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NON-EXISTENCE OF A NORMAL CONDITIONAL EXPECTATION IN A CONTINUOUS CROSSED PRODUCT

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1. Introduction. The conditional expectations of operator algebras played an important role from the outset in the theory of operator algebras. J. Diximeir [3] and H. Umegaki [14] have introduced conditional expectations in a finite von Neumann algebra onto its von Neumann subalgebras and there are abundant systematic studies concerning the conditional expectations (See for example [3], [10], $\cdots$, [17]).

Besides, we have the notion of a crossed product. It is constructed from a triple (M, G, α) where M is a von Neumann algebra, G is a locally compact group and α is an action of G on M, i. e. α is a homomorphism of G into the automorphism group of M satisfying certain continuity conditions. We call it a W^*-dynamical system. The method of construction of the crossed product $G\times_\alpha M$ from a W^*-dynamical system (M, G, α) will be made explicit in §2. Further we will call it a discrete crossed product when G is a discrete group, and a continuous crossed product when G is not discrete. Now, in the case of a discrete crossed product, there exists a faithful normal conditional expectation of $G\times_\alpha M$ onto M. But it was not known, in the case of a continuous crossed product, whether there exists a normal conditional expectation of $G\times_\alpha M$ onto M.

In this note we establish the following theorem; There is no normal conditional expectation of $G\times_\alpha M$ onto M if G is a locally compact connected group and if there is an element h in G such that α_h is an outer automorphism of M.

In spite of this result, a normal semi-finite operator valued weight from a crossed product $G\times_\alpha M$ into M can always be found. This was shown by U. Haagerup [4] prior to our result.

2. Notations and Preliminaries. Let M be a von Neumann algebra on a Hilbert space H and G be a locally compact group. The triple (M, G, α) is said a W^*-dynamical system if the mapping α of G into the group $\mathrm{Aut}(M)$ of all automorphisms of M is a homomorphism and the function $g\rightarrow\omega\alpha_g(x)$ is continuous on G for any $x\in M$ and $\omega\in M_*$ (M_* is the predual of M).

The crossed product $G\times_\alpha M$ of M with G is the von Neumann algebra on

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$L^2(G, H)$ generated by the family of the operators $\{\pi_\alpha(x), \lambda(g); x\in M, g\in G\}$;

$$(\pi_\alpha(x)\zeta)(h)=\alpha_h^{-1}(x)\zeta(h), \qquad \zeta\in L^2(G, H),$$

$$(\lambda(g)\zeta)(h)=\zeta(g^{-1}h), \qquad \zeta\in L^2(G, H).$$

The mapping π_α is then a normal isomorphism of M onto $\pi_\alpha(M)$ such that $\lambda(g)\pi_\alpha(x)\lambda(g)^*=\pi_\alpha(\alpha_g(x))$ for all $g\in G$ and $x\in M$. We often identify the von Neumann algebra M with the von Neumann algebra $\pi_\alpha(M)$.

Let T be a linear mapping of a von Neumann algebra M onto a von Neumann subalgebra N of M.

DEFINITION. 2.1 T is called a *conditional expectation of M onto N* if T has the following properties (See [3], [10], $\cdots$, [17]);

(i) $T(1)=1$, where 1 is the identity operator.

(ii) $T(axb)=a(T(x))b$, for all $a, b\in N$, $x\in M$.

Moreover T is called *normal* if ${}^tT(N_*)\subset M_*$.

Let ϕ be an automorphism of a von Neumann algebra M.

DEFINITION 2.2. ϕ is said *freely acting* if the element x of M with the property that $x\phi(y)=yx$ for any $y\in M$ is necessarily zero. For each automorphism ψ of M, there is a unique central projection q of M such that;

(i) $\psi(q)=q$

(ii) $\psi|_{M_q}$ is an inner automorphism of M_q.

(iii) $\psi|_{M_{(1-q)}}$ is a freely acting automorphism of $M_{(1-q)}$.

This central projection q will be denoted by $p(\psi)$ (cf. Kallmann [7]).

Let M be a von Neumann algebra. We also identify M_f with $fMf=\{fxf; x\in M\}$ where f is a projection of M or M'.

