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TOTALLY COMPLEX SUBMANIFOLDS OF A QUATERNIONIC KAEHLERIAN MANIFOLD

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§ 1. Introduction.

We consider some kinds of submanifolds in a quaternionic Kaehlerian manifold $\tilde{M}$ i.e., those invariant, totally real and totally complex submanifolds. These submanifolds are curvature invariant (see § 6). In $\tilde{M}$, invariant or totally real submanifolds are defined as quaternionic analogues of corresponding submanifolds in a Kaehlerian manifold ([6], [9]). It is known that each invariant submanifold of $\tilde{M}$ is totally geodesic. But in general an n-dimensional totally real submanifold of $\tilde{M}$ is not necessarily totally geodesic (see [3]). We shall define totally complex submanifold in § 2 and prove in § 2 that a totally complex submanifold is minimal.

In the present paper, totally complex submanifolds of a quaternionic Kaehlerian manifold will be studied and the so-called pinching problem of the length of the second fundamental form will be discussed as quaternionic analogues of the arguments developed in [6] and [9] for submanifolds of a Kaehlerian manifold. Our main result is stated in the following main theorem which will be given in § 5. In the following theorem and hereafter, we mean by a complex space form (resp. a quaternionic space form) a Kaehlerian manifold with constant holomorphic sectional curvature (resp. a quaternionic Kaehlerian manifold with constant curvature).

MAIN THEOREM. *Let M^{2n} $(n\geqq 2)$ be a $2n$-dimensional compact, connected and complete submanifold of a $4n$-dimensional quaternionic projective space HP^n with constant Q-sectional curvature 4. Assume that M^{2n} is totally complex in HP^n and the second fundamental form H of M^{2n} satisfies the inequality*

$$\|H\|^2 \leqq \frac{4n(n+3)}{4n-1},$$

$\|H\|$ being the length of H. Then M^{2n} is congruent to the complex projective space CP^n of complex dimension n, which is naturally imbedded in HP^n as a totally geodesic submanifold.

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We define totally complex submanifolds and give their fundamental properties in §2. We also prove in §2 that if a totally complex submanifold is totally geodesic in a quaternionic space form $\tilde{M}$, then it is a complex space form, and that if a totally complex submanifold is totally umbilical in M, then it is totally geodesic. §3 is devoted to obtain algebraic properties of the second fundamental tensor and the structure equations of Gauss, Codazzi and Ricci of a totally complex submanifold of a quaternionic space form. Moreover we prove in §3 that if for an n-dimensional totally complex submanifold M of a quaternionic space form $\tilde{M}$ of dimension $4n$ the connection in the normal bundle is flat, then both M and $\tilde{M}$ are also flat. In §4, the Laplacian of the square of the length of the second fundamental tensor is obtained by a straightforward computation and a pinching theorem is proved by using the well known inequality given in [2]. We remark in §5 that the complex projective space imbedded naturally in HP^n is the standard model of totally complex submanifolds. In the last §6, we give an algebraic theorem of a kind of subspaces in a quaternionic Hermitian vector space (This theorem was announced by S. Ishihara in 1975). This algebraic theorem is very useful in studying submanifolds of a quaternionic Kaehlerian manifold.

For geometric objects concerning quaternionic Kaehlerian manifolds, we use the same notations as in [3] and [4] (for detailed discussions, see Ishihara [4]). Manifolds, mappings and geometric objects under discussions are assumed to be of class c^∞. Unless otherwise, we use the following conventions of indices:

$$h, i, j, k=1, \cdots, 4n\,;\ x, y, z=2m+1, \cdots, 4n\,;$$
$$a, b, c, d, e, f=1, \cdots, n\,;\ \bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}=\bar{1}, \cdots, \bar{n}\,;$$
$$a^*, b^*, c^*, d^*, e^*, f^*=1^*, \cdots, n^*\,;\ \bar{a}^*, \bar{b}^*, \bar{c}^*, \bar{d}^*, \bar{e}^*, \bar{f}^*=\bar{1}^*, \cdots, \bar{n}^*\,;$$
$$s, t=1^*, \cdots, n^*, \bar{1}^*, \cdots, \bar{n}^*.$$

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§2. Totally complex submanifolds.

Let $\tilde{M}^{4n}$ be a $4n$-dimensional quaternionic Kaehlerian manifold covered by a system of coordinate neighborhoods $\{\tilde{U}\}$ and its quaternionic Kaehlerian structure be denoted by $(\tilde{g}, \tilde{V})$ (see, [4]). Then there exists a canonical local basis $\{\tilde{F}, \tilde{G}, \tilde{H}\}$ of the 3-dimensional vector bundle $\tilde{V}$ consisting of tensors of type (1, 1) over $\tilde{M}^{4n}$ such that

$$\begin{gathered}\tilde{F}^2=\tilde{G}^2=\tilde{H}^2=-\tilde{I}\,,\\ \tilde{G}\tilde{H}=-\tilde{H}\tilde{G}=\tilde{F}\,,\quad \tilde{H}\tilde{F}=-\tilde{F}\tilde{H}=\tilde{G}\,,\quad \tilde{F}\tilde{G}=-\tilde{G}\tilde{F}=\tilde{H}\end{gathered}\tag{2.1}$$

in each local coordinate neighborhood $\tilde{U}$, where $\tilde{I}$ is the identity tensor field of type (1, 1) on $\tilde{M}^{4n}$. Moreover, the local tensor fields $\tilde{F}$, $\tilde{G}$ and $\tilde{H}$ are almost Hermitian with respect to $\tilde{g}$ and the equations

$$\begin{aligned}\tilde{\nabla}_{\tilde{X}}\tilde{F} &= \phantom{-\tilde{r}(\tilde{X})\tilde{F}} \quad \tilde{r}(\tilde{X})\tilde{G} - \tilde{q}(\tilde{X})\tilde{H},\\ \tilde{\nabla}_{\tilde{X}}\tilde{G} &= -\tilde{r}(\tilde{X})\tilde{F} \phantom{\tilde{r}(\tilde{X})\tilde{G}} + \tilde{p}(\tilde{X})\tilde{H},\\ \tilde{\nabla}_{\tilde{X}}\tilde{H} &= \tilde{q}(\tilde{X})\tilde{F} - \tilde{p}(\tilde{X})\tilde{G}\end{aligned} \tag{2.2}$$

are satisfied for any vector field $\tilde{X}$ on $\tilde{M}^{4n}$, $\tilde{\nabla}$ denoting the Riemannian connection determined by the Riemannian metric $\tilde{g}$, where $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ are local 1-forms denfined on $\tilde{U}$.

We take arbitrary intersecting coordinate neighborhoods $\tilde{U}$ and $\tilde{U}'$ and denote by $\{\tilde{F}, \tilde{G}, \tilde{H}\}$ and $\{\tilde{F}', \tilde{G}', \tilde{H}'\}$ canonical local basis of $\tilde{V}$ in $\tilde{U}$ and $\tilde{U}'$ respectively. Then we have in $\tilde{U}\cap\tilde{U}'$

$$\begin{pmatrix}\tilde{F}'\\ \tilde{G}'\\ \tilde{H}'\end{pmatrix} = (\tilde{s}_{uv})\begin{pmatrix}\tilde{F}\\ \tilde{G}\\ \tilde{H}\end{pmatrix}, \tag{2.3}$$

where the (3, 3)-matrix $S_{\tilde{U},\tilde{U}'} = (\tilde{s}_{uv})$, $(u, v=1, 2, 3)$ is a function defined on $\tilde{U}\cap\tilde{U}'$ and taking its values in the special orthogonal group $SO(3)$ of degree 3.

Since the vector bundle $\tilde{V}$ admits the fibre metric $\langle\,,\rangle$, we may consider the unit sphere bundle $\tilde{S}$ which consists of unit elements with respect to $\langle\,,\rangle$. Let M be a Riemannian submanifold immersed in $\tilde{M}^{4n}$ by an isometric immersion $f: M \to \tilde{M}^{4n}$. We denote by ∇ the connection induced on $T(M)\oplus N(M)$, $T(M)$ and $N(M)$ being the tangent bundle and the normal bundle of M respectively. When we restrict ∇ to $T(M)$, this connection ∇ coincides with the Riemannian connection on M. Consider the pullback $f^*\tilde{S}$ of $\tilde{S}$. Then the following diagram is commutative, where $\tilde{\pi}$ is the restriction of the projection $\tilde{V} \to \tilde{M}^{4n}$.

$$\begin{tikzcd} f^*\tilde{S} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\pi}_f"] & \tilde{S} \arrow[d, "\tilde{\pi}"] \\ M \arrow[r, "f"] & \tilde{M}^{4n} \end{tikzcd}$$

A submanifold M is called a *totally complex submanifold* of $\tilde{M}^{4n}$ if the following two conditions (i) and (ii) are satisfied:

(i) There is a global section $\tilde{J}$ of $f^*\tilde{S}$ satisfying $\tilde{\nabla}_X\tilde{J}=0$ for any vector field X tangent to M.

(ii) For each point x in M and each element $\tilde{K}_x$ of the fibre $(f^*\tilde{S})_x$ such that $\tilde{K}_x$ is orthogonal to $\tilde{J}_x$, the algebraic conditions

$$\tilde{J}_x(T_x(M)) = T_x(M), \quad \tilde{K}_x(T_x(M)) \perp T_x(M) \tag{2.4}$$

are satisfied, where the symbol $\perp$ means to be orthogonal and the tangent space

$T_x(M)$ is identified with its image under the differential f_* of the isometric immersion f.

Remark. By a plane section of a manifold, we mean a 2-dimensional linear subspace of a tangent space. Then the condition (ii) is equivalent to the conditions:

$$\tilde{J}_x\sigma\subset T_x(M), \quad \tilde{K}_x\sigma\perp T_x(M) \tag{2.5}$$

for any plane section σ at each point x in M because of (2.3).

We define in M a tensor field J of type (1, 1) by

$$JY=\tilde{J}Y, \tag{2.6}$$

Y being an arbitrary vector field on M and we denote by g the Riemannian metric of M. Then M is of even dimension and a pair (g, J) is an almost Hermitian structure. Similarly, we give in $N(M)$ a tensor field $\hat{J}$ of type (1, 1) defined by

$$\hat{J}\xi=\tilde{J}\xi, \tag{2.7}$$

ξ being an arbitrary vector field normal to M and denote by $\hat{g}$ the metric induced in $N(M)$. Then the pair $(\hat{g}, \hat{J})$ is an almost Hermitian structure of $N(M)$.

Let $M^{2m}(m\leqq n)$ be a $2m$-dimensional totally complex submanifold of $\tilde{M}^{4n}$ and (g, J) be its almost Hermitian structure induced from $(\tilde{g}, \tilde{V})$. Take a coordinate neigborhood U in M^{2m} and consider cross-sections $\tilde{J}$, $\tilde{K}$ and $\tilde{L}$ are local cross sections such that the triple $\{\tilde{J}, \tilde{K}, \tilde{L}\}$ is an orthonormal basis of $f^*\tilde{S}$. For this cross-sections $\tilde{J}$, $\tilde{K}$ and $\tilde{L}$, the equation (2.2) can be represented by

$$\tilde{\nabla}_X\tilde{J}=0, \quad \tilde{\nabla}_X\tilde{K}=\tilde{p}(X)\tilde{L}, \quad \tilde{\nabla}_X\tilde{L}=-\tilde{p}(X)\tilde{K}, \tag{2.8}$$

X being an arbitrary vector field on M^{2m}.

