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Evasion and prediction III
Constant prediction and dominating reals

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Abstract. We prove that $\mathfrak{b} \leq \mathfrak{v}_2^{\mathrm{const}}$ where $\mathfrak{b}$ is as usual the unbounding number, and $\mathfrak{v}_2^{\mathrm{const}}$ is the constant prediction number, that is, the size of the least family Π of functions $\pi : 2^{<\omega} \to 2$ such that for each $x \in 2^\omega$ there are $\pi \in \Pi$ and k such that for almost all intervals I of length k, one has $\pi(x \restriction i) = x(i)$ for some $i \in I$. This answers a question of Kamo. We also include some related results.

Introduction.

This work is about evasion and prediction, a combinatorial concept originally introduced by Blass when studying set-theoretic aspects of the Specker phenomenon in abelian group theory [**Bl1**]. It is also about how hard (in a descriptive set-theoretic sense) it sometimes can be to prove *ZFC*-inequalities between cardinal invariants of the continuum.

For our purposes, call a function $\pi : 2^{<\omega} \to 2$ a *predictor*. Say π *k-constantly predicts* a real $x \in 2^\omega$ if for almost all intervals I of length k, there is $i \in I$ such that $x(i) = \pi(x \restriction i)$. In case π k-constantly predicts x for some k, say that π *constantly predicts* x. The *constant prediction number* $\mathfrak{v}_2^{\mathrm{const}}$ is the smallest size of a set of predictors Π such that every $x \in 2^\omega$ is constantly predicted by some $\pi \in \Pi$. As mentioned already, the concept of prediction is originally due to Blass [**Bl1**] who also put it into a much more general framework in [**Bl2**, Section 10]. The notion of constant prediction and the definition of $\mathfrak{v}_2^{\mathrm{const}}$, however, are due to Kamo (see [**Ka1**] and [**Ka2**]), and the notation $\mathfrak{v}_2^{\mathrm{const}}$ is due to Kada (see, e.g., [**Kad**]).

Kamo observed that $\mathfrak{v}_2^{\mathrm{const}} \geq \mathrm{cov}(\mathscr{M}), \mathrm{cov}(\mathscr{N})$ [**Ka1**]. He also proved that $\mathfrak{v}_2^{\mathrm{const}}$ may be larger than all cardinal invariants in Cichoń's diagram [**Ka1**], and smaller than the dominating number $\mathfrak{d}$ [**Ka2**]. He asked [**Ka3**] whether it can

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even be smaller than the unbounding number $\mathfrak{b}$. In 1.5 we shall show this is not possible.

Theorem. $\mathfrak{b} \leq \mathfrak{v}_2^{\text{const}}$.

Two comments concerning this result and its proof are in order. Firstly, shortly before we obtained our result, Kamo (unpublished) proved that an ω-stage iteration of Laver forcing adjoins $x \in 2^\omega$ which is not constantly predicted by any predictor from the ground model. This shows that $\mathfrak{v}_2^{\text{const}} = \aleph_2$ after adding ω_2 Laver reals with countable support over a model for CH. This was strong evidence, and also an incentive, for our 1.4 and 1.5. For Zapletal [**Za**] has proved, assuming a proper class of measurable Woodin cardinals, that the iterated Laver model is a minimal model for $\mathfrak{b}$ in the sense that whenever a cardinal invariant $\mathfrak{i}$ with a reasonably easy definition has value $\aleph_2$ in that model, then $\mathfrak{b} \leq \mathfrak{i}$ is provable. Now, $\mathfrak{v}_2^{\text{const}}$ indeed falls into Zapletal's framework. However, our result does not follow from Kamo's and Zapletal's work because the latter uses a large cardinal assumption while ours is in ZFC alone. Moreover, it turns out our proof of 1.5 is much simpler than Kamo's argument referred to above.

Secondly, Kamo [**Ka3**] showed that after adding one Laver real, every new real is still 2-predicted by a ground model predictor. It turns out this is still true for arbitrary finite stage iterations of Laver forcing, with 2 replaced by some larger k which depends on the length of the iteration (see Theorem 2.5 below). This means in particular that the standard approach to proving inequalities between cardinal invariants—which would in this case mean exhibiting Borel functions $f \mapsto x_f : \omega^\omega \to 2^\omega$ and $\pi \mapsto g_\pi : 2^{2^{<\omega}} \to \omega^\omega$ such that whenever $f \geq^* g_\pi$, then π does not (k-)constantly predict x_f—does not work here. For the latter would mean that given a model M of ZFC and a dominating real f over M, there is x_f not (k-)constantly predicted by any predictor from M—which fails in the Laver extension of M. Worse still, Theorem 2.5 says that one cannot get away with using 2 or 3 models, each containing a dominating real over the preceding one (as is usually the case when one model and one "generic enough" object over the model are not sufficient, e.g. in the Bartoszyński-Miller characterization of $\mathsf{cov}(\mathcal{M})$ where two infinitely often equal reals are needed to get a Cohen real, or in Truss' Theorem $\mathsf{add}(\mathcal{M}) \geq \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$ where a dominating real over a Cohen real is needed [**BJ**]). So the proof of $\mathfrak{b} \leq \mathfrak{v}_2^{\text{const}}$ is hard in a descriptive set-theoretic sense.

In Section 3, we dualize Kamo's consistency of $\mathfrak{v}_2^{\text{const}} < \mathfrak{d}$ [**Ka2**] to get the consistency of $\mathfrak{e}_2^{\text{const}} > \mathfrak{b}$, and give an alternative proof of Kamo's result as well. The subsequent section dualizes Kamo's $CON(\mathfrak{v}_2^{\text{const}} < \mathfrak{v}^{\text{const}})$ [**Ka1**] to $CON(\mathfrak{e}_2^{\text{const}} > \mathfrak{e}^{\text{const}})$, and, again, reproves his consistency. Further results connected with the work reported herein shall appear in [**BSh**].

We keep our notation fairly standard. For basics concerning the cardinal invariants considered here, as well as the forcing techniques, see [**BJ**] and [**Bl2**].

Apart from Section 4 (January 2001), the results in this paper were obtained in Spring 2000.

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1. The ZFC-results.

The following result is the main combinatorial ingredient for the proof of Theorem 1.5 below. By Theorem 2.5, it is optimal.