3. Main results.

THEOREM 3.1. *Let (M, G, α) be a W^*-dynamical system and we suppose that $sup\{p(\alpha_g); g\in G, g\neq e\}\neq 1$, where e is the identity of G. Then, the following statements are equivalent;*

(i) *G is a discrete group.*

(ii) *there exists a normal conditional expectation of $G\times_\alpha M$ onto M.*

Remark 3.2. That (i) implies (ii) is well known (cf. [2] Proposition 1, 4, 6, [9] § 4 and [6] § 2). In fact if G is a discrete group, the Hilbert space $L^2(G, H)$ is identified with $H\otimes l^2(G)$. On the other hand, for each g in G, put

$$\varepsilon_g(h)=\delta_g^h=\begin{cases} 1 & (g=h), \\ 0 & (g\neq h), \end{cases}$$

then the Hilbert space $L^2(G, H)$ is identifiable with the direct sum $\sum_{g\in G}\oplus H\otimes\varepsilon_g$ of subspaces $H\otimes\varepsilon_g$ $(g\in G)$. For each g in G and η in H, put $J_g\eta=\eta\otimes\varepsilon_g$, then J_g is an isometry of H onto $H\otimes\varepsilon_g$. Every x in $\mathcal{L}(L^2(G, H))$ has a matrix representation with an operator on H as each element

$$(x)_{g,h}=J_g^* x J_h\,,$$

where $\mathcal{L}(\mathfrak{H})$ is the algebra of all bounded linear operators on a Hilbert space $\mathfrak{H}$. Especially, we have

$$(\pi_\alpha(x))^{g,}{}_h=\delta_g^h\alpha_{g^{-1}}(x) \qquad (x\in M,\ g,\ h\in G)\,,$$
$$(\lambda(k))_{g,h}=\delta_g^{kh} \qquad (g,\ h,\ k\in G)\,.$$

Put $T(y)=(y)_{e,e}$ for $y\in G\times_\alpha M$. Then T is a faithful normal conditional expectation of $G\times_\alpha M$ onto M.

Before we prove the implication (ii)$\Rightarrow$(i), we will give two lemmas. Lemma 3.3 will be used repeatedly in the whole of our study.

LEMMA 3.3. *Let T be a conditional expectation of $G\times_\alpha M$ onto M. We then have $T(\lambda(g))_{(1-p(\alpha_g))}=0$ for any $g\in G$.*

Proof. For each $y\in M_{(1-p(\alpha_g))}$, we have;

$$yT(\lambda(g)^*)=T(y\lambda(g)^*)=T(\lambda(g)^*\lambda(g)y\lambda(g)^*)\,.$$

Since $\lambda(g)y\lambda(g)^*=\alpha_g(y)$ is an element of M,

$$yT(\lambda(g)^*)=T(\lambda(g)^*)\alpha_g(y)\,.$$

Therefore $T(\lambda(g)^*)_{(1-p(\alpha_g))}=0$ because α_g is a freely acting automorphism of $M_{(1-p(\alpha_g))}$.

LEMMA 3.4. $\sup\{p(\alpha_g);\ g\in G,\ g\neq e\}$ *is a G-invariant central projection of M.*

Proof. For any $y\in M$, g, $h\in G$ with $g\neq e$, we have

$$\alpha_{hgh^{-1}}(y\alpha_h(p(\alpha_g)))=\alpha_h(U)y\alpha_h(p(\alpha_g))\alpha_h(U)^*\,,$$

where U is an element of M such that $\alpha_{g|M_{p(\alpha_g)}}=AdU$, $U^*U=p(\alpha_g)$ and $UU^*=p(\alpha_g)$ $(AdU(x)=UxU^*$ for $x\in M_{p(\alpha_g)})$.

Therefore we get $\alpha_h(p(\alpha_g))\leqq p(\alpha_{hgh^{-1}})$, so that

$$\alpha_h(\sup\{p(\alpha_g);\ g\in G,\ g\neq e\})\leqq\sup\{p(\alpha_g);\ g\in G,\ g\neq e\}\,.$$

Hence $\sup\{p(\alpha_g);\ g\in G,\ g\neq e\}$ is a G-invariant central projection of M.

[*The proof of Theorem* 3.1.]. By Lemma 3.4, it is sufficient to prove the Theorem in the case when $p(\alpha_g)=0$ for all $g\in G$ except the identity e. It follows that $T(\lambda(g))=0$ for all $g\in G$ except e by Lemma 3.3.

