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ON THE MULTIPLICITY OF THE IMAGE OF SIMPLE CLOSED CURVES VIA HOLOMORPHIC MAPS BETWEEN COMPACT RIEMANN SURFACES

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Abstract

Every non-trivial closed curve C on a compact Riemann surface R is freely homotopic to the r-fold iterate C_0^r of some primitive closed geodesic C_0 on R. We call r the multiplicity of C, and denote it by $N_R(C)$. Let f be a non-constant holomorphic map of a compact Riemann surface R_1 of genus g_1 onto another compact Riemann surface R_2 of genus g_2 with $g_1 \geq g_2 > 1$, and C a simple closed geodesic of hyperbolic length $l_{R_1}(C)$ on R_1. In this paper, we give an upper bound for $N_{R_2}(f(C))$ depending only on g_1, g_2 and $l_{R_1}(C)$.

1. Introduction

1.1. Let R be a Riemann surface of analytically finite type, that is, a Riemann surface obtained by removing n distinct points from a compact Riemann surface of genus g. Take a non-trivial closed curve C on R. Denote by $N_R(C) > 0$ the maximum of all numbers r such that for some non-trivial closed curve C_0 on R, the r-fold iterate C_0^r of C_0 is freely homotopic to C on R. We define $N_R(C) = 0$ for any trivial closed curve C on R (cf. Buser [1], 9.2.6). In this paper, we call $N_R(C)$ the *multiplicity* of C on R. A non-trivial closed curve C on R is said to be *primitive* if $N_R(C) = 1$.

Let f be a non-constant holomorphic map of a compact Riemann surface R_1 of genus g_1 onto another compact Riemann surface R_2 of genus g_2 with $g_1 \geq g_2 > 1$. Let C be a simple closed geodesic on R_1. The purpose of this paper is to obtain an upper bound for $N_{R_2}(f(C))$.

1.2. Assume that f has no branch point. Then $f : R_1 \to R_2$ is a holomorphic unbranched covering. Since C is a closed geodesic on R_1, the image $f(C)$ is also a closed geodesic on R_2. Set $r = N_{R_2}(f(C))$, and let C_0 be the primitive closed geodesic on R_2 such that the r-fold iterate C_0^r is freely homotopic

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to $f(C)$ on R_2. Then we have $C_0^r = f(C)$ for suitable parametrizations. On the other hand, the Riemann-Hurwitz relation (see 1.2.7 of Farkas and Kra [2] for example) yields

$$2(g_1 - 1) = 2d_f(g_2 - 1) + B(f),$$

where $B(f)$ is the total branching number of f and d_f is the degree of f. Thus, in this case, we conclude that

$$N_{R_2}(f(C)) = r \le d_f = \frac{g_1 - 1}{g_2 - 1}.$$

A natural question that occurs at this point is the following: In the general case where f may have branch points, does there exist an upper bound for $N_{R_2}(f(C))$ depending only on g_1 and g_2? The answer is "No.". In fact, the example which will be given in the last section asserts that there is no upper bound for $N_{R_2}(f(C))$ depending only on g_1, g_2 and f.

In this paper, we obtain the following.

Theorem. *Let f be a non-constant holomorphic map of a compact Riemann surface R_1 of genus g_1 onto another compact Riemann surface R_2 of genus g_2 with $g_1 \ge g_2 > 1$. Let C be a simple closed geodesic on R_1. Then*

$$N_{R_2}(f(C)) \le \max\left\{\frac{g_1 - 1}{g_2 - 1}, \Lambda(g_1, g_2, l_{R_1}(C))\right\},$$

where

$$\Lambda(g_1, g_2, l) = \sinh \frac{\pi^2(1 + 2(g_1 - g_2)(g_1 - 1)/(g_2 - 1))}{\eta(l)},$$

$$\eta(l) = \frac{2\pi}{l}\left(\pi - 4 \arctan\left(\tanh \frac{l}{4}\right)\right),$$

and $l_{R_1}(C)$ is the length of C with respect to the hyperbolic metric of constant Gaussian curvature -1 on R_1.

Note that the function Λ satisfies

$$\log \Lambda(g_1, g_2, l) < M g_1^2 l e^{l/2} \quad \text{for all } l > 0,$$

where M is some positive constant.

1.3. Let $\mathrm{Hol}(R_1, R_2)$ be the set of all non-constant holomorphic maps of R_1 onto R_2, and assume that $\mathrm{Hol}(R_1, R_2)$ is not empty. In 1978, Martens [3] showed that $f \in \mathrm{Hol}(R_1, R_2)$ is determined by the homology map

$$f_* : H_1(R_1; \boldsymbol{Z}) \to H_1(R_2; \boldsymbol{Z})$$

induced naturally from f, where $H_1(R_j; \boldsymbol{Z})$ is the first homology group of R_j with integer coefficients. This is called Martens' rigidity theorem. The result was strengthened by Tanabe [4] in 1996.

Let $\mathrm{FH}(R_j)$ denote the set of all free homotopy classes of closed curves on R_j. Then f also induces a map

$$\psi(f) : \mathrm{FH}(R_1) \ni [c] \mapsto [f(c)] \in \mathrm{FH}(R_2).$$

Fix a homology basis $\{\langle a_1\rangle, \langle a_2\rangle, \dots, \langle a_{2g_1}\rangle\}$ on R_1, where $\langle a_j\rangle$ is a homology class represented by a closed curve a_j on R_1 for each j. The rigidity theorem described above yields that $f \in \mathrm{Hol}(R_1, R_2)$ is completely determined by

$$\{\psi(f)([a_1]), \psi(f)([a_2]), \dots, \psi(f)([a_{2g_1}])\}.$$

Our theorem gives a necessary condition for a map $\psi : \mathrm{FH}(R_1) \to \mathrm{FH}(R_2)$ to be induced from some $f \in \mathrm{Hol}(R_1, R_2)$, and, for example, it is applicable to the problem on estimating the number of elements of $\mathrm{Hol}(R_1, R_2)$. Furthermore, the auther hope that the results and the method are also applicable to problems on estimating numbers of objects for Mordell conjecture and Shafarevich conjecture in the function field case.