THEOREM 1.1. *Fix $k \in \omega$. Let $\ell = 2^k - 1$. Assume there are ZFC-models $M_0 \subset M_1 \subset \cdots \subset M_\ell$ and reals $f_0, \dots, f_{\ell-1} \in \omega^\omega$ such that $f_i \in M_{i+1}$ is dominating over M_i. Then there is $x \in 2^\omega \cap M_\ell$ which is not k-constantly predicted by any predictor from M_0.*

PROOF. Assume without loss all f_i are strictly increasing, $f_i(0) > 0$ and $f_i(n+1) > f_i(n) + k$. Define $h_i \in \omega^\omega \cap M_{i+1}$ by the recursion $h_i(0) = f_i(0)$ and $h_i(n+1) = f_i(h_i(n))$. Without loss we may assume $\mathrm{ran}(h_{i+1}) \subseteq \mathrm{ran}(h_i)$ for all i. Clearly $h_i \geq f_i$ for all i. List $\{s \in 2^k; s \neq 0\}$ (where 0 denotes the sequence with constant value 0) as $\{s_i; i < \ell\}$. Define $x \in 2^\omega$ as follows:

$$x(n) = \begin{cases} 0 & \text{if } n \notin \{h_0(m) + j; m \in \omega \text{ and } j < k\} \\ s_i(j) & \text{if } n \text{ is of the form } h_i(m) + j, i < \ell - 1 \text{ and } j < k, \\ & \text{and } h_i(m) \notin \mathrm{ran}(h_{i+1}) \\ s_{\ell-1}(j) & \text{if } n \text{ is of the form } h_{\ell-1}(m) + j, j < k \end{cases}$$

We also define, for each $t \in 2^{<\omega}$ and each $i \leq \ell$, a real $x_{t,i} \in 2^\omega \cap M_i$:

$$x_{t,0} = t\hat{\ }0 \quad (\text{this means } x_{t,0} \text{ is constantly 0 past } |t|)$$

$$x_{t,i}(n) = \begin{cases} t(n) & \text{if } n \in |t| \\ 0 & \text{if } n \notin \{h_0(m) + j; m \in \omega \text{ and } j < k\} \cup |t| \\ s_{i'}(j) & \text{if } n \text{ is of the form } h_{i'}(m) + j, i' < i - 1 \text{ and } j < k, \\ & h_{i'}(m) \notin \mathrm{ran}(h_{i'+1}), \text{ and } n \notin |t| \\ s_{i-1}(j) & \text{if } n \text{ is of the form } h_{i-1}(m) + j, j < k, \text{ and } n \notin |t| \end{cases}$$

for $i > 0$. So $x = x_{\langle\ \rangle,\ell}$. Moreover, the $x_{t,i}$ can be thought of as approximations to x with initial segment t in the intermediate models M_i.

Fix a predictor $\pi \in M_0$. In M_i, $i < \ell$, define $g_i \in \omega^\omega$ by

$$g_0(n) = \min\{m;\ \text{for all } t \in 2^n\text{: if there is } m' \geq n \text{ such that}$$
$$\pi(x_{t,0} \restriction m' + j) \neq x_{t,0}(m' + j) \text{ for all } j < k, \text{ then } m > m' + k\} \text{ and}$$
$$g_i(n) = \min\{m;\ \text{for all } t \in 2^n\text{: if there is } m' \in \operatorname{ran}(h_{i-1}),\ m' \geq n, \text{ such that}$$
$$\pi(x_{t,i} \restriction m' + j) \neq x_{t,i}(m' + j) \text{ for all } j < k, \text{ then } m > m' + k\}$$

for $i > 0$. Now, there is n_0 such that for all $i < \ell$ and all $n \geq n_0$ we have $f_i(n) > g_i(n + k)$. The following is clear from the way things were set up.

CLAIM 1.2. *For all $i < \ell$, all $n, n' > n_0$, all $t \in 2^{n+k}$ such that n and n' are consecutive members of* $\operatorname{ran}(h_i)$: *if there is no $m' \in \operatorname{ran}(h_{i-1}) \cap [n + k, n' - k]$ ($m' \in [n + k, n' - k]$ in case $i = 0$) such that $\pi(x_{t,i} \restriction m' + j) \neq x_{t,i}(m' + j)$ for all j, then it's not true that $\pi(x_{t,i} \restriction n' + j) \neq x_{t,i}(n' + j)$ for all j.*

PROOF. If n, n' are consecutive members of $\operatorname{ran}(h_i)$, we must have $n' = f_i(n)$. Since $g_i(n + k) < f_i(n)$, the claim follows. □

Put $s_{-1} = 0$ (the sequence in 2^k with constant value 0).

CLAIM 1.3. *For all i, all $n, n' > n_0$, all t as in Claim* 1.2: *if there is no $m' \in [n + k, n' - k]$ such that $\pi(x_{t,i} \restriction m' + j) \neq x_{t,i}(m' + j)$ for all j, then for all $i' < i$, it's not true that $\pi(x_{t,i} \restriction n'^\frown s_{i'} \restriction j) \neq (x_{t,i} \restriction n'^\frown s_{i'})(n' + j)$ for all j.*

PROOF. We make induction on i: the case $i = 0$ is clear from Claim 1.2. $i \to i + 1$. n and n' are consecutive members of $\operatorname{ran}(h_{i+1})$. So there is $n^* \geq n$ such that n^* and n' are consecutive members of $\operatorname{ran}(h_i)$. Let $t^* := x_{t,i+1} \restriction n^* + k \in 2^{n^*+k}$. Note that $x_{t^*,i} \restriction n' = x_{t,i+1} \restriction n'$. So we may apply the induction hypothesis to get the conclusion of the claim for all $i' < i$. The case $i' = i$, however, follows from Claim 1.2 (for $i + 1$). □

Applying Claim 1.3 to $i = \ell - 1$, we see that if $n, n' > n_0$ are consecutive members of $\operatorname{ran}(h_{\ell-1})$ and $t \in 2^{n+k}$, then there is $m' \in [n + k, n']$ such that $\pi(x_{t,\ell} \restriction m' + j) \neq x_{t,\ell}(m' + j)$ for all j. (Using that $x_{t,\ell} \restriction n' = x_{t,\ell-1} \restriction n'$, we see that if there is no $m' \in [n + k, n' - k]$ with this property, then, by the claim, $\pi(x_{t,\ell-1} \restriction n'^\frown s_{\ell-1} \restriction j) \neq (x_{t,\ell-1} \restriction n'^\frown s_{\ell-1})(n' + j)$ for all j. However, $x_{t,\ell-1} \restriction n'^\frown s_{\ell-1} \restriction k = x_{t,\ell} \restriction n' + k$.) This completes the proof of the theorem. □

LEMMA 1.4. *Assume there are ZFC-models $M_0 \subset M_1 \subset \cdots \subset M_i \subset \cdots$ and reals $f_0, \ldots, f_i, \ldots \in \omega^\omega$ such that $f_i \in M_{i+1}$ is dominating over M_i. Also assume $N_0 \subset N_1$ are ZFC-models containing $\langle M_i; i \in \omega\rangle, \langle f_i; i \in \omega\rangle$ and $f \in N_1$ is dominating over N_0. Then there is $x \in 2^\omega \cap N_1$ which is not constantly predicted by any predictor from M_0.*

Proof. Assume f is strictly increasing with $f(0) = 0$, and the f_i are as in the previous proof. For $k \in \omega$, let $x_k \in M_{2^k-1}$ be the real from the previous theorem. Let I_k be the intervals of ω defined by consecutive members of $\mathrm{ran}(f)$. Define $x \in 2^\omega$ by $x \restriction I_k = x_k \restriction I_k$. So $x \in N_1$.