Suppose that T is a normal conditional expectation of $G\times_{\alpha}M$ onto M. Let $K(G, M)$ be the family of M-valued, σ-weakly continuous functions on G with compact support. By [5] Lemma 2.3, $K(G, M)$ is an involutive algebra and a *-representation μ of $K(G, M)$ is defined,

$$\mu(\xi)=\int_{G}\lambda(g)\pi_{\alpha}(\xi(g))d\nu(g),$$

where $\xi\in K(G, M)$ and ν is a left Haar measure of G. Moreover the representation μ maps $K(G, M)$ onto a σ-weakly dense subalgebra of $G\times_{\alpha}M$. Since T is normal and $T(\lambda(g))=0$ for all $g\in G$ except e, we have

$$T(\mu(\xi))=\int_{G}T(\lambda(g))\pi_{\alpha}(\xi(g))d\nu(g)=\pi_{\alpha}(\xi(e))\nu(\{e\}).$$

Therefore $\nu(\{e\})$ must be a positive number, so G must be a discrete group.

Remark 3.5. Let (M, G, α) be a W^*-dynamical system. Let V be a strongly continuous unitary representation of G into M such that $\alpha_{g}=AdV_{g}$ for any $g\in G$.

We define a unitary operator W on $L^{2}(G, H)=H\otimes L^{2}(G)$

$$(W\xi)(g)=V_{g}\xi(g)$$

for all $\xi\in L^{2}(G, H)$. We get;

$$W\pi_{\alpha}(x)W^{*}=x\otimes 1 \qquad \text{for any } x\in M$$

$$W\lambda(g)W^{*}=V_{g}\otimes\rho(g) \quad \text{for any } g\in G.$$

where ρ is the left regular representation of G on $L^{2}(G)$. Therefore we get

$$W(G\times_{\alpha}M)W^{*}=M\otimes\rho(G)'', \quad W\pi_{\alpha}(M)W^{*}=M\otimes 1.$$

Whence we know that there are many normal conditional expectations of $G\times_{\alpha}M$ onto M, according to the result of [13] Theorem 1.1.

We will have a decisive result about the existence of a normal conditional expectation in case of a connected group.

THEOREM 3.6. *Let G be a locally compact connected group and (M, G, α) be a W^*-dynamical system. If there is an element h in G such that α_{h} is an outer automorphism of M, then there does not exist any normal conditional expectation of $G\times_{\alpha}M$ onto M.*

Proof. We suppose that there exists a normal conditional expectation T of $G\times_{\alpha}M$ onto M.

Assume first that there is an element g in G such that g is on a one-parameter subgroup $x(t)$ at $t=s$ and $\alpha_{g}=\alpha_{x(s)}$ is an outer automorphism of M.

$p(\alpha_{x(s)})$ is then a central projection of M which is not the identity operator of M. For any $n \in \boldsymbol{N}$, we get

$$p(\alpha_{x(s/n)}) \leqq p(\alpha_{x(s)})$$

because $(\alpha_{x(s/n,)})^n = \alpha_{x(s)}$. From Lemma 3.3, $T(\lambda(x(s/n)))_{(1-p(\alpha_{x(s/n)}))} = 0$, so we have

$$T\left(\lambda\left(x\left(\frac{s}{n}\right)\right)\right)_{(1-p(\alpha_{x(s)}))} = 0$$

for any $n \in \boldsymbol{N}$. Therefore we get,

$$T(\lambda(e))_{(1-p(\alpha_{x(s)}))} = w-\lim_{n\to\infty} T\left(\lambda\left(x\left(\frac{s}{n}\right)\right)\right)_{(1-p(\alpha_g))} = 0\,,$$

so we get $1 = p(\alpha_g)$, which is a contradiction. So the assumed situation does not take place.

When an element g in G is on a one-parameter subgroup of G, we write $e \sim g$. By the above argument, α_g must be an inner automorphism of M for any g in $\{g \in G\,;\, e \sim g\}$. Now, G is equal to the closed subgroup K generated by $\{g \in G\,;\, e \sim g\}$. Indeed, suppose that there are an element g in G and an open neighborhood U of e in G such that the intersection of gU and K is empty. By [8] Theorem 4.6, there exists in U a compact normal subgroup H such that G/H is a Lie group. Then there is a neighborhood V of e in G such that V contains H and each point of V/H is on a one-parameter subgroup in G/H. Since G/H is also a connected group, G/H is the group generated by V/H, so that there are a finite subset $\{g_i H\,;\, i=1, 2, \cdots, n\}$ in G/H, and one-parameter subgroups $x_i(t)$ $(i=1, 2, \cdots, n)$ in G/H such that $\prod_{i=1}^{n} g_i H = gH$, $g_i H$ is on the one-parameter subgroup $x_i(t)$ of G/H at $t = s_i$ $(i=1, 2, \cdots, n)$ and $g_i \in V$ $(i=1, 2, \cdots, n)$. By [8] Theorem 4.15, there are one-parameter subgroups $y_i(t)$ of G $(i=1, 2, \cdots, n)$ such that $y_i(t)H = x_i(t)$ for any $t \in \boldsymbol{R}$ $(i=1, 2, \cdots, n)$. The element $g^{-1}\prod_{i=1}^{n} y_i(s_i)$ is contained in $H \subset U$ because $\prod_{i=1}^{n} y_i(s_i)H = gH$. Then the intersection of K and gU is not empty since $\prod_{i=1}^{n} y_i(s_i)$ is contained in both K and gU, which is a contradiction.

