other components of $\boldsymbol{y}(t)$ in the same way as in Case S-(ii).

On this occasion, we can prove the condition

$$\operatorname{rank}\boldsymbol{y}(t)=1 \quad \text{on } J_2 \quad \text{and} \quad \operatorname{rank}(X(t),\ \boldsymbol{y}(t))=h+1 \quad \text{on } J_2,$$

on the same lines as in Case S-(ii).

We wish next to verify that

$$\operatorname{rank}\boldsymbol{y}(t)=1 \quad \text{on } J_1 \quad \text{and} \quad \operatorname{rank}(X(t),\ \boldsymbol{y}(t))=h+1 \quad \text{on } J_1.$$

We easily see rank $\boldsymbol{y}(t)=1$ on J_1, because $y_{k'_{h+1}}(t)\neq 0$ on J_1.

Moreover, taking the relation (3.3) and the condition (3.7) into account and putting

$$\check{y}_{k'_{h+1}}(t)=y_{k'_{h+1}}(t)-\theta_{h+1,\rho(2)}(t)y_{m'_{h+1}}(t)\,,$$

we can transform the matrix $(X(t),\ \boldsymbol{y}(t))$ on J_1, by means of elementary operations, in the following manner:

$$(X(t),\ \boldsymbol{y}(t)) \longrightarrow \begin{pmatrix} \hat{\boldsymbol{x}}_{k_1}(t) & 0 \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{k_{\rho(2)}}(t) & y_{m'_{h+1}}(t) \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{k_h}(t) & 0 \\ \hat{\boldsymbol{x}}_{k'_{h+1}}(t) & y_{k'_{h+1}}(t) \\ \vdots & 0 \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{k'_n}(t) & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \hat{\boldsymbol{x}}_{k_1}(t) & 0 \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{k_{\rho(2)}}(t) & y_{m'_{h+1}}(t) \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{k_h}(t) & 0 \\ \boldsymbol{o} & \check{y}_{k'_{h+1}}(t) \\ \boldsymbol{o} & -\theta_{h+2,\rho(2)}y_{m'_{h+1}}(t) \\ \vdots & \vdots \\ \boldsymbol{o} & -\theta_{n,\rho(2)}y_{m'_{h+1}}(t) \end{pmatrix} \longrightarrow \begin{pmatrix} \hat{\boldsymbol{x}}_{k_1}(t) & 0 \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{k_{\rho(2)}}(t) & \vdots \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{k_h}(t) & 0 \\ \boldsymbol{o} & \check{y}_{k'_{h+1}}(t) \\ \vdots & 0 \\ \vdots & \vdots \\ \boldsymbol{o} & 0 \end{pmatrix}.$$

Hence we obtain the condition (3.5), in virtue of the condition (3.7).

By repeating the process employed above for each pair $\{J_\iota, J_{\iota+1}\}$ $(\iota=1, 2, \cdots, \iota_0-1)$ of intervals, we get the desired vector $\boldsymbol{y}(t)=\mathrm{col}(y_1(t), y_2(t), \cdots, y_n(t))$ satisfying the condition (1.3).

In the accomplishment of this proof, we must examine which of Casess S-(i) $\sim$S-(iii) occurs, and if necessary, we choose the interval $[\delta^*_{\iota+1}, \gamma^*_\iota]$ corresponding to the interval $[\delta^*_2, \gamma^*_1]$ taken at the begining of the consideration for Cases S-(ii) and S-(iii), and we must adopt $\delta^*_{\iota+1}$ and γ^*_ι anew for $\delta_{\iota+1}$ and γ_ι.

Furthermore we use the functions $e(t\,;\gamma_\iota, \delta_\iota)$ for J_ι $(\iota=2, 3, \cdots, \iota_0-1)$, $e_-(t\,;\delta_1)$ for J_1 and $e_+(t, \gamma_{\iota_0})$ for J_{ι_0}.

§ 4. Summary about solutions of a linear matrix equation.

In this section, we shall summarize the matters which are used for the proof of Theorem 3.

Let I_1 and I_2 be two intervals such that $I_1=[\alpha_1, \beta_1)$ or $I_1=(\alpha_1, \beta_1)$, and $I_2=(\alpha_2, \beta_2)$ or $I_2=(\alpha_2, \beta_2]$ and further $\alpha_1<\alpha_2<\beta_1<\beta_2$.

Let $B(t)$ be the square matrix of degree n, which is given in § 1. Assume that for a positive integer $s: 2\leqq s\leqq n-1$, the condition (1.7) is satisfied on $\bar{I}_1\cup\bar{I}_2$

and further that a condition

$$B\begin{pmatrix} j_1 & j_2 & \cdots & j_r \\ k_1 & k_2 & \cdots & k_r \end{pmatrix} \neq 0 \tag{4.1}$$

is satisfied on $\bar{I}_1$ and a condition

$$B\begin{pmatrix} l_1 & l_2 & \cdots & l_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix} \neq 0 \tag{4.2}$$

is satisfied on $\bar{I}_2$.

