J. Math. Soc. Japan
Vol. 71, No. 2 (2019) pp. 349–382
doi: 10.2969/jmsj/77007700

Gluing construction of compact Spin(7)-manifolds

By Mamoru Doi and Naoto Yotsutani

(Received Dec. 27, 2016)
(Revised Oct. 12, 2017)

Abstract. We give a differential-geometric construction of compact manifolds with holonomy Spin(7) which is based on Joyce's second construction of compact Spin(7)-manifolds and Kovalev's gluing construction of compact G_2-manifolds. We provide several examples of compact Spin(7)-manifolds, at least one of which is new. Here in this paper we need orbifold admissible pairs $(\overline{X}, D)$ consisting of a compact Kähler orbifold $\overline{X}$ with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, and a smooth anticanonical divisor D on $\overline{X}$. Also, we need a compatible antiholomorphic involution σ on $\overline{X}$ which fixes the singular points on $\overline{X}$ and acts freely on the anticanoncial divisor D. If two orbifold admissible pairs $(\overline{X}_1, D_1)$, $(\overline{X}_2, D_2)$ and compatible antiholomorphic involutions σ_i on $\overline{X}_i$ for $i = 1, 2$ satisfy the gluing condition, we can glue $(\overline{X}_1 \setminus D_1)/\langle\sigma_1\rangle$ and $(\overline{X}_2 \setminus D_2)/\langle\sigma_2\rangle$ together to obtain a compact Riemannian 8-manifold (M, g) whose holonomy group $\mathrm{Hol}(g)$ is contained in Spin(7). Furthermore, if the $\widehat{A}$-genus of M equals 1, then M is a compact Spin(7)-manifold, i.e. a compact Riemannian manifold with holonomy Spin(7).

1. Introduction.

According to the Berger–Simons classification of holonomy groups of irreducible simply-connected Riemannian manifolds, the exeptional Lie group Spin(7) arises as the 'maximal' Lie group among the holonomy groups corresponding to simply-connected Ricci-flat Riemannian manifolds of dimensions less than or equal to 8; if an m-dimensional ($m \leq 8$) simply-connected Riemannian manifold (M, g) satisfies $\mathrm{Ric}(g) \equiv 0$ and $\mathrm{Hol}(g) \subsetneq \mathrm{SO}(m)$, then $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$. For example, any complex three- and four-dimensional Calabi–Yau manifold has a Kähler metric with holonomy SU(3) and SU(4) respectively, where $\mathrm{SU}(3) \subset \mathrm{SU}(4) \subset \mathrm{Spin}(7)$. Since a huge number of examples of Calabi–Yau manifolds have been discovered by mathematicians and physicists, we can expect that there are enormous examples of compact Spin(7)-manifolds also.

However, there are only a little over 200 examples of compact Spin(7)-manifolds so far, which are obtained by Joyce [**14**] and Clancy [**2**]: Joyce constructed the first compact manifolds with holonomy group Spin(7) by a generalized Kummer construction [**12**]. Later he gave another method starting from Calabi–Yau 4-orbifold in weighted projective spaces and provided further examples [**13**]. Following Joyce's second construction, Clancy systematically investigated such a Calabi–Yau 4-orbifold with particular singularities admitting an antiholomorphic involution, which fixes the singularities [**2**]. Eventually he discovered more new examples of compact Spin(7)-manifolds.

2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 14J32.
Key Words and Phrases. Ricci-flat metrics, Spin(7)-structures, gluing, doubling.

In the present paper we glue two asymptotically cylindrical Spin(7)-*orbifolds* to construct a compact Spin(7)-orbifold $M^{\triangledown}$, and then resolve the singularities of $M^{\triangledown}$ to obtain a compact Spin(7)-manifold. Such an asymptotically cylindrical Spin(7)-orbifold is obtained by setting $(\overline{X} \setminus D)/\langle\sigma\rangle$ for an *orbifold admissible pairs* $(\overline{X}, D)$ with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, where σ is a compatible antiholomorphic involution on $\overline{X}$. Another technical difficulty to deal with Spin(7)-manifolds stems from these singularities on $\overline{X}$. Although our primary joint research project has aimed to construct compact Spin(7)-manifolds, we first constructed Calabi–Yau manifolds in order to avoid such technical difficulties. In a word, we constructed Calabi–Yau threefolds [**5**] and fourfolds [**6**] by gluing together two asymptotically cylindrical Ricci-flat Kähler manifolds, using the gluing technique which Kovalev used in constructing compact G_2-manifolds [**15**]. Recall that asymptotically cylindrical Ricci-flat Kähler manifolds X are obtained from *smooth* admissible pairs $(\overline{X}, D)$ by setting $X = \overline{X} \setminus D$ with $\operatorname{Sing} \overline{X} = \emptyset$. Furthermore in [**6**], we used the $\widehat{A}$-genera of the resulting compact Riemannian 8-manifold (M, g) with $\operatorname{Hol}(g) \subseteq \operatorname{Spin}(7)$ in order to conclude $\operatorname{Hol}(g) = \operatorname{SU}(4)$ (see Theorem 2.8). This is a reason why we first considered Calabi–Yau constructions before Spin(7) cases. On the other hand, our construction of compact Spin(7)-manifolds which would be the main part of our joint research project, has been accomplished building upon Joyce's second construction of compact Spin(7)-manifolds. Originally, Joyce resolved $X = \overline{X} \setminus D$ to obtain compact Spin(7)-manifolds when $\overline{X}$ is a four-dimensional Calabi–Yau orbifold and $D = \emptyset$, so that $X = \overline{X}$ is *compact*: Beginning with a compact four-dimensional Calabi–Yau orbifold $\overline{X}$ with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, and an antiholomorphic involution σ on $\overline{X}$ with $(\overline{X})^{\sigma} = \operatorname{Sing} \overline{X}$, Joyce proved that $Z = \overline{X}/\langle\sigma\rangle$ admits a torsion-free Spin(7)-structure. Since the associated Riemannian metric is flat (Euclidean) around the singularities of Z, he then replaced the neighborhood of each singularity of Z with a suitable asymptotically locally Euclidean (ALE) Spin(7)-manifold to obtain a family of simply-connected, smooth 8-manifolds $\{ M^{\epsilon} \}$ for $\epsilon \in (0, 1]$ with a Spin(7)-structure Φ^{ϵ} *with small torsion*, which satisfies $d\Phi^{\epsilon} \to 0$ as $\epsilon \to 0$ in a suitable sense. Finally, Joyce proved that Φ^{ϵ} can be deformed to a torsion-free Spin(7)-structure for sufficiently small ϵ using the analysis on Spin(7)-structures. Hence $M = M^{\epsilon}$ admits a Riemannian metric with holonomy Spin(7).

In addition to the doubling method presented in previous papers [**5**], [**6**], one important benefit of the present paper is that we can successfully glue *different* pieces $(\overline{X}_1 \setminus D_1)/\langle\sigma_1\rangle$ and $(\overline{X}_2 \setminus D_2)/\langle\sigma_2\rangle$ together to obtain practical examples of compact Spin(7)-manifolds (see Section 6), whereas we only construct examples from two copies of admissible pairs $(\overline{X}_1, D_1) = (\overline{X}_2, D_2) = (\overline{X}, D)$ in our previous papers [**5**], [**6**]. Eventually we discovered a new example of compact Spin(7)-manifolds in our gluing construction which we already announced at Math Society of Japan Autumn Meeting 2011 and described in our abstract [**7**]. We note that asymptotically cylindrical Spin(7)-*manifolds* are recently constructed by Kovalev in [**16**] by resolving $(\overline{X} \setminus D)/\langle\sigma\rangle$.

To be specific, we begin in our construction with two orbifold admissible pairs $(\overline{X}_1, D_1)$ and $(\overline{X}_2, D_2)$, consisting of a compact Kähler orbifold $\overline{X}_i$ and a smooth anticanonical divisor D_i on $\overline{X}_i$. Also, we consider an antiholomorphic involution σ_i acting on each $\overline{X}_i$. As in Joyce's second construction, we require that $\overline{X}_i$ have isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, and $(\overline{X}_i)^{\sigma} = \operatorname{Sing} \overline{X}_i$ (see Definitions 3.6 and 3.10). In addition, we suppose that σ preserves and acts freely on D. Then by the existence result of an asymptotically cylindrical Ricci-flat Kähler form on $\overline{X}_i \setminus D_i$, each $\overline{X}_i \setminus D_i$ has a natural σ_i-invariant asymptotically cylindrical torsion-free Spin(7)-structure, which pushes down to a torsion-free Spin(7)-structure Φ_i on $(\overline{X}_i \setminus D_i)/\langle\sigma_i\rangle$. Now suppose the *asymptotic models* $((D_i \times S^1)/\langle\sigma_{D_i\times S^1,\mathrm{cyl}}\rangle \times \mathbb{R}_+, \Phi_{i,\mathrm{cyl}})$ of $((\overline{X}_i \setminus D_i)/\langle\sigma_i\rangle, \Phi_i)$ are isomorphic in a suitable sense, which is ensured by the *gluing condition* defined later (see Section 3.4.1). Then as in Kovalev's construction in [**15**], we can glue together $(\overline{X}_1 \setminus D_1)/\langle\sigma_1\rangle$ and $(\overline{X}_2 \setminus D_2)/\langle\sigma_2\rangle$ along their cylindrical ends $(D_1 \times S^1)/\langle\sigma_{D_1\times S^1,\mathrm{cyl}}\rangle \times (T-1, T+1)$ and $(D_2 \times S^1)/\langle\sigma_{D_2\times S^1,\mathrm{cyl}}\rangle \times (T-1, T+1)$, to obtain a compact Riemannian 8-*orbifold* $M_T^{\triangledown}$. Also, we can glue together the torsion-free Spin(7)-structures Φ_i on $(\overline{X}_i \setminus D_i)/\langle\sigma_i\rangle$ to construct a d-closed 4-form $\widetilde{\Phi}_T$ on $M_T^{\triangledown}$. Furthermore, replacing each neighborhood of singular points on $M_T^{\triangledown}$ with a certain ALE Spin(7)-manifold, we construct a family $(M_T^{\epsilon}, \widetilde{\Phi}_T^{\epsilon})$ of simply-connected, smooth 8-manifolds with a d-closed 4-form for sufficiently small $\epsilon > 0$. Here each $\widetilde{\Phi}_T^{\epsilon}$ is projected to a Spin(7)-structure $\Phi_T^{\epsilon} = \Theta(\widetilde{\Phi}_T^{\epsilon})$, with $\Phi_T^{\epsilon} \to 0$ as $T \to \infty$ or $\epsilon \to 0$ in a suitable sense. Now set $\epsilon = e^{-\gamma T}$ for some $\gamma > 0$, and consider a family $(M^{\epsilon}, \Phi^{\epsilon}) = (M_T^{\epsilon}, \Phi_T^{\epsilon})$ of compact 8-manifolds with a Spin(7)-structure with small torsion. Then using the analysis on Spin(7)-structures by Joyce [**14**], we shall prove that Φ^{ϵ} can be deformed into a torsion-free Spin(7)-structure for sufficiently small ϵ, that is, the resulting compact manifold M^{ϵ} admits a Riemannian metric with holonomy contained in Spin(7). Since $M = M^{\epsilon}$ is simply-connected, the $\widehat{A}$-genus $\widehat{A}(M)$ of M is $1, 2, 3$ or 4, and the holonomy group is determined as Spin(7), SU(4), Sp(2), Sp(1) $\times$ Sp(1) respectively (see Theorem 2.8). Hence if $\widehat{A}(M) = 1$, then M is a compact Spin(7)-manifold.

Finally we describe a remarkable difference between our previous works [**5**], [**6**] and the present paper to provide interesting examples of compact Spin(7)-manifolds, at least one of which is topologically *new*. For a given orbifold admissible pair $(\overline{X}_1, D_1)$ with a compatible antiholomorphic involution σ_1, it is difficult in general to find another admissible pair $(\overline{X}_2, D_2)$ with σ_2 such that both $(\overline{X}_i \setminus D_i)/\langle\sigma_i\rangle$ have the same asymptotic model. One way to solve this is the 'doubling' method used in [**5**], [**6**], in which we take $(\overline{X}_1, D_1) = (\overline{X}_2, D_2)$ and $\sigma_1 = \sigma_2$. There is another solution, which we discuss in Section 6.

In the present paper, we shall give 3 topologically distinct compact Spin(7)-manifolds, at least one of which is new. Each of the examples satisfies $b^2(M) = b^3(M) = 0$ and $\widehat{A}(M) = 1$. In order to show $\widehat{A}(M) = 1$, we reduce the problem to computations on the cohomology groups of D and S. Betti numbers (b^2, b^3, b^4) of the compact Spin(7)-manifolds in our construction are $(0, 0, 910), (0, 0, 1294)$ and $(0, 0, 1678)$. Of these compact Spin(7)-manifolds, the resulting manifold M with $\chi(M) = 1680$ is at least one new example which is not diffeomorphic to the known ones (see Theorem 5.1).

This paper is organized as follows. Section 2 is a brief review on Spin(7)-structures. In Section 3 we define orbifold admissible pairs which will be ingredients in our gluing construction of compact Spin(7)-manifolds. This section is the heart of the present paper. We consider compatible antiholomorphic involutions σ on orbifold admissible pairs $(\overline{X}, D)$ and glue together two orbifold admissible pairs with $\dim_{\mathbb{C}} \overline{X} = 4$ divided by σ. The gluing theorems are stated in Section 3.5 including both cases of Spin(7)-manifolds and Calabi–Yau fourfolds. Giving a quick review of basics on weighted projective spaces in Section 4.1, we obtain in Section 4.3 orbifold admissible pairs from complete intersections in weighted projective spaces. Then in Section 5 we give a new example of compact Spin(7)-manifolds M. In the last section we shall give other examples of compact Spin(7)-manifolds taking weighted complete intersections in $\mathbb{C}P^5(1,1,1,1,4,4)$. All the resulting compact Spin(7)-manifolds are listed in Table 6.5. Finally we shall provide a criterion for finding compact Spin(7)-manifolds (Proposition 6.2).

Acknowledgements. The authors are grateful to Professors Xiuxiong Chen and Bin Xu for valuable discussions in University of Science and Technology of China, Hefei in April, 2011. The second author is also grateful to Professors Shengli Kang and Haozhao Li for their useful comments.

2. Geometry of Spin(7)-structures.

Here we shall recall some basic facts about Spin(7)-structures on oriented 8-manifolds. For more details, see [**14**, Chapter 10].

We begin with the definition of Spin(7)-structures on oriented real vector spaces of dimension 8.

Definition 2.1. Let V be an oriented real vector space of dimension 8. Let $\{\boldsymbol{\theta}^1, \dots, \boldsymbol{\theta}^8\}$ be an oriented basis of V. Set

$$\begin{aligned}\boldsymbol{\Phi}_0 &= \boldsymbol{\theta}^{1234} + \boldsymbol{\theta}^{1256} + \boldsymbol{\theta}^{1278} + \boldsymbol{\theta}^{1357} - \boldsymbol{\theta}^{1368} - \boldsymbol{\theta}^{1458} - \boldsymbol{\theta}^{1467} \\ &\quad - \boldsymbol{\theta}^{2358} - \boldsymbol{\theta}^{2367} - \boldsymbol{\theta}^{2457} + \boldsymbol{\theta}^{2468} + \boldsymbol{\theta}^{3456} + \boldsymbol{\theta}^{3478} + \boldsymbol{\theta}^{5678}, \\ \boldsymbol{g}_0 &= \sum_{i=1}^{8} \boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^i,\end{aligned}$$

where $\boldsymbol{\theta}^{ij\dots k} = \boldsymbol{\theta}^i \wedge \boldsymbol{\theta}^j \wedge \dots \wedge \boldsymbol{\theta}^k$. Define the $\mathrm{GL}_+(V)$-orbit spaces

$$\begin{aligned}\mathcal{A}(V) &= \{\, a^*\boldsymbol{\Phi}_0 \mid a \in \mathrm{GL}_+(V) \,\}, \\ \mathcal{M}et(V) &= \{\, a^*\boldsymbol{g}_0 \mid a \in \mathrm{GL}_+(V) \,\}.\end{aligned}$$

We call $\mathcal{A}(V)$ the set of *Cayley* 4*-forms* (or the set of Spin(7)*-structures*) on V. On the other hand, $\mathcal{M}et(V)$ is the set of positive-definite inner products on V, which is also a homogeneous space isomorphic to $\mathrm{GL}_+(V)/\mathrm{SO}(V)$, where $\mathrm{SO}(V)$ is defined by

$$\mathrm{SO}(V) = \{\, a \in \mathrm{GL}_+(V) \mid a^*\boldsymbol{g}_0 = \boldsymbol{g}_0 \,\}.$$

Now the group Spin(7) is defined as the isotropy of the action of $\mathrm{GL}(V)$ (in place of $\mathrm{GL}_+(V)$) on $\mathcal{A}(V)$ at $\boldsymbol{\Phi}_0$:

$$\mathrm{Spin}(7) = \{\, a \in \mathrm{GL}(V) \mid a^*\boldsymbol{\Phi}_0 = \boldsymbol{\Phi}_0 \,\}.$$

Then one can show that Spin(7) is a compact Lie group of dimension 27 which is a Lie subgroup of $\mathrm{SO}(V)$ (see [**10**]). Thus we have a natural projection

$$\mathcal{A}(V) \cong \mathrm{GL}_+(V)/\operatorname{Spin}(7) \twoheadrightarrow \mathrm{GL}_+(V)/\operatorname{SO}(V) \cong \mathcal{M}et(V)\ ,$$

so that each Cayley 4-form (or Spin(7)-structure) $\boldsymbol{\Phi} \in \mathcal{A}(V)$ defines a positive-definite inner product $\boldsymbol{g_\Phi} \in \mathcal{M}et(V)$ on V.