Let π be a predictor from M_0. Assume the $g_i^k \in M_i$ are defined as in the proof of Theorem 1.1, $i < 2^k - 1$. So there is n_k such that for all $i < 2^k - 1$ and all $n \geq n_k$, $f_i(n) > g_i^k(n+k)$. The sequence of n_k is constructed in N_0 and therefore eventually dominated by f. Similarly, the intervals $I_k = [f(k), f(k+1)]$ eventually contain two members of $\mathrm{ran}(h_{2^k-2})$. Now, if k is such that $f(k) \geq n_k$ and there are two members of $\mathrm{ran}(h_{2^k-2})$ in I_k, then we find $n \in [f(k)+k, f(k+1)-k]$ such that $\pi(x \restriction n+j) \neq x(n+j)$ for all $j < k$ by the previous proof. So we're done. $\square$

Theorem 1.5. $\mathfrak{b} \leq \mathfrak{v}_2^{\mathrm{const}}$.

Proof. For indeed, if we had $\mathfrak{v}_2^{\mathrm{const}} < \mathfrak{b}$, we could find first a model M_0 of size $\mathfrak{v}_2^{\mathrm{const}}$, and then M_i $(i > 0)$, f_i, N_0, N_1, and f which satisfy the hypotheses of the previous lemma. Thus we reach a contradiction. $\square$

2. Finite iterations of Laver forcing.

Recall that *Laver forcing* $\boldsymbol{L}$ is forcing with trees $p \subseteq \omega^{<\omega}$ such that every node below the *stem* is an *ω-splitting node*, ordered by inclusion. A node $\sigma \in p$ is called ω-splitting if $\sigma\hat{\ }\langle n\rangle \in p$ for infinitely many n. In this case we let $\mathrm{succ}_p(\sigma) = \{n; \sigma\hat{\ }\langle n\rangle \in p\}$, the *successor nodes of σ*. The stem of p, denoted by $\mathrm{stem}(p)$, is the unique ω-splitting node which is comparable with every node of p. Given $\sigma \in p$ let $p_\sigma = \{\tau \in p; \tau \text{ is comparable with } \sigma\}$, the *restriction of p to σ*. If $\mathrm{stem}(p) \subseteq \sigma$, one has $\mathrm{stem}(p_\sigma) = \sigma$. For $p, q \in \boldsymbol{L}$, $p \leq_0 q$ means $p \leq q$ and $\mathrm{stem}(p) = \mathrm{stem}(q)$. For simplicity, think of the generic Laver real ℓ as a strictly increasing function from ω to ω. (This means we force with p containing only strictly increasing σ.)

Let $k \in \omega$ and $f \in \omega^\omega$ be strictly increasing. A tree $T \subseteq 2^{<\omega}$ is called an *(f,k)-tree* if there is $A = A_f^T \subseteq T$ such that

(i) all $s \in A$ are splitting nodes,

(ii) if $s \in A$, $1 \leq |2^{f(|s|)} \cap \{t \in T; s \subset t\}| \leq k$,

(iii) if $s \in A$ and $s \subset t \in 2^{f(|s|)} \cap T$, then $|2^{f(|t|)} \cap \{u \in T; t \subset u\}| = 1$,

(iv) if $s \in A$, $\{t \supset s; |t| \leq f^2(|s|)\} \cap A = \emptyset$,

(v) if $t \in T$ is a splitting node, then there is $s \in A$ such that $s \subseteq t$ and $|t| < f(|s|)$.

(Notice (iii) actually follows from (iv) and (v). We state it just for the sake of clarity.) It is easy to see that A_f^T witnessing T is an (f,k)-tree is unique. Also if $f \leq g$ everywhere and T is both an (f,k)-tree and a (g,k)-tree, $A_f^T \supseteq A_g^T$.

If $\sigma \in \omega^{<\omega}$ is strictly increasing, call T a (σ, k)-*tree* if (i) to (v) are satisfied with f replaced by σ, and $A \subseteq 2^{<|\sigma|}$. (Of course, this means T has only finitely many splitting nodes.) Note that an $(f, 1)$-tree is nothing but a real number.

MAIN LEMMA 2.1. *Let $\dot{T}$ be an $\mathbf{L}$-name for an $(\dot{h}, k)$-tree where $\dot{h}$ is forced to dominate $\dot{\ell}$, the $\mathbf{L}$-generic real, everywhere. Also let $p \in \mathbf{L}$ and $f \in \omega^\omega$ be arbitrary. Then there are $q \leq p$, $g \geq^* f$, and a $(g, k+1)$-tree S such that $q \Vdash \dot{T} \subseteq S$.*

PROOF. We may assume that for all $\sigma \in p$ with $\operatorname{stem}(p) \subseteq \sigma$, there are a number $a_\sigma \leq |\sigma|$ and a sequence $\upsilon_\sigma \in \omega^{a_\sigma}$ such that p_σ decides $\dot{h} \restriction a_\sigma$ to be υ_σ and for all $i \in \operatorname{succ}_p(\sigma)$, $p_{\sigma\hat{}\langle i\rangle} \Vdash \dot{h}(a_\sigma) > n_i$ where $n_i \to \infty$ as $i \to \infty$.

Let $p' \leq p$ arbitrary and observe:

CLAIM 2.2. *Given $\sigma \in p'$ there are a tree $S_\sigma \subseteq 2^{<\omega}$ and a condition $q' \leq_0 p'_\sigma$ such that for all $i \in \operatorname{succ}_{q'}(\sigma)$,*

$$q'_{\sigma\hat{}\langle i\rangle} \Vdash \dot{T} \restriction m_i = S_\sigma \restriction m_i$$

where $m_i \to \infty$ as $i \to \infty$. Furthermore, given τ such that $|\tau| \geq a_\sigma$ and $\tau \restriction a_\sigma \leq \upsilon_\sigma$, and $A \subseteq S_\sigma \restriction |\tau|$ such that $\tau^2(|s|) < |\tau| - 1$ for all $s \in A$ and A witnesses $S_\sigma \restriction |\tau|$ is a (τ, k)-tree, there are $\tau' \supseteq \tau$ with $\tau'(j) \geq f(j)$ for all $|\tau| \leq j < |\tau'|$ and $A_\sigma \subseteq S_\sigma$ containing A such that A_σ witnesses S_σ is a (τ', k)-tree and any node $t \in 2^{|\tau|-1} \cap S_\sigma$ has at most one extension in A_σ.