We define an $(n-r)$-tuple $(k'_{r+1}, k'_{r+2}, \cdots, k'_n)$ for $1 \leqq k_1 < k_2 < \cdots < k_r \leqq n$ in such a way that $1 \leqq k'_{r+1} < k'_{r+2} < \cdots < k'_n \leqq n$ and $\{k_1, \cdots, k_r, k'_{r+1}, \cdots, k'_n\} = \{1, 2, \cdots, n\}$. An $(n-r)$-tuple $(m'_{r+1}, m'_{r+2}, \cdots, m'_n)$ is also defined for $1 \leqq m_1 < m_2 < \cdots < m_r \leqq n$ in the same way.

Let us consider an $n \times s_1$ matrix $(1 \leqq s_1 \leqq s)$ $P(t)$ whose components all belong to $K(\bar{I}_1)$:

$$P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1s_1}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2s_1}(t) \\ \vdots & \vdots & & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{ns_1}(t) \end{pmatrix},$$

and put

$$\boldsymbol{p}_{k_\rho}(t) = (p_{k_\rho 1}(t), p_{k_\rho 2}(t), \cdots, p_{k_\rho s_1}(t)) \quad (\rho = 1, 2, \cdots, r);$$

$$\boldsymbol{p}_{k'_\sigma}(t) = (p_{k'_\sigma 1}(t), p_{k'_\sigma 2}(t), \cdots, p_{k'_\sigma s_1}(t)) \quad (\sigma = r+1, r+2, \cdots, n).$$

Then, in virtue of Cramer's rule, we recall the following fact.

The matrix $P(t)$ satisfies a linear equation

$$B(t)P(t) = O \tag{4.3}$$

on $\bar{I}_1$, if and only if the vectors $\boldsymbol{p}_{k_\rho}(t)$ $(\rho = 1, 2, \cdots, r)$ can be represented as linear combinations of the vectors $\boldsymbol{p}_{k'_\sigma}(t)$ $(\sigma = r+1, r+2, \cdots, n)$:

$$\boldsymbol{p}_{k_\rho}(t) = \sum_{\sigma=r+1}^{n} \xi_{\rho\sigma}(t) \boldsymbol{p}_{k'_\sigma}(t) \quad (\rho = 1, 2, \cdots, r) \tag{4.4}$$

with coefficients $\xi_{\rho\sigma}(t)$ which belong to $K(\bar{I}_1)$ and are expressed by

$$\xi_{\rho\sigma}(t) = -\frac{B_{\rho\sigma}\begin{pmatrix} j_1 & j_2 & \cdots & j_r \\ k_1 & k_2 & \cdots & k_r \end{pmatrix}}{B\begin{pmatrix} j_1 & j_2 & \cdots & j_r \\ k_1 & k_2 & \cdots & k_r \end{pmatrix}} \quad \begin{pmatrix} \rho = 1, 2, \cdots, r; \\ \sigma = r+1, r+2, \cdots, n \end{pmatrix}, \tag{4.5}$$

where

$$B_{\rho\sigma}\begin{pmatrix} j_1 & j_2 & \cdots & j_r \\ k_1 & k_2 & \cdots & k_r \end{pmatrix} = \begin{vmatrix} b_{j_1 k_1}(t) & b_{j_1 k_2}(t) & \cdots & \overset{\rho\text{-th column}}{b_{j_1 k'_\sigma}(t)} & \cdots & b_{j_1 k_r}(t) \\ b_{j_2 k_1}(t) & b_{j_2 k_2}(t) & \cdots & b_{j_2 k'_\sigma}(t) & \cdots & b_{j_2 k_r}(t) \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{j_r k_1}(t) & b_{j_r k_2}(t) & \cdots & b_{j_r k'_\sigma}(t) & \cdots & b_{j_r k_r}(t) \end{vmatrix}.$$

Therefore, we obtain the following proposition:

PROPOSITION 1. *Let $B(t)$ be the matrix given in* § 1 *and let $P(t)$ be an $n\times s_1$ matrix* $(1\leqq s_1\leqq s)$, *whose components all belong to $K(\bar{I}_1)$ and which satisfies the equation* (4.3) *on $\bar{I}_1$. Then a condition*

$$\operatorname{rank} P(t)=s_1$$

is satisfied on $\bar{I}_1$, if and only if a condition

$$\operatorname{rank}\begin{pmatrix}\hat{\boldsymbol{p}}_{k'_{r+1}}(t)\\ \vdots\\ \hat{\boldsymbol{p}}_{k'_n}(t)\end{pmatrix}=s_1$$

is satisfied on $\bar{I}_1$.

