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LEIBNIZ ALGEBRAS ASSOCIATED WITH FOLIATIONS

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Abstract

Certain types of singular foliations on a manifold have Leibniz algebra structures on the space of multivector fields. Each of them has a structure of a central extension of a Lie algebra in the sense of Leibniz algebra. To a specific Leibniz cohomology class, there corresponds an isomorphism class of central extension of a Leibniz algebra similarly as in the case of Lie algebra.

1. Introduction

Recently, a lot of interests have been taken in Leibniz algebra, which is introduced by Loday [10, 11] as a non-commutative variation of Lie algebra. A Leibniz algebra $\mathfrak{g}$ is an R-module, where R is a commutative ring, endowed with a bilinear map $[\,,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[g_1, [g_2, g_3]] = [[g_1, g_2], g_3] + [g_2, [g_1, g_3]].$$

Note that we do not require the anti-symmetricity of $[\,,]$.

In this paper, we consider Leibniz algebra associated with a certain type of singular foliations on a manifold. More precisely, we observe that when an integrable and locally decomposable q-form ω on a manifold M is given, there yields a foliation $\mathscr{F}$ of M whose leaves are either of dimension $n - q$ or 0. Any transversely oriented regular foliation of codimension q is defined by such a q-form. We show that the bundle of $(q+1)$-vectors $\bigwedge^{q+1} TM$ on M has a Leibniz algebroid structure whose anchor map is a interior product by ω and whose bracket is given by

$$[\![X, Y]\!]_\omega = [\iota_\omega X, Y] + (-1)^q \langle X \mid d\omega \rangle Y$$

for any $X, Y \in \mathscr{X}^{q+1}(M)$, where $[\,,]$ denotes the Schouten bracket, $\langle | \rangle$ the natural pairing and $\mathscr{X}^{q+1}(M)$ the space of $(q+1)$-vector fields. We see that the isomorphism class of the algebra is determined by the foliation $\mathscr{F}$. It is not a Lie algebra in general unless $q = 0$ or $q = n - 2$. Considering the difference of $\mathscr{X}^{q+1}(M)$ from Lie algebra, it is shown that $\mathscr{X}^{q+1}(M)$ is, as a Leibniz algebra, a central extension of the Lie algebra of vector fields tangent to $\mathscr{F}$.

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As it is known, central extensions of a Lie algebra $\mathfrak{g}$ with the center A are described by $H^2_{\mathrm{Lie}}(\mathfrak{g}; A)$ where $H^*_{\mathrm{Lie}}(\mathfrak{g}; A)$ denotes the Lie algebra cohomology with coefficients in A. One can ask the question: how about the case of Leibniz algebras? We see that the "usual" cohomology of Leibniz algebra does not work, but a slightly different cohomology $H^*(\mathfrak{g}; A)$ makes a similar one-to-one correspondence between equivalent classes of central extensions and elements in $H^2(\mathfrak{g}; A)$. It means that, when $\mathfrak{g}'$ is a central extension of a Leibniz algebra $\mathfrak{g}$ with the center A, Leibniz algebra structures of $\mathfrak{g}'$ is determined by an element in $H^2(\mathfrak{g}; A)$. Applying it to Leibniz algebras associated with foliations, we can obtain a lot of geometric examples of central extensions of Leibniz algebras.

The (co)homology of Leibniz algebra is studied by Loday and Pirashvili [12]. Lodder [14] extends the Leibniz cohomology from a Lie algebra invariant to an invariant for a differential manifold. The notion of Leibniz algebroid over a manifold was defined in [9] as a vector bundle with certain additional conditions as in the case of Lie algebroid, and it was proved that the bundle of $(p-1)$-forms on a Nambu-Poisson manifold has a Leibniz algebroid structure. In [6], one of the author discovered an alternative Leibniz algebroid structure which is a natural generalization of the Lie algebroid associated with a Poisson manifold. Description of all Leibniz algebras of dimension three is given in [1].

2. Leibniz algebras and cohomologies

First we review the notion of Leibniz algebra defined by Loday [10, 11, 12]. Let R be a commutative ring and $\mathfrak{g}$ an R-module endowed with a bilinear map $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[g_1, [g_2, g_3]] = [[g_1, g_2], g_3] + [g_2, [g_1, g_3]] \tag{2.1}$$

for $g_1, g_2, g_3 \in \mathfrak{g}$. The map $[\,,\,]$ is called the Leibniz bracket on $\mathfrak{g}$ and (2.1) the Leibniz identity. We remark that if $[\,,\,]$ is additionally skew-symmetric, then the Leibniz identity is just the Jacobi identity and $(\mathfrak{g}, [\,,\,])$ is a Lie algebra. Therefore, a Leibniz algebra is a non-commutative variant of Lie algebra.

Now we consider the cohomology of a Leibniz algebra with values in a module [12]. Suppose that $(\mathfrak{g}, [\,,\,])$ is a Leibniz algebra and A an R-module equipped with bilinear actions of $\mathfrak{g}$

$$[\,,\,] : \mathfrak{g} \times A \to A, \quad [\,,\,] : A \times \mathfrak{g} \to A$$

such that

$$[a, [g_1, g_2]] = [[a, g_1], g_2] + [g_1, [a, g_2]] \tag{2.2}$$

$$[g_1, [a, g_2]] = [[g_1, a], g_2] + [a, [g_1, g_2]] \tag{2.3}$$

$$[g_1, [g_2, a]] = [[g_1, g_2], a] + [g_2, [g_1, a]] \tag{2.4}$$

for $g_1, g_2 \in \mathfrak{g}$ and $a \in A$. We also use the notations $ga = l_g(a) = [g, a]$ and $ag = r_g(a) = [a, g]$. The condition (2.2)–(2.4) above is equivalent to that

$$l_{[g_1, g_2]} = [l_{g_1}, l_{g_2}] \tag{2.5}$$

$$r_{[g_1, g_2]} = [l_{g_1}, r_{g_2}] \tag{2.6}$$

$$r_{g_2} \circ l_{g_1} = -r_{g_2} \circ r_{g_1} \tag{2.7}$$

where $[\,,]$ in the right-hand side of (2.5) and (2.6) denotes the commutator of operators.