Definition 2.2. Let V be an oriented vector space of dimension 8. If $\boldsymbol{\Phi} \in \mathcal{A}(V)$, then we have the orthogonal decomposition

$$\wedge^4 V^* = T_{\boldsymbol{\Phi}}\mathcal{A}(V) \oplus T_{\boldsymbol{\Phi}}^{\perp}\mathcal{A}(V) \tag{2.1}$$

with respect to the induced inner product $\boldsymbol{g_\Phi}$. We define a neighborhood $\mathcal{T}(V)$ of $\mathcal{A}(V)$ in $\wedge^4 V^*$ by

$$\mathcal{T}(V) = \{\, \boldsymbol{\Phi} + \boldsymbol{\alpha} \mid \boldsymbol{\Phi} \in \mathcal{A}(V) \text{ and } \boldsymbol{\alpha} \in T_{\boldsymbol{\Phi}}^{\perp}\mathcal{A}(V) \text{ with } |\boldsymbol{\alpha}|_{\boldsymbol{g_\Phi}} < \rho \,\}\,.$$

We choose and fix a small constant ρ so that any $\boldsymbol{\chi} \in \mathcal{T}(V)$ is uniquely written as $\boldsymbol{\chi} = \boldsymbol{\Phi} + \boldsymbol{\alpha}$ with $\boldsymbol{\alpha} \in T_{\boldsymbol{\Phi}}^{\perp}\mathcal{A}(V)$. Thus we can define the projection

$$\Theta : \mathcal{T}(V) \longrightarrow \mathcal{A}(V), \quad \boldsymbol{\chi} \longmapsto \boldsymbol{\Phi}.$$

Lemma 2.3 (Joyce [**14**, Proposition 10.5.4]). *Let $\boldsymbol{\Phi} \in \mathcal{A}(V)$ and $\wedge^4 V^* = \wedge^4_+ V^* \oplus \wedge^4_- V^*$ be the orthogonal decomposition with respect to $\boldsymbol{g_\Phi}$, where $\wedge^4_+ V^*$* (*resp.* $\wedge^4_- V^*$) *is the set of self-dual* (*resp. anti-self-dual*) *4-forms on V. Then we have the following inclusion*:

$$\wedge^4_- V^* \subset T_{\boldsymbol{\Phi}}\mathcal{A}(V).$$

Now we define Spin(7)-structures on oriented 8-manifolds.

Definition 2.4. Let M be an oriented 8-manifold. We define $\mathcal{A}(M) \longrightarrow M$ to be the fiber bundle whose fiber over x is $\mathcal{A}(T_x^*M) \subset \wedge^4 T_x^*M$. Then $\Phi \in C^\infty(\wedge^4 T^*M)$ is a *Cayley 4-form* or a Spin(7)*-structure* on M if $\Phi \in C^\infty(\mathcal{A}(M))$, i.e., Φ is a smooth section of $\mathcal{A}(M)$. If Φ is a Spin(7)-structure on M, then Φ induces a Riemannian metric g_Φ since $\Phi|_x$ for each $x \in M$ induces a positive-definite inner product $g_{\Phi|_x}$ on T_xM. A Spin(7)-structure Φ on M is said to be *torsion-free* if it is parallel with respect to the induced Riemannian metric g_Φ, i.e., $\nabla_{g_\Phi}\Phi = 0$, where ∇_{g_Φ} is the Levi-Civita connection of g_Φ.

Definition 2.5. Let Φ be a Spin(7)-structure on an oriented 8-manifold M. We define $\mathcal{T}(M)$ to be the fiber bundle whose fiber over x is $\mathcal{T}(T_x^*M) \subset \wedge^4 T_x^*M$. Then for the constant ρ given in Definition 2.2, we have the well-defined projection $\Theta : \mathcal{T}(M) \longrightarrow \mathcal{A}(M)$. Also, we see from Lemma 2.3 that $\wedge^4_- T^*M \subset T_\Phi\mathcal{A}(M)$ as subbundles of $\wedge^4 T^*M$.

Lemma 2.6 (Joyce [**14**, Proposition 10.5.9]). *Let Φ be a* Spin(7)*-structure on M. There exist $\epsilon_1, \epsilon_2, \epsilon_3$ independent of M and Φ, such that the following is true.*

If $\eta \in C^\infty(\wedge^4 T^*M)$ satisfies $\|\eta\|_{C^0} \le \epsilon_1$, then $\Phi + \eta \in \mathcal{T}(M)$. For this η, $\Theta(\Phi + \eta)$ is well-defined as a Spin(7)*-structure on M, and expanded as*

$$\Theta(\Phi+\eta)=\Phi+p(\eta)-F(\eta), \tag{2.2}$$

where $p(\eta)$ is the linear term and $F(\eta)$ is the higher order term in η, and for each $x \in M$, $p(\eta)|_x$ is the $T_\Phi\mathcal{A}(V)$-component of $\eta|_x$ in the orthogonal decomposition (2.1) *for $V=T_x^*M$. Also, we have the following pointwise estimates for any $\eta,\eta' \in C^\infty(\wedge^4T^*M)$ with $|\eta|,|\eta'|\le\epsilon_1$:*

$$\begin{aligned}|F(\eta)-F(\eta')| &\le \epsilon_2|\eta-\eta'|(|\eta|+|\eta'|),\\ |\nabla(F(\eta)-F(\eta'))| &\le \epsilon_3\{|\eta-\eta'|(|\eta|+|\eta'|)|\mathrm{d}\Phi|+|\nabla(\eta-\eta')|(|\eta|+|\eta'|)\\ &\quad+|\eta-\eta'|(|\nabla\eta|+|\nabla\eta'|)\}.\end{aligned}$$

Here all norms are measured by g_Φ.

The following result is important in that it relates the holonomy contained in Spin(7) with the d-closedness of the Spin(7)-structure.

Theorem 2.7 (Salamon [**18**, Lemma 12.4]). *Let M be an oriented* 8*-manifold. Let Φ be a* Spin(7)*-structure on M and g_Φ the induced Riemannian metric on M. Then the following conditions are equivalent.*

(1) *Φ is a torsion-free* Spin(7)*-structure, i.e., $\nabla_{g_\Phi}\Phi=0$.*

(2) $\mathrm{d}\Phi=0$.

(3) *The holonomy group* $\mathrm{Hol}(g_\Phi)$ *of g_Φ is contained in* Spin(7).

Now suppose $\widetilde{\Phi}\in C^\infty(\mathcal{T}(M))$ with $\mathrm{d}\widetilde{\Phi}=0$. We shall construct such a form $\widetilde{\Phi}$ in Section 3.4.2. Then $\Phi=\Theta(\widetilde{\Phi})$ is a Spin(7)-structure on M. If $\eta\in C^\infty(\wedge^4T^*M)$ with $\|\eta\|_{C^0}\le\epsilon_1$, then $\Theta(\Phi+\eta)$ is expanded as in (2.2). Setting $\phi=\widetilde{\Phi}-\Phi$ and using $\mathrm{d}\widetilde{\Phi}=0$, we have

$$\mathrm{d}\Theta(\Phi+\eta)=-\mathrm{d}\phi+\mathrm{d}p(\eta)-\mathrm{d}F(\eta).$$

Thus the equation $\mathrm{d}\Theta(\Phi+\eta)=0$ for $\Theta(\Phi+\eta)$ to be a torsion-free Spin(7)-structure is equivalent to

$$\mathrm{d}p(\eta)=\mathrm{d}\phi+\mathrm{d}F(\eta). \tag{2.3}$$

In particular, we see from Lemma 2.3 that if $\eta\in C^\infty(\wedge^4_-T^*M)$ then $p(\eta)=\eta$, so that Equation (2.3) becomes

$$\mathrm{d}\eta=\mathrm{d}\phi+\mathrm{d}F(\eta). \tag{2.4}$$

Joyce proved by using the iteration method and $\mathrm{d}C^\infty(\wedge^4_-T^*M)=\mathrm{d}C^\infty(\wedge^4T^*M)$ that Equation (2.4) has a solution $\eta\in C^\infty(\wedge^4_-T^*M)$ if ϕ is sufficiently small with respect to certain norms (see Theorem 3.25).

For an oriented 8-manifold M satisfying one of the conditions (1)–(3) in Theorem 2.7, the following therem completely determines the holonomy of M from its topological invariants.

Theorem 2.8 (Joyce [**14**, Theorem 10.6.1]). *Let (M, g) be a compact Riemannian 8-manifold such that its holonomy group $\mathrm{Hol}(g)$ is contained in $\mathrm{Spin}(7)$. Then the $\widehat{A}$-genus $\widehat{A}(M)$ of M satisfies*

$$48\widehat{A}(M) = 3\tau(M) - \chi(M), \tag{2.5}$$

where $\tau(M)$ and $\chi(M)$ are the signature and the Euler characteristic of M respectively. Moreover, if M is simply-connected, then $\widehat{A}(M)$ is $1, 2, 3$ or 4, and the holonomy group of (M, g) is determined as follows:

$$\mathrm{Hol}(g) = \begin{cases} \mathrm{Spin}(7) & \text{if } \widehat{A}(M) = 1, \\ \mathrm{SU}(4) & \text{if } \widehat{A}(M) = 2, \\ \mathrm{Sp}(2) & \text{if } \widehat{A}(M) = 3, \\ \mathrm{Sp}(1) \times \mathrm{Sp}(1) & \text{if } \widehat{A}(M) = 4. \end{cases}$$

3. The gluing procedure.

3.1. Compact complex manifolds with an anticanonical divisor.

We suppose that $\overline{X}$ is a compact complex manifold of dimension m, and D is a smooth irreducible anticanonical divisor on $\overline{X}$. We recall some results in [**4**, Sections 3.1–3.2], and [**5**, Sections 3.1–3.2].

Lemma 3.1. *Let $\overline{X}$ and D be as above. Then there exists a local coordinate system $\{U_\alpha, (z_\alpha^1, \ldots, z_\alpha^{m-1}, w_\alpha)\}$ on $\overline{X}$ such that*

(i) *w_α is a local defining function of D on U_α, i.e., $D \cap U_\alpha = \{w_\alpha = 0\}$, and*

(ii) *the m-forms $\Omega_\alpha = (\mathrm{d}w_\alpha / w_\alpha) \wedge \mathrm{d}z_\alpha^1 \wedge \cdots \wedge \mathrm{d}z_\alpha^{m-1}$ on $U_\alpha \setminus D$ together yield a holomorphic volume form Ω on $X = \overline{X} \setminus D$.*

Next we shall see that $X = \overline{X} \setminus D$ is a cylindrical manifold whose structure is induced from the holomorphic normal bundle $N = N_{D/\overline{X}}$ to D in $\overline{X}$, where the definition of cylindrical manifolds is given as follows.

Definition 3.2. Let X be a noncompact differentiable manifold of dimension r. Then X is called a *cylindrical manifold* or a *manifold with a cylindrical end* if there exists a diffeomorphism $\pi : X \setminus X_0 \longrightarrow \Sigma \times \mathbb{R}_+ = \{\, (p, t) \mid p \in \Sigma, 0 < t < \infty \,\}$ for some compact submanifold X_0 of dimension r with boundary $\Sigma = \partial X_0$. Also, extending t smoothly on X so that $t \leq 0$ on X_0, we call t a *cylindrical parameter* on X.

Let (x_α, y_α) be local coordinates on $V_\alpha = U_\alpha \cap D$, such that x_α is the restriction of z_α to V_α and y_α is a coordinate in the fiber direction. Then one can see easily that $\mathrm{d}x_\alpha^1 \wedge \cdots \wedge \mathrm{d}x_\alpha^{m-1}$ on V_α together yield a holomorphic volume form Ω_D, which is also called the *Poincaré residue* of Ω along D. Let $\|\cdot\|$ be the norm of a Hermitian bundle metric on N. We can define a cylindrical parameter t on N by $t = (-1/2)\log\|s\|^2$ for $s \in N \setminus D$. Then the local coordinates (z_α, w_α) on X are asymptotic to the local coordinates (x_α, y_α) on $N \setminus D$ in the following sense.

Lemma 3.3. *There exists a diffeomorphism φ from a neighborhood V of the zero section of N containing $t^{-1}(\mathbb{R}_+)$ to a tubular neighborhood U of D in X such that φ can be locally written as*

$$z_\alpha = x_\alpha + O(|y_\alpha|^2) = x_\alpha + O(e^{-t}),$$
$$w_\alpha = y_\alpha + O(|y_\alpha|^2) = y_\alpha + O(e^{-t}),$$

where we multiply all z_α and w_α by a single constant to ensure $t^{-1}(\mathbb{R}_+) \subset V$ if necessary.

Hence X is a cylindrical manifold with the cylindrical parameter t via the diffeomorphism Φ given in the above lemma. In particular, when $H^0(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$ and $N_{D/\overline{X}}$ is trivial, we have a useful coordinate system near D.

Lemma 3.4 ([**5**, Lemma 3.4]). *Let $(\overline{X}, D)$ be as in Lemma 3.1. If $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$ and the normal bundle $N_{D/\overline{X}}$ is holomorphically trivial, then there exist an open neighborhood U_D of D and a holomorphic function w on U_D such that w is a local defining function of D on U_D. Also, we may define the cylindrical parameter t with $t^{-1}(\mathbb{R}_+) \subset U_D$ by writing the fiber coordinate y of $N_{D/\overline{X}}$ as $y = \exp(-t - \sqrt{-1}\theta)$.*

3.2. Admissible pairs and asymptotically cylindrical Ricci-flat Kähler manifolds.

Definition 3.5. Let X be a cylindrical manifold such that $\pi : X \backslash X_0 \longrightarrow \Sigma \times \mathbb{R}_+ = \{(p, t)\}$ is a corresponding diffeomorphism. If g_Σ is a Riemannian metric on Σ, then it defines a cylindrical metric $g_{\mathrm{cyl}} = g_\Sigma + \mathrm{d}t^2$ on $\Sigma \times \mathbb{R}_+$. Then a complete Riemannian metric g on X is said to be *asymptotically cylindrical* (*to* $(\Sigma \times \mathbb{R}_+, g_{\mathrm{cyl}})$) if g satisfies for some cylindrical metric $g_{\mathrm{cyl}} = g_\Sigma + \mathrm{d}t^2$

$$|\nabla^j_{g_{\mathrm{cyl}}}(g - g_{\mathrm{cyl}})|_{g_{\mathrm{cyl}}} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty \quad \text{for all } j \geq 0,$$

where we regarded g_{cyl} as a Riemannian metric on $X \backslash X_0$ via the diffeomorphism π. Also, we call (X, g) an *asymptotically cylindrical manifold* and $(\Sigma \times \mathbb{R}_+, g_{\mathrm{cyl}})$ the *asymptotic model* of (X, g).

Definition 3.6. Let $\overline{X}$ be a complex orbifold with isolated singular points $\operatorname{Sing}\overline{X} = \{ p_1, \ldots, p_k \}$ and D a divisor on $\overline{X}$. Then $(\overline{X}, D)$ is said to be an *orbifold admissible pair* if the following conditions hold:

(a) $\overline{X}$ is a compact Kähler orbifold.

(b) D is a smooth anticanonical divisor on $\overline{X}$ with $D \cap \operatorname{Sing}\overline{X} = \emptyset$.

(c) the normal bundle $N_{D/\overline{X}}$ is trivial.

(d) $\overline{X}$ and $\overline{X} \setminus (D \sqcup \operatorname{Sing}\overline{X})$ are simply-connected.

(e) Each $p \in \operatorname{Sing}\overline{X}$ has a neighborhood U_p such that there exists a crepant resolution $\widetilde{U}_p \dashrightarrow U_p$ at p.

Throughout this paper, we shall consider the action of $\mathbb{Z}_4$ on $\mathbb{C}^4$ generated by

$$(z_1, z_2, z_3, z_4) \longmapsto (\sqrt{-1}z_1, \sqrt{-1}z_2, \sqrt{-1}z_3, \sqrt{-1}z_4) \quad \text{for} \quad (z_1, z_2, z_3, z_4) \in \mathbb{C}^4.$$

Under the above action, it can be shown that $\mathbb{C}^4/\mathbb{Z}_4$ has a unique crepant resolution. If each U_p in condition (e) is isomorphic to $\mathbb{C}^4/\mathbb{Z}_4$, then we shall call $(\overline{X}, D)$ an orbifold admissible pair with isolated singular points *modelled on* $\mathbb{C}^4/\mathbb{Z}_4$. This kind of orbifold admissible pair plays an important role later in constructing compact Spin(7)-manifolds.

If $\overline{X}$ is smooth, then $\operatorname{Sing}\overline{X} = \emptyset$ and condition (e) is empty, so that the above conditions reduce to the definition of admissible pairs which originates in Kovalev [**15**] and is also used in our papers [**5**], [**6**]. From the above conditions, we see that Lemmas 3.1 and 3.4 apply to admissible pairs. Also, from conditions (a) and (b), we see that D is a compact Kähler manifold with trivial canonical bundle. The following result holds for orbifold admissible pairs $(\overline{X}, D)$, which uses a generalization of Tian–Yau's theorem [**19**] by Haskins–Hein–Nördstrom.