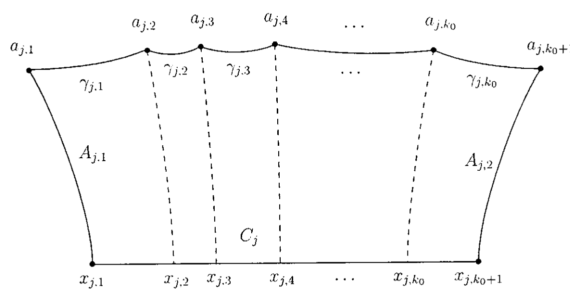

FIGURE 1. a geodesic polygon $\mathscr{P}_j$

The essential tool of our proof is the estimation of hyperbolic length of closed geodesic loops on Riemann surfaces (Lemma 5, Lemma 6).

1.4. This paper is organized as follows. In Section 2, we will see several results on hyperbolic geometry of Riemann surfaces. The proof of Theorem will be given in Section 3. In the last section, we will construct a holomorphic branched covering $f : R_1 \to R_2$ and an infinite sequence $\{C_r\}_{r=1}^{\infty}$ of simple closed geodesics on R_1 satisfying $N_{R_2}(f(C_r)) = r$ for every r.

2. Several results on hyperbolic geometry

2.1. First we see a few property of hyperbolic geodesic polygons (piecewize geodesic simple closed curves) on the open unit disk Δ endowed with the hyperbolic metric of constant negative curvature -1.

For each $j = 1, 2$, let $\mathscr{P}_j$ be a geodesic polygon on Δ satisfying the following conditions:

(1) $\mathscr{P}_j$ consists of $k_0 + 3$ sides $A_{j,1}$, $A_{j,2}$, C_j, $\gamma_{j,1}, \dots, \gamma_{j,k_0}$ and $k_0 + 3$ vertexes $x_{j,1}$, x_{j,k_0+1}, $a_{j,1}$, $a_{j,2}, \dots, a_{j,k_0+1}$ as illustrated in Figure 1,

(2) for each $k = 1, 2$, a side $A_{j,k}$ intersects C_j at right angle, and
(3) $d_\Delta(a_{j,1}, C_j) = d_\Delta(a_{j,2}, C_j) = \cdots = d_\Delta(a_{j,k_0+1}, C_j)$, where $d_\Delta(a_{j,k}, C_j)$ is the hyperbolic distance between $a_{j,k}$ and C_j.

For each $j = 1, 2$ and $k = 1, 2, \ldots, k_0$, we set

$$l_{j,k} = l_\Delta(\gamma_{j,k}),$$
$$h_j = d_\Delta(a_{j,1}, C_j) = \cdots = d_\Delta(a_{j,k_0+1}, C_j),$$

where $l_\Delta(\gamma_{j,k})$ is the hyperbolic length of $\gamma_{j,k}$.

Lemma 1. *If $h_1 \geq h_2$ and $l_{1,k} \leq l_{2,k}$ for all $k = 1, 2, \ldots, k_0$, then*

$$d_\Delta(a_{1,1}, a_{1,k_0+1}) \leq d_\Delta(a_{2,1}, a_{2,k_0+1}).$$

Proof. Without loss of generality, we may assume that $k_0 = 2$. Let $x_{j,2}$ be the intersection point of C_j and the unique perpendicular from $a_{j,2}$ to C_j. Then for each $j = 1, 2$ and $k = 1, 2$, the relationship

$$\sinh \frac{l_{j,k}}{2} = \cosh h_j \sinh \frac{d_\Delta(x_{j,k}, x_{j,k+1})}{2},$$
$$\sinh \frac{d_\Delta(a_{j,1}, a_{j,3})}{2} = \cosh h_j \sinh \frac{d_\Delta(x_{j,1}, x_{j,3})}{2}$$

follows from hyperbolic trigonometry (see Buser [1], 2.3.1 and Figure 4.1.1). This yields

$$\sinh \frac{d_\Delta(a_{j,1}, a_{j,3})}{2} = \sinh \frac{l_{j,2}}{2} \sqrt{1 + \frac{\sinh^2(l_{j,1}/2)}{\cosh^2 h_j}} + \sinh \frac{l_{j,1}}{2} \sqrt{1 + \frac{\sinh^2(l_{j,2}/2)}{\cosh^2 h_j}}.$$

Hence we obtain $d_\Delta(a_{1,1}, a_{1,3}) \leq d_\Delta(a_{2,1}, a_{2,3})$. □

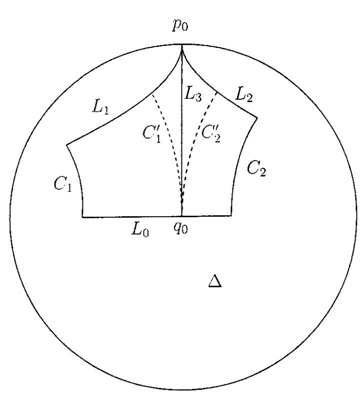

Figure 2. a degenarate right-angled geodesic hexagon $\mathscr{H}$ in the unit disk Δ

2.2. Let $\mathscr{H}$ be a degenarate right-angled geodesic hexagon in the unit disk

Δ as illustrated in Figure 2. The hexagon $\mathscr{H}$ consists of five geodesic sides L_0, L_1, L_2, C_1, C_2, and the remaining side of $\mathscr{H}$ is degenerated into a point p_0 at infinity. Let L_3 be the unique perpendicular from p_0 to L_0, and q_0 the intersection point of L_0 and L_3. Denote by G the compact subset of Δ bounded by $\mathscr{H}$. For each $j = 1, 2$, we set

$$C_j' = \left\{ p \in G \mid d_\Delta(p, C_j) = \operatorname{arcsinh} \frac{1}{\sinh(l_\Delta(C_j))} \right\}.$$

Then C_j' intersects L_0 at q_0 (see Buser [1], 2.3.1). For each $q \in L_3 \cup \{p_0\}$ and $j = 1, 2$, let P_j^q denote the unique perpendicular from q to C_j, and $a_j(q)$ the intersection point of P_j^q and C_j' (see Figure 3).