The Leibniz cohomology of $\mathfrak{g}$ with coefficients in A is the homology of the cochain complex $C^k(\mathfrak{g}; A) = \mathrm{Hom}_R(\otimes^k \mathfrak{g}, A)$ $(k \geq 0)$ whose coboundary operator $\partial^k : C^k(\mathfrak{g}; A) \to C^{k+1}(\mathfrak{g}; A)$ is defined by

$$\begin{aligned} &\partial^k c^k(g_1, \dots, g_{k+1}) \\ &\quad = \sum_{i=1}^{k} (-1)^{i-1} g_i(c^k(g_1, \dots, \widehat{g_i}, \dots, g_{k+1})) + (-1)^k (c^k(g_1, \dots, g_k)) g_{k+1} \\ &\qquad + \sum_{1 \leq i < j \leq k+1} (-1)^i c^k(g_1, \dots, \widehat{g_i}, \dots, g_{j-1}, [g_i, g_j], g_{j+1}, \dots, g_{k+1}) \end{aligned} \tag{2.8}$$

where $(g_1, \dots, g_{k+1})$ denotes $g_1 \otimes \cdots \otimes g_{k+1}$. The condition $\partial \circ \partial = 0$ is proved in [12].

When the left action and the (-1) times of the right action agree, we get the following "usual" Leibniz cohomology:

Proposition 2.1. *Let $\mathfrak{g}$ be a Leibniz algebra and A a $\mathfrak{g}$-module with respect to the representation of $\mathfrak{g}$ on A, that is, A is endowed with a bilinear map $\mathfrak{g} \times A \to A$ such that $[g_1, g_2]a = g_1(g_2 a) - g_2(g_1 a)$. Then the operator $\partial^k : C^k(\mathfrak{g}; A) \to C^{k+1}(\mathfrak{g}; A)$ given by*

$$\begin{aligned} \partial^k c^k(g_1, \dots, g_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i-1} g_i(c^k(g_1, \dots, \widehat{g_i}, \dots, g_{k+1})) \\ &+ \sum_{1 \leq i < j \leq k+1} (-1)^i c^k(g_1, \dots, \widehat{g_i}, \dots, g_{j-1}, \\ &\qquad [g_i, g_j], g_{j+1}, \dots, g_{k+1}) \end{aligned} \tag{2.9}$$

defines a Leibniz cohomology of $\mathfrak{g}$ with coefficients in A.

In most of the cases, we consider the Leibniz cohomology of this type, which is denoted by $HL^*(\mathfrak{g}; A)$. If $(\mathfrak{g}, [\,,])$ is a Lie algebra, we obtain the subcomplex of $(C^*(\mathfrak{g}; A), \partial)$ that consists of the skew-symmetric cochains. The cohomology of this subcomplex is just the usual cohomology $H^*_{\mathrm{Lie}}(\mathfrak{g}; A)$ of the Lie algebra $(\mathfrak{g}, [\,,])$ with coefficients in A. Thus there is a natural homomorphism

$$\iota : H^*_{\mathrm{Lie}}(\mathfrak{g}; A) \to HL^*(\mathfrak{g}; A).$$

The followings are several examples of Leibniz cohomology we have in mind.

Example 2.2 ([2, 4, 16]). Let (M, Π) be a Nambu-Poisson manifold of order p, that is, Π is a p-vector field satisfying

$$[\Pi(df_1, \dots, df_{p-1}), \Pi] = 0$$

for $f_1, \dots, f_{p-1} \in C^\infty(M)$, where $[\,,]$ denotes the Schouten bracket. It holds that $\bigwedge^{p-1} C^\infty(M)$ is a Leibniz algebra by the bracket $[\![\,,]\!]$ defined by

$$\begin{aligned}&[\![f_1 \wedge \cdots \wedge f_{p-1}, g_1 \wedge \cdots \wedge g_{p-1}]\!]\\&\quad = \sum_{i=1}^{p-1} g_1 \wedge \cdots \wedge \Pi(df_1, \dots, df_{p-1}, dg_i) \wedge \cdots \wedge g_{p-1}\end{aligned}$$

for $f_1, \dots, f_{p-1}, g_1, \dots, g_{p-1} \in C^\infty(M)$. Furthermore, by the natural action

$$[f_1 \wedge \cdots \wedge f_{p-1}, f] = \Pi(df_1, \dots, df_{p-1}, df),$$

we obtain the Leibniz cohomology $HL^*(\bigwedge^{p-1} C^\infty(M); C^\infty(M))$.

Example 2.3 ([9]). Let (M, Π) be a Nambu-Poisson manifold of order $p \geq 3$. The space of $(p-1)$-forms $\Omega^{p-1}(M)$ is a Leibniz algebra by the bracket $[\![\,,]\!]$ defined by

$$[\![\alpha, \beta]\!] = \mathscr{L}_{\Pi(\alpha)}\beta + (-1)^p(\Pi(d\alpha))\beta \tag{2.10}$$

for $\alpha, \beta \in \Omega^{p-1}(M)$. By the action of $\Omega^{p-1}(M)$ on $C^\infty(M)$

$$[\alpha, f] = \Pi(\alpha, df),$$

we obtain the Leibniz cohomology $HL^*(\Omega^{p-1}(M); C^\infty(M))$. The cochain complex $C^k(\Omega^{p-1}(M); C^\infty(M))$ has the subcomplex $C^k(\bigwedge^{p-1} dC^\infty(M); C^\infty(M))$, and there exist a natural map from $C^k(\bigwedge^{p-1} C^\infty(M); C^\infty(M))$ (Example 2.2) to $C^k(\Omega^{p-1}(M); C^\infty(M))$ whose image is this subcomplex.

Example 2.4 ([6]). In case of Nambu-Poisson manifold of order 2, the bracket (2.10) gives a Leibniz algebra structure only if the Poisson structure is decomposable (that is, rank $\Pi \leq 2$), and then agrees with the Lie algebra bracket on the space of 1-forms. One of the authors proved that there is a different Leibniz bracket

$$[\![\alpha, \beta]\!]' = \mathscr{L}_{\Pi(\alpha)}\beta - \imath_{\Pi(\beta)}\, d\alpha \tag{2.11}$$

on $\Omega^{p-1}(M)$ for $p \geq 2$, which agrees with the Lie bracket when $p = 2$. It defines a different Leibniz cohomology from that in Example 2.3, but it holds similarly that the cochain complex $C^k(\Omega^{p-1}(M); C^\infty(M))$ has the subcomplex $C^k(\bigwedge^{p-1} dC^\infty(M); C^\infty(M))$ and there exist a natural map from $C^k(\bigwedge^{p-1} C^\infty(M); C^\infty(M))$ to it.