PROOF. Given $i \in \operatorname{succ}_{p'}(\sigma)$, find $q^i \leq_0 p'_{\sigma\hat{}\langle i\rangle}$ and a finite tree $T^i \subseteq 2^{<\omega}$ of height i such that $q^i \Vdash \dot{T} \restriction i = T^i$. By König's lemma (or, alternatively, by a compactness argument), there are an infinite $B \subseteq \operatorname{succ}_{p'}(\sigma)$, a tree $S_\sigma \subseteq 2^{<\omega}$, and m_i for $i \in B$ with $m_i \to \infty$ such that $T^i \restriction m_i = S_\sigma \restriction m_i$ for all $i \in B$. Now define q' by $\operatorname{stem}(q') = \sigma$, $\operatorname{succ}_{q'}(\sigma) = B$, and $q'_{\sigma\hat{}\langle i\rangle} = q^i$.

We may assume there are $A^i \subseteq T^i$ such that $q^i \Vdash A^{\dot{T}}_{\dot{h}} \restriction i = A^i$. We may also suppose there is $\bar{A}_\sigma \subseteq S_\sigma$ such that $A^i \restriction m_i = \bar{A}_\sigma \restriction m_i$ for all $i \in B$. We must have $A \supseteq \bar{A}_\sigma \restriction |\tau| - 1$ (because $\dot{h}$ is above τ). Consider $t \in S_\sigma \cap 2^{|\tau|-1}$. To be able to construct the required A_σ and τ' it suffices to show that t has at most k extensions in S_σ on any level $\geq |t|$.

To this end, let $s \subset t$ be maximal with $s \in \bar{A}_\sigma$. If $|s| \geq a_\sigma$ or $|s| < a_\sigma$ and $\upsilon_\sigma(|s|) \geq a_\sigma$, s can have at most k extensions on any level $\geq |s|$ (by (iv) and because q' forces no bound on $\dot{h}(a_\sigma)$ there can be no $s' \supset s$ belonging to $\bar{A}_\sigma$).

So assume $|s| < a_\sigma$ and $\upsilon_\sigma(|s|) < a_\sigma$. Then either the set of nodes in S_σ extending t form a branch or there is $s' \supseteq t$ belonging to $\bar{A}_\sigma$ and no splitting occurs between t and s'. Again s' can have at most k extensions on any level $\geq |s'|$, and we're done. $\square$

Let $\{\tau_n; n \in \omega\}$ be a canonical enumeration of $\omega^{<\omega}$, that is, such that

- $\tau_n \subset \tau_m$ implies $n < m$,
- $\tau_n = \tau\hat{}\langle i\rangle$ and $\tau_m = \tau\hat{}\langle j\rangle$ and $i < j$ imply $n < m$.

By recursion on n, define nodes σ_n, trees $S_n \subseteq 2^{<\omega}$, conditions p^n, numbers j_n, the strictly increasing function $g \upharpoonright j_n$, and finite sets A_n such that

(a) $\sigma_n \subseteq \sigma_m$ if and only if $\tau_n \subseteq \tau_m$,

(b) $\mathrm{stem}(p^n) = \sigma_n$; in fact if $m < n+1$ and i are such that $\sigma_{n+1} = \sigma_m\hat{}\langle i\rangle$, then $p^{n+1} \leq_0 p^m_{\sigma_{n+1}}$,

(c) there are $m_i \to \infty$ such that for all $i \in \mathrm{succ}_{p^n}(\sigma_n)$, $p^n_{\sigma_n\hat{}\langle i\rangle} \Vdash \dot{T} \upharpoonright m_i = S_n \upharpoonright m_i$,

(d) if $m < n+1$ and i are such that $\sigma_{n+1} = \sigma_m\hat{}\langle i\rangle$, then $S_m \upharpoonright j_n = S_{n+1} \upharpoonright j_n$,

(e) $A_n \subseteq S_n \cap 2^{<j_n}$ witnesses S_n is a (g,k)-tree and $g^2(|s|) < j_n - 1$ for all $s \in A_n$,

(f) if $m < n+1$ and i are such that $\sigma_{n+1} = \sigma_m\hat{}\langle i\rangle$, then $A_m \subseteq A_{n+1}$ and each $t \in 2^{j_n - 1}$ has at most one extension in A_{n+1},

(g) $g(j) \geq f(j)$ for all $j \geq j_0$,

(h) if $m < n+1$ and i are such that $\sigma_{n+1} = \sigma_m\hat{}\langle i\rangle$, $g(j_n)$ is larger than the level of any splitting node of $S_m \cup S_{n+1}$.

Basic step $n = 0$. Let $\sigma_0 = \mathrm{stem}(p)$. Applying the claim with $p' = p$ and $\sigma = \sigma_0$, we get $S_\sigma = S_0$ and $q' = p^0$ satisfying (b) and (c). By an argument like in the claim find A_0 and $\tau \supseteq \upsilon_\sigma$ such that $\tau^2(|s|) < |\tau| - 1$ for all $s \in A_0$ and A_0 witnesses S_0 is a (τ,k)-tree. Put $j_0 = |\tau|$ and let $g \upharpoonright j_0 = \tau$. So (e) holds.

Recursion step $n \to n+1$. Fix $m \leq n$ such that $\tau_m = \tau_{n+1} \upharpoonright (|\tau_{n+1}| - 1)$. By (c) for m, we can choose $\sigma_{n+1} \supset \sigma_m$ with $|\sigma_{n+1}| = |\sigma_m| + 1$ such that

$$p^m_{\sigma_{n+1}} \Vdash \dot{T} \upharpoonright j_n = S_m \upharpoonright j_n.$$

So (a) holds. Applying the claim with $p' = p^m_{\sigma_{n+1}}$ and $\sigma = \sigma_{n+1}$, we get $S_\sigma = S_{n+1}$ and $q' = p^{n+1}$ satisfying (b) and (c). Since $p^{n+1} \leq_0 p^m_{\sigma_{n+1}}$, we must have $S_{n+1} \upharpoonright j_n = S_m \upharpoonright j_n$, i.e. (d). Let $\tau = g \upharpoonright j_n$ and $A = A_m$. Then $\tau^2(|s|) < |\tau| - 1$ for all $s \in A$ and A witnesses $S_m \upharpoonright |\tau|$ is a (τ,k)-tree (by (e) for m) so that we can use the claim to get $A_\sigma = A_{n+1}$ and $\tau' = g \upharpoonright |\tau'|$ witnessing S_{n+1} is a (τ',k)-tree as well as satisfying (f), (g) and (h) (by choosing $g(j_n)$ large enough). Extending τ', if necessary, we may assume $(\tau')^2(|s|) < |\tau'| - 1$ for all $s \in A_{n+1}$ so that, letting $j_{n+1} = |\tau'|$, we have (e).