Lemma 2. *If $l_\Delta(C_1) \leq l_\Delta(C_2)$, then*

$$d_\Delta(a_1(q_1), a_1(q_2)) \leq d_\Delta(a_2(q_1), a_2(q_2))$$

for any $q_1, q_2 \in L_3 \cup \{p_0\}$.

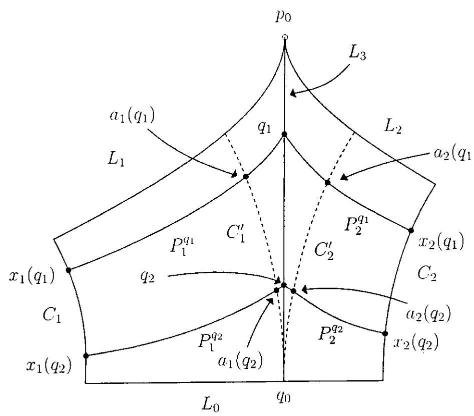

Figure 3. a degenarate right-angled geodesic hexagon $\mathscr{H}$

Proof. For every $q \in L_3 \cup \{p_0\}$, we denote by $x_j(q)$ the intersection point of P_j^q and C_j.

Fix $q_1, q_2 \in L_3 \cup \{p_0\}$ arbitrarily. It is sufficient to consider the case where $q_1, q_2 \in L_3$ and $d_\Delta(q_0, q_1) \geq d_\Delta(q_0, q_2)$. Assume that $l_\Delta(C_1) \leq l_\Delta(C_2)$. Set

$$s_k = \coth^2 d_\Delta(q_0, q_k),$$

$$t_{j,k} = \cosh^{-2} \frac{d_\Delta(x_j(q_0), x_j(q_k))}{2}, \quad \text{and}$$

$$u_j = \tanh^2 l_\Delta(C_j)$$

for $j, k = 1, 2$. Then we have $0 < u_1 \leq u_2 < 1 < s_1 \leq s_2$. By hyperbolic trigonometry (see Buser [1], 2.3.1), we obtain

$$\frac{1}{t_{j,k}} = \frac{1}{2}\left(\frac{1}{\sqrt{1-u_j/s_k}}+1\right), \quad \text{and}$$

$$\sinh\frac{d_\Delta(a_j(q_1),a_j(q_2))}{2} = \frac{1}{\sqrt{u_j}}\left\{\sqrt{\frac{1}{t_{j,2}}\left(\frac{1}{t_{j,1}}-1\right)}-\sqrt{\frac{1}{t_{j,1}}\left(\frac{1}{t_{j,2}}-1\right)}\right\}$$

for $j,k=1,2$. This yields

$$\sinh\frac{d_\Delta(a_j(q_1),a_j(q_2))}{2} = \lambda(s_1,s_2,u_j) \quad \text{for } j=1,2,$$

where

$$\lambda(x,y,z) = \lambda_1(x,y,z)-\lambda_1(y,x,z),$$

$$\lambda_1(x,y,z) = \sqrt{\frac{\lambda_2(x,y,z)}{z}}, \quad \text{and}$$

$$\lambda_2(x,y,z) = \frac{1}{4}\left(\frac{1}{\sqrt{1-z/y}}+1\right)\left(\frac{1}{\sqrt{1-z/x}}-1\right).$$

By calculation, we have

$$\frac{\partial}{\partial z}\lambda_1(x,y,z) = \frac{\sqrt{\lambda_2(x,y,z)}}{4z^{3/2}}\left(\frac{1-\sqrt{1-z/y}}{1-z/y}+\frac{1+\sqrt{1-z/x}}{1-z/x}-2\right),$$

and obtain

$$\begin{aligned}\frac{\partial}{\partial z}\lambda(x,y,z) &= \frac{\partial}{\partial z}\lambda_1(x,y,z)-\frac{\partial}{\partial z}\lambda_1(y,x,z)\\ &= \frac{1}{4z^{3/2}}\left\{(\sqrt{\lambda_2(x,y,z)}-\sqrt{\lambda_2(y,x,z)})\left(\frac{1}{1-z/x}+\frac{1}{1-z/y}-2\right)\right.\\ &\qquad\left.+(\sqrt{\lambda_2(x,y,z)}+\sqrt{\lambda_2(y,x,z)})\left(\frac{1}{\sqrt{1-z/x}}-\frac{1}{\sqrt{1-z/y}}\right)\right\}\\ &\geq 0\end{aligned}$$

for any $x,y,z\in\boldsymbol{R}$ with $0<z<1<x\leq y$. Hence $\lambda(s_1,s_2,\cdot)|_{(0,1)}$ is an increasing function, and $d_\Delta(a_1(q_1),a_1(q_2))\leq d_\Delta(a_2(q_1),a_2(q_2))$. □

2.3. Let G' be a copy of G. By pasting G and G' together along the sides L_0, L_1 and L_2, we obtain a degenerate pair of pants Y which has two boundary geodesics and one puncture. Conversely, every degenerate pair of pants Y with two boundary geodesics and one puncture can be obtained by the above construction for a suitable G (see Buser [1], 3.1 and 4.4).

2.4. Next we recall several facts of hyperbolic geometry on Riemann surfaces. Let R be a hyperbolic Riemann surface of analytically finite type endowed with the hyperbolic metric of constant negative curvature -1, and L a closed geodesic on R. We shall use the same symbol for a curve (a continuous map of an interval into a Riemann surface) and its image if there is no fear of confusion.

