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MAXIMUM MODULUS, CHARACTERISTIC, DEFICIENCY AND GROWTH OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction and statements of results

We take for granted the usual notation of Nevanlinna theory (see [5]). For a set $F \subset \boldsymbol{R}^+$, let $m(F)$ and $m_l(F) := \int_F dt/t$ denote the linear and the logarithmic measure of F respectively. The upper and the lower logarithmic density of F are defined by

$$\overline{\log dens}\, F := \limsup_{r\to\infty} \frac{m_l(F\cap[1,r])}{\log r}, \quad \underline{\log dens}\, F := \liminf_{r\to\infty} \frac{m_l(F\cap[1,r])}{\log r}.$$

Consider the second order linear differential equation

$$(*) \qquad f'' + A(z)f' + B(z)f = 0,$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. Let $\rho(g)$ denote the order of an entire function g. It is known that if either $\rho(A) < \rho(B)$ or $\rho(B) < \rho(A) \le 1/2$, then every solution $f \not\equiv 0$ of $(*)$ is of infinite order [3, 7, 12].

For the case that $\rho(A) > 1/2$ and $\rho(B) < \rho(A)$, I. Laine and P. Wu recently proved

Theorem A[11]. *Suppose that $\rho(B) < \rho(A) < \infty$ and that $T(r, A) \sim \log M(r, A)$ as $r \to \infty$ outside a set of finite logarithmic measure. Then every solution $f \not\equiv 0$ of $(*)$ is of infinite order.*

We extend Theorem A by allowing bigger exceptional sets on which restrictive condition about the growth of $A(z)$ is made.

Theorem 1. *Suppose that $\rho(B) < \rho(A) < \infty$ and that $T(r, A) \sim \log M(r, A)$ as $r \to \infty$ outside a set of upper logarithmic density less than $\{\rho(A) - \rho(B)\}/\rho(A)$. Then every solution $f \not\equiv 0$ of $(*)$ is of infinite order.*

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It is shown [6] that $A(z)$ has no finite deficient value under the hypothesis of the theorems. Complementing these theorems, we prove

Theorem 2. *Suppose that $A(z)$ and $B(z)$ are transcendental entire functions with $\rho(B) \le 1/2$ and $\rho(B) < \rho(A)$, and that $A(z)$ has a finite deficient value. Then every solution $f \not\equiv 0$ of $(*)$ is of infinite order.*

Corollary 3. *Let $B(z)$ be a transcendental entire function of order $\rho(B) \le 1/2$. Suppose that $A(z)$ is an entire function of genus $q \ge 1$, and that all the zeros of $A(z)$ lie in the angular sector $\theta_1 \le \arg z \le \theta_2$ satisfying*

$$\theta_2 - \theta_1 \le \frac{\pi}{q+1}.$$

Then every solution $f \not\equiv 0$ of $(*)$ is of infinite order.

This corollary is an immediate consequence of Theorem 2 since $A(z)$ satisfying the hypothesis of the corollary has zero as a deficient value [9]. This improves our previous work [10, Theorem 1] in which coefficient functions of $(*)$ have more restricted conditions.

2. Preliminary lemmas

We need the following known lemmas in the proofs of theorems.

Lemma A[4]. *Let $f(z)$ be a nontrivial entire function, and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then there exist a constant $c > 0$ and a set $E_1 \subset [0, \infty)$ of finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have*

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \le c[T(\alpha r, f) r^{\varepsilon} \log T(\alpha r, f)]^k, \quad k \in \boldsymbol{N}.$$

Lemma B[2]. *Let $f(z)$ be a meromorphic function of finite order ρ. Given $\zeta > 0$ and l, $0 < l < 1/2$, there exist a constant $K(\rho, \zeta)$ and a set $E_\zeta \subset [0, \infty)$ of lower logarithmic density greater than $1 - \zeta$ such that for all $r \in E_\zeta$ and for every interval J of length l*

$$r \int_J \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta < K(\rho, \zeta)\left(l \log \frac{1}{l}\right) T(r, f).$$

Lemma C[8]. *Suppose that $f(z)$ is an entire function of order $\rho \le 1/2$. Then one of the following two statements is true*:

(i) *for every $\lambda < \rho$, there exists $r_m \to \infty$ such that*

$$\log|f(z)| > r_m^{\lambda}$$

for all z satisfying $|z| = r_m$.