Let denote by D the connection induced in the normal bundle $N(M^{2m})$. Then the induced metric $\hat{g}$ of $N(M^{2m})$ is parallel with respect to D and the Gauss-Weingarten formulas are given by

$$\tilde{\nabla}_XY=\nabla_XY+H(X, Y), \quad \tilde{\nabla}_X\xi=-A_\xi X+D_X\xi \tag{2.9}$$

for any vector fields X and Y on M^{2m} and each normal vector ξ, where H is the second fundamental form of M^{2m} and A_ξ is a local field of symmetric linear transformation of the tangent space of M^{2m} defined by $g(A_\xi X, Y)=g(H(X, Y), \xi)$. Then we have $H(X, Y)=H(Y, X)$. Applying ∇_X to (2.6) and (2.7) and taking account of (2.9), we easily see that

$$\begin{gathered} \nabla_XJ=0, \quad \nabla_X\hat{J}=0, \\ H(JX, Y)=H(X, JY)=\tilde{J}H(X, Y), \end{gathered} \tag{2.11}$$

because M^{2m} is totally complex. Thus we have

PROPOSITION 2.1. *Let $(\tilde{M}^{4n}, \tilde{g}, \tilde{V})$ be a quaternionic Kaehlerian manifold of dimension $4n$ and M^{2m} $(m \leqq n)$ a $2m$-dimensional totally complex submanifold of $\tilde{M}^{4n}$. Then M^{2m} admits a Kaehlerian structure (g, J) induced naturally from a quaternionic Kaehlerian structure $(\tilde{g}, \tilde{V})$ of $\tilde{M}^{4n}$ and the normal bundle $N(M^{2m})$ also admits a complex structure $\hat{J}$ induced from $(\tilde{g}, \tilde{V})$ such that $\nabla_X \hat{J}=0$, X being a vector field tangent to M^{2m}.*

The equation (2.11) implies

$$H(JX, JY)=-H(X, Y). \tag{2.12}$$

Thus we have

PROPOSITION 2.2. *Let $(\tilde{M}^{4n}, \tilde{g}, \tilde{V})$ be a quaternionic Kaehlerian manifold of dimension $4n$ and M^{2m} $(m \leqq n)$ a $2m$-dimensional totally complex submanifold of $\tilde{M}^{4n}$. Then M^{2m} is minimal.*

When $\tilde{M}^{4n}$ is a $4n$-dimensional quaternionic space form of constant Q-sectional curvature $\tilde{c}$, we denote such a space by $\tilde{M}^{4n}(\tilde{c})$. As is well known (cf. [4]), the curvature tensor $\tilde{R}$ of $\tilde{M}^{4n}(\tilde{c})$ has the form

$$\begin{aligned}\tilde{R}(\tilde{X}, \tilde{Y})=\tilde{c}/4\{&\tilde{X}\wedge\tilde{Y}+\tilde{F}\tilde{X}\wedge\tilde{F}\tilde{Y}+\tilde{G}\tilde{X}\wedge\tilde{G}\tilde{Y}+\tilde{H}\tilde{X}\wedge\tilde{H}\tilde{Y}\\ &-2\tilde{g}(\tilde{F}\tilde{X}, \tilde{Y})\tilde{F}-2\tilde{g}(\tilde{G}X, \tilde{Y})\tilde{G}-2\tilde{g}(\tilde{H}\tilde{X}, \tilde{Y})\tilde{H}\}\end{aligned} \tag{2.13}$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ on $\tilde{M}^{4n}(\tilde{c})$, where $\{\tilde{F}, \tilde{G}, \tilde{H}\}$ is a canonical local basis of $\tilde{M}^{4n}(\tilde{c})$ and $\tilde{X}\wedge\tilde{Y}$ is defined as $(\tilde{X}\wedge\tilde{Y})\tilde{Z}=\tilde{g}(\tilde{Y}, \tilde{Z})\tilde{X}-\tilde{g}(\tilde{X}, \tilde{Z})\tilde{Y}$ for any vector field $\tilde{Z}$ in $\tilde{M}^{4n}(\tilde{c})$ (see [4]).

Now we take on U a local field of orthonormal frame $\{e_1, \cdots, e_{2m}, e_{2m+1}, \cdots, e_{4n}\}$ of $\tilde{M}^{4n}$ such that $e_1, \cdots, e_{2m}$ are tangent to M^{2m} and $e_{2m+1}, \cdots, e_{4n}$ normal to M^{2m}. Let $A_x=A_{e_x}$ $(x=2m+1, \cdots, 4n)$. Using (2.13), the structure equation of Gauss is given by

$$\begin{aligned}R(X, Y, Z, W)=\tilde{c}/4\{&g((X\wedge Y)Z, W)+g((JX\wedge JY)Z, W)\\ &-2g(JX, Y)g(JZ, W)\}+\sum_{x=2m+1}^{4n} g((A_xX\wedge A_xY)Z, W)\end{aligned} \tag{2.14}$$

for any vector fields X, Y, Z and W of M^{2m}, R being the curvature tensor of M^{2m}, where we have put $R(X, Y, Z, W)=g(R(X, Y)Z, W)$. We can see from (2.14) that the general sectional curvature $k(X, Y)$ of M^{2m} determined by orthonormal vectors X and Y tangent to M^{2m} is given by

$$k(X, Y)=\tilde{c}/4\{1+3g(JX, Y)^2\}+\tilde{g}(H(X, X), H(Y, Y))-\tilde{g}(H(X, Y), H(X, Y)). \tag{2.15}$$

Thus the holomorphic sectional curvature $\tau(X)$ of M^{2m} determined by unit vector X is given as

$$\tau(X)=c-2\tilde{g}(H(X, X), H(X, X)). \tag{2.16}$$

Let S be the Ricci tensor of M^{2m}. Then we have from (2.14)

$$S(X, Y)=\frac{1}{2}(m+1)\tilde{c}g(X, Y)-\sum_{x=2m+1}^{4n} g(A_x X, A_x Y) \tag{2.17}$$

for any vectors X and Y of M^{2m} because M^{2m} is minimal. This implies the scalar curvature ρ of M^{2m} is given by

$$\rho=m(m+1)\tilde{c}-\|H\|^2, \tag{2.18}$$

where $\|H\|^2$ is the length of the second fundamental form of M^{2m} so that

$$\|H\|^2=\sum_{x=2m+1}^{4n} \operatorname{tr} A_x^2. \tag{2.19}$$

Therefore, we have from (2.16), (2.17) and (2.18)

PROPOSITION 2.3. *Let $\tilde{M}^{4n}(\tilde{c})$ be a 4n-dimensional quaternionic space from of constant Q-sectional curvature $\tilde{c}$ and M^{2m} $(m\leqq n)$ a 2m-dimensional totally complex submanifold of $\tilde{M}^{4n}(\tilde{c})$. Then*

(1) *$\tau(X)\leqq\tilde{c}$ for each unit vector X tangent to M^{2m},*

(2) *$S-\frac{1}{2}(m+1)\tilde{c}g$ is negative semidefinite,*

(3) *$\rho\leqq m(m+1)\tilde{c}$.*

Moreover we have from the same equations (2.16), (2.17) and (2.18)

THEOREM 2.4. *Let $\tilde{M}^{4n}(\tilde{c})$ be a 4n-dimensional quaternionic space form of constant Q-sectional curvature $\tilde{c}$ and M^{2m} $(m\leqq n)$ a 2m-dimensional totally complex submanifold of $\tilde{M}^{4n}(\tilde{c})$. Then M^{2m} is totally geodesic if and only if M^{2m} satisfies one of the following conditions* (1)$\sim$(3):

(1) *$\tau(X)=\tilde{c}$ for each unit vector X tangent to M^{2m};*

(2) *$S=\frac{1}{2}(m+1)\tilde{c}g$;*

(3) *$\rho=m(m+1)\tilde{c}$.*

As is well known, a totally umbilical submanifold is totally geodesic if and only if it is minimal. Then we have from Proposition 2.2 and 2.4

PROPOSITION 2.5. *Let $\tilde{M}^{4n}(\tilde{c})$ be a 4n-dimensional quaternionic space form of constant Q-sectional curvature $\tilde{c}$ and M^{2m} $(m\leqq n)$ a 2m-dimensional totally complex submanifold of $\tilde{M}^{4n}(\tilde{c})$. Suppose that M^{2m} is totally umbilical. Then M^{2m} is totally geodesic and hence a complex space form of constant holomorphic sectional curvature $\tilde{c}$.*

§ 3. Second fundamental tensors and structure equations.

Let $(\tilde{M}^{4n}, \tilde{g}, \tilde{V})$ be a quaternionic Kaehlerian manifold of dimension $4n$ and M^{2n} a totally complex submanifold of $\tilde{M}^{4n}$. We take a coordinate neighborhood

U in M^{2n}. Let $\{\tilde{J}, \tilde{K}, \tilde{L}\}$ be a triplet of local cross sections of M^{2n} satisfying the conditions (i) and (ii) and the same equations as (2.4) stated in §2. We take now a local field of symplectic frame $\{e_1, \cdots, e_n; e_{\bar{1}}=\tilde{J}e_1, \cdots, e_{\bar{n}}=\tilde{J}e_n; e_{1^*}=\tilde{K}e_1, \cdots, e_{n^*}=\tilde{K}e_n; e_{\bar{1}^*}=\tilde{L}e_1, \cdots, e_{\bar{n}^*}=\tilde{L}e_n\}$ of $\tilde{M}^{4n}$ such that $e_1, \cdots, e_n, e_{\bar{1}}, \cdots, e_{\bar{n}}$ are tangent to M^{2n} and $e_{1^*}, \cdots, e_{n^*}, e_{\bar{1}^*}, \cdots, e_{\bar{n}^*}$ normal to M^{2n}. With respect to this symplectic frame field $\{e_a, e_{\bar{a}}, e_{a^*}, e_{\bar{a}^*}\}$, the complex structure J of M^{2n} is represented as

$$J=\begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}, \tag{3.1}$$

E_n being the identity matrix of degree n. The induced metrics g of M^{2n} and $\hat{g}$ of the normal bundle $N(M^{2n})$ are given respectively by

$$g=\begin{pmatrix} g_{ba} & g_{b\bar{a}} \\ g_{\bar{b}a} & g_{\bar{b}\bar{a}} \end{pmatrix}=\begin{pmatrix} \delta_{ba} & 0 \\ 0 & \delta_{ba} \end{pmatrix}, \tag{3.2}$$

$$\hat{g}=\begin{pmatrix} g_{b^*a^*} & g_{b^*\bar{a}^*} \\ g_{\bar{b}^*a^*} & g_{\bar{b}^*\bar{a}^*} \end{pmatrix}=\begin{pmatrix} \delta_{ba} & 0 \\ 0 & \delta_{ba} \end{pmatrix}. \tag{3.3}$$