For an arbitrary simple closed geodesic L on R, we set

$$\mathscr{C}_R(L) = \left\{ p \in R \mid d_R(p, L) \leq \operatorname{arcsinh} \frac{1}{\sinh(l_R(L)/2)} \right\},$$

where $d_R(p, L)$ is the distance between L and p with respect to the hyperbolic metric on R. The set $\mathscr{C}_R(L)$ is called the *collar* around L. The interior of $\mathscr{C}_R(L)$ is conformally equivalent to an annulus (see Buser [1], 4.1.1).

LEMMA 3. *Let L be an arbitrary simple closed geodesic on R, and $C : I = [0,1] \to R$ a closed geodesic loop freely homotopic to the r-fold iterate L^r of L with some $r \geq 1$. If C is included in the collar $\mathscr{C}_R(L)$ and $C(0) = C(1) \in \partial\mathscr{C}_R(L)$, then*

$$\sinh\left(\frac{l_R(C)}{2}\right) = \sinh\left(\frac{rl_R(L)}{2}\right) \coth\left(\frac{l_R(L)}{2}\right) > r.$$

Proof. We first note that

$$\sinh\left(\frac{sl}{2}\right) \coth\left(\frac{l}{2}\right) > s \tag{2.1}$$

holds for all $s \geq 1$ and $l > 0$.

Let $\tilde{C}$ be a lift of the curve $C : [0,1] \to R$ in the universal covering surface Δ of R, and h the covering transformation which corresponds to $\tilde{C}$. Denote by $\tilde{p}_j$ $(j = 1, 2)$ the endpoints of $\tilde{C}$, and by A_j the perpendicular from $\tilde{p}_j$ to the axis $\mathrm{Axis}(h)$ of h. Then, $\tilde{C}$, A_1, A_2, $\mathrm{Axis}(h)$ together bound a geodesic quadrangle $\mathscr{Q}$. Dropping the common perpendicular between $\tilde{C}$ and $\mathrm{Axis}(h)$, we obtain two isometric trirectangle $\mathscr{T}_1, \mathscr{T}_2$ (see Figure 4). By 2.3.1 of Buser [1], we have

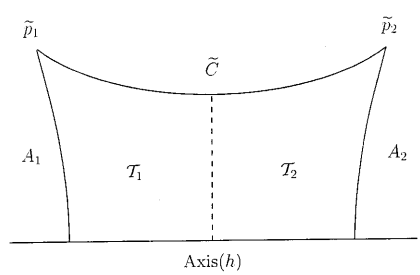

FIGURE 4. a quadrangle $\mathscr{Q}$

$$\sinh^2 l_\Delta(A_1) = \sinh^2 \frac{l_R(C)}{2} \coth^2 \frac{rl_R(L)}{2} - \cosh^2 \frac{l_R(C)}{2}. \tag{2.2}$$

If C is included in the collar $\mathscr{C}_R(L)$ and $C(0) = C(1) \in \partial\mathscr{C}_R(L)$, then

$$\sinh l_\Delta(A_1) = \sinh^{-1} \frac{l_R(L)}{2},$$

and we obtain

$$\sinh\left(\frac{l_R(C)}{2}\right) = \sinh\left(\frac{rl_R(L)}{2}\right) \coth\left(\frac{l_R(L)}{2}\right)$$

by (2.2). □

Lemma 4. *Let L be an arbitrary closed geodesic on R, and C a rectifiable closed curve on R which is freely homotopic to the r-fold iterate L^r of L for some $r \geq 1$. If the hyperbolic length $l_R(C)$ of C satisfies $l_R(C) < 2 \operatorname{arcsinh} r$, then L is simple and C is included in the interior of $\mathscr{C}_R(L)$.*

Proof. Since $r \geq 1$ and $l_R(C) < 2 \operatorname{arcsinh} r$, we have

$$rl_R(L) \leq l_R(C) < 2 \operatorname{arcsinh} r < 4r \operatorname{arcsinh} 1.$$

Hence, by Lemma 7 of Yamada [5], L is simple.

Let p be an arbitrary point of C. We may assume that $C(0) = C(1) = p$. There exists a geodesic loop $C' : I \to R$ such that $C'(0) = C'(1) = p$ and C' is homotopic to C rel the base point. Similarly as the proof of Lemma 3, we take a lift $\widetilde{C'}$ of C' in the universal covering surface Δ of R. Denote by $\tilde{p}_1$, $\tilde{p}_2$ the endpoints of $\widetilde{C'}$. For $j = 1, 2$, let A_j be the perpendicular from $\tilde{p}_j$ to the axis of the covering transformation which corresponds to $\widetilde{C'}$. The inequality $l_R(C') \leq l_R(C) < 2 \operatorname{arcsinh} r$, (2.1) and (2.2) together yield

$$\begin{aligned}\sinh^2 d_R(p, L) \leq \sinh^2 l_\Delta(A_1) \\ &= \sinh^2 \frac{l_R(C')}{2} \coth^2 \frac{rl_R(L)}{2} - \cosh^2 \frac{l_R(C')}{2} \\ &< r^2 \sinh^{-2} \frac{rl_R(L)}{2} - 1 \\ &< \coth^2 \frac{l_R(L)}{2} - 1 = \sinh^{-2} \frac{l_R(L)}{2}.\end{aligned}$$

Thus, we obtain $p \in \operatorname{Interior}(\mathscr{C}_R(L))$. □

3. Proof of Theorem

3.1. Before proceeding to the proof of Theorem, we must establish two preliminary results.

Let R be a hyperbolic Riemann surface of analytically finite type, and p_0 a point of R. Assume that there exists a subset Y of R such that Y contains p_0 and $\dot{Y} = Y \backslash \{p_0\}$ is a degenerate pair of pants which has two distinct boundary geodesics with respect to the hyperbolic metric on $\dot{R} = R \backslash \{p_0\}$. Let C_1, C_2 denote the boundary geodesics of $\dot{Y}$, and L_0 the unique simple common perpendicular between C_1 and C_2 in $\dot{Y}$. Then, there exist simple curves $A_j : I \to Y$ $(j = 1, 2)$ and $D : I \to Y$ such that