Let us next consider an $n\times s_1$ matrix $Q(t)$ whose components all belong to $K(\bar{I}_2)$:

$$Q(t)=\begin{pmatrix}q_{11}(t) & q_{12}(t) & \cdots & q_{1s_1}(t)\\ q_{21}(t) & q_{22}(t) & \cdots & q_{2s_1}(t)\\ \vdots & \vdots & & \vdots\\ q_{n1}(t) & q_{n2}(t) & \cdots & q_{ns_1}(t)\end{pmatrix},$$

and put

$$\hat{\boldsymbol{q}}_{m_\rho}(t)=(q_{m_\rho 1}(t),\ q_{m_\rho 2}(t),\ \cdots,\ q_{m_\rho s_1}(t))\quad(\rho=1,\ 2,\ \cdots,\ r),$$

$$\hat{\boldsymbol{q}}_{m'_\sigma}(t)=(q_{m'_\sigma 1}(t),\ q_{m'_\sigma 2}(t),\ \cdots,\ q_{m'_\sigma s_1}(t))\quad(\sigma=r+1,\ r+2,\ \cdots,\ n).$$

Then, on the same ground as for $P(t)$, we know the following fact.

The matrix $Q(t)$ satisfies a linear equation

$$B(t)Q(t)=O \tag{4.6}$$

on $\bar{I}_2$, if and only if the vectors $\hat{\boldsymbol{q}}_{m_\rho}(t)$ $(\rho=1,\ 2,\ \cdots,\ r)$ can be represented as linear combinations of the vectors $\hat{\boldsymbol{q}}_{m'_\sigma}(t)$ $(\sigma=r+1,\ r+2,\ \cdots,\ n)$:

$$\hat{\boldsymbol{q}}_{m_\rho}(t)=\sum_{\sigma=r+1}^{n}\eta_{\rho\sigma}(t)\hat{\boldsymbol{q}}_{m'_\sigma}(t)\quad(\rho=1,\ 2,\ \cdots,\ r) \tag{4.7}$$

with coefficients $\eta_{\rho\sigma}(t)$ which belong to $K(\bar{I}_2)$ and are expressed by

$$\eta_{\rho\sigma}(t)=-\frac{B_{\rho\sigma}\begin{pmatrix}l_1 & l_2 & \cdots & l_r\\ m_1 & m_2 & \cdots & m_r\end{pmatrix}}{B\begin{pmatrix}l_1 & l_2 & \cdots & l_r\\ m_1 & m_2 & \cdots & m_r\end{pmatrix}}\quad\begin{pmatrix}\rho=1,\ 2,\ \cdots,\ r;\\ \sigma=r+1,\ r+2,\ \cdots,\ n\end{pmatrix}, \tag{4.8}$$

where

$$B_{\rho\sigma}\begin{pmatrix} l_1 & l_2 & \cdots & l_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix} = \begin{vmatrix} b_{l_1m_1}(t) & b_{l_1m_2}(t) & \cdots & b_{l_1m'_\sigma}(t) & \cdots & b_{l_1m_r}(t) \\ b_{l_2m_1}(t) & b_{l_2m_2}(t) & \cdots & b_{l_2m'_\sigma}(t) & \cdots & b_{l_2m_r}(t) \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{l_rm_1}(t) & b_{l_rm_2}(t) & \cdots & b_{l_rm'_\sigma}(t) & \cdots & b_{l_rm_r}(t) \end{vmatrix} .$$

(ρ-th column)

In this case, we get also, for $Q(t)$, a proposition similar to Proposition 1.

Concerning the relation between the matrices $P(t)$ and $Q(t)$, we have the following lemma:

LEMMA 3. *Let $B(t)$ be the matrix given in* §1. *Let $P(t)$ and $Q(t)$ be the $n\times s$ matrices—that is, $s_1=s$—, which are given above and satisfy the equation* (4.3) *on $\bar{I}_1$ and the equation* (4.6) *on $\bar{I}_2$ respectively. Suppose further that conditions*

$$\operatorname{rank} P(t)=s \quad \textit{and} \quad \operatorname{rank} Q(t)=s$$

are satisfied on $\bar{I}_1$ and on $\bar{I}_2$ respectively.

Then there exists a square matrix $C(t)$ of degree s such that

(I) *Every component of $C(t)$ belongs to $K(\bar{I}_1\cap\bar{I}_2)$;*
(II) $\operatorname{rank} C(t)=s$ *on* $\bar{I}_1\cap\bar{I}_2$;
(III) $P(t)=Q(t)C(t)$ *on* $\bar{I}_1\cap\bar{I}_2$.

For the proof of this lemma, see the proof of Lemma 2 in the previous paper [1].

§5. Proof of Theorem 3.

Let $X(t)$ denote the matrix:

$$X(t)=(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_{s'}(t)),$$

where $\boldsymbol{x}_k(t)=\mathrm{col}(x_{1k}(t), x_{2k}(t), \cdots, x_{nk}(t))$ $(k=1, 2, \cdots, s')$ are s' prescribed vectors belonging to $W(I)$.

