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Superharmonic functions of Schrödinger operators and Hardy inequalities

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Abstract. Given a Dirichlet form with generator $\mathcal{L}$ and a measure μ, we consider superharmonic functions of the Schrödinger operator $\mathcal{L}+\mu$. We probabilistically prove that the existence of superharmonic functions gives rise to the Hardy inequality. More precisely, the L^2-Hardy inequality is derived from Itô's formula applied to the superharmonic function.

1. Introduction.

Let $\mathbb{M}=(X_t,\mathbb{P}_x)$ be an m-symmetric Hunt process on a locally compact separable metric space E. Here m is a positive Radon measure with full topological support. $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ denotes the Dirichlet form on $L^2(E;m)$ generated by $\mathbb{M}$.

Let μ be a positive smooth measure and $\mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ the set of functions locally in $\mathcal{D}(\mathcal{E})$ in the ordinary sense. A function $h\in\mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ is said to be *superharmonic* with respect to the Schrödinger operator $\mathcal{L}^\mu:=\mathcal{L}+\mu$ if

$$\mathcal{E}(h,\varphi)-\int_E h\varphi\,d\mu\geq 0\quad\text{for all }\ \varphi\in\mathcal{D}(\mathcal{E})\cap C_0(E)\ \text{ with }\ \varphi\geq 0.$$

Here $\mathcal{L}$ is the generator of the process $\mathbb{M}$ and $C_0(E)$ is the set of continuous functions with compact support. We remark that $\mathcal{E}(h,\varphi)$ is not well-defined for $h\in\mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ and $\varphi\in\mathcal{D}(\mathcal{E})\cap C_0(E)$ in general if $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ has a jumping part. For this reason, we assume that every superharmonic function belongs to the subclass $\mathcal{D}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$ of $\mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ (see Section 2 for the definition). The class $\mathcal{D}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$ was introduced by Kuwae [**16**] and satisfies the following property: for any $u\in\mathcal{D}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$ and $\varphi\in\mathcal{D}(\mathcal{E})\cap C_0(E)$, $\mathcal{E}(u,\varphi)$ is well-defined by

$$\mathcal{E}(u,\varphi)=\mathcal{E}^{(c)}(u,\varphi)+\int_{E\times E}(u(x)-u(y))(\varphi(x)-\varphi(y))J(dx,dy)+\int_E u\varphi\,d\kappa$$

(the definitions of $\mathcal{E}^{(c)}$, J and κ are found in Section 2).

It is known that superharmonic functions play an important role in the study of (L^2-)Hardy's inequality:

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$$\int_E u^2 d\mu \le \mathcal{E}(u,u), \quad u \in \mathcal{D}(\mathcal{E})$$

(see [**4**] and [**9**] for example). One of objectives is to show that if there exists a superharmonic function h of $\mathcal{L}^\mu$, then the following equality holds true

$$\mathcal{E}(u,u) - \int_E u^2 d\mu = \mathcal{E}^h\left(\frac{u}{h}, \frac{u}{h}\right) + \int_E \frac{u^2}{h} d\nu, \quad u \in \mathcal{D}(\mathcal{E}). \tag{1}$$

Note that the equality (1) is a refinement of L^2-Hardy's inequality because the right-hand side is nonnegative. Here $\mathcal{E}^h$ is the Dirichlet form generated by the Girsanov transformed process defined by h (see Section 4 for details) and ν is a positive smooth measure satisfying the relation

$$\mathcal{E}(h,\varphi) - \int_E h\varphi\, d\mu = \int_E \varphi\, d\nu, \quad \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(E).$$

Our proof is obtained by applying Itô's formula to Fukushima's decompositions of superharmonic functions. Kuwae [**16**] and [**17**] proves that every $u \in \mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E})$ admits Fukushima's decomposition: $u(X_t) - u(X_0)$ is decomposed into a martingale additive functional locally of finite energy and a continuous additive functional locally of zero energy. It is known that the 0-energy part in Fukushima's decomposition is not always of bounded variation, in particular, Itô's formula is not always applicable. From [**13**, Chapter 5], we know sufficient conditions for the 0-energy part of a function in $\mathcal{D}(\mathcal{E})$ being of locally bounded variation. We extend those conditions to the class $\mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E})$ (Theorem 3.2, Corollary 3.3) and show that the 0-energy part of superharmonic function in $\mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E})$ is of locally bounded variation (Lemma 4.1). By combining this result with Itô's formula, we prove that the equality (1) holds whenever there exists a positive continuous superharmonic function in $\mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E})$.

We consider the Dirichlet form $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^\alpha))$ associated with the symmetric α-stable process on $\mathbb{R}^d$. Assume $0 < \alpha < 2 \wedge d$, that is, $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^\alpha))$ is transient. We show that $|x|^{-p}$, $p \in (0, (d/2) \wedge (d-\alpha))$ is a superharmonic function of $-1/2(-\Delta)^{\alpha/2} + C_{d,\alpha,p} \cdot |x|^{-\alpha}$, and derive the following equality as an application of (1):

$$\begin{aligned} &\mathcal{E}^{(\alpha)}(u,u) - C_{d,\alpha,p} \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^\alpha} dx \\ &\quad = \frac{1}{2}\mathcal{A}(d,\alpha) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{u(x)}{|x|^{-p}} - \frac{u(y)}{|y|^{-p}}\right)^2 \frac{|x|^{-p}|y|^{-p}}{|x-y|^{d+\alpha}} dxdy, \quad u \in \mathcal{D}(\mathcal{E}^{(\alpha)}) \end{aligned} \tag{2}$$

(the definitions of constants $C_{d,\alpha,p}$, $\mathcal{A}(d,\alpha)$ are found in Section 6). The representation (2) has been already proved by Bogdan, Dyda and Kim [**5**] (see also [**2**], [**12**]). We would like to emphasize that although the proof in [**5**] is analytic, our proof is probabilistic, that is, L^2-Hardy's inequality follows from Itô's formula.

We can characterize superharmonic functions by using excessive functions. Let μ be a positive measure in the local Kato class and $\{p^\mu_t\}_{t\ge 0}$ the Feynman–Kac semigroup defined by

$$p_t^\mu f(x) = \mathbb{E}_x[\exp(A_t^\mu)f(X_t)],$$

where $\{A_t^\mu\}_{t\geq 0}$ is a positive continuous additive functional with Revuz measure μ. Takeda [**20**] shows that under the local property assumption, a strictly positive function h in $\mathcal{D}_{\rm loc}(\mathcal{E}) \cap C(E)$ is superharmonic with respect to $\mathcal{L}^\mu$ if and only if h is p_t^μ*-excessive*, that is, $p_t^\mu h \leq h$. We extend this result to more general Dirichlet forms with non-local part (Theorem 5.1).

2. Preliminaries on Dirichlet forms.

Let E be a locally compact separable metric space and m a positive Radon measure with full topological support on E. Denote by $E_\Delta := E \cup \{\Delta\}$ the one point compactification of E. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form on $L^2(E;m)$. We denote $\mathcal{D}_e(\mathcal{E})$ by the family of m-measurable functions u on E such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of $\mathcal{D}(\mathcal{E})$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. We call $\mathcal{D}_e(\mathcal{E})$ the *extended Dirichlet space* of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Let $\mathbb{M} = (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in E}, \{X_t\}_{t\geq 0}, \zeta)$ be the symmetric Hunt process generated by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, where $\{\mathscr{F}_t\}_{t\geq 0}$ is the augmented filtration and $\zeta := \inf\{t \geq 0 \mid X_t = \Delta\}$ is the lifetime of $\mathbb{M}$. Denote by $\{p_t\}_{t\geq 0}$ and $\{R_\beta\}_{\beta\geq 0}$ the semigroup and resolvent of $\mathbb{M}$:

$$p_t f(x) = \mathbb{E}_x[f(X_t)], \quad R_\beta f(x) = \int_0^\infty e^{-\beta t} p_t f(x) dt, \quad f \in \mathfrak{B}_b(E),$$

where $\mathfrak{B}_b(E)$ is the space of bounded Borel functions on E.

For a closed subset F of E, we define

$$\mathcal{D}(\mathcal{E})_F := \{u \in \mathcal{D}(\mathcal{E}) \mid u = 0 \ m\text{-a.e. on } E \setminus F\}.$$

An increasing sequence $\{F_n\}_{n\geq 1}$ of closed sets of E is said to be an $\mathcal{E}$*-nest* if $\bigcup_{n\geq 1} \mathcal{D}(\mathcal{E})_{F_n}$ is dense in $\mathcal{D}(\mathcal{E})$ with respect to the norm $\sqrt{\mathcal{E}_1}$ $\left(:= \sqrt{\mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)_m}\right)$, where $(\cdot,\cdot)_m$ denotes the inner product on $L^2(E;m)$.