Theorem 3.7 (Haskins–Hein–Nördstrom [**11**]). *Let $(\overline{X}, \omega')$ be a compact Kähler manifold and $m = \dim_{\mathbb{C}} \overline{X}$. If $(\overline{X}, D)$ is an orbifold admissible pair, then the following is true.*

It follows from Lemmas 3.1 *and* 3.4*, there exist a local coordinate system $(U_{D,\alpha}, (z_\alpha^1, \ldots, z_\alpha^{m-1}, w))$ on a neighborhood $U_D = \cup_\alpha U_{D,\alpha}$ of D and a holomorphic volume form Ω on $\overline{X} \setminus D$ such that*

$$\Omega = \frac{\mathrm{d}w}{w} \wedge \mathrm{d}z_\alpha^1 \wedge \cdots \wedge \mathrm{d}z_\alpha^{m-1} \quad \text{on } U_{D,\alpha} \setminus D.$$

Let κ_D be the unique Ricci-flat Kähler form on D in the Kähler class $[\omega'|_D]$. Also let (x_α, y) be local coordinates of $N_{D/\overline{X}} \setminus D$ as in Section 3.1 *and write y as $y = \exp(-t - \sqrt{-1}\theta)$. Now define a holomorphic volume form Ω_{cyl} and a cylindrical Ricci-flat Kähler form ω_{cyl} on $N_{D/\overline{X}} \setminus D$ by*

$$\Omega_{\mathrm{cyl}} = \frac{\mathrm{d}y}{y} \wedge \mathrm{d}x_\alpha^1 \wedge \cdots \wedge \mathrm{d}x_\alpha^{m-1} = (\mathrm{d}t + \sqrt{-1}\mathrm{d}\theta) \wedge \Omega_D,$$
$$\omega_{\mathrm{cyl}} = \kappa_D + \frac{\sqrt{-1}}{2}\frac{\mathrm{d}y \wedge \mathrm{d}\overline{y}}{|y|^2} = \kappa_D + \mathrm{d}t \wedge \mathrm{d}\theta. \tag{3.1}$$

Then there exist a holomorphic volume form Ω and an asymptotically cylindrical Ricci-flat Kähler form ω on $X = \overline{X} \setminus D$ such that

$$\Omega - \Omega_{\mathrm{cyl}} = \mathrm{d}\zeta, \quad \omega - \omega_{\mathrm{cyl}} = \mathrm{d}\xi \quad \textit{for some } \zeta \textit{ and } \xi \textit{ with}$$
$$|\nabla^j_{g_{\mathrm{cyl}}}\zeta|_{g_{\mathrm{cyl}}} = O(e^{-\beta t}), \quad |\nabla^j_{g_{\mathrm{cyl}}}\xi|_{g_{\mathrm{cyl}}} = O(e^{-\beta t})$$
$$\textit{for all } j \geq 0 \textit{ and } \beta \in (0, \min\{1/2, \sqrt{\lambda_1}\}),$$

where λ_1 is the first eigenvalue of the Laplacian $\Delta_{g_D + \mathrm{d}\theta^2}$ acting on $D \times S^1$ with g_D the metric associated with κ_D.

A pair (Ω, ω) consisting of a holomorphic volume form Ω and a Ricci-flat Kähler form ω on an m-dimensional Kähler manifold normalized so that

$$\frac{\omega^m}{m!} = \frac{(\sqrt{-1})^{m^2}}{2^m}\Omega \wedge \overline{\Omega} \text{ (= the volume form)}$$

is called a *Calabi–Yau structure*. The above theorem states that there exists a Calabi–Yau structure (Ω, ω) on X asymptotic to a cylindrical Calabi–Yau structure $(\Omega_{\mathrm{cyl}}, \omega_{\mathrm{cyl}})$ on $N_{D/\overline{X}} \setminus D$ if we multiply Ω by some constant.

3.3. Kähler orbifolds with an antiholomorphic involution and Spin(7) manifolds.

3.3.1. Two basic examples of ALE Spin(7)-manifolds.

Let Φ_0 be the standard Spin(7)-structure on $\mathbb{R}^8 = \{(x_1, x_2, \dots, x_8)\}$. Let α, β act on $\mathbb{R}^8$ by

$$\begin{aligned} \alpha &: (x_1, x_2, \dots, x_8) \longmapsto (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7), \\ \beta &: (x_1, x_2, \dots, x_8) \longmapsto (x_3, -x_4, -x_1, x_2, x_7, -x_8, -x_5, x_6). \end{aligned}$$

Then α, β satisfy $\alpha^4 = \beta^4 = \mathrm{id}_{\mathbb{R}^8}, \alpha\beta = \beta\alpha^3$ and $\alpha^*\Phi_0 = \beta^*\Phi_0 = \Phi_0$, so that the group $G = \langle \alpha, \beta \rangle$ is a subgroup of Spin(7). Define complex coordinates (z_1, z_2, z_3, z_4) and (w_1, w_2, w_3, w_4) on $\mathbb{R}^8$ by

$$\begin{cases} z_1 = x_1 + \sqrt{-1}x_2 \\ z_2 = x_3 + \sqrt{-1}x_4 \\ z_3 = x_5 + \sqrt{-1}x_6 \\ z_4 = x_7 + \sqrt{-1}x_8, \end{cases} \qquad \begin{cases} w_1 = -x_1 + \sqrt{-1}x_3 \\ w_2 = x_2 + \sqrt{-1}x_4 \\ w_3 = -x_5 + \sqrt{-1}x_7 \\ w_4 = x_6 + \sqrt{-1}x_8. \end{cases}$$

Then the coordinates (z_1, z_2, z_3, z_4) and (w_1, w_2, w_3, w_4) define Calabi–Yau structures (ω_0, Ω_0) and (ω_0', Ω_0') on $\mathbb{R}^8$ by

$$\begin{cases} \omega_0 = (\sqrt{-1}/2)\sum_{i=1}^4 \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i \\ \Omega_0 = \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_4, \end{cases} \qquad \begin{cases} \omega_0' = (\sqrt{-1}/2)\sum_{i=1}^4 \mathrm{d}w_i \wedge \mathrm{d}\overline{w}_i \\ \Omega_0' = \mathrm{d}w_1 \wedge \mathrm{d}w_2 \wedge \mathrm{d}w_3 \wedge \mathrm{d}w_4, \end{cases}$$

both of which induce the Spin(7)-structure Φ_0 by

$$\Phi_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \mathrm{Re}\,\Omega_0 = \frac{1}{2}\omega_0' \wedge \omega_0' + \mathrm{Re}\,\Omega_0'.$$

We see that α, β act on these coodinates as

$$\begin{cases} \alpha : (z_1, z_2, z_3, z_4) \longmapsto (\sqrt{-1}z_1, \sqrt{-1}z_2, \sqrt{-1}z_3, \sqrt{-1}z_4) \\ \beta : (z_1, z_2, z_3, z_4) \longmapsto (\overline{z}_2, -\overline{z}_1, \overline{z}_4, -\overline{z}_3), \end{cases}$$

$$\begin{cases} \alpha : (w_1, w_2, w_3, w_4) \longmapsto (\overline{w}_2, -\overline{w}_1, \overline{w}_4, -\overline{w}_3) \\ \beta : (w_1, w_2, w_3, w_4) \longmapsto (\sqrt{-1}w_1, \sqrt{-1}w_2, \sqrt{-1}w_3, \sqrt{-1}w_4). \end{cases}$$

Now we resolve the singularity of $\mathbb{R}^8/G$ in two ways. Let us consider the action of α on $\mathbb{C}^4$ in the z-coordinates. Then we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\beta} \curvearrowright \quad \mathcal{Y}_1 & \longrightarrow\!\!\!\!\!\rightarrow & \mathcal{X}_1 \\ \text{crepant} \downarrow & & \downarrow \pi_1 \\ \underline{\beta} \curvearrowright \mathbb{C}^4/\langle\alpha\rangle & \longrightarrow\!\!\!\!\!\rightarrow & \mathbb{R}^8/G, \end{array}$$

where $\underline{\beta}$ is an antiholomorphic involution on $\mathbb{C}^4/\langle\alpha\rangle$ induced by β, and $\widetilde{\beta}$ is the lift of $\underline{\beta}$ which acts freely on $\mathcal{Y}_1$. Since there exists an ALE Calabi–Yau structure $(\widetilde{\omega}_1, \widetilde{\Omega}_1)$ on $\mathcal{Y}_1$ with

$$\widetilde{\beta}^*\widetilde{\omega}_1 = -\widetilde{\omega}_1, \quad \widetilde{\beta}^*\widetilde{\Omega}_1 = \overline{(\widetilde{\Omega}_1)},$$

the induced torsion-free Spin(7)-structure $\widetilde{\Phi}_1 = (1/2)\widetilde{\omega}_1 \wedge \widetilde{\omega}_1 + \mathrm{Re}\,\widetilde{\Omega}_1$ pushes down to a torsion-free Spin(7)-structure Φ_1 on $\mathcal{X}_1$. This gives a resolution of $\mathbb{R}^8/G$ by an ALE Spin(7)-manifold $(\mathcal{X}_1, \Phi_1)$. Similarly, if we consider the action of β on $\mathbb{C}^4$ in the w-coordinate, then we have

$$\begin{array}{ccc} \widetilde{\alpha} \curvearrowright \quad \mathcal{Y}_2 & \longrightarrow\!\!\!\!\!\rightarrow & \mathcal{X}_2 \\ \text{crepant} \downarrow & & \downarrow \pi_2 \\ \underline{\alpha} \curvearrowright \mathbb{C}^4/\langle\beta\rangle & \longrightarrow\!\!\!\!\!\rightarrow & \mathbb{R}^8/G. \end{array}$$

If we consider

$$\begin{aligned} \phi : (z_1, z_2, z_3, z_4) &\longmapsto (w_1, w_2, w_3, w_4), \quad \text{that is,} \\ (x_1, x_2, \ldots, x_8) &\longmapsto (-x_1, x_3, x_2, x_4, -x_5, x_7, x_6, x_8), \end{aligned}$$

then ϕ induces an isomorphism $\mathbb{C}^4/\langle\alpha\rangle \xrightarrow{\cong} \mathbb{C}^4/\langle\beta\rangle$, which lifts to an isomorphism $\widetilde{\phi} : \mathcal{Y}_1 \xrightarrow{\cong} \mathcal{Y}_2$. Let Φ_2 be a Spin(7)-structure on $\mathcal{X}_2$ to which the Spin(7)-structure $(\widetilde{\phi^{-1}})^*\widetilde{\Phi}_1$ on $\mathcal{Y}_2$ pushes down. Then $(\mathcal{X}_2, \Phi_2)$ is another ALE Spin(7)-manifold which resolves $\mathbb{R}^8/G$, but *topologically distinct* because ϕ does not commute with α, β, so that the isomorphism ϕ acts nontrivially on $\mathbb{R}^8/G$.

Proposition 3.8 (Joyce [**14**, Section 15.1.1]). *Let $(\mathcal{X}_s, \Phi_s)$ for $s = 1, 2$ be ALE* Spin(7)*-manifolds as above. Then the fundamental group of $\mathcal{X}_s$ is $\mathbb{Z}_2$, and*

$$b^i(\mathcal{X}_s) = \begin{cases} 1 & \textit{if } i = 0, 4 \\ 0 & \textit{otherwise,} \end{cases} \quad \textit{so that} \quad \chi(\mathcal{X}_s) = 2. \tag{3.2}$$

3.3.2. Compatible antiholomorphic involutions on orbifold admissible pairs.

Proposition 3.9. *Let X be a complex orbifold and $\sigma : X \longrightarrow X$ be an antiholomorphic involution. Suppose S is a complex submanifold of X such that σ preserves and* *acts freely on S. Then σ lifts to a unique antiholomorphic involution $\widetilde{\sigma}$ on the blow-up $\varpi : \mathrm{Bl}_S(X) \dashrightarrow X$ of X along S such that $\widetilde{\sigma}$ preserves and acts freely on $\varpi^{-1}(S)$.*

Proof. Let $m = \dim_{\mathbb{C}} X$ and $k = \dim_{\mathbb{C}} S$. Fix a point $x \in S$. It is enough to find a lift $\widetilde{\sigma}$ of σ acting on a neighborhood of $\varpi^{-1}(x)$ in $\mathrm{Bl}_S(X)$.

First we consider local coordinates near x and $\sigma(x)$ in X. We can choose a neighborhood U of $x \in S$ and local coordinates $(\boldsymbol{y}, \boldsymbol{z}) = (y_1, \ldots, y_k, z_1, \ldots, z_{m-k})$ on U such that $S \cap U = \{ \boldsymbol{z} = \boldsymbol{0} \}$. We can similarly choose local coordinates $(\boldsymbol{y}', \boldsymbol{z}') = (y'_1, \ldots, y'_k, z'_1, \ldots, z'_{m-k})$ on $\sigma(U)$ such that $\sigma(S \cap U) = \{ \boldsymbol{z}' = \boldsymbol{0} \}$ and

$$(\boldsymbol{y}', \boldsymbol{z}') = \sigma(\boldsymbol{y}, \boldsymbol{z}) = (\alpha(\boldsymbol{y}, \boldsymbol{z}), \beta(\boldsymbol{y}, \boldsymbol{z}))$$

for some antiholomorphic functions $\alpha : \mathbb{C}^m \longrightarrow \mathbb{C}^k$ and $\beta : \mathbb{C}^m \longrightarrow \mathbb{C}^{m-k}$. Also, $\sigma(S) = S$ yields that for $(\boldsymbol{y}, \boldsymbol{0}) \in S \cap U$ we have

$$\sigma(\boldsymbol{y}, \boldsymbol{0}) = (\alpha(\boldsymbol{y}, \boldsymbol{0}), \boldsymbol{0}), \quad \text{that is,} \quad \beta(\boldsymbol{y}, \boldsymbol{0}) = \boldsymbol{0}. \tag{3.3}$$

Next we consider local coordinates near $\varpi^{-1}(x)$ and $\varpi^{-1}(\sigma(x))$ in $\mathrm{Bl}_S(X)$. Local coordinates of $\mathrm{Bl}_S(X)$ on $\varpi^{-1}(U)$ are written as

$$\{ (\boldsymbol{y}, \boldsymbol{z}, [\boldsymbol{\zeta}]) \in \mathbb{C}^m \times \mathbb{C}P^{m-k-1} \mid z_i \zeta_j = z_j \zeta_i \text{ for all } i, j \in \{ 1, \ldots, m-k \} \},$$

where $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_{m-k}) \in \mathbb{C}^{m-k}$. Similarly, local coordinates of $\mathrm{Bl}_S(X)$ on $\varpi^{-1}(\sigma(U))$ are written as

$$\{ (\boldsymbol{y}', \boldsymbol{z}', [\boldsymbol{\zeta}']) \in \mathbb{C}^m \times \mathbb{C}P^{m-k-1} \mid z'_i \zeta'_j = z'_j \zeta'_i \text{ for all } i, j \in \{ 1, \ldots, m-k \} \}.$$

Thus we have

$$\begin{aligned} \varpi^{-1}(\boldsymbol{y}, \boldsymbol{z}) &= \{ (\boldsymbol{y}, \boldsymbol{z}, [\boldsymbol{z}]) \} && \text{for} \quad (\boldsymbol{y}, \boldsymbol{z}) \in U \setminus S \quad (\text{and so } \boldsymbol{z} \neq \boldsymbol{0}), \\ \varpi^{-1}(\boldsymbol{y}, \boldsymbol{0}) &= \{ (\boldsymbol{y}, \boldsymbol{0}, [\boldsymbol{\zeta}]) \mid [\boldsymbol{\zeta}] \in \mathbb{C}P^{m-k-1} \} && \text{for} \quad (\boldsymbol{y}, \boldsymbol{0}) \in S \cap U. \end{aligned}$$

Now we shall find a lift $\widetilde{\sigma}$ of σ acting on $\varpi^{-1}(U)$. For $(\boldsymbol{y}, \boldsymbol{z}) \in U \setminus S$, we must have

$$\widetilde{\sigma}(\boldsymbol{y}, \boldsymbol{z}, [\boldsymbol{z}]) = (\sigma(\boldsymbol{y}, \boldsymbol{z}), [\beta(\boldsymbol{y}, \boldsymbol{z})]).$$

Then $\widetilde{\sigma}$ extends naturally to $\varpi^{-1}(S \cap U)$ by continuity as

$$\begin{aligned} \widetilde{\sigma}(\boldsymbol{y}, \boldsymbol{0}, [\boldsymbol{\zeta}]) &= \lim_{\lambda \to 0} \widetilde{\sigma}(\boldsymbol{y}, \lambda\boldsymbol{\zeta}, [\lambda\boldsymbol{\zeta}]) \\ &= \lim_{\lambda \to 0} (\alpha(\boldsymbol{y}, \lambda\boldsymbol{\zeta}), \beta(\boldsymbol{y}, \lambda\boldsymbol{\zeta}), [\beta(\boldsymbol{y}, \lambda\boldsymbol{\zeta})]) \\ &= \left(\alpha(\boldsymbol{y}, \boldsymbol{0}), \boldsymbol{0}, \left[\sum_{i=1}^{m-k} \overline{\mathrm{D}}_{k+i} \beta(\boldsymbol{y}, \boldsymbol{0}) \overline{\zeta}_i \right] \right), \end{aligned} \tag{3.4}$$

where $\overline{\mathrm{D}}_j$ is the antiholomorphic partial differentiation with respect to the j-th variable. Since σ is an antiholomorphic diffeomorphism on X, the matrix $(\overline{\mathrm{D}}_i \sigma_j(\boldsymbol{y}, \boldsymbol{z}))_{1 \leq i,j \leq m}$ is invertible for all $(\boldsymbol{y}, \boldsymbol{z}) \in U$. In particular, the invertiblility of $(\overline{\mathrm{D}}_i \sigma_j(\boldsymbol{y}, \boldsymbol{0}))_{1 \leq i,j \leq m}$ leads to the invertiblity of $(\overline{\mathrm{D}}_{k+i}\beta_j(\boldsymbol{y},\mathbf{0}))_{1\le i,j\le m-k}$. Hence (3.4) gives the desired action of $\widetilde{\sigma}$ on the neighborhood $\varpi^{-1}(U)$ of $\varpi^{-1}(x)$ in $\mathrm{Bl}_S(X)$. □

DEFINITION 3.10. Let $\overline{X}$ be a four-dimensional compact Kähler orbifold with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, such that $(\overline{X}, D)$ is an orbifold admissible pair. An antiholomorphic involution σ on $\overline{X}$ is said to be *compatible with* $(\overline{X}, D)$ if the following conditions hold:

(f) We can choose a defining function w on a neighborhood U_D of D given in Lemma 3.4 so that

$$\sigma^* w = \overline{w}, \tag{3.5}$$

where the complex conjugate $\overline{f}$ for a complex function f is defined by $\overline{f}(x) = \overline{f(x)}$.