Moreover the Gauss-Weingarten formulas (2.9) are represented as

$$\tilde{\nabla}_{e_c}e_b=\nabla_{e_c}e_b+\sum_a (H_{cb}{}^{a^*}e_{a^*}+H_{cb}{}^{\bar{a}^*}e_{\bar{a}^*}), \tag{3.4}$$

$$\tilde{\nabla}_{e_c}e_{\bar{b}}=J\nabla_{e_c}e_{\bar{b}}+\sum_a (H_{c\bar{b}}{}^{a^*}e_{a^*}+H_{c\bar{b}}{}^{\bar{a}^*}e_{\bar{a}^*}), \tag{3.5}$$

$$\tilde{\nabla}_{e_{\bar{c}}}e_b=\nabla_{e_{\bar{c}}}e_b+\sum_a (H_{\bar{c}b}{}^{a^*}e_{a^*}+H_{\bar{c}b}{}^{\bar{a}^*}e_{\bar{a}^*}), \tag{3.6}$$

$$\tilde{\nabla}_{e_{\bar{c}}}e_{\bar{b}}=\nabla_{e_{\bar{c}}}e_{\bar{b}}+\sum_a (H_{\bar{c}\bar{b}}{}^{a^*}e_{a^*}+H_{\bar{c}\bar{b}}{}^{\bar{a}^*}e_{\bar{a}^*}), \tag{3.7}$$

$$\tilde{\nabla}_{e_c}e_{b^*}=-A_{b^*}e_c+D_{e_c}e_{b^*}, \tag{3.8}$$

$$\tilde{\nabla}_{e_c}e_{\bar{b}^*}=-A_{\bar{b}^*}e_c+D_{e_c}e_{\bar{b}^*}, \tag{3.9}$$

$$\tilde{\nabla}_{e_{\bar{c}}}e_{b^*}=-A_{b^*}e_{\bar{c}}+D_{e_{\bar{c}}}e_{b^*}, \tag{3.10}$$

$$\tilde{\nabla}_{e_{\bar{c}}}e_{\bar{b}^*}=-A_{\bar{b}^*}e_{\bar{c}}+D_{e_{\bar{c}}}e_{\bar{b}^*}, \tag{3.11}$$

where we have put $H(e_c, e_b)=\sum_{\bar{a}^*}(H_{cb}{}^{a^*}e_{a^*}+H_{cb}{}^{\bar{a}^*}e_{\bar{a}^*})$ and so on. Taking account of (2.8) and using (3.4) and (3.5), we get

$$\sum_a (H_{c\bar{b}}{}^{a^*}e_{a^*}+H_{c\bar{b}}{}^{\bar{a}^*}e_{\bar{a}^*})=\sum_a (-H_{cb}{}^{\bar{a}^*}e_{a^*}+H_{cb}{}^{a^*}e_{\bar{a}^*}),$$

which implies

$$H_{cb\bar{a}^*}=-H_{c\bar{b}a^*}, \qquad H_{cba^*}=H_{c\bar{b}\bar{a}^*}, \tag{3.12}$$

where we have put $H_{cba^*}=H_{cb}{}^{d^*}g_{d^*a^*}$, $H_{cb\bar{a}^*}=H_{cb}{}^{\bar{d}^*}g_{\bar{d}^*\bar{a}^*}$ and so on. By similar devices, we have from (3.7)

$$H_{\bar{c}\bar{b}a^*}=-H_{\bar{c}b\bar{a}^*},\ H_{\bar{c}\bar{b}\bar{a}^*}=H_{\bar{c}ba^*}. \tag{3.13}$$

Taking account of (2.8) and (3.8), we get

$$A_{b^*}e_c=\sum_a (H_{cb}{}^{a^*}e_a-H_{cb}{}^{\bar{a}^*}e_{\bar{a}})$$

which implies

$$H_{cba^*}=H_{cab^*},\qquad H_{cb\bar{a}^*}=-H_{c\bar{a}b^*}. \tag{3.14}$$

By similar computaions for (3.9), (3.10) and (3.11), we have

LEMMA 3.1. *Let $(\tilde{M}^{4n}, \tilde{g}, \tilde{\nabla})$ be a quaternionic Kaehlerian manifold of dimension $4n$ and M^{2n} a $2n$-dimensional totally complex submanifold of $\tilde{M}^{4n}$. Then*

$$\begin{aligned} H_{cba^*}&=H_{c\bar{b}\bar{a}^*}=H_{\bar{a}c\bar{b}^*}=-H_{\bar{a}\bar{c}b^*}\\ &=H_{cab^*}=H_{c\bar{a}\bar{b}^*}=H_{\bar{a}b\bar{c}^*}=-H_{\bar{a}\bar{b}c^*}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} H_{cb\bar{a}^*}&=-H_{c\bar{b}a^*}=-H_{\bar{a}cb^*}=-H_{\bar{a}\bar{c}\bar{b}^*},\\ &=H_{ca\bar{b}^*}=-H_{c\bar{a}b^*}=-H_{\bar{a}bc^*}=-H_{\bar{a}\bar{b}\bar{c}^*}. \end{aligned} \tag{3.16}$$

If we define symmetric matrices H_{a^*} and $H_{\bar{a}^*}$ of degree n by

$$H_{a^*}=(H_{cba^*}),\qquad H_{\bar{a}^*}=(H_{cb\bar{a}^*}), \tag{3.17}$$

then the local fields A_{a^*} and $A_{\bar{a}^*}$ of symmetric linear transformations assosiated to the fundamental form H are represented respectively by

$$A_{a^*}=\begin{pmatrix} H_{cba^*} & H_{c\bar{b}a^*}\\ H_{\bar{c}ba^*} & H_{\bar{c}\bar{b}a^*}\end{pmatrix}=\begin{pmatrix} H_{a^*} & -H_{\bar{a}^*}\\ -H_{\bar{a}^*} & -H_{a^*}\end{pmatrix}, \tag{3.18}$$

$$A_{\bar{a}^*}=\begin{pmatrix} H_{cb\bar{a}^*} & H_{c\bar{b}\bar{a}^*}\\ H_{\bar{c}b\bar{a}^*} & H_{\bar{c}\bar{b}\bar{a}^*}\end{pmatrix}=\begin{pmatrix} H_{\bar{a}^*} & H_{a^*}\\ H_{a^*} & -H_{\bar{a}^*}\end{pmatrix} \tag{3.19}$$

because of Lemma 3.1 and hence the suquare of the length of the second fundamental form H is given by

$$\|H\|^2=4\sum_{a=1}^{n}\operatorname{tr}(H_{a^*}{}^2+H_{\bar{a}^*}{}^2). \tag{3.20}$$

As an immediately consequence of (3.18) and (3.19), we have

$$\operatorname{tr} A_{a^*}A_{b^*}=\operatorname{tr} A_{\bar{a}^*}A_{\bar{b}^*}=2\operatorname{tr}(H_{a^*}H_{b^*}+H_{\bar{a}^*}H_{\bar{b}^*}), \tag{3.21}$$

$$\operatorname{tr} A_{a^*}A_{\bar{b}^*}=-\operatorname{tr} A_{\bar{a}^*}A_{b^*}=2\operatorname{tr}(H_{a^*}H_{\bar{b}^*}-H_{\bar{a}^*}H_{b^*}), \tag{3.22}$$

which imply

$$\sum_{a,b=1}^{n}(\mathrm{tr}\, A_{a^*}A_{b^*})^2=4\sum_{a,b=1}^{n}(\mathrm{tr}\, H_{a^*}H_{b^*}+\mathrm{tr}\, H_{\bar a^*}H_{\bar b^*})^2\,, \tag{3.23}$$

$$\sum_{a,b=1}^{n}(\mathrm{tr}\, A_{a^*}A_{\bar b^*})^2=8\sum_{a,b=1}^{n}\{(\mathrm{tr}\, H_{\bar a^*}H_{b^*})^2-(\mathrm{tr}\, H_{\bar a^*}H_{b^*})(\mathrm{tr}\, H_{a^*}H_{\bar b^*})\}\,. \tag{3.24}$$

Next, from (3.1), (3.18) and (3.19), we obtain

$$JA_{a^*}=-A_{a^*}J=A_{\bar a^*} \tag{3.25}$$

for each index a. Therefore we have

$$A_{\bar a^*}{}^2=A_{a^*}{}^2, \qquad (A_{a^*}A_{\bar b^*})^2=-(A_{a^*}A_{b^*})^2\,. \tag{3.26}$$

We prepare the following three lemmas for later use.

LEMMA 3.2. *The following equations hold*:

$$\sum_{s\neq t}\mathrm{tr}\,(A_sA_t-A_tA_s)^2=-8\sum_{a,b=1}^{n}\mathrm{tr}\, A_{a^*}{}^2A_{b^*}{}^2 \tag{3.27}$$
$$=-4\sum_{a,b=1}^{n}\{(\mathrm{tr}\, A_{a^*}A_{b^*})^2+(\mathrm{tr}\, A_{a^*}A_{\bar b^*})^2\}\,,$$

$$\mathrm{tr}\,(\sum_s A_s{}^2)^2=4\sum_{a,b=1}^{n}\mathrm{tr}\, A_{a^*}{}^2A_{b^*}{}^2 \tag{3.28}$$
$$=2\sum_{a,b=1}^{n}\{(\mathrm{tr}\, A_{a^*}A_{b^*})^2+(\mathrm{tr}\, A_{a^*}A_{\bar b^*})^2\}=\sum_{s,t}(\mathrm{tr}\, A_sA_t)^2\,.$$

Proof. Using (3.26), we have

$$\sum_{s\neq t}\mathrm{tr}\,(A_sA_t-A_tA_s)^2=-2\sum_{s\neq t}\mathrm{tr}\,(A_s{}^2A_t{}^2-(A_sA_t)^2)=-8\sum_{a,b}\mathrm{tr}\, A_{a^*}{}^2A_{b^*}{}^2\,.$$

On the other hand, we have using (3.15), (3.16), (3.18), (3.21) and (3.22)

$$\begin{aligned}\sum_{a,b=1}^{n}\mathrm{tr}\, A_{a^*}{}^2A_{b^*}{}^2=&\,2\sum_{a,b}\mathrm{tr}\,(H_{a^*}{}^2H_{b^*}{}^2+H_{a^*}{}^2H_{\bar b^*}{}^2+H_{\bar a^*}{}^2H_{b^*}{}^2+H_{\bar a^*}{}^2H_{\bar b^*}{}^2)\\&+2\sum_{a,b}[\mathrm{tr}\,(H_{a^*}H_{\bar a^*}H_{\bar b^*}H_{b^*}-H_{a^*}H_{\bar a^*}H_{b^*}H_{\bar b^*}\\&-H_{\bar a^*}H_{a^*}H_{\bar b^*}H_{b^*}+H_{\bar a^*}H_{a^*}H_{b^*}H_{\bar b^*})]\\=&\,2\sum_{a,b,c,d,e,f}\{(H_{fea^*}H_{eda^*}H_{dcb^*}H_{cfb^*}+H_{fea^*}H_{eda^*}H_{dc\bar b^*}H_{cf\bar b^*}\\&+H_{fe\bar a^*}H_{ed\bar a^*}H_{dcb^*}H_{cfb^*}+H_{fe\bar a^*}H_{ed\bar a^*}H_{dc\bar b^*}H_{cf\bar b^*}\\&+H_{fea^*}H_{ed\bar a^*}H_{dc\bar b^*}H_{cfb^*}-H_{fea^*}H_{ed\bar a^*}H_{dcb^*}H_{cf\bar b^*}\\&-H_{fe\bar a^*}H_{eda^*}H_{dc\bar b^*}H_{cfb^*}+H_{fe\bar a^*}H_{eda^*}H_{dcb^*}H_{cf\bar b^*})\}\end{aligned}$$

$$=2\sum_{d,f}\{(\mathrm{tr}\,H_{d^*}H_{f^*})^2+2(\mathrm{tr}\,H_{d^*}H_{f^*})(\mathrm{tr}\,H_{\bar d^*}H_{\bar f^*})+(\mathrm{tr}\,H_{\bar d}H_{\bar f^*})^2$$

$$+(\mathrm{tr}\,H_{\bar d^*}H_{f^*})^2-2(\mathrm{tr}\,H_{\bar d^*}H_{f^*})(\mathrm{tr}\,H_{d^*}H_{\bar f^*})+(\mathrm{tr}\,H_{d^*}H_{\bar f^*})^2\}$$

$$=2\sum_{d,f}[\{\mathrm{tr}\,(H_{d^*}H_{f^*}+H_{\bar d^*}H_{\bar f^*})\}^2+\{\mathrm{tr}\,(H_{f^*}H_{\bar d^*}-H_{\bar f^*}H_{d^*})\}^2]$$

$$=\frac{1}{2}\sum_{d,f=1}^{n}\{(\mathrm{tr}\,A_{f^*}A_{d^*})^2+(\mathrm{tr}\,A_{f^*}A_{\bar d^*})^2\}\,.$$

Therefore, we have proved (3.27). Since we have from (3.26)

$$\mathrm{tr}\,(\sum_s A_s{}^2)^2=4\sum_{a,b}\mathrm{tr}\,A_{a^*}{}^2A_{b^*}{}^2\,,$$

we see easily that (3.28) is obtained from the equation mentioned above.