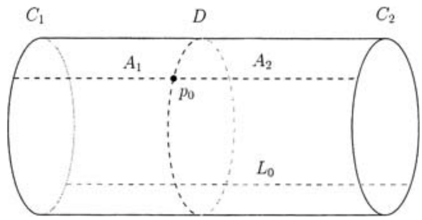

FIGURE 5. a figure of Y

(1) $A_j(0) = p_0$ and $A_j(1)$ is a point of C_j for each $j = 1, 2$,
(2) D is a simple closed curve freely homotopic to C_1 on Y satisfying $D(0) = D(1) = p_0$,
(3) $A_1|_{(0,1]}$, $A_2|_{(0,1]}$, $D|_{(0,1)}$ are geodesics with respect to the hyperbolic metric on $\dot{R}$,
(4) each $A_j|_{(0,1]}$ $(j = 1, 2)$ is a perpendicular to C_j, and
(5) $D|_{(0,1)}$ intersects L_0 at right angle.

We set $L_j = A_j|_{(0,1]}$ for $j = 1, 2$ and $L_3 = D|_{(0,1)}$ (see Figure 5). The three perpendiculars L_0, L_1, L_2 together decompose $\dot{Y}$ into two isometric degenerate right-angled geodesic hexagons G, G'.

We first state the following assertion.

LEMMA 5. *Let $C : I \to \dot{Y}$ be an arbitrary rectifiable closed curve. Assume that the hyperbolic length $l_{\dot{R}}(C)$ of C satisfies*

$$l_{\dot{R}}(C) < 2 \operatorname{arcsinh} N_R(C). \tag{3.1}$$

Then C does not intersect L_3.

Proof. Let $C : I \to \dot{Y}$ be an arbitrary rectifiable closed curve. Assume that C intersects L_3. We shall prove $l_{\dot{R}}(C) \geq 2 \operatorname{arcsinh} N_R(C)$.

There exists a unique geodesic loop $C' : I \to \dot{R}$ with respect to the hyperbolic metric on $\dot{R}$ such that $C'(0) = C'(1) = C(0) = C(1)$ and C' is homotopic to C rel the base point. Then the geodesic loop C' is included in $\dot{Y}$ and satisfies $l_{\dot{R}}(C') \leq l_{\dot{R}}(C)$. Hence, we may assume without loss of generality that C is a closed geodesic loop with respect to the hyperbolic metric on $\dot{R}$ satisfying $C(0) = C(1) \in L_3$. It is sufficient to consider the case where

$$l_{\dot{R}}(C_1) \leq l_{\dot{R}}(C_2). \tag{3.2}$$

For each $j = 1, 2$, we define the half-collar $\mathscr{C}_j$ around C_j by

$$\mathscr{C}_j = \left\{ p \in \dot{Y} \mid d_{\dot{R}}(p, C_j) \le \operatorname{arcsinh} \frac{1}{\sinh(l_{\dot{R}}(C_j)/2)} \right\}.$$

Let C_j' be the simple closed boundary curve of $\mathscr{C}_j$ lying on the interior of $\dot{Y}$.

For any $q \in L_3$ and $j = 1, 2$, let $P_j^q : I \to \dot{Y}$ denote the unique perpendicular from q to C_j such that $P_j^q|_{(0,1]}$ does not intersect L_3. For each $j = 1, 2$, we define a projection $a_j : \dot{Y} \to C_j'$ as follows:

(1) For any $p \in \dot{Y} \backslash (L_1 \cup L_2)$, there exists a unique point q on L_3 such that $p \in P_1^q \cup P_2^q$. We let $a_j(p)$ be the unique intersection point of P_j^q and C_j'.

(2) For any $p \in L_1 \cup L_2$, we let $a_j(p)$ be the unique intersection point of L_j and C_j'.

Set $\Sigma = C_1' \cup C_2' \cup L_3 \cup ((L_0 \cup L_1 \cup L_2) \backslash (\mathscr{C}_1 \cup \mathscr{C}_2))$. Since C is a closed geodesic loop in $\dot{Y}$ with $C(0) = C(1) \in L_3$, there exist points $t_0, t_1, \dots, t_n \in I$ with $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $C(t) \in \Sigma$ if and only if $t = t_k$ for some $k = 0, 1, \dots, n$. For each $k = 0, 1, \dots, n-1$, we set $\alpha_k = C|_{[t_k, t_{k+1}]}$, and let β_k denote the unique geodesic curve from $a_1(C(t_k))$ to $a_1(C(t_{k+1}))$ homotopic to the curve $a_1 \circ \alpha_k : [t_k, t_{k+1}] \to \dot{Y}$ rel $a_1(C(t_k))$, $a_1(C(t_{k+1}))$. Fix k arbitrarily, then α_k satisfies either $\alpha_k \subset \overline{\dot{Y} \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)}$ or $\alpha_k \subset \mathscr{C}_1 \cup \mathscr{C}_2$. If $\alpha_k \subset \overline{\dot{Y} \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)}$, then Lemma 2 and (3.2) yield

$$l_{\dot{R}}(\alpha_k) \ge l_{\dot{R}}(\beta_k). \tag{3.3}$$

In the case of $\alpha_k \subset \mathscr{C}_1 \cup \mathscr{C}_2$, we also obtain (3.3) by Lemma 1, Lemma 2 and (3.2). Hence (3.3) holds for all k. Denote by $\beta : I \to \dot{Y}$ the unique closed geodesic loop homotopic to the closed curve $a_1 \circ C : I \to \dot{Y}$ rel $\beta(0) = \beta(1) = a_1(C(0)) = a_1(C(1))$. Then, by Lemma 3, we have $l_{\dot{R}}(\beta) > 2 \operatorname{arcsinh} N_R(C)$ and conclude that

$$\begin{aligned} l_{\dot{R}}(C) &= \sum_{k=0}^{n-1} l_{\dot{R}}(\alpha_k) \\ &\ge \sum_{k=0}^{n-1} l_{\dot{R}}(\beta_k) \\ &\ge l_{\dot{R}}(\beta) \\ &> 2 \operatorname{arcsinh} N_R(C). \end{aligned}$$