(g) $(\overline{X})^\sigma = \operatorname{Sing}\overline{X}$, where $(\overline{X})^\sigma$ is the fixed point set of the action of σ on $\overline{X}$.

Note that (3.5) in condition (f) implies $\sigma(D) = D$, and $\sigma_D = \sigma|_D$ yields an antiholomorphic involution on D.

LEMMA 3.11. *Let σ_{cyl} be an antiholomorphic involution on $N_{D/\overline{X}}$ defined by*

$$\sigma_{\mathrm{cyl}}(x_\alpha, y) = (\sigma_D(x_\alpha), \overline{y}) \quad \textit{for} \quad (x_\alpha, y) \in (U_\alpha \cap D) \times \mathbb{C} \subset N_{D/\overline{X}}. \tag{3.6}$$

Then we have

$$\sigma(z_\alpha, w) = \sigma_{\mathrm{cyl}}(x_\alpha, y) + O(e^{-t}).$$

PROOF. Using (3.5), we can write $\sigma(z_\alpha, w)$ as

$$\sigma(z_\alpha, w) = (\sigma_1(z_\alpha, w), \overline{w}) \quad \text{with} \quad \sigma_1(x_\alpha, 0) = \sigma_D(x_\alpha). \tag{3.7}$$

Thus the assertion follows from Lemma 3.3. □

Since the cylindrical parameter t is defined by $y = \exp(-t - \sqrt{-1}\theta)$, we have

$$\sigma_{\mathrm{cyl}}^* t = t, \quad \sigma_{\mathrm{cyl}}^* \theta = -\theta$$

and thus

$$(N_{D/\overline{X}} \setminus D)/\langle\sigma_{\mathrm{cyl}}\rangle \simeq \big((D \times S^1)/\langle\sigma_{D\times S^1,\mathrm{cyl}}\rangle\big) \times \mathbb{R}_+, \tag{3.8}$$

where $\sigma_{D\times S^1,\mathrm{cyl}}$ acts on $D \times S^1$ as

$$\sigma_{D\times S^1,\mathrm{cyl}}(x_\alpha, \theta) = (\sigma_D(x_\alpha), -\theta). \tag{3.9}$$

One can prove the following result by Theorem 3.7 and an argument as used in the proof of [**14**, Proposition 15.2.2].

Theorem 3.12. *Let $(\overline{X}, \omega')$ be a four-dimensional Kähler orbifold with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, such that $(\overline{X}, D)$ is an orbifold admissible pair with a compatible antiholomorphic involution σ. Then there exists an asymptotically cylindrical Calabi–Yau structure (ω, Ω) on $X = \overline{X} \setminus D$ asymptotic to $(\omega_{\mathrm{cyl}}, \Omega_{\mathrm{cyl}})$ on $N \setminus D$, such that*

$$\sigma^* g = g, \quad \sigma^* \omega = -\omega, \quad \sigma^* \Omega = \overline{\Omega},$$

where $N = N_{D/\overline{X}}$ and g is the Riemannian metric on X associated with (ω, Ω). Thus the torsion-free $\mathrm{Spin}(7)$*-structure* $(1/2)\omega \wedge \omega + \mathrm{Re}\, \Omega$ *on X pushes down to a torsion-free* $\mathrm{Spin}(7)$*-structure Φ on $X/\langle \sigma \rangle$. Also, an antiholomorphic involution σ_{cyl} defined in* (3.9) *satisfies*

$$\sigma_{\mathrm{cyl}}^* g_{\mathrm{cyl}} = g_{\mathrm{cyl}}, \quad \sigma_{\mathrm{cyl}}^* \omega_{\mathrm{cyl}} = -\omega_{\mathrm{cyl}}, \quad \sigma_{\mathrm{cyl}}^* \Omega_{\mathrm{cyl}} = \overline{\Omega_{\mathrm{cyl}}},$$

so that the torsion-free $\mathrm{Spin}(7)$*-structure* $(1/2)\omega_{\mathrm{cyl}} \wedge \omega_{\mathrm{cyl}} + \mathrm{Re}\, \Omega_{\mathrm{cyl}}$ *pushes down to a torsion-free* $\mathrm{Spin}(7)$*-structure Φ_{cyl}. We have*

$$\begin{aligned} &\Phi - \Phi_{\mathrm{cyl}} = \mathrm{d}\Xi, \quad \text{for some } \Xi \text{ with} \\ &|\nabla^j_{g_{\mathrm{cyl}}} \Xi|_{g_{\mathrm{cyl}}} = O(e^{-\beta t}), \quad \text{for all } j \geq 0 \text{ and } 0 < \beta < \min\{ 1/2, \sqrt{\lambda_1} \}, \end{aligned} \tag{3.10}$$

where λ_1 is the constant given in Theorem 3.7*. Hence $(X/\langle \sigma \rangle, \Phi)$ is an asymptotically cylindrical* $\mathrm{Spin}(7)$*-manifold, with the asymptotic model $((N \setminus D)/\langle \sigma_{\mathrm{cyl}} \rangle, \Phi_{\mathrm{cyl}})$, with*

$$\begin{aligned} &(N \setminus D)/\langle \sigma_{\mathrm{cyl}} \rangle \simeq \big((D \times S^1)/\langle \sigma_{D \times S^1, \mathrm{cyl}} \rangle \big) \times \mathbb{R}_+ = \{ ([x_\alpha, \theta], t) \}, \\ &\text{where} \quad [x_\alpha, \theta] = [\sigma_D(x_\alpha), -\theta] \quad \text{in} \quad (D \times S^1)/\langle \sigma_{D \times S^1, \mathrm{cyl}} \rangle. \end{aligned}$$

Theorem 3.13 (Joyce [**14**, Proposition 15.2.3 and Corollary 15.2.4]). *All isolated singular points of $X/\langle \sigma \rangle$ are modelled on $\mathbb{R}^8/G$ given in Section* 3.3.1*. For each $p \in \mathrm{Sing}\, X/\langle \sigma \rangle$ there exists an isomorphism $\iota_p : \mathbb{R}^8/G \longrightarrow T_p(X/\langle \sigma \rangle)$, which identifies the* $\mathrm{Spin}(7)$*-structures Φ_0 on $\mathbb{R}^8$ and Φ on $T_p(X/\langle \sigma \rangle)$.*

3.4. Gluing orbifold admissible pairs divided by compatible antiholomorphic involutions.

In this subsection we will only consider orbifold admissible pairs $(\overline{X}, D)$ with $\dim_{\mathbb{C}} \overline{X} = 4$. Also, we will denote $N = N_{D/\overline{X}}$ and $X = \overline{X} \setminus D$.

3.4.1. The gluing condition.

Let $(\overline{X}, \omega')$ be a four-dimensional compact Kähler orbifold with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, and $(\overline{X}, D)$ be an orbifold admissible pair with a compatible antiholomorphic involution σ. Then we obtained in Theorem 3.12 an asymptotically cylindrical, torsion-free Spin(7)-manifold (X, Φ), with the asymptotic model $(N \setminus D, \Phi_{\mathrm{cyl}})$.

Next we consider the condition under which we can glue together $X_1/\langle \sigma_1 \rangle$ and $X_2/\langle \sigma_2 \rangle$ obtained from orbifold admissible pairs $(\overline{X}_1, D_1)$ and $(\overline{X}_2, D_2)$ with antiholomorphic involutions σ_i. For gluing $X_1/\langle \sigma_1 \rangle$ and $X_2/\langle \sigma_2 \rangle$ to obtain a manifold with a Spin(7)-structure with small torsion, we would like $(X_1/\langle \sigma_1 \rangle, \Phi_1)$ and $(X_2/\langle \sigma_2 \rangle, \Phi_2)$ to have the same asymptotic model. Thus we put the following

Gluing condition. There exists an isomorphism $\widetilde{f} : D_1 \longrightarrow D_2$ between the cross-sections of the cylindrical ends of $\overline{X}_i \setminus D_i$ with

$$\widetilde{f} \circ \sigma_1|_{D_1} = \sigma_2|_{D_2} \circ \widetilde{f},$$

such that

$$\widetilde{f}_T^* \left(\frac{1}{2}\omega_{2,\mathrm{cyl}} \wedge \omega_{2,\mathrm{cyl}} + \operatorname{Re}\Omega_{2,\mathrm{cyl}} \right) = \frac{1}{2}\omega_{1,\mathrm{cyl}} \wedge \omega_{1,\mathrm{cyl}} + \operatorname{Re}\Omega_{1,\mathrm{cyl}}, \tag{3.11}$$

where $\widetilde{f}_T : D_1 \times S^1 \times (0, 2T) \longrightarrow D_2 \times S^1 \times (0, 2T)$ is defined by

$$\widetilde{f}_T(x_1, \theta_1, t) = (\widetilde{f}(x_1), -\theta_1, 2T - t) \quad \text{for } (x_1, \theta_1, t) \in D_1 \times S^1 \times (0, 2T).$$

Lemma 3.14. *If $\widetilde{f} : D_1 \longrightarrow D_2$ is an isomorphism satisfying $\widetilde{f} \circ \sigma_1|_{D_1} = \sigma_2|_{D_2} \circ \widetilde{f}$ and $\widetilde{f}^* \kappa_{D_2} = \kappa_{D_1}$. Then the gluing condition* (3.11) *holds, where we change the sign of* $\Omega_{2,\mathrm{cyl}}$ (*and also the sign of* Ω_2 *correspondingly*).

Proof. It follows by a straightforward calculation using (3.1) and Lemma 3.11. □

The above $\widetilde{f}$ and $\widetilde{f}_T$ pushes down to maps

$$f : D_1/\langle \sigma_{D_1} \rangle \longrightarrow D_2/\langle \sigma_{D_2} \rangle,$$
$$f_T : \big((D_1 \times S^1)/\langle \sigma_{D_1 \times S^1,\mathrm{cyl}} \rangle\big) \times (0, 2T) \longrightarrow \big((D_2 \times S^1)/\langle \sigma_{D_2 \times S^1,\mathrm{cyl}} \rangle\big) \times (0, 2T),$$
$$\text{with} \quad f([x_1]) = [\widetilde{f}(x_1)], \quad f_T([x_1, \theta_1], t) = ([\widetilde{f}(x_1), -\theta_1], 2T - t)$$

such that

$$f_T^* \Phi_{2,\mathrm{cyl}} = \Phi_{1,\mathrm{cyl}}.$$

3.4.2. Spin(7)-structures with small torsion.

Now we shall glue $X_1/\langle \sigma_1 \rangle$ and $X_2/\langle \sigma_2 \rangle$ under the gluing condition (3.11). Let $\rho : \mathbb{R} \longrightarrow [0, 1]$ denote a smooth cut-off function

$$\rho(x) = \begin{cases} 1 & \text{if } x \le 0, \\ 0 & \text{if } x \ge 1, \end{cases}$$

and define $\rho_T : \mathbb{R} \longrightarrow [0, 1]$ by

$$\rho_T(x) = \rho(x - T + 1) = \begin{cases} 1 & \text{if } x \le T - 1, \\ 0 & \text{if } x \ge T. \end{cases}$$

Setting an approximating Calabi–Yau structure $(\Omega_{i,T}, \omega_{i,T})$ on X_i by

$$\Omega_{i,T} = \begin{cases} \Omega_i - \mathrm{d}(1 - \rho_{T-1})\zeta_i & \text{on } \{t_i \le T - 1\}, \\ \Omega_{i,\mathrm{cyl}} + \mathrm{d}\rho_{T-1}\zeta_i & \text{on } \{t_i \ge T - 2\} \end{cases}$$

and similarly

$$\omega_{i,T} = \begin{cases} \omega_i - \mathrm{d}(1-\rho_{T-1})\xi_i & \text{on } \{t_i \le T-1\}, \\ \omega_{i,\mathrm{cyl}} + \mathrm{d}\rho_{T-1}\xi_i & \text{on } \{t_i \ge T-2\}, \end{cases}$$

we can define a d-closed 4-form $\widetilde{\Phi}_{i,T}$ on each $X_i/\langle\sigma_i\rangle$ by

$$\widetilde{\Phi}_{i,T} = \pi_{i*}\left(\frac{1}{2}\omega_{i,T}\wedge\omega_{i,T} + \mathrm{Re}\,\Omega_T\right),$$

where $\pi_i : X_i \longrightarrow X_i/\langle\sigma_i\rangle$ are projections. We see that $\widetilde{\Phi}_{i,T}$ satisfies

$$\widetilde{\Phi}_{i,T} = \begin{cases} \Phi_i & \text{on } \{t_i < T-2\}, \\ \Phi_{i,\mathrm{cyl}} & \text{on } \{t_i > T-1\} \end{cases}$$

and from (3.10) that

$$|\widetilde{\Phi}_{i,T} - \Phi_{i,\mathrm{cyl}}|_{g_{\Phi_{i,\mathrm{cyl}}}} = O(e^{-\beta T}) \quad \text{for all } \beta \in (0, \min\{\,1/2, \sqrt{\lambda_1}\,\}). \tag{3.12}$$

Let $X_{1,T} = \{t_1 < T+1\} \subset X_1$ and $X_{2,T} = \{t_2 < T+1\} \subset X_2$. We glue $X_{1,T}/\langle\sigma_1\rangle$ and $X_{2,T}/\langle\sigma_2\rangle$ along $((D_1\times S^1)/\langle\sigma_{D_1\times S^1,\mathrm{cyl}}\rangle)\times\{T-1<t_1<T+1\}\subset X_{1,T}/\langle\sigma_1\rangle$ and $((D_2\times S^1)/\langle\sigma_{D_2\times S^1,\mathrm{cyl}}\rangle)\times\{T-1<t_2<T+1\}\subset X_{2,T}/\langle\sigma_2\rangle$ to construct a compact 8-orbifold using the gluing map f_T (more precisely, $F_T = \varphi_2\circ f_T\circ\varphi_1^{-1}$, where φ_1 and φ_2 are the diffeomorphisms given in Lemma 3.3). We denote this orbifold by $M_T^\triangledown$ (the upper index $\triangledown$ indicates singularities to be resolved). Also, we can glue together $\widetilde{\Phi}_{1,T}$ and $\widetilde{\Phi}_{2,T}$ to obtain a d-closed 4-form $\widetilde{\Phi}_T$ on $M_T^\triangledown$ by Lemma 3.14. There exists a positive constant T_* such that $\widetilde{\Phi}_T \in C^\infty(\mathcal{T}(M_T^\triangledown))$ for any T with $T > T_*$. This $\widetilde{\Phi}_T$ is what was discussed right after Theorem 2.7, from which we can define a Spin(7)-structure Φ_T *with small torsion* by $\Phi_T = \Theta(\widetilde{\Phi}_T)$. Letting $\phi_T = \widetilde{\Phi}_T - \Phi_T$, we have $\mathrm{d}\phi_T + \mathrm{d}\Phi_T = 0$.

Proposition 3.15. *Let $T > T_*$. Then there exist constants $A_{p,k,\beta}$ independent of T such that for $\beta \in (0, \min\{\,1/2, \sqrt{\lambda_1}\,\})$ we have*

$$\|\phi_T\|_{L^p_k} \le A_{p,k,\beta}\, e^{-\beta T},$$

where all norms are measured using g_{Φ_T}.

Proof. These estimates follow in a straightforward way from Theorem 3.7 and (3.12) by an argument similar to those in [**4**, Section 3.5]. □

3.4.3. Resolving $M_T^\triangledown$ by ALE Spin(7)-manifolds $\mathcal{X}_1$ and $\mathcal{X}_2$.

Let $p \in \mathrm{Sing}\, M_T^\triangledown$ and $\iota_p : \mathbb{R}^8/G \longrightarrow T_pM_T^\triangledown$ as in Theorem 3.13. Let $\exp_p : T_pM_T^\triangledown \longrightarrow M_T^\triangledown$ be the exponential map. Then $\psi_p = \exp_p\circ\iota_p$ maps each ball $B_{2\zeta}(\mathbb{R}^8/G)$ of 2ζ in $\mathbb{R}^8/G$ to a neighborhood of $p \in M_T^\triangledown$. Choose $\zeta > 0$ small so that $U_p = \exp_p\circ\iota_p(B_{2\zeta}(\mathbb{R}^8/G))$ satisfy $U_p\cap U_{p'} = \emptyset$ and $U_p \cap \{\,T-2<t_i<T+1\,\} = \emptyset, i = 1,2$ for any $p, p' \in M_T^\triangledown$ with $p \ne p'$ and for any $T > T_*$.