Example 2.5 ([13]). Let M be a smooth manifold and $(\mathscr{X}(M), [\,,])$ the Lie algebra of smooth vector fields on M. It is obvious that $C^\infty(M)$ is a $\mathscr{X}(M)$-module with respect to the representation by the Lie derivation. The Leibniz cohomology $HL^*(\mathscr{X}(M); C^\infty(M))$ is, by definition ([13]), the homology of the complex of continuous cochains $\mathrm{Hom}_{\boldsymbol{R}}^{\mathrm{cont}}(\otimes^k \mathscr{X}(M), C^\infty(M))$ $(k \geq 0)$ in the C^∞ topology. The coboundary operator is given as the exterior differential, that is,

$$d^k c^k(X_1, \dots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} X_i(c^k(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})) + \sum_{1 \leq i < j \leq k+1} (-1)^i c^k(X_1, \dots, \widehat{X_i}, \dots, X_{j-1}, [X_i, X_j], X_{j+1}, \dots, X_{k+1}) \tag{2.12}$$

for $X_1, \dots, X_{k+1} \in \mathscr{X}(M)$. The de Rham cohomology $H_{DR}^*(M)$ of M is just the cohomology of the subcomplex of the skew-symmetric and $C^\infty(M)$-linear cochains. Then the diagram

$$\begin{tikzcd} H_{DR}^*(M) \arrow[r, "\iota"] \arrow[d, "\pi \circ \iota"'] & H_{GF}^*(\mathscr{X}(M); C^\infty(M)) \arrow[dl, "\pi"] \\ HL^*(\mathscr{X}(M); C^\infty(M)) & \end{tikzcd} \tag{2.13}$$

commutes where $H_{GF}^*(\mathscr{X}(M); C^\infty(M))$ denotes the Gel'fand-Fuks cohomology. The map $\iota : H_{DR}^*(M) \to H_{GF}^*(\mathscr{X}(M); C^\infty(M))$ is induced by the inclusion

$$\mathrm{Hom}_{C^\infty(M)}^{\mathrm{cont}}(\mathscr{X}^k(M); C^\infty(M)) \to \mathrm{Hom}_{\boldsymbol{R}}^{\mathrm{cont}}(\mathscr{X}^k(M); C^\infty(M))$$

where $\mathscr{X}^k(M)$ denotes the space of k-vector fields on M and $\pi : H_{GF}^*(\mathscr{X}(M); C^\infty(M)) \to HL^*(\mathscr{X}(M); C^\infty(M))$ is induced by the projection $\otimes^k \mathscr{X}(M) \to \mathscr{X}^k(M)$.

3. Leibniz algebras associated with foliations

The notion of Leibniz algebroid is introduced in [9] as a generalization of the Lie algebroid:

Definition 3.1. A Leibniz algebroid is a smooth vector bundle $\pi : A \to M$ with a Leibniz algebra structure $[\![\,,]\!]$ on $\Gamma(A)$ (the space of smooth sections of A) and a bundle map $\rho : A \to TM$, called an anchor, such that the induced map $\rho : \Gamma(A) \to \mathscr{X}(M)$ satisfies the following properties:

(1) (Leibniz algebra homomorphism)

$$\rho([\![x, y]\!]) = [\rho(x), \rho(y)]$$

(2) (derivation law)

$$[\![x, fy]\!] = (\rho(x)f)y + f[\![x, y]\!]$$

for all $x, y \in \Gamma(A)$ and $f \in C^\infty(M)$.

Example 3.2. If the bracket $[\![\,,]\!]$ is skew-symmetric, we recover the Lie algebroid.

Example 3.3. The bundle of $(p-1)$-forms $\bigwedge^{p-1} T^*M$ over a Nambu-Poisson manifold (M,Π) of order p is a Leibniz algebroid with the anchor map $\Pi : \bigwedge^{p-1} T^*M \to TM$ and the bracket either (2.10) ($p \geq 2$ which we assume Π is decomposable when $p=2$) or (2.11) ($p \geq 2$).

Example 3.4 ([5]). There is a different generalization of Lie algebroid. A Filippov p-algebroid, or p-Lie algebroid, $(E, \pi, [\,,\dots,])$ over a manifold M is a vector bundle E endowed with a p-Lie bracket $[\,,\dots,]$ on $\Gamma(E)$, that is, the skew-symmetric bracket satisfying the Filippov (or Fundamental) identity

$$[a_1,\dots,a_{p-1},[b_1,\dots,b_p]] = \sum_{i=1}^{p} [b_1,\dots,[a_1,\dots,a_{p-1},b_i],\dots,b_p]$$

for any $a_1,\dots,a_{p-1},b_1,\dots,b_p \in \Gamma(E)$, and a bundle map $\pi : \bigwedge^{p-1} E \to TM$, called an anchor, such that the induced map $\pi : \Gamma(\bigwedge^{p-1} E) \to \mathscr{X}(M)$ satisfies the following properties:

$$\begin{aligned}
&[\pi(a_1 \wedge \cdots \wedge a_{p-1}), \pi(b_1 \wedge \cdots \wedge b_{p-1})] \\
&\quad = \sum_{i=1}^{p-1} \pi(b_1 \wedge \cdots \wedge [a_1,\dots,a_{p-1},b_i] \wedge \cdots \wedge b_{p-1}),
\end{aligned}$$

$$[a_1,\dots,a_{p-1},fb] = f[a_1,\dots,a_{p-1},b] + (\pi(a_1 \wedge \cdots \wedge a_{p-1})f)b$$

for all $a_1,\dots,a_{p-1},b_1,\dots,b_{p-1},b \in \Gamma(E)$ and $f \in C^\infty(M)$. In this case, it is shown that $\bigwedge^{p-1} E$ is a Leibniz algebroid with the anchor π and the bracket

$$[\![a_1 \wedge \cdots \wedge a_{p-1}, b_1 \wedge \cdots \wedge b_{p-1}]\!] = \sum_{i=1}^{p-1} b_1 \wedge \cdots \wedge [a_1,\dots,a_{p-1},b_i] \wedge \cdots \wedge b_{p-1}.$$

In the recent paper [20], it has been shown that any Nambu-Poisson manifold has an associated Filippov algebroid.