The proof of Lemma 5 is finished. □

3.2. We also need the following estimation.

Lemma 6. *Let R be a hyperbolic Riemann surface of analytically finite type. Take $k > 0$ distinct points $p_1, p_2, \dots, p_k$ of R and set $\dot{R} = R \backslash$*

$\{p_1, p_2, \dots, p_k\}$. *Let C be an arbitrary simple closed geodesic with hyperbolic length $l_R(C)$ on R.*

Then there exists a simple closed geodesic C' on $\dot{R}$ such that

(a) *C' is freely homotopic to C on R, and*

(b) *the hyperbolic length $l_{\dot{R}}(C')$ of C' satisfies*

$$l_{\dot{R}}(C') \le \frac{2\pi^2(k+1)}{\eta(l_R(C))},$$

where

$$\eta(l) = \frac{2\pi}{l}\left(\pi - 4\arctan\left(\tanh\frac{l}{4}\right)\right).$$

Proof. Take an annulur cover

$$\rho : \mathscr{A}_0 = \{z \in \boldsymbol{C} \mid 1 < |z| < r_0\} \to R$$

of R with respect to C, i.e., ρ is a holomorphic unbranched covering of R such that $\rho(\{z \in \mathscr{A}_0 \mid |z| = \sqrt{r_0}\}) = C$ and $\rho|_{\{z \in \mathscr{A}_0 \mid |z| = \sqrt{r_0}\}}$ is an injection. Set $l_0 = l_R(C)$. Then, by calculation, we have

$$l_0 \log r_0 = 2\pi^2. \tag{3.4}$$

Let $\mathscr{C}_R(C)$ be the collar around C, i.e.,

$$\mathscr{C}_R(C) = \left\{p \in R \mid d_R(p, C) \le \operatorname{arcsinh}\left(\frac{1}{\sinh(l_0/2)}\right)\right\}.$$

Then, by the collar theorem, there exists a number $r_1 \in [1, \sqrt{r_0}]$ such that

(1) $\mathscr{A}_1 = \{z \in \boldsymbol{C} \mid r_1 < |z| < r_0/r_1\} \subset \mathscr{A}_0$ is a component of the interior of $\rho^{-1}(\mathscr{C}_R(C))$, and

(2) the restricted map $\rho|_{\mathscr{A}_1}$ is an injection.

By calculation, we obtain a relation

$$\log r_1 = \frac{4\pi}{l_0}\arctan\left(\tanh\frac{l_0}{4}\right). \tag{3.5}$$

By (3.4) and (3.5), the conformal modulus $M(\mathscr{A}_1) = \log(r_0/r_1^2)$ of $\mathscr{A}_1$ satisfies

$$\begin{aligned} M(\mathscr{A}_1) &= \log\frac{r_0}{r_1^2} \\ &= \log r_0 - 2\log r_1 \\ &= \frac{2\pi}{l_0}\left(\pi - 4\arctan\left(\tanh\frac{l_0}{4}\right)\right) = \eta(l_0). \end{aligned}$$

Let $\{z_1, z_2, \dots, z_{k'}\} \subset \mathscr{A}_1$ be the finite set of distinct points such that $\{z_1, z_2, \dots, z_{k'}\} = \mathscr{A}_1 \cap \rho^{-1}(\{p_1, p_2, \dots, p_k\})$ and $r_1 = x_0 \le x_1 \le x_2 \le \cdots \le x_{k'} \le x_{k'+1} = r_0/r_1$ $(x_j = |z_j|, j = 1, 2, \dots, k')$. Since

$$\frac{x_1}{x_0} \times \frac{x_2}{x_1} \times \cdots \times \frac{x_{k'}}{x_{k'-1}} \times \frac{r_{k'+1}}{x_{k'}} = \frac{r_0}{r_1^2} = \exp(M(\mathscr{A}_1)) = \exp(\eta(l_0)),$$

there exists a number j_0 such that

$$\frac{x_{j_0+1}}{x_{j_0}} \geq \left(\frac{r_0}{r_1^2}\right)^{1/(k'+1)} = (\exp(\eta(l_0)))^{1/(k'+1)}. \tag{3.6}$$

Set

$$\mathscr{A}_2 = \{z \in \boldsymbol{C} \mid x_{j_0} < |z| < x_{j_0+1}\} \subset \mathscr{A}_1.$$

Then $\rho(\mathscr{A}_2) \subset \dot{R}$ and

$$L = \{z \in \boldsymbol{C} \mid |z| = \sqrt{x_{j_0} x_{j_0+1}}\} \subset \mathscr{A}_2$$

is the closed geodesic of $\mathscr{A}_2$. The hyperbolic length $l_{\mathscr{A}_2}(L)$ of L satisfies

$$l_{\mathscr{A}_2}(L) = \frac{2\pi^2}{\log(x_{j_0+1}/x_{j_0})}. \tag{3.7}$$

Let C' be the simple closed geodesic of $\dot{R}$ freely homotopic to $\rho(L)$ on $\dot{R}$. Then C' is freely homotopic to C on R. By (3.6) and (3.7), the hyperbolic length $l_{\dot{R}}(C')$ on $\dot{R}$ satisfies

$$\begin{aligned} l_{\dot{R}}(C') &\leq l_{\mathscr{A}_2}(L) \\ &= \frac{2\pi^2}{\log(x_{j_0+1}/x_{j_0})} \\ &\leq \frac{2\pi^2(k'+1)}{\eta(l_0)} \\ &\leq \frac{2\pi^2(k+1)}{\eta(l_0)}. \end{aligned}$$