This completes the recursive construction. Letting $q = \{\sigma_n; n \in \omega\} \cup \{\sigma_0 \upharpoonright i; i < |\sigma_0|\}$, $q \leq_0 p$ is immediate by (a). (g) entails $g \geq^* f$. Putting $S = \bigcup\{S_n; n \in \omega\}$, $q \Vdash \dot{T} \subseteq S$ is also straightforward (use (c)). So it remains to see S is a $(g,k+1)$-tree. Construct the set of witnesses A^S_g by recursion on j_n. Assume $A^S_g \cap j_n$ has been produced and witnesses $\bigcup_{m \leq n} S_m$ is a $(g \upharpoonright j_n, k+1)$-tree. So consider j_{n+1}. Let $m \leq n$ be such that $\sigma_m\hat{}\langle i\rangle = \sigma_{n+1}$ for some i. By (f), each $t \in 2^{j_n - 1}$ has at most one extension in A_{n+1}, say s. In case $s \in S_m$, put s into A_g^S. Since S_m is not branching anymore and S_{n+1} branches to at most k incompatible nodes beyond s, (ii) above is OK for $k+1$. In case $s \notin S_m$ there is a maximal $s' \subset s$ with $|s'| \geq j_n$ belonging to S_m (by (d)). So put s' into A_g^S. Again (ii) is satisfied, and (i) is because s' must be a splitting node of $S_m \cup S_{n+1}$. (v) holds in both cases because we made $g(|s|)$ for new $s \in A_g^S$ go beyond all splitting levels of $S_m \cup S_{n+1}$ (by (h) and because g is strictly increasing), and (iv) holds because we chose j_{n+1} beyond all $g^2(|s|)$ for new $s \in A_g^S$ (by (e) and because g is strictly increasing). This completes the proof of the main lemma. □

Let $\boldsymbol{L}_k$ denote the finite iteration of $\boldsymbol{L}$ of length k. It generically adds a sequence $\langle \ell_j; j < k \rangle$ of Laver reals.

LEMMA 2.3. *Let G_k be $\boldsymbol{L}_k$-generic over V, and let $x \in 2^\omega \cap V[G_k]$. Then there are $f \in \omega^\omega \cap V$ and an $(f, k+1)$-tree $T \in V$ such that $x \in [T]$.*

PROOF. Repeatedly applying the previous lemma, we find, by backwards recursion on $j < k$, reals $f_j \in V[G_j]$ and $(f_j, k+1-j)$-trees $T_j \in V[G_j]$ such that

- f_j eventually dominates ℓ_{j-1}, the j-th Laver real (in case $j > 0$),
- $x \in [T_{k-1}] \subseteq \cdots \subseteq [T_j] \subseteq \cdots \subseteq [T_0]$.

This is done in straightforward fashion. The only thing to notice is Main Lemma 2.1 also holds for functions eventually dominating the Laver-generic. □

LEMMA 2.4. *Given $f \in \omega^\omega$ strictly increasing, $k \in \omega$, an (f, k)-tree T, and j with $2^j > k$, there is a predictor $\pi : 2^{<\omega} \to 2$ which j-constantly predicts every $x \in [T]$.*

PROOF. Let $A = A^T$ witness T is an (f, k)-tree. Recursively define π. Assume $s \in A$ and $\pi \restriction \{t \in T; t \subset s\}$ has been defined already. Then define π for all $t \in T$ with $s \subseteq t$ and $|t| < f(|s|)$ such that

$$\pi(t) = i \quad \text{if and only if} \quad \frac{|2^{f(|s|)} \cap \{u \in T; t^\frown\langle i \rangle \subseteq u\}|}{|2^{f(|s|)} \cap \{u \in T; t \subset u\}|} \geq \frac{1}{2}.$$

Next define π for all $t \in T$ with $s \subseteq t$, $|t| \geq f(|s|)$, and $t \restriction m \notin A$ for all $|s| < m \leq |t|$ such that $\pi(t)$ is the unique i such that $t^\frown\langle i \rangle \in T$.

To see π j-constantly predicts all of $[T]$, fix $x \in [T]$ and let $n \in \omega$. Assume $\pi(x \restriction n+m) \neq x(n+m)$ for all $m < j$. By the fact T is an (f, k)-tree, $|2^{n+j} \cap \{t \in T; x \restriction n \subseteq t\}| \leq k$. By definition of π and the fact that π mispredicts x on the interval $[n, n+j)$, we see that $|2^{n+j} \cap \{t \in T; x \restriction n+m \subseteq t\}| \leq k/2^m$ for all $0 \leq m \leq j$. For $m = j$, $k < 2^j$ contradicts $x \restriction n+j \in T$, and we're done. □

THEOREM 2.5. *Let G_k be $\boldsymbol{L}_k$-generic over V, and let $x \in 2^\omega \cap V[G_k]$. Given j with $2^j > k+1$, there is a predictor $\pi : 2^{<\omega} \to 2$ in V which j-constantly predicts x.*

PROOF. This is immediate by Lemmata 2.3 and 2.4. $\square$

By Theorem 1.1, this result is best possible. Namely, if j is such that $2^j \leq k+1$, then there is $x \in V[G_k]$ which is not constantly j-predicted by any predictor $\pi \in V$.

3. Duality and consistency.

The *constant evasion number* $\mathfrak{e}_2^{\mathrm{const}}$ is the size of the least family $F \subseteq 2^\omega$ of reals such that for each predictor π there is $x \in F$ which is not constantly predicted by π (see also [**Kad**]). $\mathfrak{e}_2^{\mathrm{const}}$ is dual to $\mathfrak{v}_2^{\mathrm{const}}$ in a natural sense. This means the dual version of Theorem 1.5, namely the inequality $\mathfrak{e}_2^{\mathrm{const}} \leq \mathfrak{d}$, should be a result of *ZFC*. Yet, since Lemma 1.4 involved an ω-sequence of models, we have no proof for this.

CONJECTURE 3.1 (Kada, [**Kad**]). $\mathfrak{e}_2^{\mathrm{const}} \leq \mathfrak{d}$.

However, the other results concerning $\mathfrak{v}_2^{\mathrm{const}}$ which we have mentioned do dualize. Namely, $\mathfrak{e}_2^{\mathrm{const}} \leq \mathsf{non}(\mathscr{M}), \mathsf{non}(\mathscr{N})$ [**Ka1**], $\mathfrak{e}_2^{\mathrm{const}}$ is consistently smaller than all cardinal invariants in Cichoń's diagram [**BSh**], and $\mathfrak{e}_2^{\mathrm{const}}$ is consistently larger than $\mathfrak{b}$. To show the latter, define the following p.o. $\boldsymbol{P}^\omega$. Conditions are triples (k, σ, F) such that $k \in \omega$, $\sigma : \omega^{<\omega} \to \omega$ is a finite partial function, and $F \subseteq \omega^\omega$ is finite, and such that the following requirements are met:

- $|s| \leq k$ for all $s \in \mathrm{dom}(\sigma)$,
- $f \restriction n \in \mathrm{dom}(\sigma)$ for all $f \in F$ and all $n \leq k$,
- $f \restriction k \neq g \restriction k$ for all $f \neq g$ belonging to F,
- $\sigma(f \restriction k) = f(k)$ for all $f \in F$.