Q. E. D.

We now consider the following matrix B of degree $2n$:

$$B=\begin{pmatrix}\mathrm{tr}\,A_{a^*}A_{b^*} & \mathrm{tr}\,A_{a^*}A_{\bar b^*}\\ \mathrm{tr}\,A_{\bar a^*}A_{b^*} & \mathrm{tr}\,A_{\bar a^*}A_{\bar b^*}\end{pmatrix}=\begin{pmatrix}\mathrm{tr}\,A_{a^*}A_{b^*} & \mathrm{tr}\,A_{a^*}A_{\bar b^*}\\ -\mathrm{tr}\,A_{a^*}A_{\bar b^*} & \mathrm{tr}\,A_{a^*}A_{b^*}\end{pmatrix}.$$

Then the quantity

$$\sum_{s,t}(\mathrm{tr}\,A_sA_t)=2\sum_{a,b=1}^{n}\{(\mathrm{tr}\,A_{a^*}A_{b^*})^2+(\mathrm{tr}\,A_{a^*}A_{\bar b^*})^2\}$$

is a scalar invariant, that is, it does not depend on the choice of the normal vectors $e_{1^*},\cdots,e_{n^*},e_{\bar 1^*},\cdots,e_{\bar n^*}$ consisting the symplectic frame $\{e_a, e_{\bar a}, e_{a^*}, e_{\bar a^*}\}$. Moreover we have

LEMMA 3.3. *For a suitable choice of a symplectic frame* $\{e_a, e_{\bar a}, e_{a^*}, e_{\bar a^*}\}$ *of* $\tilde M^{4n}$, *the equation* $\mathrm{tr}\,A_sA_t=0$ *for* $s\neq t$ *holds and consequently we have*

$$\sum_{s,t}(\mathrm{tr}\,A_sA_t)^2\leqq\frac{1}{2}\|H\|^4\,. \tag{3.29}$$

Proof. Consider the matrix B mentioned above. Then B is a Hermitian matrix. Take an another symplectic frame $\{e'_a, e'_{\bar a}, e'_{a^*}, e'_{\bar a^*}\}$ and deote by B' the corresponding matrix for this symplectic frame. Then we have $B'={}^tUBU$, U being a unitary matrix of degree $2n$ such that its real representation satisfies

$$e'_{b^*}=\sum_{a=1}^{n}(e_{a^*}U_{ab}-e_{\bar a^*}U_{\bar ab})\,,\qquad e'_{\bar b^*}=\sum_{a=1}^{n}(e_{a^*}U_{\bar ab}+e_{\bar a^*}U_{ab})\,.$$

Therefore, the matrix B can be diagonalized for a suitable normal basis $e_{1^*},\cdots$, $e_{n^*},e_{\bar 1^*},\cdots,e_{\bar n^*}$, that is,

$${}^tUBU=\begin{pmatrix}\mathrm{tr}\,A'^2_{a^*} & 0\\ 0 & \mathrm{tr}\,A'^2_{a^*}\end{pmatrix},$$

U being the real representation of a unitary matrix. Thus we obtain

$$\sum_{s,t}(\operatorname{tr} A_sA_t)^2=2\sum_a(\operatorname{tr} A'^2_{a^*})^2,$$

which implies

$$\sum_{s,t}(\operatorname{tr} A_sA_t)^2\leqq 2(\sum_a \operatorname{tr} A'^2_{a^*})^2=\frac{1}{2}\|H\|^4.$$

This proves Lemma 3.3. Q. E. D.

We are now in position to give the structure equations of totally complex submanifold of a quaternionic space form. Let $\tilde{M}^{4n}(\tilde{c})$ be a quaternionic space form of dimension $4n$ and of constant Q-sectional curvature $\tilde{c}$ and M^{2n} a $2n$-dimensional totally complex submanifold of $\tilde{M}^{4n}(\tilde{c})$. With respect to the orthonormal frame $\{e_1, \cdots, e_n, e_{\bar{1}}, \cdots, e_{\bar{n}}\}$, we put

$$R_{dcba}=R(e_d, e_c, e_b, e_a), \qquad R_{dcb\bar{a}}=R(e_d, e_c, e_b, e_{\bar{a}}),$$
$$R_{d\bar{c}\bar{b}\bar{a}}=R(e_d, e_{\bar{c}}, e_{\bar{b}}, e_{\bar{a}}), \qquad R_{\bar{d}cb\bar{a}}=R(e_{\bar{d}}, e_c, e_b, e_{\bar{a}}), \cdots,$$

and

$$S_{cb}=S(e_c, e_b), \qquad S_{c\bar{b}}=S(e_c, e_{\bar{b}}), \qquad S_{\bar{c}\bar{b}}=S(e_{\bar{c}}, e_{\bar{b}}),$$

where R and S are the cuvature tensor and the Ricci tensor of M^{2n} respectively. We have then from (2.14) and Lemma 3.1 the following structure equations of Gauss:

$$R_{dcba}=R_{dc\bar{b}\bar{a}}=R_{\bar{d}\bar{c}\bar{b}\bar{a}}=\frac{\tilde{c}}{4}(\delta_{da}\delta_{cb}-\delta_{db}\delta_{ca}) \tag{3.30}$$
$$+\sum_e(H_{dae^*}H_{cbe^*}+H_{da\bar{e}^*}H_{cb\bar{e}^*}-H_{dbe^*}H_{cae^*}-H_{db\bar{e}^*}H_{ca\bar{e}^*}),$$

$$R_{dcb\bar{a}}=\sum_e(H_{dae^*}H_{cb\bar{e}^*}-H_{da\bar{e}^*}H_{cbe^*}+H_{dbe^*}H_{ca\bar{e}^*}-H_{db\bar{e}^*}H_{cae^*}), \tag{3.31}$$

$$R_{d\bar{c}\bar{b}\bar{a}}=\sum_e(H_{da\bar{e}^*}H_{cbe^*}-H_{dae^*}H_{cb\bar{e}^*}+H_{dbe^*}H_{ca\bar{e}^*}-H_{db\bar{e}^*}H_{cae^*}), \tag{3.32}$$

$$R_{\bar{d}cb\bar{a}}=\frac{\tilde{c}}{4}(\delta_{da}\delta_{cb}+\delta_{db}\delta_{ca}+2\delta_{dc}\delta_{ba}) \tag{3.33}$$
$$-\sum_e(H_{dae^*}H_{cbe^*}+H_{da\bar{e}^*}H_{cb\bar{e}^*}+H_{dbe^*}H_{cae^*}+H_{db\bar{e}^*}H_{ca\bar{e}^*}).$$

Using these equations above, the Ricci tensor S of M^{2n} has the following components.

$$S_{cb}=S_{\bar{c}\bar{b}}=\frac{1}{2}(n+1)\tilde{c}\delta_{cb}-2\sum_{d,e}(H_{dce^*}H_{bde^*}+H_{dc\bar{e}^*}H_{bd\bar{e}^*}), \tag{3.34}$$

$$S_{c\bar{b}}=-S_{\bar{c}b}=2\sum_{d,e}(H_{db\bar{e}^*}H_{cde^*}-H_{dbe^*}H_{cd\bar{e}^*}). \tag{3.35}$$

For any vectors X, Y and Z tangent to M^{2n}, we have from (2.13)

$$\tilde{R}(X, Y)Z=\frac{\tilde{c}}{4}\{(X\wedge Y)Z+(\tilde{J}X\wedge\tilde{J}Y)Z-2\tilde{g}(\tilde{J}X, Y)\tilde{J}X\}$$

because M^{2n} is totally complex. Therefore $[\tilde{R}(X, Y)Z]^N=0$, where the left hand side is the normal component of $\tilde{R}(X, Y)Z$. Thus we obtain the following equation of Codazzi.

$$(\nabla_X H)(Y, Z)-(\nabla_Y H)(X, Z)=0 \tag{3.36}$$

for any vectors X, Y and Z tangent to M^{2n}, that is,

$$\begin{aligned} &\nabla_d H_{cb}{}^{a^*}-\nabla_c H_{db}{}^{a^*}=0, \qquad \nabla_{\bar{d}} H_{cb}{}^{a^*}-\nabla_c H_{\bar{d}b}{}^{a^*}=0, \\ &\nabla_d H_{cb}{}^{\bar{a}^*}-\nabla_c H_{db}{}^{\bar{a}^*}=0, \qquad \nabla_{\bar{d}} H_{cb}{}^{\bar{a}^*}-\nabla_c H_{\bar{d}b}{}^{\bar{a}^*}=0, \\ &\qquad\cdots\cdots \qquad\qquad\qquad\qquad \cdots\cdots, \end{aligned} \tag{3.37}$$

where we have put

$$(\nabla H)(e_d, e_c, e_b)=\sum_a ((\nabla_d H_{cb}{}^{a^*})e_{a^*}+(\nabla_d H_{cb}{}^{\bar{a}^*})e_{\bar{a}^*})$$

and so on.

Moreover we shall give the equation of Ricci. Let X and Y be arbitrary vectors tangent to M^{2n} and ξ and η arbitrary vectors normal to M^{2n}. If we denote by R^N the curvature tensor of the connection D induced in the normal bundle $N(M^{2n})$, that is,

$$R^N(X, Y)\xi=D_X D_Y\xi-D_Y D_X\xi-D_{[X,Y]}\xi,$$

then we have

$$R^N(X, Y, \xi, \eta)=\tilde{R}(X, Y, \xi, \eta)+g([A_\xi, A_\eta](X), Y), \tag{3.38}$$

where $R^N(X, Y, \xi, \eta)=\tilde{g}(R^N(X, Y)\xi, \eta)$ and $[A_\xi, A_\eta]=A_\xi A_\eta-A_\eta A_\xi$. For our symplectic frame $\{e_a, e_{\bar{a}}, e_{a^*}, e_{\bar{a}^*}\}$, we put

$$R^N{}_{dcb^*a^*}=R^N(e_d, e_c, e_{b^*}, e_{a^*}), \qquad R^N{}_{\bar{d}cb^*\bar{a}^*}=R^N(e_{\bar{d}}, e_c, e_{b^*}, e_{\bar{a}^*})$$

and so on. Using (2.13) and Lemma 3.1, the equation (3.38) of Ricci reduces to

$$\begin{aligned} R^N{}_{dcb^*a^*}=R^N{}_{dc\bar{b}^*\bar{a}^*}=R^N{}_{\bar{d}\bar{c}b^*a^*}=R^N{}_{\bar{d}\bar{c}\bar{b}^*\bar{a}^*}=\frac{\tilde{c}}{4}(\delta_{da}\delta_{cb}-\delta_{db}\delta_{ca}) \\ +\sum_e (H_{dea^*}H_{ceb^*}-H_{cea^*}H_{deb^*}+H_{de\bar{a}^*}H_{ce\bar{b}^*}-H_{ce\bar{a}^*}H_{de\bar{b}^*}), \end{aligned} \tag{3.39}$$

$$\begin{aligned} R^N{}_{\bar{d}cb^*\bar{a}^*}=-\frac{\tilde{c}}{4}(\delta_{da}\delta_{cb}+\delta_{db}\delta_{ca}-2\delta_{dc}\delta_{ba}) \\ +\sum_e (H_{dea^*}H_{ceb^*}+H_{ce\bar{a}^*}H_{de\bar{b}^*}+H_{de\bar{a}^*}H_{ce\bar{b}^*}+H_{cea^*}H_{deb^*}), \end{aligned} \tag{3.40}$$

$$\begin{aligned} R^N{}_{dcb^*\bar{a}^*}&=R^N{}_{\bar{d}\bar{c}b^*\bar{a}^*} \\ &=\sum_e (H_{de\bar{a}^*}H_{ceb^*}-H_{ce\bar{a}^*}H_{deb^*}-H_{dea^*}H_{ce\bar{b}^*}+H_{cea^*}H_{de\bar{b}^*}), \end{aligned} \tag{3.41}$$

$$(3.42) \qquad R^N{}_{\bar{d}cb^*a^*} = R^N{}_{\bar{d}c\bar{b}^*\bar{a}^*} = \sum_e (H_{dea^*}H_{ce\bar{b}^*} - H_{ce\bar{a}^*}H_{deb^*} - H_{de\bar{a}^*}H_{ceb^*} + H_{cea^*}H_{de\bar{b}^*}).$$