Now, by assumption, we can choose a set $\{I_\kappa\}_{\kappa=1}^{\kappa_0}$ of intervals possessing the following properties:

(i) $I=\bigcup_{\kappa=1}^{\kappa_0} I_\kappa$;
(ii) $I_1=[\alpha_1, \beta_1)$, $I_{\kappa_0}=(\alpha_{\kappa_0}, \beta_{\kappa_0}]$, $\alpha_1=\alpha$, $\beta_{\kappa_0}=\beta$,
$I_\kappa=(\alpha_\kappa, \beta_\kappa)$ $(\kappa=2, 3, \cdots, \kappa_0-1)$;
(iii) $I_\kappa\cap I_{\kappa+1}\neq\emptyset$ $(\kappa=1, 2, \cdots, \kappa_0-1)$,
$I_\kappa\cap I_{\kappa'}=\emptyset$ $(\kappa+1<\kappa', \kappa=1, 2, \cdots, \kappa_0-2)$,
that is, $\alpha_1<\alpha_2<\beta_1<\cdots<\alpha_\kappa<\beta_{\kappa-1}<\alpha_{\kappa+1}<\beta_\kappa<\cdots<\beta_{\kappa_0-2}<\alpha_{\kappa_0}<\beta_{\kappa_0-1}<\beta_{\kappa_0}$
$(\kappa=2, 3, \cdots, \kappa_0-1)$;
(iv) For each I_κ, there exists a minor of degree r of $B(t)$ which does not vanish on the closure $\bar{I}_\kappa$ of I_κ.

For the details about the existence of such a set $\{I_\kappa\}_{\kappa=1}^{\kappa_0}$ of intervals, see the proof of Theorem in the previous paper [1].

Let us assume that a condition

$$B\begin{pmatrix} j_1 & j_2 & \cdots & j_r \\ k_1 & k_2 & \cdots & k_r \end{pmatrix} \neq 0 \tag{5.1}$$

is satisfied on $\bar{I}_1$, and a condition

$$B\begin{pmatrix} l_1 & l_2 & \cdots & l_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix} \neq 0 \tag{5.2}$$

is satisfied on $\bar{I}_2$.

We here remark that we are able, without loss of generality, to take the closures $\bar{I}_1=[\alpha_1, \beta_1]$ and $\bar{I}_2=[\alpha_2, \beta_2]$ of I_1 and I_2, for the conditions (5.1) and (5.2), instead of I_1 and I_2 which were taken for the similar conditions in the proof of Theorem in the previous paper [1].

We put

$$\hat{\boldsymbol{x}}_{k_\rho}(t)=(x_{k_\rho 1}(t),\ x_{k_\rho 2}(t),\ \cdots,\ x_{k_\rho s'}(t)) \quad (\rho=1, 2, \cdots, r),$$

$$\hat{\boldsymbol{x}}_{k'_\sigma}(t)=(x_{k'_\sigma 1}(t),\ x_{k'_\sigma 2}(t),\ \cdots,\ x_{k'_\sigma s'}(t)) \quad (\sigma=r+1, r+2, \cdots, n).$$

Then, in virtue of the conditions (1.7) and (5.1), the vectors $\hat{\boldsymbol{x}}_{k_\rho}(t)$ $(\rho=1, 2, \cdots, r)$ can be represented as combinations of the vectors $\hat{\boldsymbol{x}}_{k'_\sigma}(t)$ $(\sigma=r+1, r+2, \cdots, n)$:

$$\hat{\boldsymbol{x}}_{k_\rho}(t)=\sum_{\sigma=r+1}^{n}\xi_{\rho\sigma}(t)\hat{\boldsymbol{x}}_{k'_\sigma}(t) \quad (\rho=1, 2, \cdots, r), \tag{5.3}$$

where $\xi_{\rho\sigma}(t)$ are the same as in the linear combinations (4.4).

Furthermore, it follows from the condition (1.9) and Proposition 1 given in § 4, that

$$\operatorname{rank}\begin{pmatrix} \hat{\boldsymbol{x}}_{k'_{r+1}}(t) \\ \vdots \\ \hat{\boldsymbol{x}}_{k'_n}(t) \end{pmatrix} = s' \quad \text{on } \bar{I}_1. \tag{5.4}$$

The condition (5.4) and Theorem 2 imply, therefore, that there exist vectors $\hat{\boldsymbol{y}}_{k_\sigma}(t)$ $(\sigma=r+1, r+2, \cdots, n)$ such that

$$\hat{\boldsymbol{y}}_{k'_\sigma}(t)=(y_{k'_\sigma s'+1}(t),\ y_{k'_\sigma s'+2}(t),\ \cdots,\ y_{k'_\sigma s}(t))\,;$$

$$y_{k'_\sigma g}(t)\in K(\bar{I}_1) \quad (g=s'+1, s'+2, \cdots, s\,;\ \sigma=r+1, r+2, \cdots, n),$$

and

$$\operatorname{rank}\begin{pmatrix} \hat{\boldsymbol{y}}_{k'_{r+1}}(t) \\ \vdots \\ \hat{\boldsymbol{y}}_{k'_n}(t) \end{pmatrix} = s-s' \quad \text{on } \bar{I}_1, \tag{5.5}$$

and

$$\operatorname{rank}\begin{pmatrix} \hat{\boldsymbol{x}}_{k'_{r+1}}(t) & \hat{\boldsymbol{y}}_{k'_{r+1}}(t) \\ \vdots & \vdots \\ \hat{\boldsymbol{x}}_{k'_n}(t) & \hat{\boldsymbol{y}}_{k'_n}(t) \end{pmatrix} = s \quad \text{on } \bar{I}_1. \tag{5.6}$$