A subset N of E is said to be $\mathcal{E}$*-exceptional* if there is an $\mathcal{E}$-nest $\{F_n\}_{n\geq 1}$ such that $N \subset \bigcap_{n\geq 1}(E \setminus F_n)$. A statement depending on $x \in E$ is said to hold *quasi-everywhere* (q.e. in abbreviation) on E if there exists an $\mathcal{E}$-exceptional set N such that the statement is true for every $x \in E \setminus N$. A function u is said to be *quasi-continuous* if there exists an $\mathcal{E}$-nest $\{F_n\}_{n\geq 1}$ such that $u|_{F_n}$ is finite and continuous on F_n for each n. Here $u|_{F_n}$ is the restriction of u to F_n. Each function $u \in \mathcal{D}_e(\mathcal{E})$ admits a quasi-continuous m-version $\widetilde{u}$, that is $u = \widetilde{u}$ m-a.e. In the sequel, we always take a quasi-continuous m-version for every element of $\mathcal{D}_e(\mathcal{E})$.

A positive Borel measure ν on E is said to be *smooth* if it satisfies the following two conditions:

(i) ν charges no $\mathcal{E}$-exceptional set,

(ii) there exists an $\mathcal{E}$-nest $\{F_n\}_{n\geq 1}$ such that $\nu(F_n) < \infty$ for each n.

A function u is said to be *locally in* $\mathcal{D}(\mathcal{E})$ *in the ordinary sense* ($u \in \mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ in notation) if for any relatively compact open set G, there exists a function $v \in \mathcal{D}(\mathcal{E})$ such that $u = v$ m-a.e. on G.

We define the family Θ of finely open sets by

$$\Theta = \bigl\{\{G_n\}_{n\geq 1} \bigm| G_n \text{ is finely open and Borel for all } n, G_n \subset G_{n+1},\ \textstyle\bigcup_{n=1}^{\infty} G_n = E \text{ q.e.}\bigr\}.$$

(The definition of a finely open set is found in [**13**].) For two subsets A, B of E, $A = B$ q.e. means $A\Delta B := (A \setminus B) \cup (B \setminus A)$ is $\mathcal{E}$-exceptional. Note that for an $\mathcal{E}$-nest $\{F_n\}$ of closed sets, $\{G_n\} \in \Theta$ by setting $G_n := F_n^{f\text{-int}}$, where $F_n^{f\text{-int}}$ means the fine interior of F_n. A function u on E is said to be *locally in* $\mathcal{D}(\mathcal{E})$ *in the broad sense* ($u \in \dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E})$ in notation) if there exists $\{G_n\} \in \Theta$ and $\{u_n\} \subset \mathcal{D}(\mathcal{E})$ such that $u = u_n$ m-a.e. on G_n for each $n \in \mathbb{N}$. Clearly, $\mathcal{D}_{\mathrm{loc}}(\mathcal{E}) \subset \dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E})$.

For $u, v \in \mathcal{D}_e(\mathcal{E})$, the following Beurling–Deny formula holds:

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \int_{E\times E} (u(x)-u(y))(v(x)-v(y))J(dx,dy) + \int_E uv\, d\kappa \tag{3}$$

([**13**, Theorem 4.5.2]). Here J is a symmetric Radon measure on $E \times E$ and κ is a Radon measure on E. $\mathcal{E}^{(c)}$ is a symmetric form possessing the strong local property, i.e., $\mathcal{E}^{(c)}(u,v) = 0$ whenever u has a compact support and v is constant on a neighborhood of supp$[u]$. Moreover, we see by [**13**, Lemma 3.2.3] that for $u, v \in \mathcal{D}_e(\mathcal{E})$, there exists a signed measure $\mu^c_{\langle u,v\rangle}$ such that $\mathcal{E}^{(c)}(u,v) = 2^{-1}\mu^c_{\langle u,v\rangle}(E)$. Set $\mu^c_{\langle u\rangle} := \mu^c_{\langle u,u\rangle}$. We can extend $\mu^c_{\langle u,v\rangle}$ to $u, v \in \dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E})$.

Lemma 2.1. *For any* $\{G_n\} \in \Theta$*, there exists an* $\mathcal{E}$*-nest* $\{F_n\}$ *such that* $F_n \subset G_n$ *q.e. and* $J(F_n \times (E \setminus G_n)) < \infty$ *for each* n.

Proof. The proof is based on an idea in the proof of [**16**, Lemma 2.2]. Take $g \in L^2(E;m)$ with $0 < g \leq 1$ on E and define

$$R_1^{G_n} g(x) := \mathbb{E}_x\left[\int_0^{\tau_{G_n}} e^{-s} g(X_s)ds\right],$$

where τ_{G_n} is the first exit time from the set G_n. Then $R_1^{G_n}g(x) > 0$ on G_n and $R_1^{G_n}g$ is quasi-continuous for each n. Take a common $\mathcal{E}$-nest $\{K_j\}$ such that all $R_1^{G_n}g$, $n \geq 1$ are continuous on each K_j. Set $F_n := \{x \in K_n \mid R_1^{G_n}g(x) \geq 1/n\}$. Then since $B_n := \{R_1^{G_n}g > 1/n\}$ is increasing and $E \setminus \bigcup_{n\geq 1} B_n$ is $\mathcal{E}$-exceptional, $\{F_n\}$ is an $\mathcal{E}$-nest by [**15**, Lemma 3.3]. For each n, $(E \setminus G_n)^r \subset E \setminus F_n$, where $(E \setminus G_n)^r = \{x \in E \mid R_1^{G_n}g(x) = 0\}$ is the set of regular points for $E \setminus G_n$. Hence,

$$F_n \setminus G_n \subset F_n \cap \bigl((E \setminus G_n) \setminus (E \setminus G_n)^r\bigr).$$

Since $((E \setminus G_n) \setminus (E \setminus G_n)^r)$ is $\mathcal{E}$-exceptional, we see $F_n \subset G_n$ q.e. Moreover, since $R_1^{G_n}g \geq 1/n$ on F_n and $R_1^{G_n}g = 0$ q.e. on $E \setminus G_n$, it holds that

$$J\big(F_n \times (E \setminus G_n)\big) \le n^2 \int_{F_n \times (E\setminus G_n)} \big(R_1^{G_n} g(x) - R_1^{G_n} g(y)\big)^2 J(dx, dy).$$

The right-hand side is finite because $R_1^{G_n} g$ is an element of $\mathcal{D}(\mathcal{E})$. Hence, $\{F_n\}$ is a desired one. □

For $u \in \dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E})$, we define a Borel measure $\mu^j_{\langle u\rangle}$ on E by

$$\mu^j_{\langle u\rangle}(B) := \int_{B\times E} (u(x) - u(y))^2 J(dx, dy).$$

We introduce subclasses $\mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E})$ of $\mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ and $\dot{\mathcal{D}}^\dagger_{\mathrm{loc}}(\mathcal{E})$ of $\dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E})$ defined by

$$\mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E}) := \{u \in \mathcal{D}_{\mathrm{loc}}(\mathcal{E}) \mid \mu^j_{\langle u\rangle} \text{ is a Radon measure on } E\},$$
$$\dot{\mathcal{D}}^\dagger_{\mathrm{loc}}(\mathcal{E}) := \{u \in \dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E}) \mid \mu^j_{\langle u\rangle} \text{ is a smooth measure on } E\}.$$

Clearly, $\mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E}) \subset \dot{\mathcal{D}}^\dagger_{\mathrm{loc}}(\mathcal{E})$. It is noted in [**16**] that $\mathcal{D}(\mathcal{E}) \cup l(\mathcal{D}_{\mathrm{loc}}(\mathcal{E}))_b \subset \mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E})$ and $\mathcal{D}_e(\mathcal{E}) \cup (\dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E}))_b \subset \dot{\mathcal{D}}^\dagger_{\mathrm{loc}}(\mathcal{E})$. Here $(\mathcal{D}_{\mathrm{loc}}(\mathcal{E}))_b$ (resp. $(\dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E}))_b$) is the set of bounded functions in $\mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ (resp. $\dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E})$). For any $v \in \mathcal{D}(\mathcal{E})$ with compact support and $u \in \mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E})$, the value of $\mathcal{E}(u, v)$ defined by (3) is finite ([**11**, Theorem 3.5]).