Let $\mathscr{F}$ be a transversely oriented foliation of codimension q on M. Then we deduce, by using a partition of unity, that there exists a transverse volume form ω on M such that ω is decomposable (that is, $\omega = \omega_1 \wedge \cdots \wedge \omega_q$ for some 1-forms $\omega_1,\dots,\omega_q$) and integrable ($d\omega = \gamma \wedge \omega$ for some 1-form γ). In this paper, we call a decomposable and integrable form ω on M simply an integrable form. We remark that ω needs not to be nonsingular. When ω is nonsingular, the transversely oriented foliation $\mathscr{F}$ is recovered by $\omega_1 = \cdots = \omega_q = 0$ where $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. If ω is singular, it yields a foliation whose leaves are of codimension q where $\omega \neq 0$ and otherwise of dimension 0; we consider the foliation to be given by the interior product $\iota_\omega : \bigwedge^{q+1} TM \to TM$. Thus the equivalence class of an integrable form gives a foliation.

Now, we will prove that such a foliation given by an integrable q-form on a manifold M gives the Leibniz algebroid structure to the bundle of $(q+1)$-vectors.

THEOREM 3.5. *Let M be an n-dimensional smooth manifold endowed with a decomposable and integrable q-form ω $(q < n)$. Then $\bigwedge^{q+1} TM$ becomes a Leibniz algebroid over M whose anchor is the interior product $\iota_\omega : \bigwedge^{q+1} TM \to TM$ and whose bracket is defined by*

$$[\![X, Y]\!]_\omega = [\iota_\omega X, Y] + (-1)^q \langle X \mid d\omega \rangle Y.$$

for any $X, Y \in \mathscr{X}^{q+1}(M)$, where $[\,,]$ denotes the Schouten bracket, $\langle | \rangle$ the natural pairing and $\mathscr{X}^{q+1}(M)$ the space of $(q+1)$-vector fields.

Proof. This Leibniz algebroid is essentially the same as that in Example 3.3 with the bracket (2.10) by the correspondence $\Pi = (-1)^{nq}\Phi(\omega)$ where Φ is an arbitrary co-volume field on M (that is, a dimensional multivector field). However, we will give a direct verification in the realm of multivector fields.

We abbreviate $[\![\,,]\!]_\omega$ to $[\![\,,]\!]$. It is easy to see $[\![X, fY]\!] = ((\iota_\omega X)f)Y + f[\![X, Y]\!]$. Let us prove $\iota_\omega([\![X, Y]\!]) = [\iota_\omega X, \iota_\omega Y]$. Since ω is integrable, there is a 1-form γ such that $d\omega = \gamma \wedge \omega$. By the decomposability of ω we have $\omega(X(\omega)) = 0$. Thus

$$\iota_{X(\omega)}\, d\omega = (-1)^q \langle X \mid d\omega \rangle \omega. \tag{3.1}$$

Moreover,

$$\begin{aligned}(\mathscr{L}_{X(\omega)} Y)(\omega) &= \mathscr{L}_{X(\omega)}(Y(\omega)) - Y(\mathscr{L}_{X(\omega)}\omega)\\ &= [X(\omega), Y(\omega)] - (-1)^q \langle X \mid d\omega \rangle (Y(\omega)).\end{aligned}$$

Therefore, we get

$$\begin{aligned}\iota_\omega[\![X, Y]\!] &= [X(\omega), Y](\omega) + (-1)^q \langle X \mid d\omega \rangle (Y(\omega))\\ &= [\iota_\omega X, \iota_\omega Y].\end{aligned} \tag{3.2}$$

Now we will see that the Leibniz identity holds. Let $X, Y, Z \in \mathscr{X}^{q+1}(M)$. By (3.1),

$$d\iota_{X(\omega)}\, d\omega = \omega \wedge (d\langle X \mid d\omega \rangle) + (-1)^q \langle X \mid d\omega \rangle\, d\omega.$$

Thus we have

$$\begin{aligned}[\![X, Y]\!](d\omega) &= (\mathscr{L}_{X(\omega)} Y)(d\omega) + (-1)^q \langle X \mid d\omega \rangle \langle Y \mid d\omega \rangle\\ &= \mathscr{L}_{X(\omega)} \langle Y \mid d\omega \rangle - Y(\mathscr{L}_{X(\omega)}\, d\omega) + (-1)^q \langle X \mid d\omega \rangle \langle Y \mid d\omega \rangle\\ &= (X(\omega))\langle Y \mid d\omega \rangle - Y(d\iota_{X(\omega)}\, d\omega) + (-1)^q \langle X \mid d\omega \rangle \langle Y \mid d\omega \rangle\\ &= (X(\omega))\langle Y \mid d\omega \rangle - (Y(\omega))\langle X \mid d\omega \rangle.\end{aligned}$$

Therefore, by (3.2),

$$[\![[\![X, Y]\!], Z]\!] = [[X(\omega), Y(\omega)], Z] + (-1)^q((X(\omega))\langle Y \mid d\omega\rangle - (Y(\omega))\langle X \mid d\omega\rangle)Z.$$

Also using (3.2), we have

$$\begin{aligned}
[\![X, [\![Y, Z]\!]]\!] &= [X(\omega), [\![Y, Z]\!]] + (-1)^q\langle X \mid d\omega\rangle[\![Y, Z]\!] \\
&= [X(\omega), [Y(\omega), Z] + (-1)^q\langle Y \mid d\omega\rangle Z] \\
&\quad + (-1)^q\langle X \mid d\omega\rangle([Y(\omega), Z] + (-1)^q\langle Y \mid d\omega\rangle Z) \\
&= [X(\omega), [Y(\omega), Z]] \\
&\quad + (-1)^q((X(\omega))\langle Y \mid d\omega\rangle)Z + (-1)^q\langle Y \mid d\omega\rangle[X(\omega), Z] \\
&\quad + (-1)^q\langle X \mid d\omega\rangle[Y(\omega), Z] + \langle X \mid d\omega\rangle\langle Y \mid d\omega\rangle Z.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
[\![Y, [\![X, Z]\!]]\!] &= [Y(\omega), [X(\omega), Z]] \\
&\quad + (-1)^q((Y(\omega))\langle X \mid d\omega\rangle)Z + (-1)^q\langle X \mid d\omega\rangle[Y(\omega), Z] \\
&\quad + (-1)^q\langle Y \mid d\omega\rangle[X(\omega), Z] + \langle X \mid d\omega\rangle\langle Y \mid d\omega\rangle Z.
\end{aligned}$$