This completes the proof of Lemma 6. □

3.3. *Proof of Theorem.* Let C be an arbitrary simple closed geodesic on R_1 and f a non-constant holomorphic map of R_1 onto R_2. Assume that

$$N_{R_2}(f(C)) > \Lambda(g_1, g_2, l_{R_1}(C)). \tag{3.8}$$

We shall prove

$$N_{R_2}(f(C)) \leq \frac{g_1 - 1}{g_2 - 1}.$$

Denote by $\mathrm{BP}(f) \subset R_1$ the set of all branch points of f. Set $\dot{R}_2 = R_2 \setminus f(\mathrm{BP}(f))$, $\dot{R}_1 = f^{-1}(\dot{R}_2)$ and $\dot{f} = f|_{\dot{R}_1}$. Then $\dot{R}_1$ and $\dot{R}_2$ are Riemann surfaces of analytically finite type (g_1, n_1) and (g_2, n_2) respectively. The map $\dot{f} : \dot{R}_1 \to \dot{R}_2$ is a holomorphic unbranched covering. The Riemann-Hurwitz relation yields

$$2(g_1 - 1) = 2d_f(g_2 - 1) + B(f),$$

where $B(f)$ is the total branching number of f and d_f is the degree of f. Thus we have $n_2 \le B(f) \le 2(g_1 - g_2)$ and $n_1 \le n_2 d_f \le 2(g_1 - g_2)(g_1 - 1)/(g_2 - 1)$, and conclude by Lemma 6 that there exists a simple closed geodesic C' on $\dot{R}_1$ such that

(a) C' is freely homotopic to C on R_1, and
(b) the hyperbolic length $l_{\dot{R}_1}(C')$ of C' satisfies

$$l_{\dot{R}_1}(C') \le \frac{2\pi^2(1 + 2(g_1 - g_2)(g_1 - 1)/(g_2 - 1))}{\eta(l_{R_1}(C))}. \tag{3.9}$$

By (3.8) and (3.9), we obtain

$$\begin{aligned} l_{R_2}(f(C')) &\le l_{\dot{R}_2}(f(C')) \\ &= l_{\dot{R}_1}(C') \\ &\le \frac{2\pi^2(1 + 2(g_1 - g_2)(g_1 - 1)/(g_2 - 1))}{\eta(l_{R_1}(C))} \\ &< 2 \operatorname{arcsinh} N_{R_2}(f(C)) = 2 \operatorname{arcsinh} N_{R_2}(f(C')). \end{aligned} \tag{3.10}$$

Set $r_0 = N_{R_2}(f(C)) = N_{R_2}(f(C'))$. Let C_0 be the primitive closed geodesic of R_2 such that the r_0-fold iterate $C_0^{r_0}$ of C_0 is freely homotopic to $f(C)$ on R_2. By (3.10) and Lemma 4, we conclude that C_0 is a simple curve, and that $f(C')$ is included in the interior of the collar

$$\mathscr{C}_{R_2}(C_0) = \left\{ p \in R_2 \mid d_{R_2}(p, C_0) \le \operatorname{arcsinh}\left(\frac{1}{\sinh(l_{R_2}(C_0)/2)}\right) \right\}$$

around C_0.

First we consider the case of $(R_2 \backslash \dot{R}_2) \cap \text{Interior}(\mathscr{C}_{R_2}(C_0)) = \emptyset$. In this case, the closed geodesic $f(C')$ of $\dot{R}_2$ is freely homotopic to $C_0^{r_0}$ on $\dot{R}_2$. Then by the Riemann-Hurwitz relation we have $r_0 \le d_f \le (g_1 - 1)/(g_2 - 1)$. Next we see the case of $(R_2 \backslash \dot{R}_2) \cap \text{Interior}(\mathscr{C}_{R_2}(C_0)) \ne \emptyset$. Denote all the elements of $(R_2 \backslash \dot{R}_2) \cap \text{Interior}(\mathscr{C}_{R_2}(C_0))$ by $\{p_1, \dots, p_{n_3}\}$ $(1 \le n_3 \le n_2)$. Let B_1, B_2 be two boundary simple closed curves of $\mathscr{C}_{R_2}(C_0)$. For each $j \in \{1, 2, \dots, n_3\}$, we take a simple closed curve D_j on R_2 as follows: For each $i = 1, 2$, let C_i be the simple closed geodesics of $R_2 \backslash \{p_j\}$ freely homotopic to B_i on $R_2 \backslash \{p_j\}$. The geodesics C_1 and C_2 together bound a doubly connected domain Y_j of R_2 containing p_j. The domain $\dot{Y}_j = Y_j \backslash \{p_j\}$ is a degenerate pair of pants on $R_2 \backslash \{p_j\}$. Let L_0 be the unique simple common perpendicular between C_1 and C_2 in $\dot{Y}_j$ with respect to the hyperbolic metric on $R_2 \backslash \{p_j\}$. We take a simple closed curve $D_j : I \to Y_j$ so that

(1) D_j is a simple closed curve freely homotopic to C_1 on Y_j satisfying $D_j(0) = D_j(1) = p_j$,
(2) $D_j|_{(0,1)}$ is a geodesic of $R_2 \backslash \{p_j\}$, and
(3) D_j intersects L_0 at right angle.