3. Continuous additive functionals locally of zero energy.

A stochastic process $\{A_t\}_{t\ge 0}$ is said to be an *additive functional* (AF in abbreviation) if it satisfies the following conditions:

(i) $A_t(\cdot)$ is $\mathscr{F}_t$-measurable for all $t \ge 0$,

(ii) there exists a set $\Lambda \in \mathscr{F}_\infty = \sigma(\bigcup_{t\ge 0} \mathscr{F}_t)$ such that $\mathbb{P}_x(\Lambda) = 1$ for q.e. $x \in E$, $\theta_t \Lambda \subset \Lambda$ for all $t > 0$, and for each $\omega \in \Lambda$, $A_\cdot(\omega)$ is a function satisfying: $A_0(\omega) = 0$, $A_t(\omega) < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \ge \zeta(\omega)$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \ge 0$.

An AF $\{A_t\}_{t\ge 0}$ is said to be *continuous additive functional* (CAF in abbreviation) if $t \mapsto A_t(\omega)$ is continuous on $[0, \infty[$ for each $\omega \in \Lambda$. A $[0, \infty[$-valued CAF is called a *positive continuous additive functional* (PCAF in abbreviation). The family of all smooth measures and the set of all PCAF's are in one-to-one correspondence (*Revuz correspondence*) as follows: for each smooth measure ν, there exists a unique PCAF $\{A_t\}_{t\ge 0}$ such that for any nonnegative Borel function f and γ-excessive function h, that is, $e^{-\gamma t} p_t h \le h$,

$$\lim_{t\downarrow 0} \frac{1}{t} \mathbb{E}_{hm}\left[\int_0^t f(X_s)\, dA_s\right] = \int_E f(x) h(x) \nu(dx)$$

([**13**, Theorem 5.1.4]). Here $\mathbb{E}_{hm}[\,\cdot\,] = \int_E \mathbb{E}_x[\,\cdot\,] h(x) m(dx)$. For a smooth measure ν, we denote by $\{A^\nu_t\}_{t\ge 0}$ the PCAF corresponding to ν.

We see from [**17**, Theorem 1.2] that for $u \in \dot{\mathcal{D}}^\dagger_{\mathrm{loc}}(\mathcal{E})$, the additive functional $u(X_t) - u(X_0)$ admits the following decomposition (*Fukushima's decomposition*):

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \quad \text{for } t \in [0, \zeta[,$$

where $M_t^{[u]}$ is a martingale additive functional locally of finite energy and $N_t^{[u]}$ is a CAF locally of zero energy (see [**16**] and [**17**] for more details). A CAF $\{A_t\}_{t\geq 0}$ is said to be *of bounded variation* if A_t can be expressed as a difference of two PCAF's:

$$A_t = A_t^{(1)} - A_t^{(2)}, \quad t < \zeta.$$

It is known that the 0-energy part $N_t^{[u]}$ in Fukushima's decomposition is not necessary of bounded variation. For $u \in \mathcal{D}_e(\mathcal{E})$, sufficient conditions for $N_t^{[u]}$ being of bounded variation are given in [**13**, Chapter 5]. Our aim in this section is to extend those results to the class $\dot{\mathcal{D}}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$.

Recall that for a closed subset F of E, $\mathcal{D}(\mathcal{E})_F$ is the space defined by

$$\mathcal{D}(\mathcal{E})_F = \{u \in \mathcal{D}(\mathcal{E}) \mid u = 0 \ m\text{-a.e. on } F^c := E \setminus F\}.$$

$\mathcal{D}_e(\mathcal{E})_F$ and $\mathcal{D}_b(\mathcal{E})_F$ are defined similarly, where $\mathcal{D}_b(\mathcal{E})$ is a set of bounded functions in $\mathcal{D}(\mathcal{E})$. For a function f and a Borel set $B \subset E$, define

$$H_B f(x) := \mathbb{E}_x[f(X_{\sigma_B}); \sigma_B < \infty],$$

where σ_B is the first hitting time of B.

Following the argument in the proof of [**7**, Lemma 6.2.10], we have the next lemma.

LEMMA 3.1. *For any $u \in \dot{\mathcal{D}}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$, there exists an $\mathcal{E}$-nest $\{F_n\}$ such that for each n, F_n satisfies the following three properties*:

(i) $\displaystyle \mu^c_{\langle u\rangle}(F_n) + \int_{F_n\times E}(u(x)-u(y))^2 J(dx,dy) + \int_{F_n} u^2\, d\kappa < \infty,$
in particular, the value of $\mathcal{E}(u,v)$ defined by (3) *is finite for all $v \in \bigcup_{n\geq 1}\mathcal{D}(\mathcal{E})_{F_n}$,*

(ii) $u - H_{F_n^c}u \in \mathcal{D}_e(\mathcal{E})_{F_n}$ *and*

$$\begin{aligned}\mathcal{E}(u - H_{F_n^c}u, u - H_{F_n^c}u) \leq \frac{1}{2}\mu^c_{\langle u\rangle}(F_n) + \int_{F_n\times F_n}(u(x)-u(y))^2 J(dx,dy) \\ + 2\int_{F_n\times F_n^c}(u(x)-u(y))^2 J(dx,dy) + \int_{F_n} u^2\, d\kappa,\end{aligned}$$

(iii) $H_{F_n^c}u \in \dot{\mathcal{D}}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$ *and* $\mathcal{E}(H_{F_n^c}u, v) = 0$ *for any* $v \in \mathcal{D}_b(\mathcal{E})_{F_n}$.

PROOF. Note that $H_{F_n^c}u = u$ q.e. on F_n^c.

First we show that (i)–(iii) are satisfied for any $u \in \mathcal{D}_e(\mathcal{E})$ and closed set F instead of F_n. Clearly, (i) holds. $u - H_{F_n^c}u \in \mathcal{D}_e(\mathcal{E})_{F_n}$ and (iii) follow from [**13**, Theorem 4.6.5]. Since

$$\mathcal{E}(u - H_{F^c}u, u - H_{F^c}u) = \mathcal{E}(u,u) - \mathcal{E}(H_{F^c}u, H_{F^c}u)$$

and

$$\begin{aligned}\mathcal{E}(H_{F^c}u, H_{F^c}u) &\geq \frac{1}{2}\,\mu^c_{\langle H_{F^c}u\rangle}(F^c) + \int_{F^c\times F^c}\big(H_{F^c}u(x) - H_{F^c}u(y)\big)^2 J(dx,dy)\\ &\quad + \int_{F^c}\big(H_{F^c}u\big)^2\,d\kappa\\ &= \frac{1}{2}\,\mu^c_{\langle u\rangle}(F^c) + \int_{F^c\times F^c}(u(x)-u(y))^2 J(dx,dy) + \int_{F^c} u^2\,d\kappa,\end{aligned}$$

we attain (ii).

Suppose $u \in \dot{\mathcal{D}}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$. From the definition of $\dot{\mathcal{D}}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$, there exists an $\mathcal{E}$-nest $\{F_n^{(1)}\}$ such that

$$\int_{F_n^{(1)}\times E}(u(x)-u(y))^2 J(dx,dy) < \infty$$

for every n. By the regularity of $(\mathcal{E},\mathcal{D}(\mathcal{E}))$, we may assume that all $F_n^{(1)}$, $n \geq 1$ are compact. Take sequences $\{G_n\} \in \Theta$ and $\{u_n\} \subset \mathcal{D}(\mathcal{E})$ such that $u = u_n$ q.e. on G_n for each n. From Lemma 2.1, there exists an $\mathcal{E}$-nest $\{F_n^{(2)}\}$ such that $F_n^{(2)} \subset G_n$ q.e. and $J(F_n^{(2)} \times G_n^c) < \infty$ for each n. We define an $\mathcal{E}$-nest $\{F_n\}$ by $F_n := F_n^{(1)} \cap F_n^{(2)}$. Clearly, $\{F_n\}$ satisfies (i).