Then we have

PROPOSITION 3.5. *Let $\tilde{M}^{4n}(\tilde{c})$, $(n \geqq 2)$, be a $4n$-dimensional quaternionic space form with constant Q-sectional curvature $\tilde{c}$ and M^{2n} a totally complex submanifold $\tilde{M}^{4n}(\tilde{c})$. If the connection D induced in the normal bundle $N(M^{2n})$ is flat, then $\tilde{c}=0$ and M^{2n} is also flat.*

Proof. Assume that the connection D induced in the normal bundle $N(M^{2n})$ is flat. Then we have from (3.39), (3.41) and (3.42)

$$\frac{\tilde{c}}{4}(\delta_{da}\delta_{cb} - \delta_{db}\delta_{ca}) + \sum_e (H_{dae^*}H_{cbe^*} + H_{da\bar{e}^*}H_{cb\bar{e}^*} - H_{dbe^*}H_{cae^*} - H_{db\bar{e}^*}H_{ca\bar{e}^*}) = 0,$$

$$\sum_e (H_{dae^*}H_{cb\bar{e}^*} - H_{da\bar{e}^*}H_{cbe^*} + H_{dbe^*}H_{ca\bar{e}^*} - H_{db\bar{e}^*}H_{cae^*}) = 0,$$

$$\sum_e (H_{da\bar{e}^*}H_{cbe^*} - H_{dae^*}H_{cb\bar{e}^*} + H_{dbe^*}H_{ca\bar{e}^*} - H_{db\bar{e}^*}H_{cae^*}) = 0$$

by virtue of (3.15) and (3.16). Taking account of the equations above and the equations of Gauss (3.30), (3.31) and (3.32), we get

$$R_{dcba} = R_{dc\bar{b}\bar{a}} = R_{\bar{d}\bar{c}\bar{b}\bar{a}} = 0, \qquad R_{dcb\bar{a}} = 0, \qquad R_{d\bar{c}\bar{b}\bar{a}} = 0.$$

Since $R^N{}_{\bar{d}cb^*\bar{a}^*} = 0$, we have from (3.40)

$$\sum_e (H_{dae^*}H_{cbe^*} + H_{da\bar{e}^*}H_{cb\bar{e}^*} + H_{dbe^*}H_{cae^*} + H_{db\bar{e}^*}H_{ca\bar{e}^*}) = \frac{\tilde{c}}{4}(\delta_{da}\delta_{cb} + \delta_{db}\delta_{ca} - 2\delta_{dc}\delta_{ba}).$$

Therefore (3.33) reduces to $R_{\bar{d}cb\bar{a}} = \tilde{c}\delta_{dc}\delta_{ba}$. Using the first Bianchi's identities and $R_{dc\bar{b}\bar{a}} = 0$, we obtain

$$R_{c\bar{b}d\bar{a}} + R_{\bar{b}dc\bar{a}} = 0.$$

Hence we have

$$\tilde{c}(\delta_{bc}\delta_{da} - \delta_{bd}\delta_{ca}) = 0,$$

which implies $n(n-1)\tilde{c}=0$. Since $n \geqq 2$, we have $\tilde{c}=0$ and hence $R_{\bar{d}cb\bar{a}}=0$. This proves our assertion. Q. E. D.

§ 4. Totally complex submanifolds of a quaternionic space form.

In this section, we are going to give the basic theorem of the present paper. First we compute the Laplacian of the square of the length of the second fundamental tensor. Let $\tilde{M}^{4n}(\tilde{c})$ be a quaternionic space form of dimension $4n$ and of constant Q-sectional curvature $\tilde{c}$ and M^{2n} a totally complex submanifold of $\tilde{M}^{4n}(\tilde{c})$. As stated in the previous sections, we denote by H the second fundamental form of M^{2n}. Then the Laplacian of the length of H satisfies

$$(4.1)\qquad \frac{1}{2}\Delta\|H\|^2=4[(\nabla^d\nabla_d H_{cba^*}+\nabla^{\bar{d}}\nabla_{\bar{d}}H_{cba^*})H^{cba^*} +(\nabla^d\nabla_d H_{cb\bar{a}^*}+\nabla^{\bar{d}}\nabla_{\bar{d}}H_{cb\bar{a}^*})H^{cb\bar{a}^*}]+\|\nabla H\|^2$$

because of (3.20) and Lemma 3.1, where $\nabla^d=g^{de}\nabla_e$ and $\nabla^{\bar{d}}=g^{\bar{d}\bar{e}}\nabla_{\bar{e}}$ and we have used Einstein summation convention. Since M^{2n} is minimal (see, Proposition 2.2), we have

$$\begin{aligned}\nabla^d\nabla_d H_{cba^*}+\nabla^{\bar{d}}\nabla_{\bar{d}}H_{cba^*}&=S_{cd}H^d{}_{ba^*}-S_{c\bar{d}}H^d{}_{b\bar{a}^*}\\ &-R_{ecb}{}^dH^e{}_{da^*}+R_{ecb}{}^{\bar{d}}H^e{}_{d\bar{a}^*}+R_{\bar{e}cb}{}^dH^e{}_{d\bar{a}^*}+R_{\bar{e}cb}{}^{\bar{d}}H^e{}_{da^*}\\ &-R_{eca^*}{}^{d^*}H^e{}_{bd^*}-R_{eca^*}{}^{\bar{d}^*}H^e{}_{b\bar{d}^*}+R_{\bar{e}ca^*}{}^{d^*}H^e{}_{b\bar{d}^*}-R_{\bar{e}ca^*}{}^{\bar{d}^*}H^e{}_{bd^*},\end{aligned}$$

where $H^d{}_{b\bar{a}^*}=g^{de}H_{eb\bar{a}^*}$, $R_{ecb}{}^d=R_{ecba}g^{ad}$, $\cdots$ and where we have used the equation of codazzi (3.37) and Lemma 3.1. Moreover, substituting the equations of Gauss ((3.30), (3.31), (3.33), (3.34), (3.35)), and the equations of Ricci ((3.39), (3.40), (3.41), (3.42)) into the equation mentioned above, we have

$$(4.2)\qquad \begin{aligned}(\nabla^d\nabla_d H_{cba^*}+\nabla^{\bar{d}}\nabla_{\bar{d}}H_{cba^*})H^{cba^*}&=\frac{1}{2}(n+3)\tilde{c}\sum_a \operatorname{tr} H_{a^*}{}^2\\ &-6\sum_{a,b}[(\operatorname{tr} H_{a^*}H_{b^*})^2+(\operatorname{tr} H_{a^*}H_{b^*})(\operatorname{tr} H_{\bar{a}^*}H_{\bar{b}^*})\\ &+(\operatorname{tr} H_{\bar{a}^*}H_{b^*})^2-(\operatorname{tr} H_{\bar{a}^*}H_{b^*})(\operatorname{tr} H_{a^*}H_{\bar{b}^*})].\end{aligned}$$

By the same way, we have

$$(4.3)\qquad \begin{aligned}(\nabla^d\nabla_d H_{cb\bar{a}^*}+\nabla^{\bar{d}}\nabla_{\bar{d}}H_{cb\bar{a}^*})H^{cb\bar{a}^*}&=\frac{1}{2}(n+3)\tilde{c}\sum_a \operatorname{tr} H_{\bar{a}^*}{}^2\\ &-6\sum_{a,b}[(\operatorname{tr} H_{\bar{a}^*}H_{\bar{b}^*})^2+(\operatorname{tr} H_{a^*}H_{b^*})(\operatorname{tr} H_{\bar{a}^*}H_{\bar{b}^*})\\ &+(\operatorname{tr} H_{\bar{a}^*}H_{b^*})^2-(\operatorname{tr} H_{\bar{a}^*}H_{b^*})(\operatorname{tr} H_{a^*}H_{\bar{b}^*})].\end{aligned}$$

Substituting these identities (4.2) and (4.3) into (4.1), we obtain

$$(4.4)\qquad \begin{aligned}\frac{1}{2}\Delta\|H\|^2=\frac{1}{2}(n+3)\tilde{c}\|H\|^2&-24\sum_{a,b}(\operatorname{tr} H_{a^*}H_{b^*}+\operatorname{tr} H_{\bar{a}^*}H_{\bar{b}^*})^2\\ &-48\sum_{a,b}[(\operatorname{tr} H_{\bar{a}^*}H_{b^*})^2-(\operatorname{tr} H_{\bar{a}^*}H_{b^*})(\operatorname{tr} H_{a^*}H_{\bar{b}^*})]+\|\nabla H\|^2.\end{aligned}$$

Taking accout of (3.23), (3.24), (3.27), (3.28) and Lemma 3.3, and substituting these equations into (4.4), we have

PROPOSITION 4.1. *Let $\tilde{M}^{4n}(\tilde{c})$ be a quaternionic space form of dimension $4n$*

with constant Q-sectional curvature $\tilde{c}$ *and* M^{2n} *a totally complex submanifold of* $\tilde{M}^{4n}(\tilde{c})$. *Then*

$$\frac{1}{2}\Delta\|H\|^2=\frac{1}{2}(n+3)\tilde{c}\|H\|^2-6\sum_{a,b}[(\operatorname{tr} A_{a^*}A_{b^*})^2+(\operatorname{tr} A_{a^*}A_{\bar{b}^*})^2]+\|\nabla H\|^2 \tag{4.5}$$

$$=\frac{1}{2}(n+3)\tilde{c}\|H\|^2+\sum_{s\neq t}\operatorname{tr}(A_sA_t-A_tA_s)^2-\sum_{s,t}(\operatorname{tr} A_sA_t)^2+\|\nabla H\|^2$$

$$=\frac{1}{2}(n+3)\tilde{c}\|H\|^2+\sum_{s\neq t}\operatorname{tr}(A_sA_t-A_tA_s)^2-\sum_{s}(\operatorname{tr} A_s^2)^2+\|\nabla H\|^2 .$$

Now we refer to the following well known lemma.

LEMMA 4.2 ([2]). *Let A and B be symmetric matrices of degree n. Then*

$$-\operatorname{tr}(AB-BA)^2\leqq 2\operatorname{tr} A^2\operatorname{tr} B^2 ,$$

and equality holds for nonzero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiplies of the following matrices $\bar{A}$ *and* $\bar{B}$ *respectively;*

$$\bar{A}=\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline 0 & & 0 \end{array}\right), \qquad \bar{B}=\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline 0 & & 0 \end{array}\right).$$

Moreover, if A_1, A_2, A_3 are symmetric matrices of degree n such that

$$-\operatorname{tr}(A_\alpha A_\beta-A_\beta A_\alpha)^2=2\operatorname{tr} A_\alpha^2\operatorname{tr} A_\beta^2 ,\qquad 1\leqq\alpha ,\quad \beta\leqq 3 ,\quad \alpha\neq\beta ,$$

then at least one of the matrices A_α must be zero.