Further, we define vectors

$$\hat{\boldsymbol{y}}_{k_\rho}(t) = (y_{k_\rho s'+1}(t),\ y_{k_\rho s'+2}(t),\ \cdots,\ y_{k_\rho s}(t)) \quad (\rho=1, 2, \cdots, r)$$

by means of

$$\hat{\boldsymbol{y}}_{k_\rho}(t) = \sum_{\sigma=r+1}^{n} \xi_{\rho\sigma}(t)\hat{\boldsymbol{y}}_{k'_\sigma}(t),$$

where $\xi_{\rho\sigma}(t)$ are the same as in the linear combinations (4.4), and we put

$$\begin{cases} \hat{\boldsymbol{p}}_{k_\rho}(t) = (\hat{\boldsymbol{x}}_{k_\rho}(t),\ \hat{\boldsymbol{y}}_{k_\rho}(t)) & (\rho=1, 2, \cdots, r), \\ \hat{\boldsymbol{p}}_{k'_\sigma}(t) = (\hat{\boldsymbol{x}}_{k'_\sigma}(t),\ \hat{\boldsymbol{y}}_{k'_\sigma}(t)) & (\sigma=r+1, r+2, \cdots, n). \end{cases} \tag{5.7}$$

Rearranging the rows of the matrix:

$$\hat{P}(t) = \begin{pmatrix} \hat{\boldsymbol{p}}_{k_1}(t) \\ \vdots \\ \hat{\boldsymbol{p}}_{k_r}(t) \\ \hat{\boldsymbol{p}}_{k'_{r+1}}(t) \\ \vdots \\ \hat{\boldsymbol{p}}_{k'_n}(t) \end{pmatrix}$$

in the original order:

$$P^{(1)}(t) = \begin{pmatrix} \hat{\boldsymbol{p}}_1(t) \\ \vdots \\ \hat{\boldsymbol{p}}_n(t) \end{pmatrix} = (X(t), Y(t)); \tag{5.8}$$

$$X(t) = \begin{pmatrix} \hat{\boldsymbol{x}}_1(t) \\ \vdots \\ \hat{\boldsymbol{x}}_n(t) \end{pmatrix}, \quad Y(t) = \begin{pmatrix} \hat{\boldsymbol{y}}_1(t) \\ \vdots \\ \hat{\boldsymbol{y}}_n(t) \end{pmatrix},$$

we have

$$\operatorname{rank} P^{(1)}(t) = s \quad \text{on } \bar{I}_1, \tag{5.9}$$

and

$$B(t)P^{(1)}(t) = O \quad \text{on } \bar{I}_1. \tag{5.10}$$

Next we consider the vectors $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_{s'}(t)$ on $\bar{I}_2$.
If we put

$$\hat{\boldsymbol{x}}_{m_\rho}(t) = (x_{m_\rho 1}(t),\ x_{m_\rho 2}(t),\ \cdots,\ x_{m_\rho s'}(t)) \quad (\rho=1, 2, \cdots, r);$$

$$\hat{\boldsymbol{x}}_{m'_\sigma}(t)=(x_{m'_\sigma 1}(t),\ x_{m'_\sigma 2}(t),\ \cdots,\ x_{m'_\sigma s'}(t))\quad(\sigma=r+1,\ r+2,\ \cdots,\ n),$$

then, in virtue of the condition (5.2), the vectors $\hat{\boldsymbol{x}}_{m_\rho}(t)$ $(\rho=1,\ 2,\ \cdots,\ r)$ can be represented as linear combinations of the vectors $\hat{\boldsymbol{x}}_{m'_\sigma}(t)$ $(\sigma=r+1,\ r+2,\ \cdots,\ n)$:

$$\hat{\boldsymbol{x}}_{m_\rho}(t)=\sum_{\sigma=r+1}^{n}\eta_{\rho\sigma}(t)\hat{\boldsymbol{x}}_{m'_\sigma}(t)\quad(\rho=1,\ 2,\ \cdots,\ r),\tag{5.11}$$

where $\eta_{\rho\sigma}(t)$ are the same as in the linear combinations (4.7).

By the same reasoning as for the condition (5.4), we have

$$\operatorname{rank}\begin{pmatrix}\hat{\boldsymbol{x}}_{m'_{r+1}}(t)\\ \vdots\\ \hat{\boldsymbol{x}}_{m'_n}(t)\end{pmatrix}=s'\quad\text{on } \bar{I}_2.$$

