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ON THE BOUNDEDNESS OF THE SOLUTIONS OF A DIFFERENTIAL EQUATION IN THE COMPLEX DOMAIN

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1. In our previous paper [1] we proved a boundedness criterion for every solution of $w''+F(z)w=0$ along a ray. In this paper we shall give an extension of our earlier result. The result which we want to prove is the following

THEOREM 1. *Let $F(z)$ be $g(r)e^{i\gamma(r)}$ along the ray l: $re^{i\theta}$ (θ: fixed) such that $X(r)=g(r)\cos(\gamma(r)+2\theta)$ is monotone increasing for $r\geqq r_0$, $X(r_0)>0$ and there is a positive constant K such that*

$$|Y(t)|\leqq KX'(t)$$

for $t\geqq r_0$ and

$$\int^{\infty}|Y(t)|X(t)^K dt<\infty,$$

where $Y(r)=g(r)\sin(\gamma(r)+2\theta)$. Further assume that $F(z)$ is regular around the ray l. Then every solution of $w''+F(z)w=0$ is bounded along the ray l.

As an application of the above theorem we shall prove the following

THEOREM 2. *Under the same notations as in the above theorem assume that there is a positive constant K such that*

$$|Y(t)|\leqq KX'(t)$$

for $t\geqq r_0$, $X(r_0)>0$,

$$\int^{\infty}|Y(t)|dt<\infty$$

and $g(r)$ is bounded along the ray l. Further assume that $F(z)$ is regular around the ray l. Then every solution of $w''+F(z)w=0$ is bounded along l and the same is true for its derivative.

2. Proof of Theorem 1. For completeness we shall give its full proof here. Let us put $w(z)=R(r)\exp(i\Theta(r))$ along l. Then the differential equation

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$w''+F(z)w=0$ gives

$$(1)\qquad \begin{cases} R''(r)+\{X(r)-\Theta'(r)^2\}R(r)=0\,, \\ \{\Theta'(r)R(r)^2\}'+Y(r)R(r)^2=0\,. \end{cases}$$

Let us consider the following quadratic functional

$$2H=R'^2+R^2\Theta'^2+XR^2\,.$$

Then

$$\begin{aligned} 2H'&=2R'R''+2R^2\Theta'\Theta''+2RR'\Theta'^2+X'R^2+2XRR' \\ &=X'R^2-2Y\Theta'R^2 \end{aligned}$$

by the equation (1). By integration from r_1 to r we have

$$2H(r)=2H(r_1)+\int_{r_1}^{r} X'R^2dt-2\int_{r_1}^{r} Y\Theta'R^2dt\,,$$

that is,

$$(2)\qquad \begin{aligned} &R'(r)^2+R(r)^2\Theta'(r)^2+X(r)R(r)^2 \\ &=R'(r_1)^2+R(r_1)^2\Theta'(r_1)^2+X(r_1)R(r_1)^2 \\ &\quad+\int_{r_1}^{r} R^2dX-2\int_{r_1}^{r} Y\Theta'R^2dt\,. \end{aligned}$$

Now we shall estimate the last integral. By the second equation of (1)

$$-\int_{r_1}^{r} YR^2dt=\Theta'(r)R(r)^2-\Theta'(r_1)R(r_1)^2\,.$$

Hence

$$\begin{aligned} &-\int_{r_1}^{r} Y(t)\Theta'(t)R(t)^2dt \\ &=-\Theta'(r_1)R(r_1)^2\int_{r_1}^{r} Y(t)dt+\int_{r_1}^{r} Y(t)\int_{r_1}^{t} Y(s)R(s)^2dsdt\,. \end{aligned}$$

Therefore

$$\left|-\int_{r_1}^{r} Y\Theta'R^2dt\right|\leqq|\Theta'(r_1)|R(r_1)^2\int_{r_1}^{r}|Y(t)|dt+\int_{r_1}^{r}|Y(t)|R(t)^2dt\int_{r_1}^{r}|Y(t)|dt\,.$$