Then the Leibniz identity

$$[\![X, [\![Y, Z]\!]]\!] = [\![[\![X, Y]\!], Z]\!] + [\![Y, [\![X, Z]\!]]\!]$$

is equivalent to

$$[X(\omega), [Y(\omega), Z]] = [[X(\omega), Y(\omega)], Z] + [Y(\omega), [X(\omega), Z]]$$

which is true since $[\mathcal{L}_{X(\omega)}, \mathcal{L}_{Y(\omega)}] = \mathcal{L}_{[X(\omega), Y(\omega)]}$ holds. $\square$

COROLLARY 3.6. (1) *$(\mathcal{X}^{q+1}(M), [\![\,,]\!])$ is a Leibniz algebra where*

$$[\![X, Y]\!] = [\iota_\omega X, Y] + (-1)^q(X(d\omega))Y. \tag{3.3}$$

The interior product ι_ω is a Leibniz algebra homomorphism from $\mathcal{X}^{q+1}(M)$ to the Lie algebra of vector fields $(\mathcal{X}(M), [\,,])$. It also holds that $[\![\ker \iota_\omega, Y]\!] = 0$ and $[\![X, \ker \iota_\omega]\!] \in \ker \iota_\omega$ where $X, Y \in \mathcal{X}^{q+1}(M)$.

(2) *For any non-zero function f, the multiplication by f induces an isomorphism from the Leibniz algebra $(\mathcal{X}^{q+1}(M), [\![\,,]\!]_{f\omega})$ to $(\mathcal{X}^{q+1}(M), [\![\,,]\!]_\omega)$. That is, the isomorphism class of Leibniz algebra structure is determined by the foliation.*

Proof. Since (1) is obvious, we will check (2). We have

$$\begin{aligned}
[\![X, Y]\!]_{f\omega} &= f[X(\omega), Y] - X(\omega) \wedge Y(df) + (X(\omega \wedge df))Y + (-1)^q\langle X \mid f\, d\omega\rangle)Y \\
&= f[\![X, Y]\!]_\omega + (X(\omega) \wedge Y)(df).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}[\![fX, fY]\!]_\omega &= (fX(\omega \wedge df))Y + f[\![fX, Y]\!]_\omega \\ &= f^2[X(\omega), Y] - fX(\omega) \wedge Y(df) \\ &\quad + (-1)^q f^2(X(d\omega))Y + f(X(\omega \wedge df))Y \\ &= f^2[\![X, Y]\!]_\omega + f(X(\omega) \wedge Y)(df).\end{aligned}$$

This is equal to $f[\![X, Y]\!]_{f\omega}$, and we obtain (2). □

In general, $(\mathscr{X}^{q+1}(M), [\![\, ,]\!])$ is not a Lie algebra unless $q = 0$ or $q = n - 2$.

Example 3.7. (1) The case $q = n - 2$ corresponds to the Lie algebra associated with a Poisson manifold of rank 2 via the isomorphism by the volume.

(2) Consider the case $q = 0$. For any function f on M, the Lie bracket is given as

$$[X, Y]_f = f[X, Y] + (Xf)Y - (Yf)X$$

where $X, Y \in \mathscr{X}(M)$. This corresponds to the Lie algebra associated with a Nambu-Poisson manifold coming from a volume form.

(3) Consider the case $q = n - 1$. Then the Leibniz bracket is given as

$$[\![f\Phi, g\Phi]\!]_\omega = (fZg - gZf + fg\langle Z|\gamma\rangle)\Phi$$

where Φ is a co-volume field, $f, g \in C^\infty(M)$, $d\omega = \gamma \wedge \omega$ and $Z = \Phi(\omega)$. Therefore, if ω is a closed $(n-1)$-form, $(\mathscr{X}^n(M), [\![\, ,]\!]_\omega)$ is a Lie algebra. This corresponds to $(C^\infty(M), [\, ,]_Z)$ defined by an arbitrary vector field Z where

$$[f, g]_Z = fZg - gZf.$$

Sometimes, we have a Lie algebra as a Leibniz subalgebra. For example, let us consider $(\mathscr{X}^2(\boldsymbol{R}^n), [\![\, ,]\!]_\omega)$. By Corollary 3.6, it is a Leibniz algebra if ω is an integrable 1-form on $\boldsymbol{R}^n$. In the following by a constant bivector field we mean the bivector field of the form

$$\sum_{i<j} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

where $a_{ij} \in \boldsymbol{R}$.

Proposition 3.8. *Let f be a quadratic function on $\boldsymbol{R}^n$. In the Leibniz algebra $(\mathscr{X}^2(\boldsymbol{R}^n), [\![\, ,]\!]_{df})$, the subset of constant bivector fields $\mathscr{X}^2_{\mathrm{const}}(\boldsymbol{R}^n)$ forms a Lie algebra.*

Proof. It follows from a direct computation. □

We can relate this Lie algebra to the Lie algebra of matrices; let (j, k) be the signature and l the nullity of any quadratic function f on $\boldsymbol{R}^n$. Denote by P_f the matrix $\operatorname{diag}(I_{j+k}, 0_l)$ where I_{j+k} is the unit matrix of size $j+k$ and 0_l is the zero matrix of size l, and by $so(j,k,l)$ the set of matrices in $gl(n)$ satisfying

$$I_{jkl}A + {}^tAI_{jkl} = 0$$

where $I_{jkl} = \operatorname{diag}(I_j, -I_k, I_l)$. Then,

Theorem 3.9. *$(\mathscr{X}^2_{\mathrm{const}}(\boldsymbol{R}^n), [\![\,,]\!]_{df})$ is isomorphic to $(so(j,k,l), \{\,,\}_{P_f})$ where $\{X, Y\}_{P_f} = XP_fY - YP_fX$ for any $X, Y \in so(j,k,l)$.*

Proof. It also follows from a direct computation. □

In case f is nondegenerate, $(\mathscr{X}^2_{\mathrm{const}}(\boldsymbol{R}^n), [\![\,,]\!]_{df})$ is isomorphic to $(so(j,k), [\,,])$.