Therefore, it follows from Theorem 2, that there exist vectors $\hat{\boldsymbol{z}}_{m'_\sigma}(t)$ $(\sigma=r+1,\ r+2,\ \cdots,\ n)$ such that

$$\hat{\boldsymbol{z}}_{m'_\sigma}(t)=(z_{m_\sigma s'+1}(t),\ z_{m'_\sigma s'+2}(t),\ \cdots,\ z_{m'_\sigma s}(t))\,;$$

$$z_{m'_\sigma g}(t)\in K(\bar{I}_2)\quad(g=s'+1,\ s'+2,\ \cdots,\ s\,;\ \sigma=r+1,\ r+2,\ \cdots,\ n),$$

and

$$\operatorname{rank}\begin{pmatrix}\hat{\boldsymbol{z}}_{m'_{r+1}}(t)\\ \vdots\\ \hat{\boldsymbol{z}}_{m'_n}(t)\end{pmatrix}=s-s'\quad\text{on } \bar{I}_2,\tag{5.12}$$

and

$$\operatorname{rank}\begin{pmatrix}\hat{\boldsymbol{x}}_{m'_{r+1}}(t) & \hat{\boldsymbol{z}}_{m'_{r+1}}(t)\\ \vdots & \vdots\\ \hat{\boldsymbol{x}}_{m'_n}(t) & \hat{\boldsymbol{z}}_{m'_n}(t)\end{pmatrix}=s\quad\text{on } \bar{I}_2.\tag{5.13}$$

We define vectors

$$\hat{\boldsymbol{z}}_{m_\rho}(t)=(z_{m_\rho s'+1}(t),\ z_{m_\rho s'+2}(t),\ \cdots,\ z_{m_\rho s}(t))\quad(\rho=1,\ 2,\ \cdots,\ r)$$

by means of

$$\hat{\boldsymbol{z}}_{m_\rho}(t)=\sum_{\sigma=r+1}^{n}\eta_{\rho\sigma}(t)\hat{\boldsymbol{z}}_{m'_\sigma}(t)\,,$$

where $\eta_{\rho\sigma}(t)$ are the same as in the linear combinations (4.7), and we put

$$\begin{cases}\hat{\boldsymbol{q}}_{m_\rho}(t)=(\hat{\boldsymbol{x}}_{m_\rho}(t),\ \hat{\boldsymbol{z}}_{m_\rho}(t))\quad(\rho=1,\ 2,\ \cdots,\ r),\\ \hat{\boldsymbol{q}}_{m'_\sigma}(t)=(\hat{\boldsymbol{x}}_{m'_\sigma}(t),\ \hat{\boldsymbol{z}}_{m'_\sigma}(t))\quad(\sigma=r+1,\ r+2,\ \cdots,\ n).\end{cases}\tag{5.14}$$

Rearranging the rows of the matrix:

$$\hat{Q}(t)=\begin{pmatrix} \hat{\boldsymbol{q}}_{m_1}(t) \\ \vdots \\ \hat{\boldsymbol{q}}_{m_r}(t) \\ \hat{\boldsymbol{q}}_{m'_{r+1}}(t) \\ \vdots \\ \hat{\boldsymbol{q}}_{m'_n}(t) \end{pmatrix}$$

in the original order:

$$Q(t)=\begin{pmatrix} \hat{\boldsymbol{q}}_1(t) \\ \vdots \\ \hat{\boldsymbol{q}}_n(t) \end{pmatrix}=(X(t),\ Z(t))\ ;$$

$$X(t)=\begin{pmatrix} \hat{\boldsymbol{x}}_1(t) \\ \vdots \\ \hat{\boldsymbol{x}}_n(t) \end{pmatrix},\quad Z(t)=\begin{pmatrix} \hat{\boldsymbol{z}}_1(t) \\ \vdots \\ \hat{\boldsymbol{z}}_n(t) \end{pmatrix},$$

we obtain

(5.15) $$\operatorname{rank} Q(t)=s \quad \text{on } \bar{I}_2,$$

and

(5.16) $$B(t)Q(t)=O \quad \text{on } \bar{I}_2.$$

In virtue of Lemma 3 given in § 4, there exists a square matrix $C(t)$ of degree s, such that

(5.17) $$P^{(1)}(t)=Q(t)C(t) \quad \text{on } \bar{I}_1\cap\bar{I}_2,$$

and

$$\operatorname{rank} C(t)=s \qquad \text{on } \bar{I}_1\cap\bar{I}_2,$$

and every component of $C(t)$ belongs to $K(\bar{I}_1\cap\bar{I}_2)$.

Moreover, we have especially

(5.18) $$\begin{pmatrix} \hat{\boldsymbol{p}}_{m'_{r+1}}(t) \\ \vdots \\ \hat{\boldsymbol{p}}_{m'_n}(t) \end{pmatrix}=\begin{pmatrix} \hat{\boldsymbol{q}}_{m'_{r+1}}(t) \\ \vdots \\ \hat{\boldsymbol{q}}_{m'_n}(t) \end{pmatrix}C(t) \quad \text{on } \bar{I}_1\cap\bar{I}_2.$$

§ 6. Proof of Theorem 3 (continued).