By the assumption

$$\int^{\infty}|Y|X^Kdt<\infty$$

and by $X(r)\geqq X(r_1)>0$ for $r\geqq r_1>r_0$,

$$\int_{r_1}^{\infty}|Y|dt\leqq\frac{1}{X(r_1)^K}\int_{r_1}^{\infty}|Y|X^Kdt<\infty\,.$$

We set

$$C_0=\int_{r_1}^{\infty}|Y(t)|dt<\infty .$$

Hence

$$\left|-\int_{r_1}^{r} Y\Theta' R^2 dt\right|\leqq C_0|\Theta'(r_1)|R(r_1)^2+C_0\int_{r_1}^{r}|Y|R^2 dt .$$

By $|Y(t)|\leqq KX'(t)$ for $t\geqq r_0$

$$C_0\int_{r_1}^{r}|Y(t)|R(t)^2 dt\leqq C_0 K\int_{r_1}^{r} R(t)^2 dX(t) .$$

Thus by (2)

$$\frac{1}{2}X(r)R(r)^2\leqq C_1+\frac{1}{2}(1+2C_0K)\int_{r_1}^{r} R(t)^2 dX(t)$$

with a positive constant C_1. By the same process as in the proof of the Gronwall inequality we have

$$X(r)R(r)^2\leqq 2C_1X(r)^{1+2C_0K}X(r_1)^{-1-2C_0K} ,$$

that is,

(3) $$R(r)^2\leqq C^*X(r)^{2C_0K}$$

with a positive constant C^*, which depends on r_1. If $X(r)$ is bounded, then $R(r)$ is bounded by (3). Hence we may assume that $X(r)$ is unbounded. Since $X(r)$ is non-decreasing, we may assume that $X(r)$ is larger than 1 for $r\geqq r_0$. We now take an r_1 sufficiently large so that $2C_0\leqq 1$, which is clearly possible. Then

$$\begin{aligned}|\Theta'(r)R(r)^2-\Theta'(r_1)R(r_1)^2|&\leqq\int_{r_1}^{r}|Y|R^2 dt\\ &\leqq C^*\int_{r_1}^{r}|Y(t)|X(t)^{2C_0K}dt\\ &\leqq C^*\int_{r_1}^{r}|Y(t)|X(t)^{K}dt\leqq C_2 .\end{aligned}$$

Hence

$$|\Theta'(r)|R(r)^2\leqq C_2+|\Theta'(r_1)|R(r_1)^2=C_3 .$$

This implies that

$$\left|\int_{r_1}^{r} Y\Theta' R^2 dt\right|\leqq C_3\int_{r_1}^{\infty}|Y|dt=C_0C_3 .$$

Therefore

$$X(r)R(r)^2\leqq D+\int_{r_1}^{r} R(t)^2 dX(t)$$

with a suitable constant $D>0$. By the Gronwall inequality $R(r)^2\leqq D/X(r_1)$, which is just the desired result.

3. Proof of Theorem 2. The inequality (3) holds in this case too. Since $X(r)$ is monotone increasing and $X(r_0)>0$ and since $g(t)$ is bounded and $X(t)\leqq g(t)$ for $t\geqq r_0$, $X(t)$ is bounded. Hence (3) implies the boundedness of R, which is the first desired result. For the second half

$$\frac{1}{2}(1+2C_0K)\int_{r_1}^{r} R(t)^2 dX(t)\leqq C^*\{X(r)^{1+2C_0K}-X(r_1)^{1+2C_0K}\}\ .$$

by (3). The right hand side term is bounded along the ray l. Therefore

$$R'(r)^2+R(r)^2\Theta'(r)^2+X(r)R(r)^2$$

is bounded. Hence

$$|w'(z)|=|R'(r)+iR(r)\Theta'(r)|$$

is bounded along the ray l.