4. Central extensions of Leibniz algebras

Let us return to the Leibniz cohomology of a Leibniz algebra $\mathfrak{g}$. The condition (2.2)–(2.4) admits the case that the right action $r_g = 0$ for any $g \in \mathfrak{g}$. If this is the case, we get a different Leibniz cohomology from "usual" one given by Proposition 2.1. In this section, we assume the right action $r_g = 0$, and we use this kind of Leibniz cohomology since it is essential when we consider the extensions of Leibniz algebras.

Proposition 4.1. *Let $\mathfrak{g}$ be a Leibniz algebra and A a $\mathfrak{g}$-module with respect to the representation of $\mathfrak{g}$ on A, that is, A is endowed with a bilinear map $\mathfrak{g} \times A \to A$ such that $[g_1, g_2]a = g_1(g_2a) - g_2(g_1a)$. Then the operator $\delta^k : C^k(\mathfrak{g}; A) \to C^{k+1}(\mathfrak{g}; A)$ given by*

$$\begin{aligned}\delta^k c^k(g_1, \dots, g_{k+1}) = {} & \sum_{i=1}^{k} (-1)^{i-1} g_i(c^k(g_1, \dots, \widehat{g_i}, \dots, g_{k+1})) \\ & + \sum_{1 \le i < j \le k+1} (-1)^i c^k(g_1, \dots, \widehat{g_i}, \dots, g_{j-1}, \\ & \qquad [g_i, g_j], g_{j+1}, \dots, g_{k+1})\end{aligned} \tag{4.1}$$

defines a Leibniz cohomology of $\mathfrak{g}$ with coefficients in A.

We denote this Leibniz cohomology by $H^*(\mathfrak{g}; A)$. Note that even though $\mathfrak{g}$ is a Lie algebra and c^k is skew-symmetric, c^{k+1} is not skew-symmetric in general.

Now, we will consider the central extensions of Leibniz algebras. A central extension $(\mathfrak{g}', [\![\,,]\!])$ of a Leibniz algebra $(\mathfrak{g}, [\,,])$ with a center A is a Leibniz algebra with a surjective homomorphism $\Pi : \mathfrak{g}' \to \mathfrak{g}$ whose kernel A is a center in the sense of $[\![A, \mathfrak{g}']\!] = 0$. This is equivalent to giving an exact sequence

$$0 \to A \xrightarrow{\iota} \mathfrak{g}' \xrightarrow{\Pi} \mathfrak{g} \to 0$$

such that A is a center of $\mathfrak{g}'$.

The next theorem shows that an analog of the case of Lie algebra holds (see also [12]).

THEOREM 4.2. *Let* $(\mathfrak{g}, [\,,])$ *be a Leibniz algebra and* A *a* $\mathfrak{g}$*-module. Then an element of* $H^2(\mathfrak{g}; A)$ *determines an equivalence class of central extensions of* $\mathfrak{g}$ *with the center* A. *The action of* $\mathfrak{g}$ *on* A *is recovered by* $g \cdot a = [\![s(g), a]\!]$ *where* $(\mathfrak{g}', [\![\,,]\!])$ *is a central extension of* $\mathfrak{g}$ *and* $s : \mathfrak{g} \to \mathfrak{g}'$ *an arbitrary linear map satisfying* $\Pi \circ s = \mathrm{id}_{\mathfrak{g}}$. *Conversely, an equivalence class of central extensions of* $\mathfrak{g}$ *with the center* A *defines the action of* $\mathfrak{g}$ *on* A *by* $g \cdot a = [\![s(g), a]\!]$ *where* s *is as above, and determines an element of* $H^2(\mathfrak{g}; A)$. *That is, a central extension of a Leibniz algebra* $\mathfrak{g}$ *with a center* A *is in one-to-one correspondence to an element of* $H^2(\mathfrak{g}; A)$ *up to isomorphisms.*

Proof. Take an arbitrary "section" s. Then $S = s(\mathfrak{g})$ has a Leibniz bracket $[\,,]_s$ induced by s. We may write $\mathfrak{g}' = S \oplus A$. Thus it may be written $g_i' = s(g_i) + a_i$ for any $g_i' \in \mathfrak{g}'$ where $\Pi(g_i') = g_i \in \mathfrak{g}, a_i \in A$ and $i = 1, 2$. We deduce that the action of $\mathfrak{g}$ on A is independent to the choice of a section map s. It holds

$$[\![g_1', g_2']\!] = [\![s(g_1), s(g_2)]\!] + [\![s(g_1), a_2]\!],$$

and from $\Pi([\![g_1', g_2']\!]) = [s(g_1), s(g_2)]_s$ it follows

$$[\![s(g_1), s(g_2)]\!] = s[g_1, g_2] + \psi_s(g_1, g_2)$$

for some linear map $\psi_s : \mathfrak{g} \otimes \mathfrak{g} \to A$. It is shown that the Leibniz identity holds if and only if ψ_s is a 2-cocycle. Now, we will see that $[\psi_s] \in H^2(\mathfrak{g}; A)$ does not depend on the choice of s. Take a section $\tilde{s}$ and let $\tilde{a}_i = g_i' - \tilde{s}(g_i)$ $(i = 1, 2)$. Then we may define a 1-cochain $t : \mathfrak{g} \to A$ by $t(g) = \tilde{s} - s$. Since

$$s([g_1, g_2]) + [\![s(g_1), \tilde{a}_2]\!] + \psi_s(g_1, g_2) = \tilde{s}([g_1, g_2]) + [\![\tilde{s}(g_1), a_2]\!] + \psi_{\tilde{s}}(g_1, g_2),$$

we have

$$(\psi_{\tilde{s}} - \psi_s)(g_1, g_2) = g_1 \cdot t(g_2) - t([g_1, g_2]).$$

The right hand of the equation is just $\delta t(g_1, g_2)$, thus we deduce that $[\psi_s] \in H^2(\mathfrak{g}; A)$ does not depend on the choice of s. We denote this element simply by $[\psi]$.