(ii) *for every* $\lambda < \rho$, *if*

$$K_r = \{\theta \in [0, 2\pi] : \log|f(re^{i\theta})| < r^{\lambda}\},$$

there exists a set $E_2 \subset [0, \infty)$ *of logarithmic density* 1 *such that for* $r \in E_2$,

$$m(K_r) \to 0, \quad r \to \infty.$$

Lemma D[8]. *Suppose* $f(z)$ *is a nonconstant entire function of finite order. For a positive number* α, *there exists a set* $E_\alpha \subset [1, \infty)$ *with finite linear measure such that*

$$m(E_\alpha \cap [r/e, er]) < \exp(-r^{\alpha}), \quad r > r_0(f),$$

and that, for $|z| = r \notin E_\alpha$, *we have*

$$\left|\frac{f'(z)}{f(z)}\right| < \exp(r^{2\alpha}), \quad r > r_0(f).$$

Lemma E[8]. *Suppose* $f(z)$ *is entire of order* $\rho < 1$ *and* $0 < \varepsilon < \min(\rho/2, 1 - \rho)$. *Suppose there exists an unbounded set of r-values such that*

$$\log|f(re^{i\theta})| > r^{\rho - \varepsilon}$$

for all $\theta \in [0, 2\pi]$. *Suppose also that* $E_3 \subset [1, \infty)$ *satisfies*

$$m(E_3 \cap [r/e, er]) < \exp(-r^{6\varepsilon}), \quad r > R_0.$$

Then there is an unbounded set of s-values with $s \notin E_3$ *such that*

$$\log|f(se^{i\theta})| > s^{\rho - 2\varepsilon}$$

for all $\theta \in [0, 2\pi]$.

3. Proofs of the theorems

Proof of Theorem 1. Suppose that $T(r, A) \sim \log M(r, A)$ as $r \to \infty$ outside a set of upper logarithmic density less than $\{\rho(A) - \rho(B)\}/\rho(A)$. For given c, $0 < c < 1/4$, let

$$I_c(r) = \{\theta \in [0, 2\pi) : \log|A(re^{i\theta})| < (1 - c) \log M(r, A)\}.$$

Then there are $\varepsilon > 0$ and a set $F_1 \subset [1, \infty)$ with

$$\underline{\log dens}\, F_1 \geq 1 - \{\rho(A) - \rho(B)\}/\rho(A) + \varepsilon$$

such that $m(I_c(r)) \to 0$, as $r \to \infty$ in F_1.

Apply Lemma B with $\zeta = \varepsilon/3$ on $A(z)$, and choose $l > 0$ so small that

$$K(\rho, \zeta)\left(l \log \frac{1}{l}\right) < c.$$

Then for every interval J of length l and for all $r \in E_\zeta$, we have

$$r \int_J \left| \frac{A'(re^{i\theta})}{A(re^{i\theta})} \right| d\theta < cT(r, A), \tag{1}$$

where E_ζ is a set of lower logarithmic density greater than $1 - \zeta$ by Lemma B. If $\phi \in [0, 2\pi)$, then for all sufficiently large $r \in F_1 \cap E_\zeta$, there is a $\psi \notin I_c(r)$ such that $|\phi - \psi| \le l$ and

$$\begin{aligned} \log|A(re^{i\phi})| &= \log|A(re^{i\psi})| + \int_\psi^\phi \frac{d}{d\theta} \log|A(re^{i\theta})|\, d\theta \\ &\ge (1-c) \log M(r, A) - r \int_\psi^\phi \left| \frac{A'(re^{i\theta})}{A(re^{i\theta})} \right| |d\theta| \\ &\ge (1 - 2c) \log M(r, A). \end{aligned} \tag{2}$$