The order is given by: $(\ell, \tau, G) \leq (k, \sigma, F)$ if and only if $\ell \geq k$, $\tau \supseteq \sigma$, $G \supseteq F$, and for all $f \in F$ and all n with $k < n < \ell - 1$, either $\tau(f \restriction n) = f(n)$ or $\tau(f \restriction n+1) = f(n+1)$. It is easy to see $\leq$ is transitive. $\boldsymbol{P}^\omega$ adds a generic predictor which 2-constantly predicts all $f \in \omega^\omega$ from the ground model in a canonical fashion.

LEMMA 3.2. *$\boldsymbol{P}^\omega$ is σ-linked.*

PROOF. Note that given k, σ and F_0, F_1, the conditions (k, σ, F_0) and (k, σ, F_1) are compatible: first find $\ell \geq k$ such that $f \restriction \ell \neq g \restriction \ell$ for all $f \neq g$ in $F_0 \cup F_1$. Then extend σ to τ such that $f \restriction n \in \mathrm{dom}(\tau)$ for all $f \in F_0 \cup F_1$ and all n with $k < n \leq \ell$, guaranteeing that

- $\tau(f \restriction \ell) = f(\ell)$ for all $f \in F_0 \cup F_1$,
- for all $f \in F_0 \cup F_1$ and all n with $k < n < \ell - 1$, either $\tau(f \restriction n) = f(n)$ or $\tau(f \restriction n+1) = f(n+1)$.

It is easy to see this can indeed be done for, given any $s \in 2^k$, there can be at most two $f, g \in F_0 \cup F_1$ with $f \restriction k = g \restriction k = s$. $\square$

In fact, the argument above shows $\boldsymbol{P}^\omega$ is $\sigma - 3$-linked (i.e. it's the union of countably many sets P_n such that for all n, any three elements of P_n have a common extension). However, it cannot possibly be $\sigma - 4$-linked [**BSh**]. See also [**Kad**] for related results.

We proceed to show a strong version of "$\boldsymbol{P}^\omega$ does not add a dominating real."

Lemma 3.3. *Given a $\boldsymbol{P}^\omega$-name $\dot{h}$ for a real in ω^ω, there is $H \in \omega^\omega$ such that whenever $x \not\leq^* H$, then $\Vdash \exists^\infty n \ (x(n) > \dot{h}(n))$.*

Proof. Given k, σ, and $\bar{\phi} = \{\phi_0, \ldots, \phi_{i-1}\} \subseteq \omega^k$, define

$$H_{k,\sigma,\bar{\phi}}(n) = \min\{m; \neg\exists(k,\sigma,F) \in \boldsymbol{P}^\omega \ \ (|F| = i \wedge \forall f \in F \ \exists j < i \ \ (f \restriction k = \phi_j)$$
$$\wedge \ (k,\sigma,F) \Vdash \dot{h}(n) \geq m)\}.$$

Clearly $H = H_{k,\sigma,\bar{\phi}} \in (\omega+1)^\omega$. The point, however, is

Claim 3.4. $H \in \omega^\omega$.

Proof. Assume not. Then there are n_0 and $(k,\sigma,F^m) \in \boldsymbol{P}^\omega$, $m \in \omega$, such that $|F^m| = i$, for all $f \in F^m$ there is $j < i$ with $f \restriction k = \phi_j$, and $(k,\sigma,F^m) \Vdash \dot{h}(n_0) \geq m$. Let $F^m = \{f_j^m; j < i\}$ where $f_j^m \restriction k = \phi_j$. Using a standard compactness argument to prune the collection of F^m's, if necessary, we may assume without loss that for all $j < i$, either

- $(*_j)$ there is $g_j \in \omega^\omega$ such that $f_j^m \to g_j$ as $m \to \infty$, or
- $(+_j)$ there are $\ell_j \geq k$ and $\psi_j \in \omega^{\ell_j}$ such that $f_j^m \restriction \ell_j = \psi_j$ for all m, and the values $f_j^m(\ell_j)$ are all distinct.

For j satisfying $(+_j)$ choose $g_j \supset \psi_j$ arbitrarily. Let $G = \{g_j; j < i\}$. Extend (k,σ,G) to (ℓ,τ,G) such that $\ell > \ell_j$ for all j which satisfy $(+_j)$ and such that prediction is correct everywhere, that is, $\tau(g_j \restriction n) = g_j(n)$ for all j and all n with $k < n \leq \ell$.

Find $(\ell',\tau',G') \leq (\ell,\tau,G)$ forcing a value to $\dot{h}(n_0)$, say $(\ell',\tau',G') \Vdash \dot{h}(n_0) = m$. Next choose m_0 such that

- $m_0 > m$,
- $f_j^{m_0} \restriction \ell' + 1 = g_j \restriction \ell' + 1$ for all j which satisfy $(*_j)$,
- $f_j^{m_0} \restriction n \notin \mathrm{dom}(\tau')$ for all j which satisfy $(+_j)$ and all $\ell_j < n \leq \ell$.

Then define $\tau_0 \supseteq \tau'$ such that for all j which satisfy $(+_j)$ and all n with $\ell_j < n \leq \ell'$, $f_j^{m_0} \restriction n \in \mathrm{dom}(\tau_0)$ and $\tau_0(f_j^{m_0} \restriction n) = f_j^{m_0}(n)$. It is straightforward to check that $(\ell',\tau_0,F^{m_0}) \in \boldsymbol{P}^\omega$ and $(\ell',\tau_0,F^{m_0}) \leq (k,\sigma,F^{m_0})$. Furthermore, $(\ell',\tau_0,G') \leq$

(ℓ', τ', G') is trivial. This means (ℓ', τ_0, F^{m_0}) and (ℓ', τ_0, G') force contradictory statements about the value of $\dot{h}(n_0)$, yet, by the argument of 3.2, they are compatible. This contradiction completes the proof of the claim. $\square$

Now choose $H \in \omega^\omega$ such that $H \geq^* H_{k,\sigma,\bar{\phi}}$ for all k, σ, and $\bar{\phi}$. Fix $x \in \omega^\omega$ with $x \not\leq^* H$. A standard argument shows x is indeed forced not to be eventually dominated by $\dot{h}$, and we're done with the lemma. $\square$