If we represent the matrix $C(t)$ in the form of a blocked matrix:

$$C(t)=\begin{pmatrix} \overbrace{C_{11}(t)}^{s'} & \overbrace{C_{12}(t)}^{s-s'} \\ C_{21}(t) & C_{22}(t) \end{pmatrix}\begin{matrix} \}s' \\ \}s-s' \end{matrix},$$

then, the relation (5.18) and the definition of the vectors $\hat{\boldsymbol{p}}_{m'_\sigma}(t)$ and $\hat{\boldsymbol{q}}_{m'_\sigma}(t)$:

$$\hat{\boldsymbol{p}}_{m'_\sigma}(t)=(\hat{\boldsymbol{x}}_{m'_\sigma}(t),\ \hat{\boldsymbol{y}}_{m'_\sigma}(t)) \quad\text{and}\quad \hat{\boldsymbol{q}}_{m'_\sigma}(t)=(\hat{\boldsymbol{x}}_{m'_\sigma}(t),\ \hat{\boldsymbol{z}}_{m'_\sigma}(t)),$$

$$(\sigma=r+1,\ r+2,\ \cdots,\ n)$$

imply

$$\begin{pmatrix}\hat{\boldsymbol{x}}_{m'_{r+1}}(t)\\ \vdots\\ \hat{\boldsymbol{x}}_{m'_n}(t)\end{pmatrix}=\begin{pmatrix}\hat{\boldsymbol{x}}_{m'_{r+1}}(t)\\ \vdots\\ \hat{\boldsymbol{x}}_{m'_n}(t)\end{pmatrix}C_{11}(t)+\begin{pmatrix}\hat{\boldsymbol{z}}_{m'_{r+1}}(t)\\ \vdots\\ \hat{\boldsymbol{z}}_{m'_n}(t)\end{pmatrix}C_{21}(t) \tag{6.1}$$

and

$$\begin{pmatrix}\hat{\boldsymbol{y}}_{m'_{r+1}}(t)\\ \vdots\\ \hat{\boldsymbol{y}}_{m'_n}(t)\end{pmatrix}=\begin{pmatrix}\hat{\boldsymbol{x}}_{m'_{r+1}}(t)\\ \vdots\\ \hat{\boldsymbol{x}}_{m'_n}(t)\end{pmatrix}C_{12}(t)+\begin{pmatrix}\hat{\boldsymbol{z}}_{m'_{r+1}}(t)\\ \vdots\\ \hat{\boldsymbol{z}}_{m'_n}(t)\end{pmatrix}C_{22}(t) \tag{6.2}$$

on $\bar{I}_1\cap\bar{I}_2$.

It is easily seen from the condition (5.13) and the relation (6.1), that

$$C_{11}(t)=E_{s'} \quad\text{and}\quad C_{21}(t)=O\,,$$

where $E_{s'}$ is the unit matrix of degree s'. Hence the matrix $C(t)$ has the following form:

$$C(t)=\begin{pmatrix}E_{s'} & C_{12}(t)\\ O & C_{22}(t)\end{pmatrix}. \tag{6.3}$$

Now let t_1 be a point belonging to $I_1\cap I_2$. Then, since $C(t_1)$ is non-singular, there exists a positive number $\hat{\varepsilon}$ such that any square matrix C of degree s, satisfying $\|C-C(t_1)\|<\hat{\varepsilon}$, is non-singular, where $\|\cdot\|$ denotes the Euclidean norm of a matrix.

We can find, in virtue of the continuity of functions, a positive number ε_0 such that $\|C(t)-C(t_1)\|<\hat{\varepsilon}$ whenever $|t-t_1|<\varepsilon_0$ and $t\in I_1\cap I_2$.

Let t'_1 be a point belonging to $I_1\cap I_2$ such that $0<t'_1-t_1<\varepsilon_0$ and let ε_1 be a small positive number satisfying the inequality $t_1+\varepsilon_1<t'_1-\varepsilon_1$.

Furthermore we prepare a real-valued function $\chi(t)$ defined and of class C^∞ on $-\infty<t<+\infty$, such that $0\leqq\chi(t)\leqq1$ for all t, $\chi(t)=1$ for $t\leqq t_1+\varepsilon_1$ and $\chi(t)=0$ for $t\geqq t'_1-\varepsilon_1$.

Let $\tilde{C}(t)$ be a square matrix of degree s, defined in the following way:

$$\tilde{C}(t)=\begin{cases} C(t) & \text{for } \alpha_2\leqq t\leqq t_1,\\ \chi(t)(C(t)-C(t'_1))+C(t'_1) & \text{for } t_1\leqq t\leqq t'_1,\\ C(t'_1) & \text{for } t'_1\leqq t<+\infty.\end{cases}$$

Since $\|\tilde{C}(t)-C(t_1)\|<\hat{\varepsilon}$ for $t_1\leqq t\leqq t'_1$, $\tilde{C}(t)$ is non-singular on $\alpha_2\leqq t<+\infty$. Further we can easily verify that every component of $\tilde{C}(t)$ is of class C^μ on $\alpha_2\leqq t<+\infty$.