Now let a and b be chosen to satisfy $\rho(B) < b < a < \rho(A)$, and

$$(a - b)/a \ge \{\rho(A) - \rho(B)\}/\rho(A) - \varepsilon/3.$$

Then there is a sequence $r_n \to \infty$ of real numbers for which

$$\log M(r_n, A) \ge r_n^a.$$

Hence for all $r \in [r_n, r_n^{a/b}]$,

$$\log M(r, A) \ge \log M(r_n, A) \ge (r_n^{a/b})^b \ge r^b.$$

Here we put $F_2 = \bigcup_n [r_n, r_n^{a/b}]$. Then the upper logarithmic density of F_2 is at least $(a - b)/a$, and it follows that for all $r \in F_2$,

$$\log M(r, A) \ge r^b. \tag{3}$$

Note that the set $F_0 = F_1 \cap E_\zeta \cap F_2$ has positive upper logarithmic density $(\ge \varepsilon/3)$. We conclude from (2) and (3) that for all z satisfying $|z| = r \in F_0$,

$$\log|A(z)| \ge (1 - 2c) r^b \ge r^b/2. \tag{4}$$

Let $f \not\equiv 0$ be a solution of $(*)$. Then we get

$$\left| \frac{f''(z)}{f'(z)} \right| \ge |A(z)| - |B(z)| \left| \frac{f(z)}{f'(z)} \right|. \tag{5}$$

We note from the fundamental theorem of calculus and the maximum modulus theorem that, for all large $r > 0$, there exist z_r with $|z_r| = r$ on which

$$\left| \frac{f(z_r)}{f'(z_r)} \right| \le r + O(1). \tag{6}$$

By Lemma A, there is a set E_1 of finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have

$$\left|\frac{f''(z)}{f'(z)}\right| \le rT(2r, f')^2. \tag{7}$$

Calculating on the unbounded points z_r, $|z_r| \in F_0 - E_1$, we conclude from (4), (5), (6) and (7) that $f'(z)$ has infinite order. Since f and f' have the same order, the conclusion of the theorem follows.

Proof of Theorem 2. Suppose that $A(z)$ has deficiency $\delta(a, f) = 2\delta > 0$ at $a \in \boldsymbol{C}$ as stated in the hypothesis. Then it follows from the definition of deficiency that for all sufficiently large r, we have

$$m\left(r, \frac{1}{A-a}\right) \ge \delta T(r, A).$$

Hence, for any sufficiently large r, there exists a point z_r such that $|z_r| = r$ and

$$\log|A(z_r) - a| \le -\delta T(r, A). \tag{8}$$

Assume first that $A(z)$ has zero as a deficient value, that is, $a = 0$. Now set $z_r = re^{i\theta_r}$ and let $\zeta > 0$ be a sufficiently small number. Then, by virtue of Lemma B and the inequalities (1) and (8), we can choose a number $\phi > 0$, $|\theta_r - \phi| \le l$ and a set $E_\zeta \subset [0, \infty)$ of lower logarithmic density greater than $1 - \zeta$ such that for given $r \in E_\zeta$,

$$\log|A(re^{i\theta})| \le 0$$

for all $\theta \in [\theta_r - \phi, \theta_r + \phi]$. In fact, if we determine c sufficiently small in (1), we have

$$\begin{aligned}\log|A(re^{i\theta})| &= \log|A(re^{i\theta_r})| + \int_{\theta_r}^{\theta} \frac{d}{dt} \log|A(re^{it})|\, dt \\ &\le -\delta T(r, A) + r\int_{\theta_r}^{\theta} \left|\frac{A'(re^{it})}{A(re^{it})}\right| |dt| \\ &\le (-\delta + c)T(r, A) \le 0.\end{aligned}$$

In general, if $A(z)$ has a finite deficient value $a \in \boldsymbol{C}$, then we can apply the same reasoning as above to the function $A(z) - a$ since it has zero as a deficient value. Hence there exist real numbers $\phi > 0$, θ_r and a set $E_\zeta \subset [0, \infty)$ of lower logarithmic density greater than $1 - \zeta$ such that for given $r \in E_\zeta$,

$$\log|A(re^{i\theta}) - a| \le 0$$

for all $\theta \in [\theta_r - \phi, \theta_r + \phi]$. Thus for these r and θ, we get

$$|A(re^{i\theta})| \le |a| + 1. \tag{9}$$

Suppose $\rho(B) = \rho$, $0 < \rho \le 1/2$. The proof is divided into two cases depending on the growth property of $B(z)$ by Lemma C. First, we assume that there exists $r_m \to \infty$ such that given ε, $0 < \varepsilon < \rho/2$,

$$\log|B(z)| > r_m^{\rho-\varepsilon} \tag{10}$$

for all z satisfying $|z| = r_m$.