Since $A_{a^*}{}^2=A_{\bar{a}^*}{}^2$ holds for each a by means of (3.26), we have

$$\sum_{s\neq t}\operatorname{tr} A_s^2\operatorname{tr} A_t^2=2\Big(\sum_{a\neq b}\operatorname{tr} A_{a^*}{}^2\operatorname{tr} A_{b^*}{}^2+\sum_{a,b}\operatorname{tr} A_{a^*}{}^2\operatorname{tr} A_{b^*}{}^2\Big)$$

$$=4\sum_{a\neq b}\operatorname{tr} A_{a^*}{}^2\operatorname{tr} A_{b^*}{}^2+2\sum_{a}(\operatorname{tr} A_{a^*}{}^2)^2 .$$

Using this equation, Lemma 3.4 and the equation (4.5), we obtain

$$\frac{1}{2}\Delta\|H\|^2=\frac{1}{2}(n+3)\tilde{c}\|H\|^2+\sum_{s\neq t}\operatorname{tr}(A_sA_t-A_tA_s)^2-\sum_{s}(\operatorname{tr} A_s^2)^2+\|\nabla H\|^2 \tag{4.6}$$

$$\geqq\frac{1}{2}(n+3)\tilde{c}\|H\|^2-2\sum_{s\neq t}\operatorname{tr} A_s^2\operatorname{tr} A_t^2-\sum_{s}(\operatorname{tr} A_s^2)^2+\|\nabla H\|^2$$

$$=\frac{1}{2}(n+3)\tilde{c}\|H\|^2-8\sum_{a\neq b}\operatorname{tr} A_{a^*}{}^2\operatorname{tr} A_{b^*}{}^2-6\sum_{a}(\operatorname{tr} A_{a^*}{}^2)^2+\|\nabla H\|^2$$

$$=\frac{1}{2}(n+3)\tilde{c}\|H\|^2-4\Big[2\sum_{a\neq b}\operatorname{tr} A_{a^*}{}^2\operatorname{tr} A_{b^*}{}^2+\frac{3}{2}\sum_{a}(\operatorname{tr} A_{a^*}{}^2)^2\Big]+\|\nabla H\|^2$$

$$=\frac{1}{2}(n+3)\tilde{c}\|H\|^2+4\Big[\frac{1}{2n}\sum_{a<b}(\operatorname{tr}A_{a^*}{}^2-\operatorname{tr}A_{b^*}{}^2)^2$$

$$-\Big(2-\frac{1}{2n}\Big)(\sum_a \operatorname{tr}A_{a^*}{}^2)^2\Big]+\|\nabla H\|^2$$

$$=\frac{1}{2}\Big[(n+3)\tilde{c}-\Big(2-\frac{1}{2n}\Big)\|H\|^2\Big]\|H\|^2$$

$$+4\Big[\frac{1}{2n}\sum_{a<b}(\operatorname{tr}A_{a^*}{}^2-\operatorname{tr}A_{b^*}{}^2)^2\Big]+\|\nabla H\|^2 .$$

Therefore, we have

THEOREM 4.3. *Let $\tilde{M}^{4n}(\tilde{c})$ be a quaternionic space form of dimension $4n$ $(n\geqq 2)$ and of constant Q-sectional curvature $\tilde{c}$ $(\tilde{c}>0)$. Let M^{2n} be a $2n$-dimensional compact totally complex submanifold of $\tilde{M}^{4n}(\tilde{c})$. If the second fundamental form H of M^{2n} satisfies the inequality*

$$\|H\|^2\leqq\frac{n(n+3)}{4n-1}\tilde{c},$$

then M^{2n} is totally geodesic and hence M^{2n} is of constant holomorphic sectional curvature $\tilde{c}$.

Proof. Assume that $\|H\|^2<[n(n+3)/(4n-1)]\tilde{c}$. Applying a well known theorem of E. Hopf to (4.6), we have $\Delta\|H\|^2=0$, so that $\|H\|^2=0$, i.e., $H=0$. This means that M^{2n} is totally geodesic. When $\|H\|^2=[n(n+3)/(4n-1)]\tilde{c}$, the square of the length of H is constant. Therefore, we obtain $\Delta\|H\|^2=0$ so that equality holds in the inequality (4.6). From this facts, we have

$$\operatorname{tr}(A_sA_t-A_tA_s)^2=-2\operatorname{tr}A_s{}^2\operatorname{tr}A_t{}^2 \quad \text{for} \quad s\neq t, \tag{4.7}$$

$$\operatorname{tr}A_{a^*}{}^2=\operatorname{tr}A_{b^*}{}^2 \quad \text{for} \quad a\neq b. \tag{4.8}$$

Since the indices s and t run over the range $1^*, \cdots, n^*, \bar{1}^*, \cdots, \bar{n}^*$, at most two of the matrices $A_{1^*}, \cdots, A_{n^*}, A_{\bar{1}^*}, \cdots, A_{\bar{n}^*}$ are nonzero because of Lemma 4.2. On the other hand, taking account of (3.25) and (3.26), we have

$$A_{a^*}=0 \quad \text{if and only if} \quad A_{\bar{a}^*}=0.$$

$$\operatorname{tr}(A_{a^*}A_{b^*}-A_{b^*}A_{a^*})^2=-2\operatorname{tr}A_{a^*}{}^2\operatorname{tr}A_{b^*}{}^2 \quad \text{if and only if}$$
$$\operatorname{tr}(A_{\bar{a}^*}A_{\bar{b}^*}-A_{\bar{b}^*}A_{\bar{a}^*})^2=-2\operatorname{tr}A_{\bar{a}^*}{}^2\operatorname{tr}A_{\bar{b}^*}{}^2 .$$

$$\operatorname{tr}(A_{a^*}A_{\bar{b}^*}-A_{\bar{b}^*}A_{a^*})^2=-2\operatorname{tr}A_{a^*}{}^2\operatorname{tr}A_{\bar{b}^*}{}^2 \quad \text{if and only if}$$
$$\operatorname{tr}(A_{\bar{a}^*}A_{b^*}-A_{b^*}A_{\bar{a}^*})^2=-2\operatorname{tr}A_{\bar{a}^*}{}^2\operatorname{tr}A_{b^*}{}^2 .$$

Hence only A_{a^*} and $A_{\bar{a}^*}$ may not vanish for some a. Without loss of generality we may assume $a^*=1^*$, $A_{b^*}=0$ and $A_{\bar{b}^*}=0$ for $b=2, \cdots, n$ and that $\operatorname{tr}(A_{1^*}A_{\bar{1}^*}-$

$A_{\bar{1}^*}A_{1^*})^2=-2\,\mathrm{tr}\,A_{1^*}{}^2\,\mathrm{tr}\,A_{\bar{1}^*}{}^2$. However, according to Lemma 4.2, A_{1^*} and $A_{\bar{1}^*}$ have the special form $A_{1^*}=\lambda\bar{A}$ $A_{\bar{1}^*}=\mu\bar{B}$ respectively. But we have from (4.8) $\mathrm{tr}\,A_{1^*}{}^2=2\lambda^2=0$, $\mathrm{tr}\,A_{\bar{1}^*}{}^2=2\mu^2=0$. Hence we have $\lambda=0$ and $\mu=0$ so that $A_{1^*}=0$ and $A_{\bar{1}^*}=0$. Therefore, in this case, M^{2n} is totally geodesic. Q. E. D.

Next we consider a totally complex submanifold which is an Einstein space. First, we prove the following lemma for later use.

LEMMA 4.4. *Let $\tilde{M}^{4n}(\tilde{c})$ be a quaternionic space form of dimension $4n$ and of constant Q-sectional curvature $\tilde{c}$ and M^{2n} a $2n$-dimensional totally complex submanifold of $\tilde{M}^{4n}(\tilde{c})$. Assume that M^{2n} is an Einstein space. Then*

$$\mathrm{tr}\,A_{a^*}A_{b^*}=\mathrm{tr}\,A_{\bar{a}^*}A_{\bar{b}^*}=\frac{\|H\|^2}{2n}\delta_{cb}\,, \tag{4.9}$$

$$\mathrm{tr}\,A_{a^*}A_{\bar{b}^*}=0 \tag{4.10}$$

for each $a, b=1, \cdots, n$.

Proof. Since M^{2n} is an Einstein space, the equation (3.34) reduces to

$$2\sum_{c,d}(H_{dac^*}H_{bdc^*}+H_{da\bar{c}^*}H_{bd\bar{c}^*})=\frac{1}{2n}(n(n+1)\tilde{c}-\rho)\delta_{ab}=\frac{\|H\|^2}{2n}\delta_{ab}\,,$$

ρ being the scalar curvature of M^{2n}, where we have used (2.18). Using Lemma 3.1, we have

$$\sum_{c,d}(H_{dac^*}H_{bdc^*}+H_{da\bar{c}^*}H_{bd\bar{c}^*})=\mathrm{tr}\,(H_{a^*}H_{b^*}+H_{\bar{a}^*}H_{\bar{b}^*})\,.$$

From this and (3.21), we obtain (4.9).

As similar way, taking account of Lemma 3.1 and (3.22), we have

$$\begin{aligned}\mathrm{tr}\,A_{a^*}A_{\bar{b}^*}&=2\,\mathrm{tr}\,(H_{a^*}H_{\bar{b}^*}-H_{\bar{a}^*}H_{b^*})=2\sum_{c,d}(H_{dca^*}H_{cd\bar{b}^*}-H_{dc\bar{a}^*}H_{cdb^*})\\&=2\sum_{c,d}(H_{db\bar{c}^*}H_{adc^*}-H_{dbc^*}H_{ad\bar{c}^*})=S_{a\bar{b}}=0\,,\end{aligned}$$

because M^{2n} is an Einstein space. Thus we obtain (4.10). Q. E. D.

Corresponding to Theorem 4.3, we have the following

THEOREM 4.5. *Let $\tilde{M}^{4n}(\tilde{c})$ be a quaternionic space form of dimension $4n$ and of constant Q-sectional curvature $\tilde{c}$ $(c>0)$ and M^{2n} a $2n$-dimensional totally complex submanifold of $\tilde{M}^{4n}(\tilde{c})$. Assume that M^{2n} is an Einstein space and that the second fundamental form H of M^{2n} satisfies the inequality*

$$\|H\|^2<\frac{n(n+3)}{3}\tilde{c}\,.$$

Then M^{2n} is totally geodesic and hence M^{2n} a complex space form of constant holomorphic sectional curvature $\tilde{c}$.

Proof. By our assumption and the previous Lemma 4.4, we have from (4.5)

$$\frac{1}{2}\Delta\|H\|^2=\frac{1}{2}(n+3)\tilde{c}\|H\|^2-6\sum_{a,b}(\mathrm{tr}\, A_{a^*}A_{b^*})^2+\|\nabla H\|^2$$

$$\geqq\frac{1}{2}[(n+3)\tilde{c}-\frac{3}{n}\|H\|^2]\|H\|^2\geqq 0\,.$$

Since M^{2n} is Einstein, we get $\Delta\|H\|^2=0$ by the theorem of E. Hopf. We have then $\|H\|=0$ so that M^{2n} is totally geodesic. Q. E. D.

§5. Totally complex submanifolds of HP^n.