Proposition 3.16 (Joyce [**14**, Proposition 15.2.6]). *There exists a smooth* 3*-form* σ_p *on* $B_{2\zeta}(\mathbb{R}^8/G)$ *for each* $p \in \operatorname{Sing} M_T^\nabla$ *and a constant* $C_1 > 0$ *independent of* $T > T_*$*, such that*

$$\psi_p^* \Phi_T - \Phi_0 = \mathrm{d}\sigma_p, \quad |\nabla^\ell \sigma_p| \leq C_1 r^{3-\ell} \quad \text{for } \ell = 0, 1, 2$$

on $B_{2\zeta}(\mathbb{R}^8/G)$*. Here* $|\cdot|$ *and* ∇ *is defined by the metric* g_0 *induced by* Φ_0*, and* r *is the radius function on* $\mathbb{R}^8/G$*.*

Let $\pi_s : \mathcal{X}_s \longrightarrow \mathbb{R}^8/G$ be the projections given in Section 3.3.1. For each $\epsilon \in (0,1]$ and $s = 1,2$ let $\mathcal{X}_s^\epsilon = \mathcal{X}_s$, define a Spin(7)-structure $\Phi_s^\epsilon = \epsilon^4 \Phi_s$ and define $\pi_s^\epsilon : \mathcal{X}_s^\epsilon \longrightarrow \mathbb{R}^8/G$ by $\pi_s^\epsilon = \epsilon \pi_s$. Then $(\mathcal{X}_s^\epsilon, \Phi_s^\epsilon)$ is an ALE Spin(7)-manifold asymptotic to $\mathbb{R}^8/G$.

Proposition 3.17 (Joyce [**14**, Equation (15.6)]). *There exist a constant* $C_2 > 0$ *independent of* $T > T_*$*, and a smooth* 3*-form* τ_s^ϵ *on* $\mathbb{R}^8/G \setminus B_{\epsilon\zeta}(\mathbb{R}^8/G)$ *such that*

$$(\pi_s^\epsilon)_* \Phi_s^\epsilon - \Phi_0 = \mathrm{d}\tau_s^\epsilon, \quad |\nabla^\ell \tau_s^\epsilon| \leq C_2 \epsilon^8 r^{-7-\ell} \quad \text{for } \ell = 0, 1, 2$$

on $\mathbb{R}^8/G \setminus B_{\epsilon\zeta}(\mathbb{R}^8/G)$*.*

Now we glue together

$$U_T^\epsilon = M_T^\nabla \setminus \bigcup_{p \in \operatorname{Sing} M_T^\nabla} \psi_p(\overline{B}_{\epsilon^{4/5}\zeta}(\mathbb{R}^8/G)) \quad \text{and}$$
$$V_p^\epsilon = (\pi_{s_p}^\epsilon)^{-1}(B_{2\epsilon^{4/5}\zeta}(\mathbb{R}^8/G)), \quad s_p \in \{1, 2\},$$

along the regions diffeomorphic to

$$B_{2\epsilon^{4/5}\zeta}(\mathbb{R}^8/G) \setminus \overline{B}_{\epsilon^{4/5}\zeta}(\mathbb{R}^8/G) \quad \text{in } \mathbb{R}^8/G,$$

to obtain a compact 8-manifold M_T^ϵ. Choosing $s_p \in \{1, 2\}$ for each $p \in \operatorname{Sing} M_T^\nabla$, we can also glue the Spin(7)-structures Φ_T on M_T^∇ and $\Phi_{s_p}^\epsilon$ on $\mathcal{X}_{s_p}^\epsilon$ to obtain a closed 4-form $\widetilde{\Phi}_T^\epsilon$ on M_T^ϵ by

$$\widetilde{\Phi}_T^\epsilon = \Phi_0 + \mathrm{d}(\rho_{\epsilon^{-4/5}r} \sigma_p) + \mathrm{d}((1 - \rho_{\epsilon^{-4/5}r}) \tau_{s_p}^\epsilon) \quad \text{on } U_T^\epsilon \cap V_p^\epsilon.$$

Now we set $\epsilon = \exp(-\gamma T)$ for some constant $\gamma > 0$ to be determined later, and define $M^\epsilon = M_T^\epsilon$, $\widetilde{\Phi}^\epsilon = \widetilde{\Phi}_T^\epsilon$ and $U^\epsilon = U_T^\epsilon$.

Proposition 3.18 (Joyce [**14**, Proposition 15.2.9]). *If* $s_p = 1$ *for all* $p \in \operatorname{Sing} M_T^\nabla$*, then the fundamental group of* M^ϵ *is* $\mathbb{Z}_2$*. Otherwise,* M^ϵ *is simply-connected.*

The following result is a consequence of Propositions 3.16 and 3.17.

Lemma 3.19 (Joyce, [**14**, Lemma 15.2.11]). *There exists a constant* $C_3 > 0$ *independent of* $T > T_*$ *such that* $\widetilde{\Phi}_T$ *satisfies*

$$|\widetilde{\Phi}^\epsilon - \Phi_0| \leq C_3 \epsilon^{8/5}, \quad |\nabla(\widetilde{\Phi}^\epsilon - \Phi_0)| \leq C_3 \epsilon^{4/5}$$

on $U^\epsilon \cap V_p^\epsilon$, *where* $|\cdot|$ *and* ∇ *is defined using the metric* g_0 *induced by* Φ_0.

Letting $\Phi^\epsilon = \Theta(\widetilde{\Phi}^\epsilon)$ and $\phi^\epsilon = \widetilde{\Phi}^\epsilon - \Phi^\epsilon$, we have $\mathrm{d}\phi^\epsilon + \mathrm{d}\Phi^\epsilon = 0$.

Theorem 3.20. *There exists a family* $(M^\epsilon, \Phi^\epsilon)$ *of smooth 8-manifolds with a* Spin(7)*-structure with small torsion and resolutions* $\pi^\epsilon : M^\epsilon \longrightarrow M^\nabla$ *for* $\epsilon \in (0,1]$ *such that we have*

(i) $\|\phi^\epsilon\|_{L^2} \leq \lambda\epsilon^{24/5}$ *and* $\|\mathrm{d}\phi^\epsilon\|_{L^{10}} \leq \lambda\epsilon^{36/25}$,

(ii) *the injectivity radius* $\delta(g)$ *satisfies* $\delta(g) \geq \mu\epsilon$, *and*

(iii) *the Riemann curvature* $R(g)$ *satisfies* $\|R(g)\|_{C^0} \leq \nu\epsilon^{-2}$,

where all norms are measured using the metric g^ϵ *on* M^ϵ *induced by* Φ^ϵ.

Proof. The proof is almost the same as that of [**14**, Proposition 15.2.13] except for the contributions from the cylinder, which is diffeomorphic to $\Sigma \times (0, 2T)$ with $\Sigma = (D \times S^1)/\langle \sigma_{D\times S^1,\mathrm{cyl}}\rangle$. Joyce proved using Lemma 3.19 that

$$\sum_{p\in \mathrm{Sing}\, M_T^\nabla} \int_{U^\epsilon\cap V_p^\epsilon} |\phi^\epsilon|^2 \leq \lambda^2\epsilon^{48/5}, \quad \sum_{p\in M_T^\nabla} \int_{U^\epsilon\cap V_p^\epsilon} |\mathrm{d}\phi^\epsilon|^2 \leq \lambda^{10}\epsilon^{72/5}.$$

Meanwhile, Proposition 3.15 gives

$$\int_{\Sigma\times(0,2T)} |\phi_T|^2 \leq 2{A_\beta}^2 e^{-2\beta T}, \quad \int_{\Sigma\times(0,2T)} |\mathrm{d}\phi_T|^{10} \leq 2{A_\beta}^{10} e^{-10\beta T},$$

where we take $\beta \in (0, \max\{1/2, \sqrt{\lambda_1}\})$ and $A_\beta = \max\{A_{2,0,\beta}, A_{10,1,\beta}\}$. Now if we choose $\gamma > 0$ for $\epsilon = e^{-\gamma T}$ so that $(24/5)\gamma \leq \beta$, then we have $e^{-2\beta T} \leq \epsilon^{48/5}$ and $e^{-10\beta T} \leq \epsilon^{72/5}$. Summing up the above contributions and redefining λ to be $\max\{(\lambda^2 + 2{A_\beta}^2)^{1/2}, (\lambda^{10} + 2{A_\beta}^{10})^{1/10}\}$, we see that condition (i) holds. Conditions (ii) and (iii) are obvious. □

3.5. Gluing theorems.

First we give a gluing and a doubling construction of Calabi–Yau fourfolds from *orbifold* admissible pairs, which are generalizations of Theorem 3.10 and Corollary 3.11 in [**6**].

Theorem 3.21. *Let* $(\overline{X}_1, \omega_1')$ *and* $(\overline{X}_2, \omega_2')$ *be compact Kähler orbifolds with* $\dim_{\mathbb{C}} \overline{X}_i = 4$ *such that* $(\overline{X}_1, D_1)$ *and* $(\overline{X}_2, D_2)$ *are orbifold admissible pairs. Suppose there exists an isomorphism* $f : D_1 \longrightarrow D_2$ *such that* $f^*\kappa_2 = \kappa_1$, *where* κ_i *is the unique Ricci-flat Kähler form on* D_i *in the Kähler class* $[\omega_i'|_{D_i}]$. *Then we can glue together the crepant resolutions of* X_1 *and* X_2 *along their cylindrical ends to obtain a compact simply-connected* 8*-manifold* M. *The manifold* M *admits a Riemannian metric with holonomy contained in* Spin(7). *Moreover, if* $\widehat{A}(M) = 2$, *then* M *is a Calabi–Yau fourfold, i.e.,* M *admits a Ricci-flat Kähler metric with holonomy* SU(4).

Corollary 3.22. *Let $(\overline{X}, D)$ be an orbifold admissible pair with $\dim_{\mathbb{C}} \overline{X} = 4$. Then we can glue two copies of the crepant resolution of X along their cylindrical ends to obtain a compact simply-connected 8-manifold M. Then M admits a Riemannian metric with holonomy contained in* Spin(7)*. If $\widehat{A}(M) = 2$, then the manifold M is a Calabi–Yau fourfold.*

Next we give a gluing and a doubling construction of compact Spin(7)-manifolds.

Theorem 3.23. *Let $(\overline{X}_1, \omega_1')$ and $(\overline{X}_2, \omega_2')$ be four-dimensional compact Kähler orbifolds with singularities such that $(\overline{X}_1, D_1)$, $(\overline{X}_2, D_2)$ are orbifold admissible pairs with a compatible antiholomorphic involution σ_i. Suppose there exists an isomorphism $\widetilde{f} : D_1 \longrightarrow D_2$ such that $\widetilde{f} \circ \sigma_1|_{D_1} = \sigma_2|_{D_2} \circ \widetilde{f}$ and $\widetilde{f}^* \kappa_2 = \kappa_1$, where κ_i is the unique Ricci-flat Kähler form on D_i in the Kähler class $[\omega_i'|_{D_i}]$. Then we can glue together $X_1/\langle\sigma_1\rangle$ and $X_2/\langle\sigma_2\rangle$ along their cylindrical ends to obtain a compact 8-orbifold $M^{\triangledown}$. There exists a compact simply-connected 8-manifold M which resolves $M^{\triangledown}$ at $(\#\operatorname{Sing}\overline{X}_1 + \#\operatorname{Sing}\overline{X}_2)$ isolated singular points such that M admits a Riemannian metric with holonomy contained in* Spin(7)*. Furthermore if $\widehat{A}(M) = 1$, then M is a compact* Spin(7)*-manifold.*

Corollary 3.24. *Let $(\overline{X}, \omega')$ be a four-dimensional Kähler orbifold with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, such that $(\overline{X}, D)$ be an orbifold admissible pair with a compatible antiholomorphic involution σ. Then we can glue together two copies of $X/\langle\sigma\rangle = (\overline{X} \setminus D)/\langle\sigma\rangle$ to obtain a compact 8-orbifold $M^{\triangledown}$. There exists a compact simply-connected 8-manifold M which resolves $M^{\triangledown}$ at $2(\#\operatorname{Sing}\overline{X})$ isolated singular points such that M admits a Riemannian metric with holonomy contained in* Spin(7)*. Furthermore if $\widehat{A}(M) = 1$, then M is a compact* Spin(7)*-manifold.*

Proof of Theorem 3.23. By Proposition 3.18, there exists a choice $\{ s_p \in \{1, 2\} \mid p \in \operatorname{Sing} M^{\triangledown} \}$ of resolutions by $\mathcal{X}_{s_p}$ such that $M = M^{\epsilon}$ is simply-connected. The assertion for $\widehat{A}(M) = 1$ in Theorem 3.23 follows directly from Theorem 2.8. Thus it remains to prove the existence of a torsion-free Spin(7)-structure on M^{ϵ} for sufficiently small $\epsilon \in (0, 1]$. This is a consequence of the following.

Theorem 3.25 (Joyce [**14**, Theorem 13.6.1]). *Let λ, μ, ν be positive constants. Then there exists a positive constant ϵ_* such that whenever $0 < \epsilon < \epsilon_*$, the following is true.*

Let M be a compact 8-manifold and Φ a Spin(7)*-structure on M. Suppose ϕ is a smooth 4-form on M with $\mathrm{d}\Phi + \mathrm{d}\phi = 0$, and*

1. $\|\phi\|_{L^2} \leq \lambda\epsilon^{13/3}$ *and* $\|\mathrm{d}\phi\|_{L^{10}} \leq \lambda\epsilon^{7/5}$,
2. *the injectivity radius $\delta(g)$ satisfies $\delta(g) \geq \mu\epsilon$, and*
3. *the Riemann curvature $R(g)$ satisfies $\|R(g)\|_{C^0} \leq \nu\epsilon^{-2}$.*

Let ϵ_1 be as in Lemma 2.6. Then there exists $\eta \in C^{\infty}(\wedge^4 T_-^* M)$ with $\|\eta\|_{C^0} < \epsilon_1$ such that $\mathrm{d}\Theta(\Phi + \eta) = 0$. Hence the manifold M admits a torsion-free Spin(7)*-structure $\Theta(\Phi + \eta)$.*

If we set $\phi = \phi^\epsilon$, then M^ϵ and ϕ^ϵ satisfy conditions (i)–(iii) in Theorem 3.20. Thus we can apply Theorem 3.25 to prove that Φ^ϵ can be deformed into a torsion-free Spin(7)-structure for sufficiently small $\epsilon \in (0, 1]$. This completes the proof of Theorem 3.23. □

4. Orbifold admissible pairs and weighted projective spaces.

In order to find orbifold admissible pairs with a compatible antiholomorphic involution in Definitions 3.6 and 3.10, we will use some algebro-geometrical approach. First we review some basics on weighted projective spaces. In Section 4.2, we explain notation on complete intersections in weighted projective spaces. (See [**8**] for more details). Later in Section 4.3, we consider a situation where the gluing condition holds naturally.

4.1. Basics on projective spaces.

First we will observe the structure of the weighted projective space as a complex orbifold. Let $a_0, \dots, a_n$ be positive integers with $\gcd(a_0, \dots, a_n) = 1$. Recall that the *weighted projective space* $\mathbb{C}P^n(a_0, \dots, a_n)$ is the quotient $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1} \setminus \{0\}$ by

$$\mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}^{n+1} \setminus \{0\}, \qquad (w_0, \dots, w_n) \longmapsto (t^{a_0} w_0, \dots, t^{a_n} w_n)$$

for $t \in \mathbb{C}^*$. Let us fix the point $p = [1, 0, \dots, 0]$ in $\mathbb{C}P^n(a_0, \dots, a_n)$. Denote the stabilizer of p in $\mathbb{C}^*$ by $(\mathbb{C}^*)_p$. Then the point $(1, 0, \dots, 0)$ in $\mathbb{C}^{n+1} \setminus \{0\}$ is taken to $(t^{a_0}, 0, \dots, 0)$ under the action of $t \in \mathbb{C}^*$. Thus we have an isomorphism

$$(\mathbb{C}^*)_p = \{\, t \in \mathbb{C}^* \mid t^{a_0} = 1 \,\} \cong \mathbb{Z}_{a_0},$$

where $\mathbb{Z}_{a_0}$ is a finite cyclic group of order a_0. Let $[z_0, \dots, z_n]$ be the weighted homogeneous coordinates on $\mathbb{C}P^n(a_0, \dots, a_n)$. Then the affine open chart

$$U_0 = \{\, [z_0, \dots, z_n] \in \mathbb{C}P^n(a_0, \dots, a_n) \mid z_0 \neq 0 \,\}$$

is isomorphic to $\mathbb{C}^n/\mathbb{Z}_{a_0}$. Furthermore $p \in \mathbb{C}P^n(a_0, \dots, a_n)$ is a quotient singular point with a finite cyclic group $\mathbb{Z}_{a_0}$ which acts on $\mathbb{C}^n$ by

$$(x_1, \dots, x_n) \longmapsto (\zeta^{a_1} x_1, \dots, \zeta^{a_n} x_n),$$

where $\zeta \in (\mathbb{C}^*)_p$ is a primitive a_0-th root of unity. In this way, we see that all singularities of $\mathbb{C}P^n(a_0, \dots, a_n)$ are cyclic quotient singularities.