In the remainder of the proof, we fix $n \geq 1$ and put $F := F_n$, $G := G_n$. For $k > n$ and $M > 0$, we set $u_k^{(M)} := (-M)\vee u_k \wedge M$, $u^{(M)} := (-M)\vee u \wedge M$. We have by applying (ii) to $u_k^{(M)} \in \mathcal{D}(\mathcal{E})$

$$\begin{aligned}&\mathcal{E}\big(u_k^{(M)} - H_{F^c}u_k^{(M)},\, u_k^{(M)} - H_{F^c}u_k^{(M)}\big)\\ &\quad\leq \frac{1}{2}\mu^c_{\langle u_k^{(M)}\rangle}(F) + \int_{F\times F}\big(u_k^{(M)}(x) - u_k^{(M)}(y)\big)^2 J(dx,dy)\\ &\qquad\qquad + 2\int_{F\times F^c}\big(u_k^{(M)}(x) - u_k^{(M)}(y)\big)^2 J(dx,dy) + \int_F \big(u_k^{(M)}\big)^2\,d\kappa.\end{aligned}$$

Noting that $u_k^{(M)} = u^{(M)}$ q.e. on G and $u^{(M)}$ is a normal contraction of u, the right-hand side is dominated by

$$\begin{aligned}&\frac{1}{2}\,\mu^c_{\langle u\rangle}(F) + \int_{F\times F}\big(u(x)-u(y)\big)^2 J(dx,dy)\\ &\quad + 2\int_{F\times(F^c\cap G)}\big(u(x)-u(y)\big)^2 J(dx,dy)\\ &\quad + 2\int_{F\times(F^c\cap G^c)}\big(u^{(M)}(x) - u_k^{(M)}(y)\big)^2 J(dx,dy) + \int_F u^2\,d\kappa.\end{aligned}$$

Since $J(F \times G^c) < \infty$, we have by the bounded convergence theorem

$$
\begin{aligned}
\limsup_{k\to\infty} \mathcal{E}\big(u_k^{(M)} - H_{F^c}u_k^{(M)}, u_k^{(M)} - H_{F^c}u_k^{(M)}\big) \\
\le \frac{1}{2}\,\mu^c_{\langle u\rangle}(F) + \int_{F\times F}(u(x)-u(y))^2 J(dx,dy) \qquad (4)\\
+ 2\int_{F\times F^c}(u(x)-u(y))^2 J(dx,dy) + \int_F u^2\,d\kappa < \infty.
\end{aligned}
$$

We see from the Banach–Saks theorem ([**7**, Theorem A.4.1]) that there exists a subsequence $\{u_{k_j}^{(M)}\}_{j\ge1}$, $k_1 > n$ such that

$$\psi_j := \frac{1}{j}\sum_{\ell=1}^{j}\big(u_{k_\ell}^{(M)} - H_{F^c}u_{k_\ell}^{(M)}\big)$$

is an $\mathcal{E}$-Cauchy sequence. Hence, we see that $\{\psi_j\}$ $\mathcal{E}$-converges to $u^{(M)} - H_{F^c}u^{(M)} \in \mathcal{D}_e(\mathcal{E})_F \cap L^\infty(E;m)$. Since F is compact, the space $\mathcal{D}_e(\mathcal{E})_F \cap L^\infty(E;m)$ is contained in $L^2(E;m)$, and thus it coincides with $\mathcal{D}_b(\mathcal{E})_F$ by [**13**, Theorem 1.5.2]. Moreover,

$$
\begin{aligned}
\mathcal{E}\big(u^{(M)} - H_{F^c}u^{(M)}, u^{(M)} - H_{F^c}u^{(M)}\big) &= \lim_{j\to\infty}\mathcal{E}(\psi_j,\psi_j)\\
&\le \limsup_{k\to\infty}\mathcal{E}\big(u_k^{(M)} - H_{F^c}u_k^{(M)}, u_k^{(M)} - H_{F^c}u_k^{(M)}\big).
\end{aligned}
$$

From the inequality (4), the right-hand side is uniformly bounded in $M > 0$. By using the Banach–Saks theorem again, we can choose an increasing sequence $\{M_j\}_{j\ge1}$ such that

$$\varphi_j := \frac{1}{j}\sum_{\ell=1}^{j}\big(u^{(M_\ell)} - H_{F^c}u^{(M_\ell)}\big)$$

is an $\mathcal{E}$-approximating sequence of $u - H_{F^c}u$, which proves (ii).

Finally, we show (iii). From (ii) and the fact $\mathcal{D}_e(\mathcal{E}) \subset \dot{\mathcal{D}}^\dagger_{\mathrm{loc}}(\mathcal{E})$, we see $H_{F^c}u \in \dot{\mathcal{D}}^\dagger_{\mathrm{loc}}(\mathcal{E})$. For $M > 0$, we take the sequences $\{u_{k_j}^{(M)}\}_{j\ge1}$, $\{\psi_j\}_{j\ge1}$ defined in the last paragraph and put $\overline{u}_j^{(M)} := (1/j)\sum_{\ell=1}^{j} u_{k_\ell}^{(M)}$. Note that $\overline{u}_j^{(M)} = u^{(M)}$ q.e. on G. For $v \in \mathcal{D}_b(\mathcal{E})_F$, the value of $\mathcal{E}(\overline{u}_j^{(M)}, v)$ equals

$$
\begin{aligned}
&\frac{1}{2}\,\mu^c_{\langle u^{(M)},v\rangle}(F) + \int_{F\times F}\big(u^{(M)}(x) - u^{(M)}(y)\big)(v(x)-v(y))J(dx,dy)\\
&\quad + 2\int_{F\times(F^c\cap G)}\big(u^{(M)}(x) - u^{(M)}(y)\big)(v(x)-v(y))J(dx,dy)\\
&\quad + 2\int_{F\times(F^c\cap G^c)}\big(u^{(M)}(x) - \overline{u}_j^{(M)}(y)\big)(v(x)-v(y))J(dx,dy) + \int_F u^{(M)} v\,d\kappa.
\end{aligned}
$$

Hence, $\mathcal{E}(\overline{u}_j^{(M)}, v)$ converges to $\mathcal{E}(u^{(M)}, v)$ as $j\to\infty$ by the bounded convergence theorem, and thus

$$\mathcal{E}\big(H_{F^c}u^{(M)}, v\big) = \lim_{j\to\infty}\Big(\mathcal{E}\big(\overline{u}_j^{(M)}, v\big) - \mathcal{E}(\psi_j, v)\Big)$$
$$= \lim_{j\to\infty}\frac{1}{j}\sum_{\ell=1}^{j}\mathcal{E}\big(H_{F^c}u_{k_\ell}^{(M)}, v\big) = 0.$$

Take the sequences $\{u^{(M_j)}\}_{j\ge 1}$, $\{\varphi_j\}_{j\ge 1}$ defined in the last paragraph and put $\overline{u}_j := (1/j)\sum_{\ell=1}^{j} u^{(M_\ell)}$. Since $\overline{u}_j$ is a normal contraction of u, $\mathcal{E}(\overline{u}_j, v)$ converges to $\mathcal{E}(u, v)$ as $j\to\infty$. Consequently, we have

$$\mathcal{E}(H_{F^c}u, v) = \lim_{j\to\infty}\Big(\mathcal{E}(\overline{u}_j, v) - \mathcal{E}(\varphi_j, v)\Big)$$
$$= \lim_{j\to\infty}\frac{1}{j}\sum_{\ell=1}^{j}\mathcal{E}\big(H_{F^c}u^{(M_\ell)}, v\big) = 0. \qquad \square$$

We can now give a sufficient condition for $u\in\dot{\mathcal{D}}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$ that the 0-energy part $N^{[u]}$ in Fukushima's decomposition is of bounded variation.

THEOREM 3.2. *Let $\nu=\nu^+-\nu^-$ be a difference of positive smooth measures on E. If $u\in\dot{\mathcal{D}}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$ satisfies*

$$\mathcal{E}(u, v) = \int_E v\, d\nu \quad \text{for all } v\in\bigcup_{n=1}^{\infty}\mathcal{D}_b(\mathcal{E})_{F_n} \tag{5}$$

for an $\mathcal{E}$-nest $\{F_n\}$ associated with ν and $\mu^j_{\langle u\rangle}$, then

$$\mathbb{P}_x\big(N_t^{[u]} = -A_t^+ + A_t^-,\ t<\zeta\big) = 1 \quad \text{q.e. } x\in E,$$

where $A_t^\pm$ is a PCAF *with Revuz measure $\nu^\pm$.*

PROOF. Suppose that u satisfies (5) for an $\mathcal{E}$-nest $\{F_n^{(1)}\}$. Take another $\mathcal{E}$-nest $\{F_n^{(2)}\}$ satisfying conditions in Lemma 3.1. Set $F_n := F_n^{(1)}\cap F_n^{(2)}$. By repeating computations in the proof of the previous lemma, we can check that the $\mathcal{E}$-nest $\{F_n\}$ also satisfies the statements in Lemma 3.1. On account of Lemma 3.1 (iii) and [**17**, Theorem 1.2], $H_{F_n^c}u(X_t) - H_{F_n^c}u(X_0)$ has Fukushima's decomposition:

$$H_{F_n^c}u(X_t) - H_{F_n^c}u(X_0) = M_t^{[H_{F_n^c}u]} + N_t^{[H_{F_n^c}u]}, \quad t<\zeta.$$

By an argument similar to that in the proof of [**7**, Lemma 5.5.5], we can show that

$$\mathbb{P}_x\Big(N_t^{[H_{F_n^c}u]} = 0,\ t<\tau_{F_n}\Big) = 1 \quad \text{q.e. } x\in E. \tag{6}$$

Here τ_{F_n} is the first exit time from F_n. Note that $u - H_{F_n^c}u\in\mathcal{D}_e(\mathcal{E})_{F_n}$ and

$$\mathcal{E}(u - H_{F_n^c}u, v) = \int_E v\, d\nu \quad \text{for all } v\in\mathcal{D}_b(\mathcal{E})_{F_n}$$

by Lemma 3.1. We then see from [**13**, Lemma 5.4.4] and (6) that

$$\mathbb{P}_x(N_t^{[u]} = -A_t^+ + A_t^-,\ t < \tau_{F_n}) = 1 \quad \text{q.e. } x \in E.$$

We have the assertion by letting $n \to \infty$. □

By the same argument as that in the proof of [**13**, Corollary 5.4.1], we have the next corollary.