A remark should be mentioned here. $X(r)\to b$ as $r\to\infty$. By

$$\int^{\infty}|Y(t)|\,dt<\infty$$

$|b|\,|\sin(\gamma(r)+2\theta)|/|\cos(\gamma(r)+2\theta)|\to 0$ as $r\to\infty$. Since $b>0$, $\sin(\gamma(r)+2\theta)\to 0$ and $\cos(\gamma(r)+2\theta)\to 1$ as $r\to\infty$. Hence $g(r)\to b$, that is, $|F(z)|\to b$ along the ray l. By the way in the case of Theorem 1 we can say that $|w'(z)|^2/X(r)$ is bounded along the ray l.

4. Taam's result. In this section we shall give a shorter proof of Taam's result [2]. There is no new idea. Let us consider the following functional

$$H=bR^2+R'^2+\Theta'^2R^2\,,$$

where b is a positive constant. Then

$$\frac{d}{dr}H=2bRR'+2R'R''+2\Theta'\Theta''R^2+2\Theta'^2RR'\,.$$

By (1) we have

$$H'=2(b-X)RR'-2Y\Theta'R^2\,.$$

Hence

$$H'\leqq\{|b-X|(bR^2+R'^2)+|Y|(\Theta'^2R^2+bR^2)\}\frac{1}{\sqrt{b}}$$

$$\leqq\frac{1}{\sqrt{b}}(|b-X|+|Y|)H\,.$$

Therefore

$$H(r)\leqq H(r_1)\exp\frac{1}{\sqrt{b}}\int_{r_1}^{r}\{|b-X|+|Y|\}\,dt\,.$$

If

$$\int^{\infty}(|b-X|+|Y|)dt<\infty\,,$$

then w and w' are bounded along the ray l. This is nothing but a result due to Taam.

5. Next we start from the following quadratic functional

$$Q=\sqrt{X}R^2+\frac{1}{\sqrt{X}}(R'^2+R^2\Theta'^2).$$

By the equation (1)

$$Q'=\frac{1}{2}\frac{X'}{\sqrt{X}}R^2-\frac{1}{2}\frac{X'}{\sqrt{X^3}}(R'^2+R^2\Theta'^2)-2\frac{Y}{\sqrt{X}}\Theta'R^2.$$

Now the last term is estimated by

$$\frac{|Y|}{\sqrt{X^3}}\left(\frac{1}{a}X^\alpha\Theta'^2R^2+aX^\beta R^2\right)$$

with a positive constant a and constants α, β satisfying $\alpha+\beta=2$, $0\leqq\alpha\leqq 2$. Assume that $X'\geqq 2|Y|X^\alpha/a$ and $X(t)>0$ for $t\geqq r_0$. Then

$$Q'\leqq\frac{1}{2}\frac{X'}{\sqrt{X}}R^2(1+a^2X^{\beta-1-\alpha})$$

and hence with a positive constant C

$$\sqrt{X(r)}\,R(r)^2\leqq C+\frac{1}{2}\int_{r_1}^{r}\frac{X'}{\sqrt{X}}R^2(1+a^2X^{\beta-1-\alpha})dt.$$

Thus

$$R(r)^2\leqq C\frac{1}{\sqrt{X(r_1)}}\exp\frac{a^2}{2}\int_{r_1}^{r}X^{\beta-2-\alpha}X'dt.$$

Assume that $\alpha>1/2$. Then $-\gamma=\beta-1-\alpha<0$. In this case

$$R(r)^2\leqq\frac{C}{\sqrt{X(r_1)}}\exp\left\{\frac{a^2}{2\gamma}(X(r_1)^{-\gamma}-X(r)^{-\gamma})\right\}$$
$$\leqq B.$$

Thus we have the following

THEOREM 3. *Suppose that $X'\geqq 2|Y|X^\alpha/a$ with positive constants a and α, $2\geqq\alpha>1/2$ and $x(t)>0$ for $t\geqq r_0$. Then every solution w of $w''+F(z)w=0$ is bounded along the ray l.*

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