Next we shall define $\mathbb{C}P^n(a_0, \dots, a_n)$ as a projective variety. Let R be the graded ring $\mathbb{C}[z_0, \dots, z_n]$. Suppose each variable z_i has the weight a_i. Then R has a natural weight decomposition $R = \bigoplus_{d=0}^{\infty} R_d$ where R_d denotes the vector space spanned by all monomials $z_0^{d_0} \dots z_n^{d_n}$ with $\sum a_i d_i = d$. Elements of R_d are said to be *weighted homogeneous polynomials* of degree d and then $\mathbb{C}P^n(a_0, \dots, a_n)$ is defined by

$$\mathbb{C}P^n(a_0, \dots, a_n) = \mathrm{Proj}(R).$$

For a given finitely generated graded ring R, $\mathrm{Proj}(R)$ denotes the projective scheme. Furthermore, if positive integers $a_1, \dots, a_n$ have a common divisor, we have an isomorphism

$$\mathbb{C}P^n(a_0,\dots,a_n) \cong \mathbb{C}P^n(a_0, a_1/q,\dots,a_n/q)$$

where $q = \gcd(a_1,\dots,a_n)$. This yields the following property.

PROPOSITION 4.1 (Fletcher [**8**, Corollary 5.9]). *Let $a_0,\dots,a_n$ be positive integers with $\gcd(a_0,\dots,a_n) = 1$. Then we have an isomorphism as varieties*

$$\mathbb{C}P^n(a_0,\dots,a_n) \cong \mathbb{C}P^n(b_0,\dots,b_n)$$

for some positive integers $b_0,\dots,b_n$ with $\gcd(b_0,\dots,\widehat{b_i},\dots,b_n) = 1$ for each i. Here the symbol $\widehat{b_i}$ means that the entry b_i is omitted.

A weighted projective space $\mathbb{C}P^n(a_0,\dots,a_n)$ is said to be *well-formed* if and only if $\gcd(a_0,\dots,\widehat{a_i},\dots,a_n) = 1$ for each i. Now we recall that the graded ring $R = \mathbb{C}[z_0,\dots,z_n]$ is given by $\deg z_i = a_i \in \mathbb{Z}_{>0}$. Let $S = \mathbb{C}[w_0,\dots,w_n]$ be the standard polynomial ring with $\deg w_i = 1$. Then we have the injective ring homomorphism

$$R \longrightarrow S, \quad z_i \longmapsto w_i^{a_i}.$$

This injective ring homomorphism induces the well-defined surjective morphism of varieties

$$\begin{aligned} \pi: \quad \mathrm{Proj}(S) = \mathbb{C}P^n &\longrightarrow \mathrm{Proj}(R) = \mathbb{C}P^n(a_0,\dots,a_n), \\ [w_0,\dots,w_n] &\longmapsto [z_0,\dots,z_n] = [w_0^{a_0},\dots,w_n^{a_n}]. \end{aligned} \tag{4.1}$$

By abuse of notation, we *also* denote by π the canonical projection from $\mathbb{C}^{n+1}\setminus\{0\}$ onto $\mathbb{C}P^n(a_0,\dots,a_n)$:

$$\pi: \quad \mathbb{C}^{n+1}\setminus\{0\} \longrightarrow \mathbb{C}P^n(a_0,\dots,a_n), \quad (w_0,\dots,w_n) \longmapsto [w_0^{a_0},\dots,w_n^{a_n}].$$

For this canonical projection π and a subvariety $X \subset \mathbb{C}P^n(a_0,\dots,a_n)$, we define the *affine cone* C_X over X to be

$$C_X = \pi^{-1}(X) \cup \{0\} \quad \text{in} \quad \mathbb{C}^{n+1}.$$

Then a subvariety X of $\mathbb{C}P^n(a_0,\dots,a_n)$ is said to be *quasismooth* if C_X is smooth except at the origin. Furthermore, let X be a subvariety of $\mathbb{C}P^n(a_0,\dots,a_n)$ of codimension k. Then X is said to be *well-formed* if $\mathbb{C}P^n(a_0,\dots,a_n)$ is well-formed and X does not contain a codimension $k+1$ singular locus of $\mathbb{C}P^n(a_0,\dots,a_n)$.

4.2. Weighted complete intersections.

Let $a_0,\dots,a_n$ be positive integers with $\gcd(a_0,\dots,a_n) = 1$ and $R = \mathbb{C}[z_0,\dots,z_n]$ be the graded ring with $\deg z_i = a_i$ as usual. Let $f_1,\dots,f_k$ with $k \le n+1$ be weighted homogeneous polynomials of the graded ring R with $\deg f_i = d_i$. Then $I = \langle f_1,\dots,f_k\rangle$ is a homogeneous ideal of R. We define X_I by

$$X_I = \mathrm{Proj}(R/I) \subset \mathbb{C}P^n(a_0,\dots,a_n).$$

Then X_I is a *weighted complete intersection* of multidegree $(d_1, \dots, d_k)$ if the defining ideal I can be generated by a regular sequence $f_1, \dots, f_k$. Here a sequence of elements $g_1, \dots, g_\ell$ with $\ell \le n+1$ in R is said to be a *regular sequence* if g_1 is not a zero-divisor in R and the class $[g_i]$ is not a zero-divisor in $R/\langle g_1, \dots, g_{i-1}\rangle$ for each $2 \le i \le \ell$. Now we will state the following results which will be needed for our arguments later on.

Lemma 4.2 (Fletcher [**8**, Lemma 7.1]). *Let $X \subset \mathbb{C}P^n(a_0, \dots, a_n)$ be a well-formed quasismooth weighted complete intersection with the defining ideal $I(X) = \langle f_1, \dots, f_k\rangle$. Suppose* $\deg f_i = d_i$. *Let A be the residue ring*

$$A = \frac{\mathbb{C}[z_0, \dots, z_n]}{\langle f_1, \dots, f_k\rangle}.$$

Since each f_i is homogeneous, the ring A decomposes into graded pieces as $A = \bigoplus_m A_m$. Then we have

$$H^q(X, \mathcal{O}_X(m)) \cong \begin{cases} A_m & \text{if } \ q = 0 \\ 0 & \text{if } \ q = 1, \dots, \dim_{\mathbb{C}} X - 1 \\ A_{\alpha - m} & \text{if } \ q = \dim_{\mathbb{C}} X \end{cases}$$

for any $m \in \mathbb{Z}$, where $\alpha = \sum_{\lambda=1}^k d_\lambda - \sum_{i=0}^n a_i$.

In particular, we have the following result for hypersurfaces.

Theorem 4.3 (Fletcher [**8**, Theorem 7.2]). *Let f be the defining polynomial of a weighted hypersurface X in $\mathbb{C}P^n(a_0, \dots, a_n)$ with* $\deg f = d$. *The* Jacobian ring $R(f)$ *of f is the quotient ring*

$$R(f) = \frac{\mathbb{C}[z_0, \dots, z_n]}{\langle \partial f/\partial z_0, \dots, \partial f/\partial z_n\rangle}.$$

Let $R(f)_m$ denote the m-th graded part of $R(f)$. Then the Hodge numbers of X are given by

$$h^{p,q}(X) = \begin{cases} 0 & \text{if } \ p+q \ne n-1, \ p \ne q \\ 1 & \text{if } \ p+q \ne n-1, \ p = q \\ \dim_{\mathbb{C}} R(f)_{qd+\alpha} & \text{if } \ p+q = n-1, \ p \ne q \\ \dim_{\mathbb{C}} R(f)_{qd+\alpha} + 1 & \text{if } \ p+q = n-1, \ p = q, \end{cases}$$

where $\alpha = d - \sum_{i=0}^n a_i$.

4.3. Orbifold admissible pairs with a compatible antiholomorphic involution from weighted complete intersections.

We first recall the following result, which provides a way of obtaining orbifold admissible pairs of *Fano type*.

Theorem 4.4 (Kovalev [**15**]). *Let V be a Fano four-orbifold with isolated singular* *points which have local crepant resolutions and $D \in |-K_V|$ a smooth Calabi–Yau divisor. We denote a smooth surface representing the self-intersection class of $D \cdot D$ by S.*

Let $\varpi : \overline{X} = \mathrm{Bl}_S(V) \dashrightarrow V$ be the blow-up of V along the surface S. If we take the proper transform D' of D under the blow-up ϖ, then $(\overline{X}, D')$ is an orbifold admissible pair. Moreover, $\varpi|_{D'}$ yields an isomorphism between D' and D, and so we may denote D' by D.

Proof. See [**15**, Proposition 6.42 and Corollary 6.43]. One can see that these results for Fano threefolds also hold for Fano four-orbifolds. □

The above orbifold admissible pair $(\overline{X}, D)$ obtained from V and D is said to be *of Fano type*.

Next we consider a well-formed weighted projective space $W = \mathbb{C}P^{k+3}(a_0, a_1, \ldots, a_{k+3})$ with $k \geq 1$. Let $f_1, \ldots, f_{k+1}$ be a regular sequence of weighted homogeneous polynomials such that

(1) $\sum_{\lambda=1}^{k} d_\lambda = \sum_{i=0}^{k+3} a_i$, where $d_\lambda = \deg f_\lambda$,

(2) V is a complete intersection defined by the ideal $I_{k-1} = \langle f_1, \ldots, f_{k-1} \rangle$, with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$ (we set $I_0 = 0$ and $V = W$ when $k = 1$),

(3) D is a *smooth* complete intersection defined by the ideal $I_k = \langle f_1, \ldots, f_k \rangle$, so that $D \cap \mathrm{Sing}\, V = \emptyset$, and

(4) S is a smooth complete intersection defined by the ideal $I_{k+1} = \langle f_1, \ldots, f_{k+1} \rangle$ with $\deg f_{k+1} = \deg f_k$.

Then V is a four-dimensional Fano orbifold with D a smooth anticanonical Calabi–Yau divisor, and S is a smooth surface in D representing $D \cdot D$ on V. Suppose there exists an antiholomorphic involution σ on W such that

(5) $\sigma^* f_i = \overline{f_i}$ for $i = 1, \ldots, k+1$ and σ acts freely on D and S, and

(6) $V^\sigma = \mathrm{Sing}\, V$, where $V^\sigma = \{ x \in V \mid \sigma(x) = x \}$.

Then by Proposition 3.9, σ lifts to an antiholomorphic involution $\widetilde{\sigma}$ on the blow-up $\varpi : \overline{X} = \mathrm{Bl}_S(V) \dashrightarrow V$ such that $\widetilde{\sigma}$ preserves and acts freely on the exceptional divisor $E = \varpi^{-1}(S)$. Let $[\boldsymbol{z}] = [z_0, \ldots, z_{k+3}]$ be weighted homogeneous coordinates on W, with $\deg z_i = a_i$ for $i = 0, \ldots, k+3$. We can describe the blow-up $\overline{X}$ of V, the exceptional divisor E and the proper transform D' of D as

$$\begin{aligned}
\overline{X} = \mathrm{Bl}_S(V) &= \{([\boldsymbol{z}], [u,v]) \in W \times \mathbb{C}P^1 \,|\, f_1(\boldsymbol{z}) = \cdots = f_{k-1}(\boldsymbol{z}) = 0, v f_k(\boldsymbol{z}) = u f_{k+1}(\boldsymbol{z})\}, \\
\varpi : \overline{X} &\dashrightarrow V, \quad ([\boldsymbol{z}], [u,v]) \longmapsto [\boldsymbol{z}], \\
E = \varpi^{-1}(S) &= \{ ([\boldsymbol{z}], [u,v]) \in W \times \mathbb{C}P^1 \mid f_1(\boldsymbol{z}) = \cdots = f_{k+1}(\boldsymbol{z}) = 0 \} \cong S \times \mathbb{C}P^1, \\
D' = \overline{\varpi^{-1}(D \setminus S)} &= \{ ([\boldsymbol{z}], [u,v]) \in W \times \mathbb{C}P^1 \mid f_1(\boldsymbol{z}) = \cdots = f_k(\boldsymbol{z}) = u = 0 \} \\
&= D \times \{ [0,1] \in \mathbb{C}P^1 \} \cong D, \\
E \cap D' &= S \times \{ [0,1] \in \mathbb{C}P^1 \} \cong S.
\end{aligned}$$

Note that the above equation $vf_k(\boldsymbol{z}) = uf_{k+1}(\boldsymbol{z})$ is well-defined because both $f_k(\boldsymbol{z})$ and $f_{k+1}(\boldsymbol{z})$ are sections of the line bundle $\mathcal{O}_W(d_k)$. Also, we can compute as

$$\begin{aligned} D' &= \varpi^* D - E, \\ K_{\overline{X}} &= \varpi^* K_V + E = \varpi^*(K_V + D) - D' = -D', \\ N_{D'/\overline{X}} &= D'|_{D'} = D' \cdot D' = 0, \end{aligned}$$

by the adjunction formula. Let $\boldsymbol{z}' = (z'_0, \dots, z'_{k+3})$ and consider the transformation

$$z'_i = z_i \quad \text{for} \quad i = 0, 1, \dots, k+1, \quad z'_{k+2} = f_k(\boldsymbol{z}) \quad \text{and} \quad z'_{k+3} = f_{k+1}(\boldsymbol{z}).$$

Then $\boldsymbol{z}'$ define well-defined coordinates on W, and we can rewrite $\overline{X}$ and D' as

$$\begin{aligned} &\overline{X} = \{ ([\boldsymbol{z}'], [u, v]) \in W \times \mathbb{C}P^1 \mid f'_1(\boldsymbol{z}') = \cdots = f'_{k-1}(\boldsymbol{z}') = 0, vz'_{k+2} = uz'_{k+3} \}, \\ &D' = \{ ([\boldsymbol{z}'], [u, v]) \in \overline{X} \mid u = 0 \}, \end{aligned}$$

where $f'_i(\boldsymbol{z}') = f_i(\boldsymbol{z})$ for $i = 1, \dots, k-1$. In this coordinate system, it follows from the proof of Proposition 3.9 that

$$\widetilde{\sigma}(\boldsymbol{z}', [u, v]) = (\sigma(\boldsymbol{z}'), [\overline{u}, \overline{v}]) \quad \text{for} \quad (\boldsymbol{z}', [u, v]) \in \overline{X}.$$

Thus we may assume that the defining function u of D' on $\overline{X}$ satisfies (3.5), so that $\widetilde{\sigma}$ is a compatible antiholomorphic involution on $\overline{X}$. Observe that V inherits a Kähler form ω_V from the ambient Kähler orbifold (W, ω_W) with $\omega_V = \omega_W|_V$, and that $\overline{X}$ is endowed with a Kähler form $\omega' = \varpi^*\omega_V - k^{-1}\omega_{[E]}$ for sufficiently large k, where $\omega_{[E]}$ is a d-closed semi-positive $(1,1)$-form which represents $c_1([E])$ and satisfies $\omega_{[E]}|_{\widetilde{D}} = 0$ (see Griffiths–Harris [**9**, pp. 186–187] and [**15**, Proof of Proposition 6.42]). Therefore $\varpi|_{\widetilde{D}} : \widetilde{D} \longrightarrow D$ is an isomorphism with $(\varpi|_{\widetilde{D}})^*\omega_V|_D = \omega'|_{\widetilde{D}}$.

Now suppose $k \geq 2$ in the above situation. Let $g_1 = f_1, \dots, g_{k-2} = f_{k-2}$ and $g_{k-1} = f_k, g_k = f_{k-1}$. Also, choose g_{k+1} so that g_{k+1} satisfies the above conditions (4) and (5). Let $(\overline{X}_1, D_1), V_1, S_1, \sigma_1$ and $(\overline{X}_2, D_2), V_2, S_2, \sigma_2$ correspond to $f_1, \dots, f_{k+1}$ and $g_1, \dots, g_{k+1}$ respectively. Then $\overline{X}_2$ and V_2 may change from $\overline{X}_1$ and V_1, but $D_2 = D_1$ and the asymptotic models of $\overline{X}_1 \setminus D_1$ and $\overline{X}_2 \setminus D_2$ are *the same*.

Setting the isomorphism $\widetilde{f} : D_1 \longrightarrow D_2$ by

$$\widetilde{f} = (\varpi_2|_{D_2})^{-1} \circ \varpi_1|_{D_1} : D_1 \longrightarrow D \longrightarrow D_2,$$

we have $\widetilde{f} \circ \sigma_1|_{D_1} = \sigma_2|_{D_2} \circ \widetilde{f}$ and $\widetilde{f}^*\omega'_2|_{D_2} = \omega'_1|_{D_1}$. Also, we have $\widetilde{f}^*\kappa_2 = \kappa_1$, where κ_i is the unique Ricci-flat Kähler form on D_i in the Kähler class $[\omega'_i|_{D_i}]$. Consequently, we have the following theorem which we shall need in Section 6.1.

THEOREM 4.5. *The above isomorphism $\widetilde{f}$ satisfies the gluing condition given in Section* 3.4.1. *Thus we can apply Theorem* 3.23 *to $(\overline{X}_i, D_i), \sigma_i$ for $i = 1, 2$, to obtain a compact simply-connected Riemannian* 8*-manifold M, which has holonomy* Spin(7) *if $\widehat{A}(M) = 1$.*

5. A new example of compact Spin(7)-manifolds.

The main theorem of this section is the following.

Theorem 5.1. *There exists a compact* Spin(7)*-manifold M whose Betti numbers, the Euler characteristic and the signature are given by*

$$\begin{cases} b^2(M) = b^3(M) = 0, \\ b^4(M) = 1678, \\ \chi(M) = 1680 \quad \textit{and} \quad \tau(M) = 576. \end{cases} \tag{5.1}$$

In particular, this is a new *example of compact* Spin(7)*-manifold.*

Remark that only a small number of examples (around 200) of compact Spin(7)-manifolds are known and all known examples of them can be found in [**14**, Tables 14.1–3, 15.1] and [**2**]. Among them, one can see that there is *no* example of compact Spin(7)-manifolds which has Betti numbers $(b^2, b^3, b^4) = (0, 0, 1678)$. Hence it suffices to construct a compact Spin(7)-manifold satisfying (5.1) by using Corollary 3.24.