By putting

$$\tilde{C}_{12}(t)=\chi(t)(C_{12}(t)-C_{12}(t_1'))+C_{12}(t_1'),$$
$$\tilde{C}_{22}(t)=\chi(t)(C_{22}(t)-C_{22}(t_1'))+C_{22}(t_1')$$

on $t_1\leqq t\leqq t_1'$, we see that the matrix $\tilde{C}(t)$ has the following form:

$$\tilde{C}(t)=\begin{cases}\begin{pmatrix}E_{s'} & C_{12}(t)\\ O & C_{22}(t)\end{pmatrix} & \text{for } \alpha_2\leqq t\leqq t_1;\\ \begin{pmatrix}E_{s'} & \tilde{C}_{12}(t)\\ O & \tilde{C}_{22}(t)\end{pmatrix} & \text{for } t_1\leqq t\leqq t_1';\\ \begin{pmatrix}E_{s'} & C_{12}(t_1')\\ O & C_{22}(t_1')\end{pmatrix} & \text{for } t_1'\leqq t<+\infty.\end{cases} \tag{6.4}$$

If we define a matrix $P^{(2)}(t)$ on $\bar{I}_1\cup\bar{I}_2$ in the following manner:

$$P^{(2)}(t)=\begin{cases}P^{(1)}(t) & \text{for } \alpha_1\leqq t\leqq\alpha_2;\\ Q(t)\tilde{C}(t) & \text{for } \alpha_2\leqq t\leqq\beta_2,\end{cases} \tag{6.5}$$

then, the matrix $P^{(2)}(t)$ satisfies a linear equation

$$B(t)P^{(2)}(t)=O \qquad \text{on } \bar{I}_1\cup\bar{I}_2,$$

and satisfies a condition

$$\operatorname{rank} P^{(2)}(t)=s \qquad \text{on } \bar{I}_1\cup\bar{I}_2, \tag{6.6}$$

and further all components of $P^{(2)}(t)$ belong to $K(\bar{I}_1\cup\bar{I}_2)$.

Since it follows from (6.4) and (6.5), that

$$P^{(2)}(t)=\begin{cases}(X(t),\ X(t)C_{12}(t)+Z(t)C_{22}(t)) & \text{for } \alpha_2\leqq t\leqq t_1;\\ (X(t),\ X(t)\tilde{C}_{12}(t)+Z(t)\tilde{C}_{22}(t)) & \text{for } t_1\leqq t\leqq t_1';\\ (X(t),\ X(t)C_{12}(t_1')+Z(t)C_{22}(t_1')) & \text{for } t_1'\leqq t\leqq\beta_2,\end{cases}$$

if we put

$$Y(t)\left(\equiv\begin{pmatrix}\hat{\boldsymbol{y}}_1(t)\\ \vdots\\ \hat{\boldsymbol{y}}_n(t)\end{pmatrix}\right)=\begin{cases}X(t)C_{12}(t)+Z(t)C_{22}(t) & \text{for } \alpha_2\leqq t\leqq t_1;\\ X(t)\tilde{C}_{12}(t)+Z(t)\tilde{C}_{22}(t) & \text{for } t_1\leqq t\leqq t_1';\\ X(t)C_{12}(t_1')+Z(t)C_{22}(t_1') & \text{for } t_1'\leqq t\leqq\beta_2,\end{cases}$$

then, by taking the form (5.8) of the matrix $P^{(1)}(t)$ into consideration, we have

$$P^{(2)}(t)=(X(t),\ Y(t)) \qquad \text{on } \bar{I}_1\cup\bar{I}_2. \tag{6.7}$$

We shall here show that

$$\operatorname{rank} Y(t)=s-s' \qquad \text{on } \bar{I}_1\cup\bar{I}_2. \tag{6.8}$$

It is already known that $\operatorname{rank} Y(t)=s-s'$ on $\bar{I}_1$, and further, as the vectors $\hat{\boldsymbol{y}}_m(t)$ $(m=1, 2, \cdots, n)$ are of $(s-s')$-dimension, we see

$$\operatorname{rank} Y(t)\leqq s-s' \qquad \text{on } \bar{I}_2.$$

If there exists a point $t_0 \in \bar{I}_2$ such that rank $Y(t_0) < s-s'$, then we get

$$\operatorname{rank} P^{(2)}(t_0) \leqq \operatorname{rank} X(t_0) + \operatorname{rank} Y(t_0)$$
$$< s' + (s-s') = s,$$

which contradicts the condition (6.6).

By repeating the above-mentioned process for each pair $\{I_\kappa, I_{\kappa+1}\}$ $(\kappa = 2, 3, \cdots, \kappa_0 - 1)$ of intervals, we obtain a matrix:

$$Y(t) = (\boldsymbol{y}_{s'+1}(t), \boldsymbol{y}_{s'+2}(t), \cdots, \boldsymbol{y}_s(t))$$

defined on the interval I, such that

$$\operatorname{rank} Y(t) = s - s' \quad \text{on } I;$$
$$\operatorname{rank} (X(t), Y(t)) = s \quad \text{on } I,$$

and $\boldsymbol{y}_{s'+1}(t)$, $\boldsymbol{y}_{s'+2}(t)$, $\cdots$, $\boldsymbol{y}_s(t)$ belong to $W(I)$.

The vectors $\boldsymbol{y}_{s'+1}(t)$, $\boldsymbol{y}_{s'+2}(t)$, $\cdots$, $\boldsymbol{y}_s(t)$ are thus the desired ones.

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