As the group generated by $\{g \in G\,;\, e \sim g\}$ was shown to be dense in G, there is a net $\{g_i\}_{i \in I}$ in this group such that it converges to h in G, h being the element in the statement of the Theorem. Since α_g are inner automorphisms of M for any g in $\{g \in G\,;\, e \sim g\}$, α_{g_i} are inner automorphisms of M for any $i \in I$. Then we get;

$$p(\alpha_{g_i^{-1}h}) = p(\alpha_h)\,,$$

$$T(\lambda(g_i^{-1}h))_{(1-p(\alpha_{g_i^{-1}h}))} = T(\lambda(g_i^{-1}h))_{(1-p(\alpha_h))} = 0\,,$$

$$T(\lambda(e))_{(1-p(\alpha_h))}=w-\lim_{\iota\in I} T(\lambda(g_\iota^{-1}h))_{(1-p(\alpha_h))}=0\,,$$

so that $1-p(\alpha_h)=0$, which is not the case. We get thus a contradiction and the proof is complete.

Remark 3.7. If the group is not supposed connected, there are W^*-dynamical systems with a non-discrete locally compact group such that there is an element h in G with the freely acting automorphism α_h of M and there is a normal conditional expectation of $G\times_\alpha M$ onto M. For instance, let G be a locally compact group $G_1\times G_2$ where G_1 is a discrete group and G_2 is a non-discrete locally compact group. Then $(L^\infty(G_1), G_1\times G_2, \alpha)$ and $(L^\infty(G_1), G_1, \sigma)$ are W^*-dynamical systems where the actions $\alpha_{(g,h)}=\sigma_g$ are the translation of $L^\infty(G_1)$ for all $(g, h)\in G_1\times G_2$. Then $G\times_\alpha L^\infty(G_1)$ is isomorphic to $G_1\times_\sigma L^\infty(G_1)\otimes\rho(G_2)''$ where ρ is the left regular representation of G_2 on $L^2(G_2)$. Let ω be a normal state of $\rho(G_2)''$, p_ω be a slice mapping associated with ω (See [13]) of $G_1\times_\sigma L^\infty(G_1)\otimes\rho(G_2)''$ onto $G_1\times_\sigma L^\infty(G_1)$. Let T be a normal conditional expectation of $G_1\times_\sigma L^\infty(G_1)$ onto $L^\infty(G_1)$ (Remark 3.2). Then $T\circ p_\omega$ is a normal conditional expectation of $G\times_\alpha L^\infty(G_1)$ onto $L^\infty(G_1)$.

PROPOSITION 3.8. *Let (M, G, α) be a W^*-dynamical system, Γ be an open subgroup of G and ω be a faithful normal semi-finite weight on M. Then there is a faithful normal conditional expectation T of $G\times_\alpha M$ onto $W^*(M, \Gamma, \alpha)\equiv\{\pi_\alpha(M), \lambda(\Gamma)\}''$ such that $\tilde{\omega}\circ T=\tilde{\omega}$ and $T(\lambda(g))=0$ if $g\notin\Gamma$ where $\tilde{\omega}$ is the dual weight associated with ω.*

Proof. By [5] Theorem 3.2 we get,

$$\sigma_t^{\tilde{\omega}}(\pi_\alpha(x))=\pi_\alpha(\sigma_t^\omega(x)) \qquad \text{for all } x\in M\,,$$

$$\sigma_t^{\tilde{\omega}}(\lambda(g))=\Delta(g)^{it}\lambda(g)\pi_\alpha((D\omega\cdot\alpha_g : D\omega)_t) \qquad \text{for all } g\in G\,.$$