Next we prove that by the equivalence class of extensions ψ is determined up to coboundaries. Suppose that $(\mathfrak{g}', [\![\,,]\!]), (\bar{\mathfrak{g}}', [\![\,,]\!]^-)$ are isomorphic central extensions of $\mathfrak{g}$, and $\psi_s, \bar{\psi}_{\bar{s}}$ are corresponding cocycles with respect to sections $s, \bar{s}$ respectively. We consider the commutating diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"'] \arrow[d, equal] & \mathfrak{g}' \arrow[r, "\Pi"'] \arrow[d, "f"] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\iota^-"'] & \bar{\mathfrak{g}}' \arrow[r, "\Pi^-"'] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

where f is a Leibniz algebra isomorphism. We define 1-cochain $t : \mathfrak{g} \to A$ by $t = f \circ s - \bar{s}$. Then, from $\bar{\psi}_{f \circ s} = f \circ \psi_s$, $f|_A = 1$ and

$$f \circ s([g_1, g_2]) + [\![f \circ s(g_1), a_2]\!]^- + \bar{\psi}_{f \circ s}(g_1, g_2)$$
$$= \bar{s}([g_1, g_2]) + [\![\bar{s}(g_1), a_2 + t(g_2)]\!]^- + \bar{\psi}_{\bar{s}}(g_1, g_2)$$

where $g_i' = s(g_i) + a_i$, it follows

$$(\psi_s - \bar{\psi}_{\bar{s}})(g_1, g_2) = g_1 \cdot t(g_2) - t([g_1, g_2]) = \delta t(g_1, g_2).$$

Hence we have $[\psi] = [\bar{\psi}]$. Conversely, it is not difficult to see that if corresponding cohomologies with two central extensions of $\mathfrak{g}$ are equal then they are isomorphic. □

We remark that we cannot develop a Leibniz generalization of the abelian extension of a Lie algebra because $[\![[\![a_1, s(g_2)]\!], s(g_3)]\!] + [\![s(g_2), [\![a_1, s(g_3)]\!]]\!]$ does not vanish in general for $g_i' = s(g_i) + a_i$ $(i = 1, 2, 3)$.

As an example of a central extension, we have the Leibniz algebras associated with foliations. For any foliation $\mathscr{F}_\omega$ given by a q-form ω, we have shown that $(\mathscr{X}^{q+1}(M), [\![\ ,\ ]\!]_\omega)$ is a Leibniz algebra. In fact, it follows from Corollary 3.6(1) that there is a central extension

$$0 \to \ker \iota_\omega \xrightarrow{\iota} \mathscr{X}^{q+1}(M) \xrightarrow{\iota_\omega} \mathscr{X}_\omega(M) \to 0.$$

where $\mathscr{X}_\omega(M)$ denotes the image of ι_ω, which yields the foliation $\mathscr{F}_\omega$. We will calculate the 2-cocycle of this extension. When ω is nonsingular, that is, the given foliation is regular, we may take a section s by $s(X) = Z \wedge X$ where Z is an arbitrary q-vector field satisfying $\omega(Z) = 1$, and then ψ_s is given by

$$\psi_s(X, Y) = \mathscr{L}_X Z \wedge Y + \langle X | \gamma \rangle (Z \wedge Y)$$

where $d\omega = \gamma \wedge \omega$. Therefore, if a foliation is given by $\omega_i = 0$ for non-zero 1-forms $\omega_1, \dots, \omega_q$,

$$\psi_s(X, Y) = \mathscr{L}_X (Z_1 \wedge \cdots \wedge Z_q) \wedge Y + \left\langle X \,\middle|\, \sum_{i=1}^{q} \gamma_{ii} \right\rangle (Y \wedge Z_1 \wedge \cdots \wedge Z_q)$$

where $d\omega_i = \sum_{k=1}^{q} \gamma_{ik} \wedge \omega_k$ and $\omega_i(Z_j) = \delta_{ij}$. For a singular q-form $\omega = f\omega'$ where f is an arbitrary function and ω' a nonsingular q-form, we take a metric g and identify the tangent space and the cotangent space. Then we may take a section

$$s(X) = \frac{1}{|\omega|^2} Z \wedge X$$

where $g(Z) = \omega$, which is well-defined since both Z and an element of $\mathscr{X}_\omega(M)$ are divisible by f. Using the metric g satisfying $|\omega'| = 1$, the corresponding cocycle with s is given by

$$\psi_s(X, Y) = \mathscr{L}_X Z' \wedge Y' + \langle X|\gamma\rangle(Z' \wedge Y')$$

where Z' and Y' denote $f^{-1}Z$ and $f^{-1}Y$, respectively.

Conversely, by the theorem above, an arbitrary element of $H^2(\mathscr{X}_\omega(M); \ker \iota_\omega)$ determines a Leibniz algebra structure on $\mathscr{X}^{q+1}(M)$.

The following consideration gives us homomorphisms between Leibniz algebras.

Proposition 4.3. *Suppose that a p-form α and a q-form β which are both integrable are given, and that $\alpha \wedge \beta \neq 0$. Then $\alpha \wedge \beta$ is also integrable, and we get the exact sequence*

$$0 \longrightarrow \ker \iota_{\alpha\wedge\beta} \xrightarrow{\iota} \mathscr{X}^{p+q+1}(M) \xrightarrow{\iota_{\alpha\wedge\beta}} \mathscr{X}_{\alpha\wedge\beta}(M) \longrightarrow 0. \tag{4.2}$$

The following diagram of Leibniz algebra commutes where $\iota'_\beta = (-1)^{pq}\iota_\beta$ and $\mathscr{X}_\alpha^{q+1}(M) \subset \mathscr{X}^{q+1}(M)$ is the image of the interior product $\iota_\alpha : \mathscr{X}^{p+q+1}(M) \to \mathscr{X}^{q+1}(M)$.

$$\begin{tikzcd}
& (\mathscr{X}^{p+q+1}(M), \llbracket\, , \rrbracket_{\alpha\wedge\beta}) \arrow[dl, "\iota_\alpha"'] \arrow[dr, "\iota'_\beta"] \arrow[dd, "\iota_{\alpha\wedge\beta}"] & \\
(\mathscr{X}_\alpha^{q+1}(M), \llbracket\, , \rrbracket_\alpha) \arrow[dr, "\iota_\beta"'] & & (\mathscr{X}_\beta^{p+1}(M), \llbracket\, , \rrbracket_\beta) \arrow[dl, "\iota_\alpha"] \\
& (\mathscr{X}_{\alpha\wedge\beta}(M), [\, ,]) &
\end{tikzcd}$$