In the previous sections, we have obtained the fundamental properties of totally complex submanifolds of a quaternionic Kaehlerian manifold. In this section, the first what we do is to give an example of totally complex submanifold of a quaternionic space form. We use the same notations as used in [3]. Let $\{\xi, \eta, \zeta\}$ be a Sasakian 3-structure on S^{4n+3} induced from the standard quaternionic structure $\{I, J, K\}$ of the $(4n+4)$-dimensional Euclidian space R^{4n+4} whose components have the form

$$I=\left(\begin{array}{cc|cc} 0 & -E & & \\ E & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & E \\ \multicolumn{2}{c|}{0} & -E & 0 \end{array}\right),\quad J=\left(\begin{array}{cc|cc} & & -E & 0 \\ \multicolumn{2}{c|}{0} & 0 & -E \\ \hline E & 0 & & \\ 0 & E & \multicolumn{2}{c}{0} \end{array}\right), \tag{5.1}$$

$$K=\left(\begin{array}{cc|cc} & & 0 & E \\ \multicolumn{2}{c|}{0} & -E & 0 \\ \hline 0 & E & & \\ -E & 0 & \multicolumn{2}{c}{0} \end{array}\right),$$

E being the unit matrix of degree $n+1$ (see [5] and [8]). Let $\tilde{\pi}: S^{4n+3}\to HP^n$ be the Hopf fibration. Then, as is well known, HP^n is a quaternionic space form of constant Q-sectional curvature 4 (see [4]).

Consider the $(2n+1)$-dimensional unit sphere S^{2n+1} defined by $S^{2n+1}=\{p=(x, y, 0, 0)\,|\, p\in S^{4n+3},\ x=(x^0, \cdots, x^n),\ y=(y^0, \cdots, y^n)\}$ which is a totally geodesic submanifold of S^{4n+3}. We denote by $\mathring{\nabla}$ the induced connection on S^{2n+1}. Since the structure vector ξ is tangent to S^{2n+1} and η and ζ are normal to S^{2n+1}, a Sasakian structure ξ_0 of S^{2n+1} is induced from ξ on S^{4n+3} and ξ_0 coincides with the Sasakian structure induced from the natural complex structure I_0 of the $(2n+2)$-dimensional Euclidian space R^{2n+2}, where I_0 has the same components as J appearing in (3.1). Moreover we consider a tensor field $\varphi_0=\mathring{\nabla}\xi_0$ of type (1, 1) in S^{2n+1}. If we denote by $\tilde{\imath}: S^{2n+1}\to S^{4n+3}$ the inclusion map, then $\tilde{\imath}_*\xi_0=\xi$.

Let $\{\varphi, \psi, \theta\}$ be the triplet of the structure tensor fields for the Sasakian 3-structure $\{\xi, \eta, \zeta\}$ (see [8]). Take an arbitrary point p in S^{2n+1}. Then $(T_p(S^{2n+1}))$ is a subspace in $T_p(S^{2n+1})$ transversal to $(\xi_0)_p$ such that $T_p(S^{2n+1})\perp\psi(T_p(S^{2n+1}))$, $T_p(S^{2n+1})\perp\theta(T_p(S^{2n+1}))$. Moreover for every point p of S^{2n+1} each of linear subspaces $\varphi(T_p(S^{2n+1}))$, $\psi(T_p(S^{2n+1}))$ and $\theta(T_p(S^{2n+1}))$ is horizontal in S^{4n+3} with respect to the Hopf fibration $\tilde{\pi}: S^{4n+3}\to HP^n$.

We now consider the Hopf fibration $\pi: S^{2n+1}\to CP^n$. Then the natural Kaehlerian structure $(\hat{g}, \hat{J})$ of CP^n is given by $\hat{g}(X, Y)\circ\pi=\mathring{g}(\pi_* X^L, \pi_* Y^L)$ and $\hat{J}X=\pi_*(\varphi_0 X^L)$ for any vector fields X and Y in CP^n, where π_* is the differential of the submersion π and X^L denotes the unique horizontal lift of X (see [1] and [7]). Therefore we have an isometric immersion $i: CP^n\to HP^n$ such that the following diagram is commutative. Then CP^n is totally geodesic, $\tilde{\imath}_*\xi_0=\xi$,

$$\begin{tikzcd} S^{2n+1} \arrow[r, "\tilde{\iota}"] \arrow[d, "\pi"] & S^{4n+3} \arrow[d, "\tilde{\pi}"] \\ CP^n \arrow[r, "\iota"] & HP^n \end{tikzcd}$$

and the almost complex structure $\hat{J}$ induced in CP^n coincides with that induced from $\tilde{F}$ belonging to a canonical local basis $\{\tilde{F}, \tilde{G}, \tilde{H}\}$ defined as $\tilde{F}\tilde{X}=\tilde{\pi}_*(\varphi\tilde{X}^L)$, $\tilde{G}\tilde{X}=\tilde{\pi}_*(\varphi\tilde{X}^L)$ and $\tilde{H}\tilde{X}=\tilde{\pi}_*(\theta\tilde{X}^L)$ for each vector field X tangent to HP^n (see [5]). Therefore, we see easily from the theory of Riemannian submersion ([5], [7]) that CP^n is a totally complex submanifold of HP^n.

Similarly, the complex projective space CP^m of complex dimension m $(m\leqq n)$ is a totally geodesic and totally complex submanifold of HP^n with constant holomorphic sectional curvature 4. We call such a submanifold CP^m $(m\leqq n)$ the *standard totally complex submanifold* of HP^n.

We now give the following rigidity theorem for totally complex submanifold of a quaternionic space form as the results analogous to those proved in [3].

THEOREM 5.1. *Let HP^n be the $4n$-dimensional quaternionic projective space and M^{2m} $(m\leqq n)$ a connected and complete $2m$-dimensional submanifold of HP^n. Assume M^{2m} is totally geodesic and totally complex. Then M^{2m} is congruent to the standard complex projective space CP^m of complex dimension m.*

As the immediate consequence of Theorems 4.3 and 5.1, we have our Main Theorem mentioned in § 1.

§ 6. Appendix; Subspaces of a quaternionic Hermitian vector space.

An algebraic theorem. We here prove an algebraic theorem classifying the subspaces of a quaternionic Hermitian vector space under the some condition.

Let $\tilde{W}^{4n}$ be a real vector space of dimension $4n$ with a positive definite inner product $\langle , \rangle$, and $\{\tilde{F}, \tilde{G}, \tilde{H}\}$ be a quaternionic structure on $\tilde{W}^{4n}$, which satisfies (2.1). Assume that each $\tilde{F}$, $\tilde{G}$ and $\tilde{H}$ are Hermitian with respect to the inner product $\langle\,,\rangle$. In such case, $\tilde{W}^{4n}$ is called a *quaternionic Hermitian vector space*. We denote by $\|\tilde{X}\|$ the length of a vector $\tilde{X}$ in $\tilde{W}^{4n}$ and by $Q(\tilde{X})$ the Q-section determined by $\tilde{X}$ which is defined in [4], that is,

$$Q(\tilde{X})=\{\tilde{Y}\,|\,\tilde{Y}=a\tilde{X}+b\tilde{F}\tilde{X}+c\tilde{G}\tilde{X}+d\tilde{H}\tilde{X}\,;\, a,\, b,\, c,\, d\in R\}\ .$$

Suppose that there is given an m-dimensional linear subspace W^m $(m\leqq 4n)$ of $\tilde{W}^{4n}$. When W^m satisfies the condition that $\tilde{F}W^m=W^m$, $\tilde{G}W^m=W^m$ and $\tilde{H}W^m=W^m$, we call W^m a *invariant subspace*. When W^m satisfies the condition that $\tilde{F}W^m=W^m$, $W^m\perp\tilde{G}W^m$ and $W^m\perp\tilde{H}W^m$, we call W^m a *totally complex subspace*. In addition, when W^m satisfies the condition that $W^m\perp\tilde{F}W^m$, $W^m\perp\tilde{G}W^m$ and $W^m\perp\tilde{H}W^m$, we call W^m a *totally real subspace*.

We now prove the following Lemmas 6.1~6.5. In this section, let's consider a linear subspace W^m of $\tilde{W}^{4n}$ having the following property

$$\langle\tilde{F}Y,\, X\rangle\tilde{F}X+\langle\tilde{G}Y,\, X\rangle\tilde{G}X+\langle\tilde{H}Y,\, X\rangle\tilde{H}X\in W^m \tag{6.1}$$

for any vectors X and Y in W^m.

LEMMA 6.1. *Let Y and Z are arbitrary vectors of W^m where the vector Z is assumed to be orthogonal to each of $\tilde{F}Y$, $\tilde{G}Y$ and $\tilde{H}Y$. Then we have*

$$\langle\tilde{F}Y,\, X\rangle\tilde{F}Z+\langle\tilde{G}Y,\, X\rangle\tilde{G}Z+\langle\tilde{H}Y,\, X\rangle\tilde{H}Z\in W^m \tag{6.2}$$

for any vector X of W^m.

Proof. Let X be an arbitrary non-zero vector of W^m. Replacing X in (6.1) by $X+Z$, we have

$$\begin{aligned}\langle\tilde{F}Y,\, X+Z\rangle\tilde{F}(X+Z)+\langle\tilde{G}Y,\, X+Z\rangle\tilde{G}(X+Z)+\langle\tilde{H}Y,\, X+Z\rangle\tilde{H}(X+Z)\\ =\langle\tilde{F}Y,\, X\rangle\tilde{F}X+\langle\tilde{G}Y,\, X\rangle\tilde{G}X+\langle\tilde{H}Y,\, X\rangle\tilde{H}X\\ +\langle\tilde{F}Y,\, X\rangle\tilde{F}Z+\langle\tilde{G}Y,\, X\rangle\tilde{G}Z+\langle\tilde{H}Y,\, X\rangle\tilde{H}Z\in W^m\end{aligned}$$

which implies from (6.1) that $\langle\tilde{F}Y,\, X\rangle\tilde{F}Z+\langle\tilde{G}Y,\, X\rangle\tilde{G}Z+\langle\tilde{H}Y,\, X\rangle\tilde{H}Z\in W^m$.

Q. E. D.

We denote by $\bar{Q}(Z)$ a linear closure generated by $\tilde{F}Z$, $\tilde{G}Z$ and $\tilde{H}Z$ for any vector Z in W^m. Let X be an arbitrary fixed non-zero vector in W^m. We now consider the following four cases:

Case 1. $\dim(W^m\cap\bar{Q}(X))=0$, i. e., $\tilde{F}X$, $\tilde{G}X$, $\tilde{H}X\notin W^m$;
Case 2. $\dim(W^m\cap\bar{Q}(X))=1$, i. e., $\tilde{F}X\in W^m$, $\tilde{G}X\notin W^m$ and $\tilde{H}X\notin W^m$;
Case 3. $\dim(W^m\cap\bar{Q}(X))=2$, i. e., $\tilde{F}X$, $\tilde{G}X\in W^m$ and $\tilde{H}X\notin W^m$;
Case 4. $\dim(W^m\cap\bar{Q}(X))=3$, i. e., $\tilde{F}X$, $\tilde{G}X$, $\tilde{H}X\in W^m$.