Therefore $W^*(M, \Gamma, \alpha)$ is $\sigma_t^{\tilde{\omega}}$-invariant for all $t\in\boldsymbol{R}$. Let $K(\Gamma, \mathfrak{A}_\omega)$ be the family of all $\mathfrak{A}_\omega$-valued continuous functions on Γ with a compact support where $\mathfrak{A}_\omega$ is the left Hilbert algebra associated with ω. We regard $K(\Gamma, \mathfrak{A}_\omega)$ as $\{f\in K(G, \mathfrak{A}_\omega)\,;\, f=0 \text{ outside } \Gamma\}$. Then by [5] Theorem 3.2, $\tilde{\omega}|_{W^*(M,\Gamma,\alpha)}$ is semi-finite. Then by [11] Theorem, there is a unique faithful normal conditional expectation T of $G\times_\alpha M$ onto $W^*(M, \Gamma, \alpha)$ such that $\tilde{\omega}\circ T=\tilde{\omega}$. Moreover we find, by the construction of T in [11], that $T(x)=\Phi(ExE)$ for all $x\in G\times_\alpha M$ where E is the projection of $L^2(G, H)$ onto $L^2(\Gamma, H)$ and Φ is the canonical automorphism of $\Gamma\times_\beta M$ onto $W^*(M, \Gamma, \alpha)$;

$$\Phi(\pi_\beta(x))=\pi_\alpha(x) \qquad \text{for all } x\in M\,,$$

$$\Phi(\lambda(g))=\lambda(g) \qquad \text{for all } g\in\Gamma$$

(where the action β is the restriction of α on Γ). For all $x(g)\in K(G, M)$, we obtain,

$$T\Big(\int_G \pi_\alpha(x(g))\lambda(g)dg\Big)=\Phi\Big(E\int_G \pi_\alpha(x(g))\lambda(g)dg\, E\Big)$$

$$=\int_\Gamma \pi_\alpha(x(g))\lambda(g)dg\,.$$

Then we get $T(\lambda(g))=0$ for all $g\notin\Gamma$ since $E\lambda(g)F=0$ for $g\notin\Gamma$.

Remark 3.9. The above proposition was proved by H. Choda in case of a discrete group ([1] Proposition 2).

4. Applications.

COROLLARY 4.1. *Let G be a locally compact connected group and (M, G, α) be a W^*-dynamical system. If there is an element $g\in G$ such that $p(\alpha_g)=0$, then the crossed product $G\times_\alpha M$ is properly infinite.*

COROLLARY 4.2. *Let (M, G, α) be a W^*-dynamical system. If the group G is not discrete and $p(\alpha_g)=0$ for all $g\in G$ except $g=e$, then the crossed product $G\times_\alpha M$ is properly infinite.*

We prove the Corollary 4.1 only, as we can prove the Corollary 4.2 in the same way.

[*Proof of Corollary* 4.1]. We suppose $G\times_\alpha M$ is not properly infinite and let $(G\times_\alpha M)_p$ be the greatest finite portion of $G\times_\alpha M$. Since p is a central projection of $G\times_\alpha M$, p is a projection of $\pi_\alpha(M)'$. Let q be the central support of p in $\pi_\alpha(M)'$. Then we get that q is a G-invariant projection of $\pi_\alpha(M)\cap\pi_\alpha(M)'$ because p is $Ad\lambda(g)$-invariant for all $g\in G$. The von Neumann algebra M_p is a von Neumann subalgebra of a finite von Neumann algebra $(G\times_\alpha M)_p$, so that there is a normal conditional expectation T_1 of $(G\times_\alpha M)_p$ onto M_p (See [3] Théorème 8 or [14] Theorem 1). We define a new normal conditional expectation T of $(G\times_\alpha M)_q$ onto M_q;

$$T(x)=\Phi(T_1(pxp))$$

for all $x\in(G\times_\alpha M)_q$ where Φ is the canonical isomorphism of M_p onto M_q.

For $a, b\in M_q$, $x\in(G\times_\alpha M)_q$, we have

$$T(axb)=\Phi(T_1(paxbp))=\Phi\{T_1((pap)(pxp)(pbp))\}$$

$$=\Phi\{pap\,T_1(pxp)pbp\}=a\{\Phi T_1(pxp)\}b=a(T(x))b\,,$$

and

$$T(q)=\Phi T_1(pqp)=\Phi(p)=q\,.$$

Therefore T is a conditional expectation of $(G\times_\alpha M)_q$ onto M_q. The normality of T is clear. On the other hand, $(G\times_\alpha M)_q$ is the crossed product with the W^*-dynamical system $(M_q, G, \alpha|_{M_q})$. This contradicts Theorem 3.6.

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