COROLLARY 3.5. *$\boldsymbol{P}^\omega$ preserves unbounded families.*

Before stating and proving the main result of this section, let us introduce the constant prediction and evasion numbers for the Baire space ω^ω. This is done in exactly the same fashion as for the Cantor space 2^ω: say $\pi : \omega^{<\omega} \to \omega$ *k-constantly predicts* $f \in \omega^\omega$ if for almost all intervals I of length k, $\pi(f \restriction i) = f(i)$ for some $i \in I$. Let $\mathfrak{v}^{\mathrm{const}}$ be the size of the least family of predictors Π such that for all $f \in \omega^\omega$ there are k and $\pi \in \Pi$ such that π k-constantly predicts f, and let $\mathfrak{e}^{\mathrm{const}}$ be the size of the least $F \subseteq \omega^\omega$ such that for each predictor π there is $f \in F$ which is not k-constantly predicted by π for any k. Clearly, $\mathfrak{e}^{\mathrm{const}} \leq \mathfrak{e}_2^{\mathrm{const}}$ and $\mathfrak{v}_2^{\mathrm{const}} \leq \mathfrak{v}^{\mathrm{const}}$. Furthermore, $\mathfrak{e}^{\mathrm{const}} \leq \mathsf{cov}(\mathscr{M})$ and $\mathfrak{v}^{\mathrm{const}} \geq \mathsf{non}(\mathscr{M})$ [**Ka1**], and $\mathfrak{v}_2^{\mathrm{const}} < \mathfrak{v}^{\mathrm{const}}$ [**Ka1**] and $\mathfrak{v}^{\mathrm{const}} < \mathfrak{d}$ [**Ka2**] are both consistent.

THEOREM 3.6. (a) $\mathfrak{e}^{\mathrm{const}} > \mathfrak{b}$ *is consistent*; *in fact*, *given $\kappa < \lambda = \lambda^{<\kappa}$ regular uncountable*, *there is a p.o. $\boldsymbol{P}$ forcing $\mathfrak{e}^{\mathrm{const}} = \lambda = \mathfrak{c}$ and $\mathfrak{b} = \kappa$.*

(b) (Kamo, [**Ka2**]) $\mathfrak{v}^{\mathrm{const}} < \mathfrak{d}$ *is consistent*; *in fact*, *given κ regular uncountable and $\lambda = \lambda^\omega > \kappa$, there is a p.o. $\boldsymbol{P}$ forcing $\mathfrak{v}^{\mathrm{const}} = \kappa$ and $\mathfrak{d} = \lambda = \mathfrak{c}$.*

Note that Kamo's original proof of (b) uses a countable support iteration of Miller's rational perfect set forcing, and thus works only in case $\kappa = \aleph_1$ and $\lambda = \aleph_2$. (In fact, in light of Zapletal's result [**Za**] that the iterated Miller model is a minimal model for $\mathfrak{d}$, Kamo's $\mathfrak{v}^{\mathrm{const}} = \aleph_1$ [**Ka2**] in the latter model follows from our result.) (a) answers another question of Kamo's [**Ka2**].

PROOF. (a) Let $\langle \boldsymbol{P}_\alpha, \dot{\boldsymbol{Q}}_\alpha; \alpha < \lambda \rangle$ be a finite support iteration of ccc forcing such that

- for even α, $\Vdash_\alpha \dot{\boldsymbol{Q}}_\alpha = \dot{\boldsymbol{P}}^\omega$, the forcing defined above,
- for odd α, $\Vdash_\alpha \dot{\boldsymbol{Q}}_\alpha$ is a subforcing of Hechler forcing of size $< \kappa$.

Guarantee that we take care of all small subforcings of Hechler forcing by a book-keeping argument. Then $\mathfrak{b} \geq \kappa$ is straightforward. $\mathfrak{e}^{\mathrm{const}} \geq \lambda \geq \mathfrak{c} \geq \mathfrak{e}^{\mathrm{const}}$ is clear because we iteratively add predictors which 2-constantly predict all ground model reals. To show $\mathfrak{b} \leq \kappa$, argue by induction that a family $F \subseteq \omega^\omega$ of size κ such that given any $G \subseteq \omega^\omega$ of size $< \kappa$ there is $f \in F$ with $f \not\leq^* g$ for all $g \in G$ (such a family is added after the first κ stages of the iteration, simply use the family of Cohen reals adjoined in the limit steps up to κ) is preserved along the iteration. For the even successor step, this follows from Lemma 3.3, for the odd successor step, use the well-known analog of 3.3 for forcing notions of size $< \kappa$, and for the limit step, use a standard argument.

(b) First add λ many Cohen reals. Then make a finite support iteration $\langle \boldsymbol{P}_\alpha, \dot{\boldsymbol{Q}}_\alpha; \alpha < \kappa \rangle$ of the forcing $\boldsymbol{P}^\omega$ defined above. Again, $\mathfrak{v}^{\text{const}} = \kappa$ is clear. $\mathfrak{d} = \mathfrak{c} = \lambda$ follows from Lemma 3.3 using standard arguments (the point is that $\mathfrak{d} = \mathfrak{c} = \lambda$ in the intermediate model, and this is preserved along the iteration because the analog of 3.3 holds for any $\boldsymbol{P}_\alpha$). $\square$

4. Baire space versus Cantor space.

To dualize Kamo's consistency of $\mathfrak{v}_2^{\text{const}} < \mathfrak{v}^{\text{const}}$ [**Ka1**], use the forcing $\boldsymbol{P}^2$ which is the analog of $\boldsymbol{P}^\omega$ in the Cantor space. That is, conditions are of the form (k, σ, F) such that $k \in \omega$, $\sigma : 2^{<\omega} \to 2$ is a finite partial function, and $F \subseteq 2^\omega$ is finite satisfying the same requirements as $\boldsymbol{P}^\omega$ in Section 3. Additionally stipulate $\text{dom}(\sigma) = 2^{\leq k}$.

Given a predictor $\pi : \omega^{<\omega} \to \omega$, say $x \in \omega^\omega$ *strongly evades* π if for all k there is an interval I of length k such that $\pi(x \restriction i) < x(i)$ for all $i \in I$. Obviously, if x strongly evades π, then π does not constantly predict x.

CRUCIAL LEMMA 4.1. *Given a $\boldsymbol{P}^2$-name $\dot{\pi} : \omega^{<\omega} \to \omega$ for a predictor, there is a predictor $\psi : \omega^{<\omega} \to \omega$ such that whenever x strongly evades ψ, then $\Vdash$ "$\dot{\pi}$ does not constantly predict x."*

PROOF. Given conditions (k, σ, F), (ℓ, τ, G), say that (ℓ, τ, G) is an *almost extension* of (k, σ, F) if there is $G_0 \subseteq G$ with $|G_0| = |F|$ such that $(k, \sigma, G_0) \geq (\ell, \tau, G)$ and for all $f \in F$ there is $g \in G_0$ such that $f \restriction \ell = g \restriction \ell$. Note that if (ℓ, τ, G) is an almost extension of (k, σ, F), then (k, σ, F) and (ℓ, τ, G) are compatible (use the argument of the proof of Lemma 3.2).