Here and hereafter, we will use the same notation as in Section 4.3. First we provide an explicit example of simply-connected 8-manifolds as follows.

5.1. Setup.

Let $W = \mathbb{C}P^4(1,1,1,1,4)$ be the weighted projective space and $[\boldsymbol{z}] = [z_0, \dots, z_4]$ be weighted homogeneous coordinates on W. Then W has an isolated singular point at $p = [0,0,0,0,1]$, which is modelled on $\mathbb{C}^4/\mathbb{Z}_4$. If we define an antiholomorphic involution σ on W by

$$[z_0, z_1, z_2, z_3, z_4] \longmapsto [-\overline{z}_1, \overline{z}_0, -\overline{z}_3, \overline{z}_2, \overline{z}_4], \tag{5.2}$$

then we have $W^\sigma = \{ p \} = \operatorname{Sing} W$. Define

$$V = W, \quad D = \{ [\boldsymbol{z}] \in W \mid f_1(\boldsymbol{z}) = 0 \} \quad \text{and} \quad S = \{ [\boldsymbol{z}] \in W \mid f_1(\boldsymbol{z}) = f_2(\boldsymbol{z}) = 0 \} \tag{5.3}$$

by weighted homogeneous polynomials

$$f_1(\boldsymbol{z}) = z_0^8 + z_1^8 + z_2^8 + z_3^8 + z_4^2 \quad \text{and} \quad f_2(\boldsymbol{z}) = az_0^8 + az_1^8 + bz_2^8 + bz_3^8 + cz_4^2, \tag{5.4}$$

where a, b and c are real coefficients. Then we see that conditions (1)–(3), (5) and (6) in Section 4.3 hold. Also, we can choose a, b and c so that condition (4) holds. Thus following Section 4.3, we have an orbifold admissible pair $(\overline{X}, D)$ from V, D and S, where $\overline{X} = \mathrm{Bl}_S(V)$ and we denote the proper transform D' of D by D again. Then Proposition 3.9 gives a lift of σ on $\overline{X}$, which satisfies conditions (f) and (g) in Definition 3.10 (we denote this lift by σ again). Hence this is a compatible antiholomorphic involution on $\overline{X}$. Applying the doubling construction in Corollary 3.24, we can resolve the orbifold $M^\triangledown = X/\langle\sigma\rangle \cup X/\langle\sigma\rangle$ to obtain a compact 8-manifold M. Hence we have the following result.

Proposition 5.2. *This simply-connected 8-manifold M admits a Riemannian metric with holonomy contained in* Spin(7).

Now it suffices to show that the above resulting manifold (M, g) with $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$ is a compact Spin(7)-manifold (i.e. $\mathrm{Hol}(g) = \mathrm{Spin}(7)$) which satisfies (5.1) to prove Theorem 5.1. We will show this in Section 5.5, while Sections 5.2–5.4 are devoted to compute the Hodge numbers of D and S.

5.2. Contributions from the singular point.

First, we observe that the branched covering of the isolated singular point $p = [0,0,0,0,1]$ on $V = \mathbb{C}P^4(1,1,1,1,4)$. Consider the surjective morphism

$$\pi : \mathbb{C}P^4 \longrightarrow V$$

defined in (4.1), and let $[\boldsymbol{w}] = [w_0, \dots, w_4]$ be the standard homogeneous coordinates on $\mathbb{C}P^4$. Then the restriction of the map π to $\widetilde{\Sigma}_4 = \{\, [\boldsymbol{w}] \in \mathbb{C}P^4 \mid w_4 = 0 \,\}$ is bijective since $\widetilde{\Sigma}_4$ can be identified with $\mathbb{C}P^3$. On the other hand, the restriction of the map π to $\widetilde{U}_p = \{\, [\boldsymbol{w}] \in \mathbb{C}P^4 \mid w_4 \neq 0 \,\} \cong \mathbb{C}^4$ is $4:1$ except at p. This is because we have $U_p = \{\, [\boldsymbol{z}] \in V \mid z_4 \neq 0 \,\} \cong \mathbb{C}^4/\mathbb{Z}_4$ as seen in Section 4.1:

$$\begin{array}{ccccc} \mathbb{C}P^4 & = & (\widetilde{\Sigma}_4 \sqcup \{\, p \,\}) & \sqcup & (\widetilde{U}_p \setminus \{\, p \,\}) \\ \Big\downarrow \pi & & 1:1 \Big\downarrow & & 4:1 \Big\downarrow \\ V & = & (\Sigma_4 \sqcup \{\, p \,\}) & \sqcup & (U_p \setminus \{\, p \,\}). \end{array} \tag{5.5}$$

Here we denote $\Sigma_4 = \pi(\widetilde{\Sigma}_4) = \{\, [\boldsymbol{z}] \in V \mid z_4 = 0 \,\}$.

A straightforward computation shows the following.

Lemma 5.3. *Let $\widetilde{F}$ be a projective subvariety of $\mathbb{C}P^4$ with $\widetilde{F} \cap \{\, p \,\} = \emptyset$, and $F = \pi(\widetilde{F})$. Then we have*

$$\chi(F) = \frac{1}{4}(\chi(\widetilde{F}) + 3\chi(\widetilde{F} \cap \widetilde{\Sigma}_4)).$$

5.3. Computing the topology of D.

In order to prove Theorem 5.1 first we need to calculate the Euler characteristic $\chi(D)$. We will find this by the following two ways.

Computing $\chi(D)$: *First way.* Let f_1 and f_2 be the weighted homogeneous polynomial defined in (5.4). Then $\widetilde{f}_i = \pi^* f_i$ for $i = 1, 2$ are homogeneous polynomials of degree 8 in $\mathbb{C}[w_0, \dots, w_4]$ given by

$$\widetilde{f}_1(\boldsymbol{w}) = w_0^8 + w_1^8 + w_2^8 + w_3^8 + w_4^8, \quad \text{and} \quad \widetilde{f}_2(\boldsymbol{w}) = a w_0^8 + a w_1^8 + b w_2^8 + b w_3^8 + c w_4^8, \tag{5.6}$$

where $[\boldsymbol{w}] = [w_0, \dots, w_4]$ are the standard homogeneous coordinates on $\mathbb{C}P^4$. Setting

$$\widetilde{D} = \{\, [\boldsymbol{w}] \in \mathbb{C}P^4 \mid \widetilde{f}_1(\boldsymbol{w}) = 0 \,\} \quad \text{and} \quad \widetilde{S} = \{\, [\boldsymbol{w}] \in \mathbb{C}P^4 \mid \widetilde{f}_1(\boldsymbol{w}) = \widetilde{f}_2(\boldsymbol{w}) = 0 \,\}, \tag{5.7}$$

we have $\pi(\widetilde{D}) = D$, $\pi(\widetilde{S}) = S$ and $\widetilde{D} \cap \{p\} = \widetilde{S} \cap \{p\} = \emptyset$. Thus the assumption of Lemma 5.3 holds for $\widetilde{D}$ and $\widetilde{S}$. Since $\widetilde{D} \cap \Sigma_4$ is given by

$$\widetilde{D} \cap \widetilde{\Sigma}_4 = \{ [\boldsymbol{w}] \in \mathbb{C}P^4 \mid \widetilde{f}_1(\boldsymbol{w}) = w_4 = 0 \} \cong \{ [\boldsymbol{w}'] \in \mathbb{C}P^3 \mid w_0^8 + w_1^8 + w_2^8 + w_3^8 = 0 \}, \tag{5.8}$$

where $[\boldsymbol{w}'] = [w_0, w_1, w_2, w_3]$ are the standard homogeneous coordinates on $\mathbb{C}P^3$, a computation of the total Chern classes gives

$$\chi(\widetilde{D}) = -2096 \quad \text{and} \quad \chi(\widetilde{D} \cap \widetilde{\Sigma}_4) = 7808.$$

Hence Lemma 5.3 yields the following.

Proposition 5.4. *This smooth Calabi–Yau divisor D on V has the Euler characteristic*

$$\chi(D) = -296.$$

Computing $\chi(D)$: Second way. Theorem 4.3 determines the Hodge numbers of D as follows. Let $R(f)$ be the Jacobian ring of f

$$R(f) = \frac{\mathbb{C}[z_0, \dots, z_4]}{\langle z_0^7, z_1^7, z_2^7, z_3^7, z_4 \rangle}.$$

Assume that a graded ring B is finitely generated over $\mathbb{C}$. Then the *Hilbert series* of the graded ring $B = \bigoplus_m B_m$ is defined to be

$$H_B(t) = \sum_{m=0}^{\infty} (\dim_{\mathbb{C}} B_m) t^m.$$

On the one hand, we can apply [**1**, Proposition 23.4] to the Jacobian ring $R(f)$. Consequently, the Hilbert series of $R(f)$ is the power series expansion at $t = 0$ of a rational function

$$H_{R(f)}(t) = \frac{(1-t^7)^4}{(1-t)^4} = 1 + 4t + 10t^2 + \cdots + 149t^8 + \mathcal{O}(t^9).$$

Then Theorem 4.3 gives

$$h^{1,1}(D) = 1, \quad h^{3,0}(D) = \dim_{\mathbb{C}} R(f)_0 = 1 \quad \text{and} \quad h^{2,1}(D) = \dim_{\mathbb{C}} R(f)_8 = 149.$$

Since the Euler characteristic $\chi(D)$ is also given by $\chi(D) = \sum_{p,q} (-1)^{p+q} h^{p,q}(D)$, the result is consistent with Proposition 5.4.

Remark 5.5. Since D is a Calabi–Yau threefold, the Lefchetz hyperplane theorem and the Euler characteristic determine the Hodge numbers in this example.

5.4. Computing the topology of S.

Analogously to Section 5.3, we shall find all Hodge numbers of the weighted complete intersection S defined in (5.3).

Recall that $f_i(\boldsymbol{z})$ and $\widetilde{f}_i(\boldsymbol{w})$ for $i = 1, 2$ are the weighted homogeneous polynomials given by (5.4) and (5.6) respectively. Let $\widetilde{S}$ be a complex surface given by (5.7). Then we have $\chi(\widetilde{S}) = 7808$. As in (5.8), we have

$$\begin{aligned}\widetilde{S} \cap \widetilde{\Sigma}_4 &= \{ [\boldsymbol{w}] \in \mathbb{C}P^4 \mid \widetilde{f}_1(\boldsymbol{w}) = \widetilde{f}_2(\boldsymbol{w}) = w_4 = 0 \} \\ &\cong \{ [\boldsymbol{w}'] \in \mathbb{C}P^3 \mid w_0^8 + w_1^8 + w_2^8 + w_3^8 = aw_0^8 + aw_1^8 + bw_2^8 + bw_3^8 = 0 \},\end{aligned}$$

which is a smooth complex curve in $\widetilde{S}$ with $\chi(\widetilde{S} \cap \widetilde{\Sigma}_4) = -768$. Again by using Lemma 5.3, we find $\chi(S) = 1376$. Also, we have $b^1(S) = 0$ by the Lefschetz hyperplane theorem. Let us consider the residue ring

$$A = \frac{\mathbb{C}[z_0, \ldots, z_4]}{\langle f_1, f_2 \rangle}.$$

Using [**1**, Proposition 23.4] again we find that the Hilbert series of A can be written as

$$H_A(t) = \frac{(1-t^8)^2}{(1-t)^4(1-t^4)} = 1 + 4t + 10t^2 + \cdots + 199t^8 + \mathcal{O}(t^9).$$

Applying Lemma 4.2 to the residue ring A for $q = 2$, $m = 0$ and $\alpha = 8$, we have

$$h^{0,2}(S) = \dim_{\mathbb{C}} A_8 = 199.$$

Since $\chi(S) = 1376$, we find $h^{1,1}(S) = 976$. By the Hodge index theorem, we also find the signature of S is

$$\tau(S) = \sum_{p,q=0}^{\dim_{\mathbb{C}} S} (-1)^q h^{p,q} = -576.$$

Summing up our argument, we conclude the following.

Proposition 5.6. *This smooth compact complex surface S has*

$$\chi(S) = 1376 \quad \textit{and} \quad \tau(S) = -576.$$

5.5. Proof of Theorem 5.1.

Our proof separates into the following two steps: In Step 1, we show that the resulting manifold in Proposition 5.2 is a compact Spin(7)-manifold by Theorem 2.8. In Step 2, we conclude that our Spin(7)-manifold M has the Betti numbers $(b^2, b^3, b^4) = (0, 0, 1678)$.

Proof of Theorem 5.1.

Step 1: First we will compute the Euler characteristic and the signature of the resulting compact simply-connected 8-manifold M. Recall that $\varpi : \overline{X} \dashrightarrow V$ is the blow-up of V along the submanifold S. It is well-known that the Euler characteristic of $\overline{X}$ satisfies the equality

$$\chi(\overline{X}) = \chi(V) + \chi(E) - \chi(S) \tag{5.9}$$

where E is the exceptional divisor of the blow-up ϖ. As seen in Section 4.3, we have $E \cong S \times \mathbb{C}P^1$, and so

$$\chi(\overline{X}) = \chi(V) + \chi(S) = 1381,$$

where we used Proposition 5.6 and $\chi(V) = 5$. Thus $\chi(X) = \chi(\overline{X}) - \chi(D) = 1677$. Since σ fixes the singular point p on X, we have

$$\chi(X/\langle\sigma\rangle) = \frac{1}{2}(\chi(X) + 1) = 839.$$

Now we construct M by resolving the orbifold $M^\triangledown = X/\langle\sigma\rangle \cup X/\langle\sigma\rangle$ with two isolated singular points. Observing from (3.2) that replacing the neighborhood of each singular point on $M^\triangledown$ with an ALE manifold $\mathcal{X}_s$ adds 1 to the Euler characteristic, we have

$$\chi(M) = \chi(M^\triangledown) + 2 = 2\chi(X/\langle\sigma\rangle) + 2 = 1680.$$

In order to find the signature $\tau(M)$, we see that $\tau(\overline{X}) = \tau(V) - \tau(S) = 577$ in the same manner as (5.9). Hence

$$\begin{aligned}\tau(M^\triangledown) &= 2\tau(X/\langle\sigma\rangle) = \tau(X) + 1\\ &= \frac{1}{2}(2\tau(\overline{X}) - \tau(D \times \mathbb{C}P^1)) + 1 = 578.\end{aligned}$$

Consequently we obtain $\tau(M) = \tau(M^\triangledown) - 2 = 576$ by taking resolutions of isolated singular points. Hence (2.5) implies that $\widehat{A}(M) = 1$, that is, M is a compact Spin(7)-manifold.

Step 2: Next we find the Betti numbers of our Spin(7)-manifold M. Consider

$$M^\triangledown = Z_1 \cup Z_2$$

where $Z_i = X/\langle\sigma\rangle$ for $i = 1, 2$. Then we have homotopy equivalences

$$M^\triangledown \sim Z_1 \cup Z_2, \quad Z_1 \cap Z_2 \sim (D \times S^1)/\langle\sigma_{D\times S^1,\mathrm{cyl}}\rangle =: Y \tag{5.10}$$

as in [**5**, Equation (4.6)]. Here the action of $\sigma_{D\times S^1,\mathrm{cyl}}$ is given by (3.9).

Lemma 5.7 (Kovalev [**16**]). *Let Z_i $(i = 1, 2)$ and Y be as above. Then we have*

$$b^1(Y) = b^2(Y) = 0 \quad \textit{and} \quad b^2(Z_i) = b^3(Z_i) = 0.$$

Once Lemma 5.7 has been proved, we conclude that

$$b^2(M^\triangledown) = b^3(M^\triangledown) = 0$$

by applying the Mayer–Vietoris theorem to (5.10). Then it follows from $\chi(M^\triangledown) = 1678$ that

$$b^4(M^\triangledown) = 1676.$$

By (15.10) in [**14**], the Betti numbers $b^j(M)$ satisfy

$$b^j(M) = b^j(M^\triangledown) \quad \text{for} \quad j = 1, 2, 3 \quad \text{and} \quad b^4(M) = b^4(M^\triangledown) + k$$

where $k = \#\,\mathrm{Sing}\, M^\triangledown$. Thus, we conclude our Spin(7)-manifold M has the Betti numbers $(b^2, b^3, b^4) = (0, 0, 1678)$. This completes the proof. □

It remains to prove Lemma 5.7.