Let $f \not\equiv 0$ be a solution of $(*)$. Then we get

$$|B(z)| \le \left|\frac{f''(z)}{f(z)}\right| + |A(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{11}$$

In order to prove the theorem by contradiction, assume that $f \not\equiv 0$ is of finite order. Then, by Lemma D, if α is a positive number, there exists a set $E_\alpha \subset [1, \infty)$ with finite linear measure such that

$$m(E_\alpha \cap [r/e, er]) < 2\exp(-r^\alpha), \quad r > r_0(f), \tag{12}$$

and that, for $|z| = r \notin E_\alpha$,

$$\left|\frac{f'(z)}{f(z)}\right| < \exp(r^{2\alpha}), \quad \left|\frac{f''(z)}{f(z)}\right| < \exp(r^{4\alpha}), \quad r > r_0(f). \tag{13}$$

Furthermore, choosing α small enough to apply Lemma E to $B(z)$ with (10) and (12), we get a sequence $s_m \to \infty$ with $s_m \notin E_\alpha$ such that for all $\theta \in [0, 2\pi]$,

$$\log|B(s_m e^{i\theta})| > s_m^{\rho-2\varepsilon}. \tag{14}$$

Hence the combination of (8), (11), (13) and (14) yield that as $s_m \to \infty$,

$$\exp(s_m^{\rho-2\varepsilon}) \le (|a| + 2)\exp(s_m^{4\alpha})$$

on the points z_r $(r = s_m)$. This inequality leads to a desired contradiction if we make ε and α sufficiently small. Therefore $f \not\equiv 0$ has infinite order.

Now let us prove the second case with respect to Lemma C. Suppose that if

$$K_r = \{\theta \in [0, 2\pi] : \log|B(re^{i\theta})| < r^\lambda\}$$

for given λ, $0 < \lambda < \rho(B)$, there exists a set $E_2 \subset [0, \infty)$ of logarithmic density 1 such that $m(K_r) \to 0$, as $r \to \infty$ in E_2.

It follows from Lemma A that there exists a set $E_1 \subset [0, \infty)$ having a finite linear measure such that for all z with $|z| = r \notin E_1$, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \le rT(2r, f)^3; \quad k = 1, 2. \tag{15}$$

Note that $F_3 = E_\zeta \cap E_2 - E_1$ has a positive lower logarithmic density, and that for all sufficiently large r in F_3, we have $[\theta_r - \phi, \theta_r + \phi] - K_r \ne \emptyset$. Hence there are unbounded points, $z = r^{i\theta}$ on which inequalities (9), (15) and $\log|B(re^{i\theta})| \geq r^{\lambda}$ hold simultaneously. On these points, these inequlities and (11) yield

$$\exp(r^{\lambda}) \leq (|a| + 2)rT(2r, f)^3$$

for some unbounded r-set. Therefore $f \not\equiv 0$ has infinite order.

Finally, we suppose that $B(z)$ is a transcendental entire function of order zero. Then there is a sequence $r_n \to \infty$ of real numbers for which

$$\log M(r_n, B) \geq n^2 \log r_n.$$

Hence for all $r \in [r_n, r_n^n]$,

$$\log M(r, B) \geq \log M(r_n, B) \geq n^2 \log r_n \geq n \log r.$$

Now, set $F_4 = \bigcup_n [r_n, r_n^n]$. Then it follows that the upper logarithmic density of F_4 is 1, and that as $r \to \infty$ in F_4,

$$\frac{\log M(r, B)}{\log r} \to \infty. \tag{16}$$

We note [1] that there exists a set $F_5 \subset [0, \infty)$ of logarithmic density 1 such that, given $r \in F_5$,

$$\log|B(re^{i\theta})| \geq \frac{1}{2} \log M(r, B) \tag{17}$$

for all $\theta \in [0, 2\pi)$.

Furthermore, from (8), (11) and (15), there is a set $F_6 \subset [0, \infty)$ of finite linear measure such that for all z_r satisfying $|z_r| = r \notin F_6$

$$|B(z_r)| \leq (|a| + 2)rT(2r, f)^3. \tag{18}$$

Therefore we conclude from (16), (17) and (18) that $f \not\equiv 0$ is of infinite order.

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