Let D_j' denote a connected component of $D_j \cap \mathrm{Interior}(\mathscr{C}_{R_2}(C_0))$ containing p_j. Then, for each j, we have $D_j' \cap f(C') = \emptyset$ as follows: Suppose that $D_j' \cap f(C')$ is not empty. Take a point x of $D_j' \cap f(C')$, and let α be the unique geodesic loop of $R_2 \backslash \{p_j\}$ homotopic rel x to $f(C')$ on $R_2 \backslash \{p_j\}$. Then α is included in $\dot{Y}_j$. Indeed, by Baer-Zieschang theorem (A.3 of Buser [1]), there exists a self-homeomorphism w of $R_2 \backslash \{p_j\}$ isotopic to the identity and there exists an isotopy $h_w : (R_2 \backslash \{p_j\}) \times I \to R_2 \backslash \{p_j\}$ such that $h_w(\cdot, 0) = \mathrm{id}$, $h_w(\cdot, 1) = w(\cdot)$, and $w(\mathrm{Interior}(\mathscr{C}_{R_2}(C_0)) \backslash \{p_j\}) = \dot{Y}_j$. The set $D_j' \backslash \{p_j\}$ consists of two components at most. Take a point $y \in D_j' \backslash \{p_j\}$ so that

(1) y is in a component of $D_j' \backslash \{p_j\}$ containing x, and
(2) for each $t \in I$, define a curve δ_t by $\delta_t(s) = h_w(y, st)$ $(s \in I)$, then δ_t is included in $\dot{Y}_j$.

Let ϵ be a curve from x to y with $\epsilon \subset D_j'$. We set

$$\zeta_t = \epsilon \delta_t h_w(\epsilon^{-1} f(C') \epsilon, t) \delta_t^{-1} \epsilon^{-1}, \quad t \in I.$$

Then $\zeta_0 = \epsilon\epsilon^{-1} f(C') \epsilon\epsilon^{-1}$ is homotopic rel x to $\zeta_1 = \epsilon \delta_1 w(\epsilon^{-1} f(C')\epsilon) \delta_1^{-1} \epsilon^{-1}$ by the homotopy ζ_t $(t \in I)$. The loop ζ_0 is homotopic rel x to α, and the loop ζ_1 is included in $\dot{Y}_j$. This implies that α is included in $\dot{Y}_j$. On the other hand, (3.10) yields

$$\begin{aligned} l_{R_2 \backslash \{p_j\}}(\alpha) &\le l_{R_2 \backslash \{p_j\}}(f(C')) \\ &\le l_{\dot{R}_2}(f(C')) \\ &< 2 \operatorname{arcsinh} N_{R_2}(f(C')) = 2 \operatorname{arcsinh} N_{R_2}(\alpha). \end{aligned}$$

This contradicts the assertion of Lemma 5. Therefore we obtain $D_j' \cap f(C') = \emptyset$.

Since each component of $\mathscr{C}_{R_2}(C_0) \backslash (D_1' \cup \cdots \cup D_{n_3}')$ is topologically a disk or an annulus, and is included in $\dot{R}_2$, the closed geodesic $f(C')$ is the r_0-fold iterate $(C_0')^{r_0}$ of some simple closed geodesic C_0' of $\dot{R}_2$. Hence, by the Riemann-Hurwitz relation we obtain $r_0 \le d_f \le (g_1 - 1)/(g_2 - 1)$. Theorem is now proved. □

4. Example

In this section, we shall give an example which asserts that there is no upper bound for $N_{R_2}(f(C))$ depending only on g_1 and g_2.

Let R_2 be a Riemann surface of genus 2. Fix four distinct points $p_1, q_1, p_2, q_2 \in R_2$ and two disjoint simple arcs α_j from p_j to q_j $(j = 1, 2)$. We cut R_2 along the arcs α_1, α_2. Each cut α_j has two edges, labeled α_j^+ edge and α_j^- edge. We take two replicas of R_2 with cuts, and call them sheet I and sheet II. To construct a Riemann surface R_1, we attach the α_j^+ edge on sheet I and the α_j^- edge on sheet II, and then attach the α_j^+ edge on sheet II and the α_j^- edge on sheet I for each $j = 1, 2$. Then we obtain a compact Riemann surface R_1 of genus 5 and two-sheeted branched covering $f : R_1 \to R_2$ which is branched over p_1, q_1, p_2, q_2 with branch order two (see Figure 6).

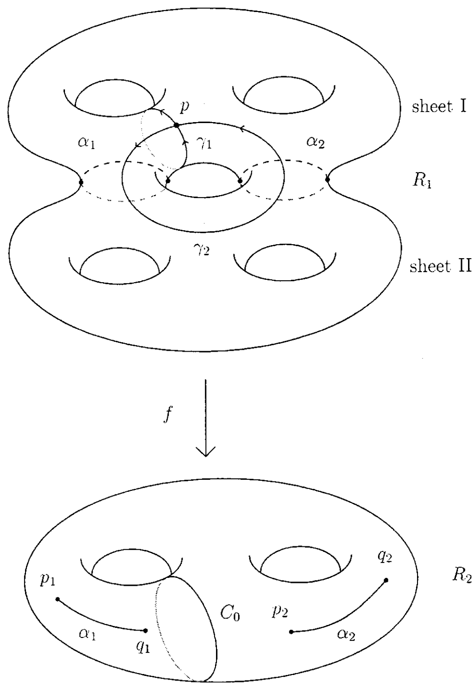

FIGURE 6. a figure of $f : R_1 \to R_2$

We take two simple closed curves γ_1 and γ_2 on R_1 with base point $p \in R_1$, as illustrated in Figure 6. For an arbitrary positive integer r, let C_r be the simple closed geodesic freely homotopic to $\gamma_1^r \gamma_2$ on R_1, where γ_1^r is the r-fold iterate of γ_1. Then the image curve $f(C_r)$ is freely homotopic to the r-fold iterate of the simple closed curve $C_0 = f(\gamma_1)$ on R_2, and we have $N_{R_2}(f(C_r)) = r$. This example implies that there is no upper bound for $N_{R_2}(f(C))$ depending only on g_1, g_2 and f.

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