Corollary 3.3. *Let $\nu = \nu^+ - \nu^-$ be a difference of positive smooth Radon measures on E. Suppose $u \in \mathcal{D}_{\mathrm{loc}}^{\dagger}(\mathcal{E})$ satisfies*

$$\mathcal{E}(u, v) = \int_E v\, d\nu \quad \textit{for all } v \in \mathcal{C}$$

for some special standard core $\mathcal{C}$. Then

$$\mathbb{P}_x\left(N_t^{[u]} = -A_t^+ + A_t^-,\ t < \zeta\right) = 1 \quad \text{q.e. } x \in E,$$

where $A_t^{\pm}$ is a PCAF *with Revuz measure $\nu^{\pm}$.*

4. Hardy inequalities.

Let μ be a smooth measure (denote by $\mu \in \mathcal{S}$). In this section, we consider the *Hardy-type inequality*:

$$\int_E u^2\, d\mu \le \mathcal{E}(u, u) \quad \text{for all } \ u \in \mathcal{D}(\mathcal{E}).$$

We shall show that if there exists a function in the space $\widetilde{\mathcal{H}}^+(\mu)$ below, then the inequality above holds.

Define

$$\Theta_0 = \{G \mid G \text{ is open and } E \setminus G \text{ is } \mathcal{E}\text{-exceptional}\}.$$

Take $G \in \Theta_0$ and let $\mathbb{M}^G = (X_t^G, \mathbb{P}_x)$ be the part process on G:

$$X_t^G = \begin{cases} X_t, & t < \tau_G, \\ \Delta, & t \ge \tau_G. \end{cases}$$

Define the Dirichlet form $(\mathcal{E}^G, \mathcal{D}(\mathcal{E}^G))$ on $L^2(G, m)$ by

$$\begin{cases} \mathcal{E}^G = \mathcal{E}, \\ \mathcal{D}(\mathcal{E}^G) = \mathcal{D}(\mathcal{E})_G. \end{cases}$$

Then $(\mathcal{E}^G, \mathcal{D}(\mathcal{E}^G))$ is a regular Dirichlet form generated by $\mathbb{M}^G$ ([**13**, Theorem 4.4.3]). Note that $\mathcal{D}(\mathcal{E}^G) = \mathcal{D}(\mathcal{E})$ because $E \setminus G$ is $\mathcal{E}$-exceptional.

For $\mu \in \mathcal{S}$, we set a function space of superharmonic functions:

$$\widetilde{\mathcal{H}}^+(\mu) := \left\{ h \,\middle|\, \begin{array}{l} \text{there exists } G \in \Theta_0 \text{ such that } h \in \mathcal{D}^\dagger_{\rm loc}(\mathcal{E}^G) \cap C(G \cup \{\Delta\}),\ h > 0 \text{ on } G \\ \text{and } \mathcal{E}^G(h,\varphi) - \int_E h\varphi\, d\mu \geq 0 \text{ for all } \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0^+(G) \end{array} \right\}.$$

Here $C_0^+(G)$ is a set of nonnegative continuous functions on G whose supports are compact and contained in G. Note that $v \in \mathcal{D}^\dagger_{\rm loc}(\mathcal{E}^G)$ implies $v \in \dot{\mathcal{D}}^\dagger_{\rm loc}(\mathcal{E})$ and $\mathcal{E}^G(v,\varphi) = \mathcal{E}(v,\varphi)$ holds for any $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(G)$.

The next lemma tells us that $h(X_t)$ is a semimartingale for any $h \in \widetilde{\mathcal{H}}^+(\mu)$.

Lemma 4.1. *For $h \in \widetilde{\mathcal{H}}^+(\mu)$, there exists a smooth measure ν_h such that*

$$N_t^{[h]} = -\int_0^t h(X_s)dA_s^\mu - A_t^{\nu_h}, \quad t < \zeta,\ \mathbb{P}_x\text{-a.s. q.e. } x \in E,$$

where $N_t^{[h]}$ is the 0-energy part in Fukushima's decomposition of $h(X_t) - h(X_0)$.

Proof. Define a functional I on $\Lambda := \mathcal{D}(\mathcal{E}) \cap C_0(G)$ by

$$I(\varphi) = \mathcal{E}^G(h,\varphi) - \int_E h\varphi\, d\mu, \quad \varphi \in \Lambda.$$

Note that Λ is a *Stone vector lattice*, i.e., $u \wedge v \in \Lambda$, $u \wedge 1 \in \Lambda$ for any $u, v \in \Lambda$. Moreover, I is *pre-integral* on the space Λ, that is, $I(\varphi_k) \downarrow 0$ whenever $\varphi_k \in \Lambda$ and $\varphi_k(x) \downarrow 0$ for all $x \in E$. Indeed, let $\psi \in \mathcal{D}(\mathcal{E}) \cap C_0^+(G)$ such that $\psi = 1$ on $\mathrm{supp}[\varphi_1]$. Then since $\|\varphi_k\|_\infty \psi - \varphi_k \in \mathcal{D}(\mathcal{E}) \cap C_0^+(G)$, it holds that

$$I(\varphi_k) \leq \|\varphi_k\|_\infty \cdot I(\psi) \downarrow 0 \quad \text{as } k \to \infty$$

by Dini's theorem. We see from [**8**, Theorem 4.5.2] that there exists a Borel measure ν on G such that

$$I(\varphi) = \int_G \varphi\, d\nu, \quad \varphi \in \Lambda. \tag{7}$$

We extend ν to a measure on E by setting $\nu(E \setminus G) = 0$.

We shall prove that ν is a smooth measure on E. Let $K \subset G$ be a compact set of zero capacity and take a relatively compact open set D such that $K \subset D \subset G$. On account of [**7**, Theorem 3.3.8(iii)], there exists a sequence $\{\varphi_n\}_{n\geq 1} \subset \mathcal{D}(\mathcal{E}) \cap C_0^+(D)$ such that $\varphi_n \geq 1$ on K and $\mathcal{E}_1(\varphi_n, \varphi_n) \to 0$ as $n \to \infty$. Let $\psi \in \mathcal{D}(\mathcal{E}) \cap C_0(G)$ such that $\psi = 1$ on D and $0 \leq \psi \leq 1$ on E. Then note that $h\psi \in \mathcal{D}(\mathcal{E}) \cap C_0(G)$ and $h\psi = h$ on D. Hence,

$$\begin{aligned}\mathcal{E}(h\psi, \varphi_n) &= \frac{1}{2}\int_E d\mu^c_{\langle h,\varphi_n\rangle} + \int_{D\times D} (h(x)-h(y))(\varphi_n(x)-\varphi_n(y))J(dx,dy) \\ &\quad + 2\int_{D\times(E\setminus D)} (h(x) - h\psi(y))\varphi_n(x)J(dx,dy) + \int_E h\varphi_n\, d\kappa \\ &\geq \mathcal{E}^G(h,\varphi_n).\end{aligned}$$

Therefore,

$$\nu(K) \leq \int_E \varphi_n \, d\nu = \mathcal{E}^G(h, \varphi_n) - \int_E h\varphi_n \, d\mu \leq \mathcal{E}(h\psi, \varphi_n)$$

and the right-hand side is dominated by

$$\mathcal{E}(h\psi, h\psi)^{1/2} \cdot \mathcal{E}(\varphi_n, \varphi_n)^{1/2}.$$

Since $\mathcal{E}(\varphi_n, \varphi_n)^{1/2}$ tends to 0 as $n \to \infty$, the measure ν charges no $\mathcal{E}$-exceptional set. For any compact subset K of G, we can see $\nu(K) < \infty$ as proved above. Let $\{K_j\}$ be an $\mathcal{E}$-nest of compact sets satisfying $E \setminus G \subset \bigcap_{j=1}^{\infty} K_j^c$. Then $\nu(K_j) < \infty$ implies the smoothness of ν.