Proof. It is easy to see that $\alpha \wedge \beta$ is an integrable $(p+q)$-form. Let us show the above diagram commutes. All the maps are well-defined since $\iota_\beta(\mathscr{X}_\alpha^{q+1}(M)), \iota_\alpha(\mathscr{X}_\beta^{p+1}(M)) \subset \mathscr{X}_{\alpha\wedge\beta}(M)$. For any $X, Y \in \mathscr{X}^{p+q+1}(M)$, we calculate

$$\begin{aligned}
[\iota_{\alpha\wedge\beta}X, Y](\alpha) &= \mathscr{L}_{X(\alpha\wedge\beta)}(Y(\alpha)) - Y(\mathscr{L}_{X(\alpha\wedge\beta)}\alpha) \\
&= [(\iota_\alpha X)(\beta), \iota_\alpha Y] - (-1)^{p+q}\langle X \,|\, d\alpha \wedge \beta\rangle(Y(\alpha)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\iota_\alpha(\llbracket X, Y\rrbracket_{\alpha\wedge\beta}) &= ([\iota_{\alpha\wedge\beta}X, Y] + (-1)^{p+q}\langle X \,|\, d(\alpha\wedge\beta)\rangle Y)(\alpha) \\
&= [(\iota_\alpha X)(\beta), \iota_\alpha Y] + (-1)^q\langle X \,|\, \alpha \wedge d\beta\rangle(Y(\alpha)) \\
&= [(\iota_\alpha X)(\beta), \iota_\alpha Y] + (-1)^q\langle \iota_\alpha X \,|\, d\beta\rangle(Y(\alpha)) \\
&= \llbracket \iota_\alpha X, \iota_\alpha Y\rrbracket_\beta
\end{aligned}$$

and thus we conclude that $\iota_\alpha : \mathscr{X}^{p+q+1}(M) \to \mathscr{X}^{q+1}(M)$ is a Leibniz homomorphism; similarly for $\iota'_\beta : \mathscr{X}^{p+q+1}(M) \to \mathscr{X}^{p+1}(M)$. □

Example 4.4. We consider the Lie algebra $sl(2, \boldsymbol{R})$ with the basis

$$e_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then it holds that

$$[e_1, e_2] = e_2, \quad [e_2, e_3] = 2e_1, \quad [e_1, e_3] = -e_3.$$

Let us take the dual e_1^*, e_2^*, e_3^* of e_1, e_2, e_3. From $de_2^* = -e_1^* \wedge e_2^*$, it follows that $(\wedge^2 sl(2, \boldsymbol{R}), [\![\, ,]\!]_{e_2^*})$ is a Leibniz algebra which is a central extension of the Lie algebra $\mathfrak{g}$ where

$$\mathfrak{g} = \mathrm{span}(e_1, e_3), \quad [e_3, e_1] = e_3.$$

As we mentioned before, we may take a section $s(X) = e_2 \wedge X$. Then the corresponding cocycle $\psi \in H^2(\mathfrak{g}; \mathfrak{g} \wedge \mathfrak{g})$ is given by

$$\psi(X, Y) = [X, e_2] \wedge Y - \langle X | e_1^* \rangle e_2 \wedge Y$$

for any $X, Y \in \mathfrak{g}$. Since it follows that

$$\psi(e_1, e_1) = \psi(e_1, e_3) = \psi(e_3, e_1) = 0, \quad \psi(e_3, e_3) = 2a$$

where $a = e_3 \wedge e_1 \in \mathfrak{g} \wedge \mathfrak{g}$, we may write $\psi = 2a e_3^* \otimes e_3^*$.

When we replace e_2^* with ce_2^* where c is a non-zero constant, which preserves the foliation, then by

$$[\![g_1, g_2]\!]_{ce_2^*} = c[\![g_1, g_2]\!]_{e_2^*}$$

we deduce that the cocycle ψ is replaced with $c^{-1}\psi$.

Now, let us elucidate all the central extension $(\wedge^2 sl(2, \boldsymbol{R}), [\![\, ,]\!])$ of $\mathfrak{g}$. The action of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$ is given by

$$e_1 \cdot a = -2a, \quad e_3 \cdot a = 0, \tag{4.3}$$

that is, $g \cdot a = 2\mathscr{L}_g a$, and any 1-cochain t is generated by

$$t_1 = a e_1^*, \quad t_3 = a e_3^*.$$

Since $\delta t(g_1, g_2) = g_1 \cdot t(g_2) - t([g_1, g_2])$, we have

$$\delta t_1(e_1, e_1) = -2a, \quad \delta t_1(e_1, e_3) = \delta t_1(e_3, e_1) = \delta t_1(e_3, e_3) = 0, \tag{4.4}$$

$$\delta t_3(e_1, e_3) = \delta t_3(e_3, e_1) = -a, \quad \delta t_3(e_1, e_1) = \delta t_3(e_3, e_3) = 0, \tag{4.5}$$

thus we may write $\delta t_1 = 2\kappa_{11}$ and $\delta t_3 = \kappa_{13} + \kappa_{31}$ where κ_{ij} denotes $-a e_i^* \otimes e_j^*$. A direct computation shows $\delta\kappa_{13}(e_1, e_3, e_3) \neq 0$, that is, κ_{13} is not a cocycle, thus we deduce that $H^2(\mathfrak{g}; \mathfrak{g} \wedge \mathfrak{g})$ is 1-dimensional and a cocycle $c\kappa_{33}$ where c is a constant determines an element in $H^2(\mathfrak{g}; \mathfrak{g} \wedge \mathfrak{g})$. The corresponding Leibniz algebra structure on $\wedge^2 sl(2, \boldsymbol{R})$ is then given by

$$[\![e_2 \wedge e_1, e_2 \wedge e_1]\!] = 0, \qquad [\![e_2 \wedge e_1, e_2 \wedge e_3]\!] = -e_2 \wedge e_3,$$

$$[\![e_2 \wedge e_3, e_2 \wedge e_1]\!] = e_2 \wedge e_3, \quad [\![e_2 \wedge e_3, e_2 \wedge e_3]\!] = -c e_3 \wedge e_1$$

(the rest is given by (4.3) and $[\![\mathfrak{g} \wedge \mathfrak{g}, \wedge^2 sl(2, \boldsymbol{R})]\!] = 0$). Thus we have shown that, on any central extension of $\mathfrak{g}$ with the center $\mathfrak{g} \wedge \mathfrak{g}$, the Leibniz algebra structure is necessarily of this type.

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