Proof of Lemma 5.7. Note that $b^j(Y) = 0$ for $j = 1, 2$ were already proved in [**16**, Proposition 6.2]. Hence it suffices to show $b^2(Z_i) = b^3(Z_i) = 0$ for our purpose. Recall that $b^2(V) = 1$ and $b^3(V) = 0$ for $V = \mathbb{C}P^4(1, 1, 1, 1, 4)$. Now $\varpi^{-1}(S) \cong S \times \mathbb{C}P^1$ where $\varpi : \overline{X} \dashrightarrow V$ is the blow-up of V along S. Then the Betti numbers $b^i(\overline{X})$ are given by the formula

$$b^i(\overline{X}) = b^i(V) + b^{i-2}(S)$$

(see [**3**, (1.10)]). This gives

$$b^2(\overline{X}) = b^2(V) + b^0(S) = 2 \quad \text{and} \quad b^3(\overline{X}) = b^3(V) + b^1(S) = 0.$$

Since there is a tubular neighborhood U of D in $\overline{X}$ such that

$$\overline{X} = X \cup U \quad \text{and} \quad X \cap U \simeq D \times S^1 \times \mathbb{R}_{>0}, \tag{5.11}$$

we apply the Mayer–Vietoris theorem to (5.11). Then we see that

$$\begin{cases} b^2(\overline{X}) = b^2(X) + 1, \\ b^3(X) = b^3(\overline{X}) + b^2(D) - b^2(X) \end{cases} \tag{5.12}$$

(see [**17**, (2.10)]). Let $b^i(X)^\sigma$ be the dimension of the σ-invariant part of $H^i(X, \mathbb{R})$. Then

$$b^2(Z_i) = b^2(X)^\sigma = 0$$

because $H^2(X, \mathbb{R})$ is generated by the Kähler form on X and is *not* σ-invariant. Similarly,

$$b^3(Z_i) = b^3(X)^\sigma = 0$$

by (5.12). The assertion is verified. □

6. Other examples.

In Section 6.1, we investigate orbifold admissible pairs $(\overline{X}, D)$ of Fano type when V is a complete intersection in a weighted projective space $W = \mathbb{C}P^{k+3}(a_0, \dots, a_{k+3})$ with $k \geq 2$. Suppose σ is an antiholomorphic involution on W and

$$V = \{\, [\boldsymbol{z}] \in W \mid f_1(\boldsymbol{z}) = \cdots = f_{k-1}(\boldsymbol{z}) = 0 \,\},$$
$$D = \{\, [\boldsymbol{z}] \in W \mid f_1(\boldsymbol{z}) = \cdots = f_k(\boldsymbol{z}) = 0 \,\},$$

where D is smooth and f_i are weighted homogeneous polynomials satisfying $\deg f_1 + \cdots + \deg f_k = a_0 + \cdots + a_{k+3}$ and $\sigma^* f_i = \overline{f_i}$ for $i = 1, \ldots, k$. Then by the adjunction formula, V is a Fano four-orbifold with an anticanonical Calabi–Yau divisor D. Choosing f_{k+1} so that

$$\begin{aligned}&\deg f_{k+1} = \deg f_k, \quad \sigma^* f_{k+1} = \overline{f_{k+1}} \quad \text{and} \\ &S = \{\,[\boldsymbol{z}] \in W \mid f_1(\boldsymbol{z}) = \cdots = f_{k+1}(\boldsymbol{z}) = 0\,\} \quad \text{represents} \quad D \cdot D,\end{aligned}$$

we have an orbifold admissible pair $(\overline{X}_1, D_1)$ with a compatible antiholomorphic involution σ_1 such that $(D_1, \sigma_1|_{D_1})$ is isomorphic to $(D, \sigma|_D)$. Meanwhile, if we exchange f_k and f_{k-1} (and choose suitable f_{k+1} correspondingly), then V may change, but D *does not change.* Hence we have another orbifold admissible pair $(\overline{X}_2, D_2)$ with σ_2 which has the same asymptotic model. This new perspective leads us to obtain practical examples in our gluing construction.

6.1. Complete intersections in $\mathbb{C}\boldsymbol{P}^5(\mathbf{1,1,1,1,4,4})$.

Suppose $k = 2$ in the above argument. We consider the weighted complete intersection of two weighted hypersurfaces in $W = \mathbb{C}P^5(1,1,1,1,4,4)$ with homogeneous coordinates $[\boldsymbol{z}] = [z_0, \ldots, z_5]$. Define an antiholomorphic involution $\sigma : W \longrightarrow W$ by

$$[z_0, \ldots, z_5] \longmapsto [-\overline{z}_1, \overline{z}_0, -\overline{z}_3, \overline{z}_2, \overline{z}_4, \overline{z}_5]. \tag{6.1}$$

Consider complete intersections

$$\begin{aligned}&V_1 = \{\,[\boldsymbol{z}] \in W \mid f_1(\boldsymbol{z}) = 0\,\}, \quad D_1 = \{\,[\boldsymbol{z}] \in W \mid f_1(\boldsymbol{z}) = f_2(\boldsymbol{z}) = 0\,\} \quad \text{and} \\ &S_1 = \{\,[\boldsymbol{z}] \in W \mid f_1(\boldsymbol{z}) = f_2(\boldsymbol{z}) = f_3(\boldsymbol{z}) = 0\,\},\end{aligned}$$

where f_1 and f_2 are defined by

$$f_1(\boldsymbol{z}) = z_0^8 + z_1^8 + z_2^8 + z_3^8 + z_4^2 - z_5^2 \quad \text{and} \quad f_2(\boldsymbol{z}) = z_0^4 + z_1^4 + z_2^4 + z_3^4 + 2z_4 + z_5,$$

and $f_3(\boldsymbol{z})$ is chosen so that $\deg f_3 = \deg f_2 = 4$, $\sigma^* f_3 = \overline{f_3}$, and S_1 is a smooth complete intersection in W. Then V_1 has two isolated singular points $p_1 = [0,0,0,0,1,1]$ and $p_2 = [0,0,0,0,1,-1]$, which are modelled on $\mathbb{C}^4/\mathbb{Z}_4$ and fixed by σ. We can see easily that conditions (1)–(6) in Section 4.3 hold, and thus following the argument in Section 4.3 we obtain an orbifold admissible pair $(\overline{X}_1, D_1)$ with a compatible antiholomorphic involution σ_1.

Similarly, we set $g_1 = f_2$, $g_2 = f_1$ and

$$\begin{aligned}&V_2 = \{\,[\boldsymbol{z}] \in W \mid g_1(\boldsymbol{z}) = 0\,\}, \quad D_2 = \{\,[\boldsymbol{z}] \in W \mid g_1(\boldsymbol{z}) = g_2(\boldsymbol{z}) = 0\,\} \quad \text{and} \\ &S_2 = \{\,[\boldsymbol{z}] \in W \mid g_1(\boldsymbol{z}) = g_2(\boldsymbol{z}) = g_3(\boldsymbol{z}) = 0\,\},\end{aligned}$$

where we choose g_3 with $\deg g_3 = \deg g_2 = 8$ so that $\sigma^* g_3 = \overline{g_3}$, and S_2 is a smooth complete intersection. Then V_2 has an isolated singular point $p_3 = [0,0,0,0,1,-2]$, which is modelled on $\mathbb{C}^4/\mathbb{Z}_4$ and fixed by σ. Conditions (1)–(6) in Section 4.3 also hold in this case, and we obtain another orbifold admissible pair $(\overline{X}_2, D_2)$ with σ_2. Note that

$(\overline{X}_i \setminus D_i)/\langle\sigma_i\rangle$ for $i = 1, 2$ have the same asymptotic model, and so can be glued together.

Now we can apply Theorem 4.5. Setting $Z_i = (\overline{X}_i \setminus D_i)/\langle\sigma_i\rangle$ and $M_{ij}^{\triangledown} = Z_i \cup Z_j$, where $i, j \in \{1, 2\}$, we can resolve orbifolds $M_{11}^{\triangledown}, M_{12}^{\triangledown}$ and $M_{22}^{\triangledown}$ to obtain compact simply-connected 8-manifolds M_{11}, M_{12} and M_{22} respectively. Then we see that $\widehat{A}(M_{ij}) = 1$ in each case. Hence we conclude that all resulting manifolds M_{ij} are compact Spin(7)-manifolds. In particular, the resulting manifold M_{22} has the same Betti numbers as the above Spin(7)-manifold M in Theorem 5.1. Finally we shall list all Hodge numbers in Table 6.4 which are needed to compute $\chi(M_{ij})$ and $\tau(M_{ij})$.

Remark 6.1. Since our examples M_{11}, M_{12} with $(b^2, b^3, b^4) = (0, 0, 910)$, $(0, 0, 1294)$ in Table 6.5 are already listed in [**14**, Table 15.1], we can not distinguish the topological types of these examples from those in [**14**].

6.2. From the viewpoint of Calabi–Yau structures.

In this subsection, we shall give a useful criterion for finding a compact Spin(7)-manifold by considering Calabi–Yau fourfolds constructed by Theorem 3.21. Let V, D and S be as in Theorem 4.4. Let $\varpi : \overline{X} \dashrightarrow V$ be the blow-up of V along S. Taking the proper transform D' of D under ϖ, we have an orbifold admissible pair $(\overline{X}, D')$ by Theorem 4.4. Then we may denote D' by D. Let $\overline{\pi} : \widehat{X} \dashrightarrow \overline{X}$ and $\pi : \widehat{V} \dashrightarrow V$ be the crepant resolutions of $\overline{X}$ and V respectively. Let $\widehat{D}$ denote the proper transform of $D \in |-K_{\overline{X}}|$ under the resolution $\overline{\pi}$. Then there is an induced map $\widehat{\varpi} : \widehat{X} \dashrightarrow \widehat{V}$ which makes the following diagram commutative:

$$\begin{array}{ccc} \widehat{X} & \overset{\widehat{\varpi}}{\dashrightarrow} & \widehat{V} \\ {\scriptstyle \overline{\pi}\,:\,\text{crepant}}\ \Big\downarrow & & \Big\downarrow\ {\scriptstyle \pi\,:\,\text{crepant}} \\ \overline{X} & \overset{\varpi}{\dashrightarrow} & V \end{array}$$

Here the vertical maps are crepant resolutions and the horizontal maps are the blow-ups of four-dimensional complex algebraic varieties along the complete intersections. Furthermore, a compatible antiholomorphic involution σ on V lifts to $\overline{X}$ by Proposition 3.9. With this notation, we consider a compact simply-connected 8-manifold $M_{\mathrm{CY}} = (\widehat{X}_1 \setminus \widehat{D}_1) \cup (\widehat{X}_2 \setminus \widehat{D}_2)$ which is obtained by Theorem 3.21. Also, let M_{Spin} be a compact simply-connected 8-manifold which is a resolution of $M_{\mathrm{Spin}}^{\triangledown} = (\overline{X}_1 \setminus D_1)/\langle\sigma_1\rangle \cup (\overline{X}_2 \setminus D_2)/\langle\sigma_2\rangle$ obtained by Theorem 3.23. Then we have the following.

Proposition 6.2. *The above M_{CY} admits a Ricci-flat Kähler metric. Moreover, if M_{CY} has no $K3$-factor, then M_{CY} is a Calabi–Yau fourfold and M_{Spin} is a compact* Spin(7)*-manifold.*

Proof. For $i = 1, 2$, let $k_i = \#\,\mathrm{Sing}\,\overline{X}_i$. In our case, each singular point is modelled on $\mathbb{C}^4/\mathbb{Z}_4$ and has a unique crepant resolution with the exceptional divisor $E = \widehat{\mathbb{C}^4/\mathbb{Z}_4} \cong K_{\mathbb{C}P^3}$. Thus we have $\chi(E) = 4$. This implies that

$$\chi(\widehat{X}_i) = \chi(\overline{X}_i) - k_i + \chi(E)k_i = \chi(\overline{X}_i) + 3k_i.$$

A straightforward calculation shows that

$$\chi(M_{\mathrm{CY}}) = 2\chi(M_{\mathrm{Spin}}) = \sum_{i=1}^{2} \left(\chi(\overline{X}_i) - \chi(D_i) + 3k_i\right) \quad \text{and}$$
$$\tau(M_{\mathrm{CY}}) = 2\tau(M_{\mathrm{Spin}}) = \sum_{i=1}^{2} \left(\tau(\overline{X}_i) - k_i\right).$$

This yields

$$\widehat{A}(M_{\mathrm{CY}}) = 2\widehat{A}(M_{\mathrm{Spin}}). \tag{6.2}$$

Now Theorem 3.23 shows that $\mathrm{Hol}(M_{\mathrm{Spin}}) \subseteq \mathrm{Spin}(7)$. Therefore we conclude that $\widehat{A}(M_{\mathrm{Spin}}) \geq 1$ by Theorem 2.8. Then $\widehat{A}(M_{\mathrm{CY}})$ is 2 or 4 by (6.2). Again by Theorem 2.8, M_{CY} admits a Ricci-flat Kähler metric. Moreover, if M_{CY} has no $K3$-factor, then $\widehat{A}(M_{\mathrm{CY}})$ must be 2, and hence $\widehat{A}(M_{\mathrm{Spin}}) = 1$. □

Finally we find an example of Calabi–Yau fourfolds using the same ingredients of the previous Spin(7)-manifold in Section 5.

EXAMPLE 6.3. Let V be $\mathbb{C}P^4(1,1,1,1,4)$. Let D and S be as in Section 5. According to the previous argument, we obtain M_{CY} by gluing two copies of $\widehat{X} \setminus \widehat{D}$ along their cylindrical ends. Then we have $\chi(\widehat{X}) = 1381 - 1 + 4 = 1384$ and $\chi(\widehat{D}) = \chi(D) = -296$. This implies

$$\chi(M_{\mathrm{CY}}) = 2(\chi(\widehat{X}) - \chi(\widehat{D})) = 3360 \neq 576 = \chi(K3 \times K3).$$

Thus M_{CY} is a Calabi–Yau fourfold by Proposition 6.2.

Table 6.4. The list of the Hodge numbers.

Index	Weighted hypersurfaces in $W = \mathbb{C}P^5(1^4, 4^2)$	Smooth Calabi–Yau divisor on V_i	Weighted complete intersection in V_i
i	V_i	$D = D_1 = D_2$	$S_i \in \lvert D_i \cdot D_i \rvert$
1	$h^{1,1}(V_1) = 1, h^{3,1}(V_1) = 35,$ $h^{2,2}(V_1) = 232$	$h^{1,1}(D) = 1,$ $h^{2,1}(D) = 149$	$h^{0,2}(S_1) = 35,$ $h^{1,1}(S_1) = 232$
2	$h^{1,1}(V_2) = h^{2,2}(V_2) = 1$	$h^{1,1}(D) = 1,$ $h^{2,1}(D) = 149$	$h^{0,2}(S_2) = 199,$ $h^{1,1}(S_2) = 976$

Table 6.5. The resulting Spin(7)-manifolds in Section 6.1.

The resulting Spin(7)-manifolds M	$\tau(M)$	$\chi(M)$	b^4
M_{11}	320	912	910
M_{12}	448	1296	1294
M_{22}	576	1680	1678

References

[1] R. Bott and L. Tu, Differential forms in algebraic topology, Graduate Texts in Math., **82**, Springer-Verlag, New York-Berlin, 1982.
[2] R. Clancy, New examples of compact manifolds with holonomy Spin(7), Ann. Global Anal. Geom., **40** (2011), 203–222.
[3] V. I. Danilov and A. G. Khovanskii, Newton polyhedra and an algorithm for computing Hodge-Deligne numbers, Math., USSR-Izv, **29** (1987), 279–298.
[4] M. Doi, Gluing construction of compact complex surface with trivial canonical bundle, J. Math. Soc. Japan, **61** (2009), 853–884.
[5] M. Doi and N. Yotsutani, Doubling construction of Calabi–Yau threefolds, New York J. Math., **20** (2014), 1203–1235.
[6] M. Doi and N. Yotsutani, Doubling construction of Calabi–Yau fourfolds from toric Fano fourfolds, Commun. Math. Stat., **3** (2015), 423–447.
[7] M. Doi and N. Yotsutani, Gluing construction of compact Spin(7)-manifolds and its new example [in Japanese], The Geometry Section of the Mathematical Society of Japan, Autumn Meeting 2011, Shinshu University.
[8] A. R. Fletcher, Working with weighted complete intersections, Explicit birational geometry of 3-folds, London Math. Society Lecture Note Series, **281**, Cambridge University Press, Cambridge 2000, 101–173.
[9] P. Griffths and J. Harris, Principles of algebraic geometry, John Wiley & Sons, New York, 1978.
[10] R. Harvey, Spinors and Calibrations, Perspectives in Mathematics, **9**, Academic Press, San Diego, 1990.
[11] M. Haskins, H.-J. Hein and J. Nördstrom, Asymptotically cylindrical Calabi–Yau manifolds, J. Differential Geom. Math., **101** (2015), 213–265.
[12] D. D. Joyce, Compact 8-manifolds with holonomy Spin(7), Invent. Math., **123** (1996), 507–552.
[13] D. D. Joyce, A new construction of compact 8-manifolds with holonomy Spin(7), J. Differ. Geom., **53** (1999), 89–130.
[14] D. D. Joyce, Compact Manifolds with Special Holonomy, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.
[15] A. Kovalev, Twisted connected sums and special Riemannian holonomy, J. Reine Angew. Math., **565** (2003), 125–160.
[16] A. Kovalev, Asymptotically cylindrical manifolds with holonomy Spin(7), I, arXiv: math.DG/1309.5027.
[17] A. Kovalev and N.-H. Lee, $K3$ surfaces with non-symplectic involution and compact irreducible G_2-manifolds, Math. Proc. Camb. Phil. Soc., **151** (2011), 193–218.
[18] S. M. Salamon, Riemannian Geometry and Holonomy Groups, Pitman Research Notes in Math., **201**, Longman, Harlow, 1989.
[19] G. Tian and S.-T. Yau, Complete Kähler manifolds with zero Ricci curvature, I, J. Amer. Math. Soc., **3** (1990), 579–609.

Mamoru Doi
11-9-302 Yumoto-cho, Takarazuka
Hyogo 665-0003, Japan
E-mail: doi.mamoru@gmail.com

Naoto Yotsutani
Faculty of Education, Mathematics
Kagawa University
Saiwai-cho, 1-1
Takamatsu, Kagawa 760-8522, Japan
E-mail: naoto-yotsutani@ed.kagawa-u.ac.jp