We see from (7) that

$$\mathcal{E}^G(h, \varphi) = \int_G \varphi \, (h \, d\mu + d\nu) \quad \text{for all } \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(G).$$

Recall that $h \in \mathcal{D}^{\dagger}_{\mathrm{loc}}(\mathcal{E}^G)$ implies $h \in \dot{\mathcal{D}}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$. By applying Theorem 3.2 to $\mathbb{M}$, it holds that

$$N_t^{[h]} = -\int_0^t h(X_s) dA_s^{\mu} - A_t^{\nu}, \quad t < \zeta, \ \mathbb{P}_x\text{-a.s. q.e. } x \in E.$$

We have the assertion by setting $\nu_h := \nu$. □

Lemma 4.2.

$$\int_E u^2 \, d\mu + \int_E \frac{u^2}{h} \, d\nu_h \leq \mathcal{E}(u, u) \quad \textit{for any } u \in \mathcal{D}(\mathcal{E}).$$

Proof. We first show the following claim:

$$\int_E \varphi \, d\mu + \int_E \frac{\varphi}{h} \, d\nu_h = \mathcal{E}\left(h, \frac{\varphi}{h}\right) \quad \text{for any } \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(G). \tag{8}$$

Let $K = \mathrm{supp}[\varphi]$ and D a relatively compact open set satisfying $K \subset D \subset \overline{D} \subset G$. Put $c := 1/(\inf_{x \in D} h(x))$. Then for $(x, y) \in D \times D$

$$\left|\frac{\varphi}{h}(x)\right| \leq c|\varphi(x)|,$$
$$\left|\frac{\varphi}{h}(x) - \frac{\varphi}{h}(y)\right| \leq 2c|\varphi(x) - \varphi(y)| + c^2|h(x)\varphi(x) - h(y)\varphi(y)|.$$

Since $\varphi, h\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(G)$, the function φ/h also belongs to $\mathcal{D}(\mathcal{E}) \cap C_0(G)$. Hence, the claim follows from (7).

Secondary, we shall show

$$\mathcal{E}\left(h, \frac{\varphi^2}{h}\right) \leq \mathcal{E}(\varphi, \varphi) \quad \text{for any } \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(G). \tag{9}$$

Put $\psi = \varphi/h$. By the derivation property, $\mathcal{E}(h, \varphi^2/h)$ is equal to

$$\mathcal{E}(h, h\psi^2) = \frac{1}{2}\int_E \psi^2 d\mu^c_{\langle h\rangle} + \int_E h\psi\, d\mu^c_{\langle h,\psi\rangle} + \mathcal{E}^{(j)}(h, h\psi^2) + \int_E (h\psi)^2 d\kappa,$$

where

$$\mathcal{E}^{(j)}(f, g) := \int_{E\times E} (f(x) - f(y))(g(x) - g(y))J(dx, dy).$$

On the other hand, $\mathcal{E}(\varphi, \varphi)$ equals

$$\mathcal{E}(h\psi, h\psi) = \frac{1}{2}\int_E \psi^2 d\mu^c_{\langle h\rangle} + \int_E h\psi\, d\mu^c_{\langle h,\psi\rangle} + \frac{1}{2}\int_E h^2 d\mu^c_{\langle \psi\rangle} + \mathcal{E}^{(j)}(h\psi, h\psi) + \int_E (h\psi)^2 d\kappa.$$

Since

$$\mathcal{E}^{(j)}(h\psi, h\psi) - \mathcal{E}^{(j)}(h, h\psi^2) = \int_{E\times E} (\psi(x) - \psi(y))^2 h(x)h(y)J(dx, dy),$$

we have

$$\mathcal{E}(h\psi, h\psi) - \mathcal{E}(h, h\psi^2) = \frac{1}{2}\int_E h^2 d\mu^c_{\langle\psi\rangle} + \int_{E\times E} (\psi(x) - \psi(y))^2 h(x)h(y)J(dx, dy).$$

Obviously, the right-hand side is nonnegative, and thus (9) holds.

Remark that $\mathcal{D}(\mathcal{E})\cap C_0(G)$ is $\mathcal{E}_1^{1/2}$-dense in $\mathcal{D}(\mathcal{E}^G) = \mathcal{D}(\mathcal{E})$. For any $u \in \mathcal{D}(\mathcal{E})$, there exists $\{u_n\} \subset \mathcal{D}(\mathcal{E}) \cap C_0(G)$ such that, $u_n \to u$ q.e. and $\mathcal{E}(u_n, u_n) \to \mathcal{E}(u, u)$ as $n \to \infty$ ([**13**, Theorem 2.1.4]). By Fatou's lemma and (8), we have

$$\int_E u^2\, d\mu + \int_E \frac{u^2}{h}\, d\nu_h \le \liminf_{n\to\infty} \mathcal{E}\left(h, \frac{u_n^2}{h}\right).$$

On account of (9), the right-hand side is dominated by

$$\liminf_{n\to\infty} \mathcal{E}(u_n, u_n) = \mathcal{E}(u, u). \qquad \square$$

Suppose $\widetilde{\mathcal{H}}^+(\mu) \neq \emptyset$ and take $h \in \widetilde{\mathcal{H}}^+(\mu)$. Define a local martingale on the random interval $[\![0, \zeta^h[\![$ by $M_t = \int_0^t (h(X_{s-}))^{-1} dM_s^{[h]}$, where

$$\zeta^h := \zeta \wedge \sigma_h, \quad \sigma_h := \inf\bigl\{t > 0 \,|\, X_t \in \{h = 0 \text{ or } h = \infty\}\bigr\}$$

and $M_t^{[h]}$ is the martingale part in Fukushima's decomposition of $h(X_t) - h(X_0)$. Let L_t^h be the solution to the following stochastic differential equation:

$$L_t^h = 1 + \int_0^t L_{s-}^h\, dM_s, \quad t < \zeta^h.$$

It is known from the Doláns-Dade formula ([**14**, Theorem 9.39]) that

$$L_t^h = \exp\left(M_t - \frac{1}{2}\langle M^c\rangle_t\right) \prod_{0<s\le t} \frac{h(X_s)}{h(X_{s-})} \exp\left(1 - \frac{h(X_s)}{h(X_{s-})}\right).$$

Since L_t^h is a positive local martingale on the random interval $[\![0, \zeta^h[\![$, so is a positive supermartingale. Define a family of probability measures on $(\Omega, \mathscr{F})$ by

$$d\mathbb{P}_x^h := L_t^h d\mathbb{P}_x \quad \text{on } \mathscr{F}_t \cap \{t < \zeta^h\}.$$

It follows from [**19**, (62.19)] that under new measures $\{\mathbb{P}_x^h\}$, $\{X_t\}_{t\ge 0}$ is a right Markov process on $\{0 < h < \infty\}$. It is known that $\mathbb{M}^h := (\Omega, \mathscr{F}_t, X_t, \mathbb{P}_x^h, \zeta^h)$ is an h^2m-symmetric process (cf. [**6**], [**18**]). Let $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ be the Dirichlet form generated by $\mathbb{M}^h$.

On account of Lemma 4.1, we have the decomposition

$$h(X_t) - h(X_0) = M_t^{[h]} - \int_0^t h(X_s) dA_s^\mu - A_t^{\nu_h}, \quad t < \zeta,\ \mathbb{P}_x\text{-a.s. q.e. } x \in E.$$

By Itô's formula applied to the semimartingale $h(X_t)$ with the function $\log x$, we have

$$\begin{aligned} L_t^h &= \frac{h(X_t)}{h(X_0)} \exp\left(-\int_0^t \frac{1}{h(X_{s-})}\, dN_s^{[h]}\right) \\ &= \frac{h(X_t)}{h(X_0)} \exp\left(A_t^\xi\right), \quad t < \zeta,\ \mathbb{P}_x\text{-a.s. q.e. } x \in E, \end{aligned} \tag{10}$$

where $\xi(dx) := \mu(dx) + (1/h(x))\nu_h(dx)$. Hence, the transition semigroup p_t^h of $\mathbb{M}^h$ is expressed by

$$\begin{aligned} p_t^h f(x) &= \mathbb{E}_x\left[L_t^h f(X_t)\,; t < \zeta^h\right] \\ &= \frac{1}{h(x)}\, \mathbb{E}_x\left[\exp(A_t^\xi) h(X_t) f(X_t)\,; t < \zeta\right] \end{aligned} \tag{11}$$

for q.e. $x \in E$. By using these expressions, we will prove the following equality. This gives a refinement of Hardy's inequality.