LEMMA 6.2. *In the case 1, the linear subspace W^m is a totally real subspace of W^{4n} provided that $m\leqq n$.*

Proof. When $m=1$, Lemma 6.2 is trivially established. So, we may assume that $2\leqq m$. By the assumption, it is clear that each of vectors $\tilde{F}X$, $\tilde{G}X$ and $\tilde{H}X$ does not belong to W^m. Take an arbitrary vector Y in W^m such that Y is orthogonal to X. We are going to show that $\dim(W^m\cap\bar{Q}(Y))=0$. To do so, taking account of (6.1), we have $\langle\tilde{F}Y, X\rangle\tilde{F}X+\langle\tilde{G}Y, X\rangle\tilde{G}X+\langle\tilde{H}Y, X\rangle\tilde{H}X\in W^m$. On the other hand, since $\dim(W^m\cap\bar{Q}(X))=0$, this implies

$$\langle X, \tilde{F}Y\rangle=\langle X, \tilde{G}Y\rangle=\langle X, \tilde{H}Y\rangle=0. \tag{6.3}$$

Assume that there is a linear combination $a\tilde{F}Y+b\tilde{G}Y+c\tilde{H}Y$ belonging to W^m for some real number a, b and c. However, as was proved by (6.3), X orthogonal to $\tilde{F}Y$, $\tilde{G}Y$ and $\tilde{H}Y$. Then we have from Lemma 6.1

$$\begin{aligned}\langle\tilde{F}Y, a\tilde{F}Y+b\tilde{G}Y+c\tilde{H}Y\rangle\tilde{F}X+\langle\tilde{G}Y, a\tilde{F}Y+b\tilde{G}Y+c\tilde{H}Y\rangle GX\\ +\langle\tilde{H}Y, a\tilde{F}Y+b\tilde{G}Y+c\tilde{H}Y\rangle\tilde{H}X=\langle Y, Y\rangle(a\tilde{F}X+b\tilde{G}X+c\tilde{H}X)\in W^m.\end{aligned}$$

Thus, since $\dim(W^m\cap\bar{Q}(X))=0$, we get $a=b=c=0$ so that $\dim(W^m\cap\bar{Q}(Y))=0$. Since $\langle\tilde{F}Y, X\rangle\tilde{F}Y+\langle\tilde{G}Y, X\rangle\tilde{G}Y+\langle\tilde{H}Y, X\rangle\tilde{H}Y\in W^m$ because of (6.1), we have $\langle\tilde{F}Y, X\rangle=\langle\tilde{G}Y, X\rangle=\langle\tilde{H}Y, X\rangle=0$. Therefore, W^m is totally real, because Y is taken arbitrary in W^m. Q. E. D.

LEMMA 6.3. *In the case* 2, *the linear subspace W^m is of even dimensions $m=2p$ ($p\leqq n$) and W^m is a totally compelx subspace of W^{4n}.*

Proof. Clearly, we have $m\geqq 2$. When $m=2$, the lemma is obviousely true. So, we may assume that $m>2$. Take any non-zero vector Y in W^m such that Y is orthogonal to X and $\tilde{F}X$, where $\tilde{F}X\in W^m$. Since $\langle\tilde{G}Y, X\rangle\tilde{G}X+\langle\tilde{H}Y, X\rangle\tilde{H}X\in W^m$, we have $\langle\tilde{G}Y, X\rangle=\langle\tilde{H}Y, X\rangle=0$ because of $\tilde{G}X\notin W^m$ and $\tilde{H}X\notin W^m$. Hence Y is also orthogonal to the Q-section $Q(X)\supset\bar{Q}(X)$. This implies by Lemma 6.1 that $\|X\|^2\tilde{F}Y\in W^m$. Thus, as $\|X\|=0$, we have $\tilde{F}Y\in W^m$. Finally, we shall show that both $\tilde{G}Y$ and $\tilde{H}Y$ do not belong to W^m. To do so, we assume that $\tilde{G}Y\in W^m$. Since Y is orthogonal to $Q(X)$, we easily see that $\tilde{G}X$ is orthogonal to $\bar{Q}(Y)$. Using Lemma 6.1, we have $\langle\tilde{F}Y, \tilde{F}Y\rangle\tilde{F}\tilde{G}X+\langle\tilde{G}Y, \tilde{F}Y\rangle\tilde{G}^2X+\langle\tilde{H}Y, \tilde{F}Y\rangle\tilde{H}\tilde{G}X=\|Y\|^2\tilde{H}X\in W^m$, which contradicts $\tilde{H}X\in W^m$, since $\|Y\|\neq 0$ and $\|X\|\neq 0$. By a similar way, we have $\tilde{G}Y\notin W^m$. Therefore, we can conclude that W^m is even dimensional and that $\tilde{F}Y\in W^m$, $\langle\tilde{G}Y, X\rangle=\langle\tilde{H}Y, X\rangle=0$ because of (6.1). Since Y is a non-zero vector arbitrary taken in W^m, the subspace W^m is totally complex. Q. E. D.

LEMMA 6.4. *The case* 3 *does not occure.*

Proof. We assume that the Case 3 occures for W^m. Suppose that $m>3$. Then we can take a non-zero vector Y in W^m such that Y is orthogonal to X, $\tilde{F}X$ and $\tilde{G}X$. On account of (6.1), we get $\langle\tilde{H}Y, X\rangle\tilde{H}X\in W^m$ which implies that $\langle\tilde{H}Y, X\rangle=0$. This means that Y is orthogonal to the Q-section $Q(X)$. Hence we have, from Lemma 6.1, $\langle\tilde{F}X, \tilde{F}X\rangle\tilde{F}Y+\langle\tilde{G}X, \tilde{F}X\rangle\tilde{G}X+\langle\tilde{H}X, \tilde{F}X\rangle\tilde{H}X=$

$\|X\|^2\tilde{F}Y\in W^m$. Thus, since $\|X\|\neq 0$, it follows that $\tilde{F}Y\in W^m$. But, as Y is orthogonal to the Q-section $Q(X)$, we get $\langle\tilde{F}Y,\tilde{F}Y\rangle\tilde{F}\tilde{G}X+\langle\tilde{G}Y,\tilde{F}Y\rangle\tilde{G}^2X+\langle\tilde{H}Y,\tilde{F}Y\rangle\tilde{H}\tilde{G}X=\|Y\|^2\tilde{H}X\in W^m$ by Lemma 6.1. Since $\tilde{H}X\notin W^m$, we have $Y=0$, which is a contradition. Next, we assume that $m=3$. Then W^m is spanned by X, $\tilde{F}X$ and $\tilde{G}X$. We now take a non-zero vector $Z=X+\tilde{F}X+\tilde{G}X$ in W^m. By a straightforward computation, we have $FZ=-X+\tilde{F}X+\tilde{H}X$, $\tilde{G}Z=-X+\tilde{G}X-\tilde{H}X$ and $\tilde{H}Z=-\tilde{F}X+\tilde{G}X+\tilde{H}X$ which do not belong to W^m. This means that dim $(W^m\cap\bar{Q}(Z))=0$. Consequently, the linear subspace W^m is totally real as a consequence of Lemma 6.2. Therefore, we have $\dim(W^m\cap\bar{Q}(X))=0$, which contradicts the assumption that $\dim(W^m\cap\bar{Q}(X))=2$. Q. E. D.

LEMMA 6.5. *In the case 4, the linear subspace W^m is a invariant subspace of W^{4n}.*

Proof. Since $\dim(W^m\cap\bar{Q}(X))=3$, we have $Q(X)\subset W^m$. When $Q(X)=W^m$, the lemma is obviously true. So, we may assume that dim $W^m>4$. Take an arbitrary non-zero vector Y in W^m which is orthogonal to the Q-section $Q(X)$. Substituting $\tilde{F}X$ for X and Y for Z in Lemma 6.1, we have $\|X\|^2\tilde{F}Y\in W^m$. Hence we obtain $\tilde{F}Y\in W^m$ because of $\|X\|\neq 0$. In addition, we easily see that both $\tilde{G}Y$ and $\tilde{H}Y$ are orthogonal to the Q-section $Q(X)$. Putting $Y=X$ and $Z=GY$ in Lemma 6.1, we have $\langle\tilde{F}X,\tilde{F}X\rangle\tilde{F}\tilde{G}Y+\langle\tilde{G}X,\tilde{F}X\rangle\tilde{G}^2Y+\langle\tilde{H}X,\tilde{F}X\rangle\tilde{H}\tilde{G}Y=\|X\|^2\tilde{H}Y\in W^m$ which implies that $\tilde{H}Y\in W^m$. Putting $Y=X$ and $Z=\tilde{H}Y$ in Lemma 6.1, by similar devices, we have $\tilde{G}Y\in W^m$. Summing up, we can see that the linear subspace W^m is invariant. Q. E. D.

Taking an element $\tilde{s}=(\tilde{s}_{uv})$ $(u, v=1, 2, 3)$ of the special orthogonal group $SO(3)$, we can defined by (1.3) the action of $\tilde{s}$ on quaternionic structures $\{\tilde{F},\tilde{G},\tilde{H}\}$ on $\tilde{W}^{4n}$. So, the group $SO(3)$ acts transitively on the set of all quaternionic structures in $\tilde{W}^{4n}$. Taking account of this fact and summing up the Lemmas 6.2$\sim$6.5 obtained above, we have

THEOREM 6.6. *Let $\tilde{W}^{4n}$ be a quaternionic Hermitian vector space with positive definite inner product $\langle\,,\rangle$ and quaternionic structure $\{\tilde{F},\tilde{G},\tilde{H}\}$. Let W^m $(m\geqq 4)$ be an m-dimensional linear subspace of $\tilde{W}^{4n}$. Then W^m satisfies the property*

$$\langle\tilde{F}Y, X\rangle\tilde{F}X+\langle\tilde{G}Y, X\rangle\tilde{G}X+\langle\tilde{H}Y, X\rangle\tilde{H}X\in W^m$$

for any vectors X and Y in W^m if and only if W^m is one of the following

(1) *W^m is a invariant subspace of $\tilde{W}^{4n}$, that is, $W^m=\overline{W}$;*

(2) *W^m is a totally real subspace of $\tilde{W}^{4n}$, that is, $W^m\perp\overline{W}$, where $\overline{W}$ denotes the direct sum $\tilde{F}W^m\oplus\tilde{G}W^m\oplus\tilde{H}W^m$ and the symbol $\perp$ shows to be orthogonal,*

(3) *W^m is a totally complex subspace of $\tilde{W}^{4n}$, that is, $\tilde{J}W^m=W^m$ for some complex structure $\tilde{J}$ such that $\tilde{J}=a\tilde{F}+b\tilde{G}+c\tilde{H}$, $a^2+b^2+c^2=1$ where a, b, c are real numbers.*

Curvature invariant submanifolds. Let $\tilde{M}^{4n}(\tilde{c})$ be a $4n$-dimensional quaternionic space form and M a submanifold of $\tilde{M}^{4n}(\tilde{c})$. If we denote by $\tilde{R}$ the curvature tensor of $\tilde{M}^{4n}(\tilde{c})$, then $\tilde{R}$ has the form of (2.13). Take an arbitrary point x in M and consider the tangent space $T_x(M)$ at x. If for any vectors X and Y in $T_x(M)$

$$\tilde{R}(X, Y)(T_x(M)) \subset T_x(M) \tag{6.4}$$

holds, then we call M a *curvature invariant submanifold* of $\tilde{M}^{4n}(\tilde{c})$. Taking account of (2.13) and (6.4), we have easily

PROPOSITION 6.7. *Let $\tilde{M}^{4n}(\tilde{c})$ be a quaternionic space form of constant Q-sectional curvature $\tilde{c}$ ($\tilde{c} \neq 0$) and of dimension $4n$ ($n \geqq 2$). Let M be a submanifold of $\tilde{M}^{4n}(\tilde{c})$. Then M is a curvature invariant submanifold of $\tilde{M}^{4n}(\tilde{c})$ if and only if $\langle \tilde{F}Y, X \rangle \tilde{F}X + \langle \tilde{G}Y, X \rangle \tilde{G}X + \langle \tilde{H}Y, X \rangle \tilde{H}X$ is tangent to M for any vector fields X and Y tangent to M, where $\langle\, ,\rangle$ is the Riemannian metric of $\tilde{M}^{4n}(\tilde{c})$.*

Let $\{\tilde{M}^{4n}, \tilde{g}, \tilde{V}\}$ be a $4n$-dimensional quaternionic Kaehlerian manifold and M a submanifold of $\tilde{M}^{4n}$. If each tangent space at each point in M is invariant, we call M an *invariant submanifold* of $\tilde{M}^{4n}$. By the same way as mentioned in [3] or §2, we see that each invariant submanifold of $\tilde{M}^{4n}$ is totally geodesic. If each tangent space at any point in M is totally real, we call M a *totally real submanifold* of $\tilde{M}^{4n}$ ([3]). We see easily that any invariant or totally real or totally complex submanifold of $\tilde{M}^{4n}(c)$ is curvature invariant.

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