THEOREM 4.3. *Suppose $\widetilde{\mathcal{H}}^+(\mu) \ne \emptyset$. Then for any $h \in \widetilde{\mathcal{H}}^+(\mu)$,*

$$\mathcal{E}(u,u) - \int_E u^2 d\mu = \mathcal{E}^h\left(\frac{u}{h}, \frac{u}{h}\right) + \int_E \frac{u^2}{h}\, d\nu_h, \quad u \in \mathcal{D}(\mathcal{E}).$$

In addition, the value of $\mathcal{E}^h(u/h, u/h)$ is equal to

$$\frac{1}{2}\int_E h^2 d\mu^c_{\langle u/h\rangle} + \int_{E\times E} \left(\frac{u}{h}(x) - \frac{u}{h}(y)\right)^2 h(x)h(y)J(dx,dy) + h(\Delta)\int_E \frac{u^2}{h}\, d\kappa. \tag{12}$$

PROOF. Let $\xi(dx) = \mu(dx) + (1/h(x))\nu_h(dx)$ and

$$\mathcal{E}^\delta(u,u) := \mathcal{E}(u,u) + \delta \int_E u^2\, d\xi, \quad \delta > 0.$$

Then it follows from Lemma 4.2 that

$$\int_E u^2\,d\xi \le \frac{1}{1+\delta}\,\mathcal{E}^\delta(u,u), \quad u\in\mathcal{D}(\mathcal{E}),$$

and thus ξ belongs to the Hardy class associated with $\mathcal{E}^\delta$. Define the subprocess $\mathbb{P}_x^\delta$ by $\mathbb{P}_x^\delta = \exp(-\delta A_t^\xi)\mathbb{P}_x$. On account of the relation (10),

$$\mathbb{E}_x^\delta\left[e^{A_t^\xi} f(X_t)\right] = h(x)\,\mathbb{E}_x^h\left[e^{-\delta A_t^\xi}\left(\frac{f}{h}(X_t)\right)\right].$$

We see from [**10**] that for $u\in\mathcal{D}(\mathcal{E})$,

$$\lim_{t\downarrow 0}\frac{1}{t}\left(u-\mathbb{E}_\cdot^\delta\left[e^{A_t^\xi}u(X_t)\right], u\right)_m = \mathcal{E}^\delta(u,u) - \int_E u^2\,d\xi.$$

On the other hand, we see from [**18**] that

$$\begin{aligned}\lim_{t\downarrow 0}\frac{1}{t}\left(u-h\,\mathbb{E}_\cdot^h\left[e^{-\delta A_t^\xi}\left(\frac{u}{h}(X_t)\right)\right], u\right)_m &= \lim_{t\downarrow 0}\frac{1}{t}\left(\frac{u}{h}-\mathbb{E}_\cdot^h\left[e^{-\delta A_t^\xi}\left(\frac{u}{h}(X_t)\right)\right], \frac{u}{h}\right)_{h^2 m}\\ &= \mathcal{E}^h\left(\frac{u}{h},\frac{u}{h}\right) + \delta\int_E\left(\frac{u}{h}\right)^2 h^2\,d\xi.\end{aligned}$$

Moreover, it is noted in [**18**] that $\mathcal{E}^h(u/h,u/h)$ equals (12). □

Assume $\mathbb{M}$ is transient. For $\mu\in\mathcal{S}$, we define its potential by $R\mu(x)=\mathbb{E}_x\left[A_\zeta^\mu\right]$. We introduce

$$\mathcal{S}^\dagger := \left\{\mu\in\mathcal{S}\;\middle|\;\begin{array}{l}\text{there exists } G\in\Theta_0 \text{ such that } \mu \text{ is a Radon measure on } G,\\ R\mu>0 \text{ on } G \text{ and } R\mu\in\mathcal{D}_{\mathrm{loc}}^\dagger(\mathcal{E}^G)\cap C(G\cup\{\Delta\})\end{array}\right\}.$$

For $\mu\in\mathcal{S}^\dagger$, the potential $R\mu$ satisfies

$$\mathcal{E}^G(R\mu,\varphi) - \int_E\varphi\,d\mu = 0 \quad \text{for all } \varphi\in\mathcal{D}(\mathcal{E})\cap C_0(G).$$

Since $\int_E\varphi\,d\mu = \int_E R\mu\cdot\varphi\,(1/R\mu)\,d\mu$, we see that $R\mu$ is in the space $\widetilde{\mathcal{H}}^+((1/R\mu)\cdot\mu)$. By applying the previous theorem, we get

COROLLARY 4.4. *Let $\mu\in\mathcal{S}^\dagger$. Then*

$$\mathcal{E}(u,u) - \int_E\frac{u^2}{R\mu}\,d\mu = \mathcal{E}^{R\mu}\left(\frac{u}{R\mu},\frac{u}{R\mu}\right), \quad u\in\mathcal{D}(\mathcal{E}).$$

5. p_t^μ-excessive functions.

We introduce some subclasses of smooth measures $\mathcal{S}$. A positive measure ν in $\mathcal{S}$ is said to be in the *Kato class* ($\mathcal{K}$ in abbreviation) if

$$\lim_{\beta\to\infty}\left\|\mathbb{E}_\cdot\left[\int_0^\infty e^{-\beta t}dA_t^\nu\right]\right\|_\infty = 0.$$

A positive measure ν in $\mathcal{S}$ is said to be in the *local Kato class* ($\mathcal{K}_{\mathrm{loc}}$ in abbreviation) if $\nu(\cdot \cap K) \in \mathcal{K}$ for any compact set K.

Let $\mu \in \mathcal{K}_{\mathrm{loc}}$ and define the Feynman–Kac semigroup $\{p_t^\mu\}_{t \geq 0}$ by

$$p_t^\mu f(x) = \mathbb{E}_x[\exp(A_t^\mu) f(X_t)].$$

Let us introduce the function space of p_t^μ-excessive functions.

$$\mathcal{H}^+(\mu) := \left\{ h \,\middle|\, \begin{array}{l} \text{there exists } G \in \Theta_0 \text{ such that } h \in \mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E}^G) \cap C(G \cup \{\Delta\}), \\ h > 0 \text{ on } G \text{ and } p_t^\mu h \leq h \ m\text{-a.e.} \end{array} \right\}.$$

The next theorem gives a characterization of p_t^μ-excessive functions in $\mathcal{H}^+(\mu)$.

THEOREM 5.1. *Let $\mu \in \mathcal{K}_{\mathrm{loc}}$. Then*

$$\mathcal{H}^+(\mu) = \widetilde{\mathcal{H}}^+(\mu)$$

PROOF. ($\mathcal{H}^+(\mu) \supset \widetilde{\mathcal{H}}^+(\mu)$): Let $\{p_t^h\}_{t \geq 0}$ be the transition semigroup of $\mathbb{M}^h$ given by (11). Then

$$p_t^\mu h(x) \leq h(x) \cdot p_t^h 1(x) \leq h(x), \quad \text{q.e. } x \in E,$$

and thus h is p_t^μ-excessive.

($\mathcal{H}^+(\mu) \subset \widetilde{\mathcal{H}}^+(\mu)$): Let $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0^+(G)$. Take an increasing sequence $\{G_n\}$ of relatively compact open sets such that $K := \mathrm{supp}[\varphi] \subset G_1$ and $G_n \uparrow G$. From the regularity of $\mathcal{E}$, there exists a sequence $\{\psi_n\} \subset \mathcal{D}(\mathcal{E}) \cap C_0(G)$ such that $0 \leq \psi_n \leq 1$ on G and $\psi_n = 1$ on G_n. Then $h\psi_n \in \mathcal{D}(\mathcal{E})$ and

$$\mathcal{E}(h\psi_n, \varphi) - \int_E h\psi_n \varphi \, d\widehat{\mu} \geq 0 \quad \text{for all } n \geq 1,$$

where $\widehat{\mu} := \mu(\cdot \cap K)$. Indeed, on account of $\widehat{\mu} \in \mathcal{K}$, the left-hand side is equal to

$$\lim_{t \downarrow 0} \frac{1}{t} \big(h\psi_n - p_t^{\widehat{\mu}}(h\psi_n), \varphi\big)_m = \lim_{t \downarrow 0} \frac{1}{t} \Big(\big(h, \varphi\big)_m - \big(p_t^{\widehat{\mu}}(h\psi_n), \varphi\big)_m \Big).$$

This limit is nonnegative because $p_t^{\widehat{\mu}}(h\psi_n) \leq p_t^\mu h \leq h$. Since $h\psi_n = h$ on G_1, the value of $\mathcal{E}(h\psi_n, \varphi)$ is equal to

$$\begin{aligned} \frac{1}{2} \int_E d\mu^c_{\langle h, \varphi \rangle} &+ \int_{K \times K} (h(x) - h(y))(\varphi(x) - \varphi(y)) J(dx, dy) \\ &+ 2 \int_{K \times (K^c \cap G_1)} (h(x) - h(y))(\varphi(x) - \varphi(y)) J(dx, dy) \\ &+ 2 \int_{K \times (K^c \cap G_1^c)} (h(x) - h\psi_n(y)) \cdot \varphi(x) \, J(dx, dy) + \int_E h\varphi \, d\kappa. \end{aligned}$$

Noting that $J(K \times G_1^c) < \infty$, the fourth term tends to

$$2\int_{K\times(K^c\cap G_1^c)}(h(x)-h(y))\cdot\varphi(x)\,J(dx,dy)$$

as $n\to\infty$ by the dominated convergence theorem. Consequently, we have

$$\begin{aligned}\mathcal{E}(h,\varphi)-\int_E h\varphi\,d\mu&=\mathcal{E}(h,\varphi)-\int_E h\varphi\,d\widehat{\mu}\\&=\lim_{n\to\infty}\left(\mathcal{E}(h\psi_n,\varphi)-\int_E h\psi_n\varphi\,d\widehat{\mu}\right)\geq 0.\end{aligned}$$ □

6. Applications and examples.

In this section, we treat the case where the Dirichlet form has the jumping part. Let $d(\cdot,\cdot)$ be the metric which induces the original topology of E. We impose the next assumption on $\mathbb{M}$.

(**J**): For some Radon measure m^* on E and non-increasing $[0,\infty)$-valued function Φ on $(0,\infty)$, the jumping measure $J(dx,dy)$ on $E\times E\setminus\boldsymbol{d}$ is expressed as

$$J(dx,dy)=\Phi(d(x,y))m^*(dx)m^*(dy),$$

where $\boldsymbol{d}$ is the diagonal set.

Firstly, we give sufficient conditions for a function in $\mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ belonging to $\mathcal{D}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$.

Lemma 6.1. *Let $u\in\mathcal{D}_{\mathrm{loc}}(\mathcal{E})\cap C(E)$. Then u belongs to $\mathcal{D}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$ if and only if for any compact set K, there exists a constant $c>0$ such that*

$$\int_{K\times\{|u|>c\}}(u(x)-u(y))^2J(dx,dy)<\infty.$$

Proof. The “only if” part is trivial.

We prove the “if” part. Take a relatively compact open set D such that $K\subset D$. Note that $J(K\times D^c)<\infty$ because of the regularity of $\mathcal{E}$. We shall show that

$$\int_{K\times E}(u(x)-u(y))^2J(dx,dy)<\infty.$$

The integral is decomposed as

$$\int_{K\times D}(u(x)-u(y))^2J(dx,dy)+\int_{K\times D^c}(u(x)-u(y))^2J(dx,dy).$$

The first term is finite because there exists $v\in\mathcal{D}(\mathcal{E})$ such that $u=v$ q.e. on D. The second term is less than or equal to

$$\begin{aligned}&\int_{K\times(D^c\cap\{|u|\le c\})}(u(x)-u(y))^2J(dx,dy)+\int_{K\times(D^c\cap\{|u|>c\})}(u(x)-u(y))^2J(dx,dy)\\&\quad\le 2\left(\|\mathbb{1}_K\cdot u\|_\infty^2+c^2\right)\cdot J(K\times D^c)+\int_{K\times\{|u|>c\}}(u(x)-u(y))^2J(dx,dy)<\infty.\end{aligned}$$ □

Lemma 6.2. *Let $u \in \mathcal{D}_{\mathrm{loc}}(\mathcal{E}) \cap C(E)$. If there exists $c > 0$ such that*

$$\int_{\{|u|>c\}} u^2 dm^* < \infty,$$

then $u \in \mathcal{D}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$.

Proof. By considering the decomposition $u = (u \vee 0) - (-u \vee 0)$, we may assume $u \geq 0$. Fix a compact set K and put $M := c \vee (\max_{x \in K} u(x))$. On account of Lemma 6.1, it is sufficient to prove that

$$\int_K m^*(dx) \int_{\{u>2M\}} (u(y) - u(x))^2 \Phi(d(x,y)) m^*(dy) < \infty.$$

Since $|u(y) - u(x)| \leq u(y)$ for $(x, y) \in K \times \{u > 2M\}$, the left-hand side is bounded by

$$\int_K m^*(dx) \int_{\{u>2M\}} u(y)^2 \Phi(d(x,y)) m^*(dy). \tag{13}$$

Let $d(x) := \inf\{d(x,y) \,|\, y \in \{u > 2M\}\}$ and $\delta := \inf\{d(x) \,|\, x \in K\}$. Then we easily see that δ is strictly positive. Hence, (13) is dominated by

$$\int_K m^*(dx) \int_{\{u>2M\}} u(y)^2 \Phi(\delta)\, m^*(dy) \leq \Phi(\delta)\, m^*(K) \int_{\{u>c\}} u^2 dm^* < \infty. \qquad \square$$

Example 6.3 (*α-stable process*). Let $\mathbb{M}^\alpha = (X_t, \mathbb{P}_x)$, $0 < \alpha < 2$, be a symmetric α-stable process on $\mathbb{R}^d$ generated by the fractional Laplacian $-1/2(-\Delta)^{\alpha/2}$. Assume $\alpha < d$, that is, $\mathbb{M}^\alpha$ is transient. Then its Green function $R(x,y)$ is given by

$$R(x,y) = C(d,\alpha) \cdot |x-y|^{\alpha-d},$$

where $C(d,\alpha) = 2^{-\alpha}\pi^{-d/2}\Gamma((d-\alpha)/2)\Gamma(\alpha/2)^{-1}$ and Γ is the Gamma function. For a Borel function f, the 0-potential of f is written as

$$Rf(x) = \int_{\mathbb{R}^d} R(x,y) f(y) dy.$$

The Dirichlet form generated by $\mathbb{M}^\alpha$ is given by

$$\begin{cases} \mathcal{E}^{(\alpha)}(u,v) = \dfrac{1}{2}\mathcal{A}(d,\alpha) \displaystyle\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\, dxdy, \\ \mathcal{D}(\mathcal{E}^{(\alpha)}) = \left\{ u \in L^2(\mathbb{R}^d; dx) \,\middle|\, \displaystyle\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\, dxdy < \infty \right\}, \end{cases}$$

where $\mathcal{A}(d,\alpha) = \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma((\alpha+d)/2)\Gamma(1-(\alpha/2))^{-1}$.

Let $w(x) = |x|^{-p}$. If $p \in (0, d/2)$, then

$$\int_{\{w>c\}} w(x)^2 dx < \infty$$

for any $c > 0$, and thus $w \in \mathcal{D}^{\dagger}_{\mathrm{loc}}(\mathcal{E}^G) \cap C(G \cup \{\Delta\})$, $G := \mathbb{R}^d \setminus \{0\}$ by Lemma 6.2. Let $v(x) = |x|^{-(p+\alpha)}$, $0 < p < (d/2) \wedge (d-\alpha)$. Then it follows from [**3**, Lemma 2.1] that

$$Rv(x) = C^{-1}_{d,\alpha,p} \cdot |x|^{-p}, \quad \text{where } C_{d,\alpha,p} := 2^{\alpha} \frac{\Gamma((p+\alpha)/2)\Gamma((d-p)/2)}{\Gamma((d-(p+\alpha))/2)\Gamma(p/2)}.$$

By applying Corollary 4.4 to Rv, we have the equality

$$\begin{aligned}&\mathcal{E}^{(\alpha)}(u,u) - C_{d,\alpha,p} \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^{\alpha}} dx \\ &\quad = \frac{1}{2}\mathcal{A}(d,\alpha) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{u(x)}{|x|^{-p}} - \frac{u(y)}{|y|^{-p}} \right)^2 \frac{|x|^{-p}|y|^{-p}}{|x-y|^{d+\alpha}} dxdy, \quad u \in \mathcal{D}(\mathcal{E}^{(\alpha)}).\end{aligned}$$

The equality above has been already shown by Bogdan, Dyda and Kim [**5**, Proposition 5] in an analytic way. The case $p = (d-\alpha)/2$ is treated in [**2**] and [**12**]. We see from [**3**, Lemma 2.2] that the maximum of a function

$$F(p) := 2^{\alpha} \frac{\Gamma((p+\alpha)/2)\Gamma((d-p)/2)}{\Gamma((d-(p+\alpha))/2)\Gamma(p/2)} \; (= C_{d,\alpha,p}), \qquad p \in (0, d-\alpha),$$

is achieved at $p = (d-\alpha)/2$. It is known in [**1**] that $C_{d,\alpha,(d-\alpha)/2} = 2^{\alpha}\Gamma((d+\alpha)/4)^2\Gamma((d-\alpha)/4)^{-2}$ is the best constant for Hardy's inequality, that is, for any $C > C_{d,\alpha,(d-\alpha)/2}$, there exists $u \in \mathcal{D}(\mathcal{E}^{(\alpha)})$ such that

$$\mathcal{E}^{(\alpha)}(u,u) < C \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^{\alpha}} dx.$$

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