Fix k, σ. Let $\bar{\phi} = \{\phi_0, \ldots, \phi_{i-1}\} \subseteq 2^k$. Define $A_{k,\sigma,\bar{\phi}} = \{(k, \sigma, F) \in \boldsymbol{P}^2;$ $|F| = i$ and $\forall f \in F\ \exists j < i\ (f \restriction k = \phi_j)\}$.

CLAIM 4.2. *Given $D \subseteq \boldsymbol{P}^2$ open dense and finitely many conditions $(\ell_0^j, \tau_0^j, G_0^j)$, $j < m_0$, such that for all $(k, \sigma, F) \in A_{k,\sigma,\bar{\phi}}$ there is j such that $(\ell_0^j, \tau_0^j, G_0^j)$ is an almost extension of (k, σ, F), there are finitely many conditions $(\ell_1^j, \tau_1^j, G_1^j) \in D$, $j < m_1$, such that*

- *each $(\ell_1^j, \tau_1^j, G_1^j)$ extends some $(\ell_0^{\bar{j}}, \tau_0^{\bar{j}}, G_0^{\bar{j}})$,*
- *for all $(k, \sigma, F) \in A_{k,\sigma,\bar{\phi}}$ there is j such that $(\ell_1^j, \tau_1^j, G_1^j)$ is an almost extension of (k, σ, F).*

PROOF. Note first that if there is some number m such that the conditions of the form (ℓ, τ, G) where $\ell \leq m$ satisfy the conclusion of the claim, then finitely many such (ℓ, τ, G) are sufficient, and we are done (this is immediate from the definition of "almost extension").

Therefore, assuming the claim is false, we may suppose there are (k, σ, F^m) such that for all m, no condition of the form (ℓ, τ, G) with $\ell \le m$ is simultaneously in D, an extension of some $(\ell_0^j, \tau_0^j, G_0^j)$ and an almost extension of (k, σ, F^m). Let $F^m = \{f_j^m; j < i\}$. Without loss there are $f_j \in 2^\omega$ such that $f_j^m \to f_j$ as $m \to \infty$. Put $F = \{f_j; j < i\}$ and consider (k, σ, F). Find $j < m_0$ such that $(\ell_0^j, \tau_0^j, G_0^j)$ is an almost extension of (k, σ, F). Choose a common extension $(\bar{\ell}, \bar{\tau}, \bar{G})$. Then find $(\ell^*, \tau^*, G^*) \le (\bar{\ell}, \bar{\tau}, \bar{G})$ with $(\ell^*, \tau^*, G^*) \in D$. Note that for large enough m, (ℓ^*, τ^*, G^*) is an almost extension of (k, σ, F^m) (because $(k, \sigma, F) \ge (\ell^*, \tau^*, G^*)$ and $f_j^m \restriction \ell^* = f_j \restriction \ell^*$ for large enough m). For $m > \ell^*$, this contradicts the choice of F^m, and the claim is proved. □

Let $\{s_n; n \in \omega\}$ list $\omega^{<\omega}$. For each n, put $D_n = \{(\ell, \tau, G) \in \boldsymbol{P}^2; (\ell, \tau, G)$ decides $\dot{\pi}(s_n)\}$. Clearly this set is open dense. Still keeping $k, \sigma, \bar{\phi}$ fixed, and using the claim we can easily construct conditions $(\ell^j_{n,k,\sigma,\bar{\phi}}, \tau^j_{n,k,\sigma,\bar{\phi}}, G^j_{n,k,\sigma,\bar{\phi}}) = (\ell_n^j, \tau_n^j, G_n^j) \in D_n$, $j < m_n$, such that

- for all n, $(\ell_{n+1}^j, \tau_{n+1}^j, G_{n+1}^j)$ extends some $(\ell_n^{\bar{j}}, \tau_n^{\bar{j}}, G_n^{\bar{j}})$,
- for all $(k, \sigma, F) \in A_{k,\sigma,\bar{\phi}}$ there is $j < m_n$ such that $(\ell_n^j, \tau_n^j, G_n^j)$ is an almost extension of (k, σ, F).

Define $\chi_{k,\sigma,\bar{\phi}}(s_n) = \max\{a;$ some $(\ell_n^j, \tau_n^j, G_n^j)$ forces $\dot{\pi}(s_n) = a\} + 1$.

Finally unfix $(k, \sigma, \bar{\phi})$, and let $\psi(s_n) = \max\{\chi_{k,\sigma,\bar{\phi}}(s_n); k \le n, \mathrm{dom}(\sigma) = 2^{\le k}$ and $\bar{\phi} \subseteq 2^k\}$.

To see this works, choose x strongly evading ψ. Also fix a condition (k, σ, F), and $k_0 \ge k$ such that for all $i \in [k_0, k_0 + k)$, we have $\psi(x \restriction i) < x(i)$. Let $\bar{\phi} = \{f \restriction k; k \in F\}$. Let n_i be such that $x \restriction i = s_{n_i}$. Without loss $k \le n_{k_0} < \cdots < n_{k_0+k-1}$. Put $n = n_{k_0+k-1}$. Find $j < m_n$ such that $(\ell_n^j, \tau_n^j, G_n^j) = (\ell^j_{n,k,\sigma,\bar{\phi}}, \tau^j_{n,k,\sigma,\bar{\phi}}, G^j_{n,k,\sigma,\bar{\phi}})$ is an almost extension of (k, σ, F). Let $(\bar{\ell}, \bar{\tau}, \bar{G})$ be a common extension. Then

$$(\bar{\ell}, \bar{\tau}, \bar{G}) \Vdash \dot{\pi}(x \restriction i) < \chi_{k,\sigma,\bar{\phi}}(x \restriction i) \le \psi(x \restriction i) < x(i)$$

for all $i \in [k_0, k_0 + k)$, as required. □

Notice the argument really showed

Lemma 4.3. *Given a $\boldsymbol{P}^2$-name $\dot{\pi} : \omega^{<\omega} \to \omega$ for a predictor, there is a predictor $\psi : \omega^{<\omega} \to \omega$ such that whenever x strongly evades ψ, then $\Vdash$ "x strongly evades $\dot{\pi}$."*

Call $F \subseteq \omega^\omega$ a *strongly evading family* if given any predictor $\pi : \omega^{<\omega} \to \omega$, there is $f \in F$ which strongly evades π.

COROLLARY 4.4. $\boldsymbol{P}^2$ *preserves strongly evading families.*

We are ready to prove the main result of this section. Part (a) answers another question of Kamo's [**Ka2**].

THEOREM 4.5. (a) $\mathfrak{e}_2^{\text{const}} > \mathfrak{e}^{\text{const}}$ *is consistent; in fact, given* $\kappa < \lambda = \lambda^{<\kappa}$ *regular uncountable, there is a p.o.* $\boldsymbol{P}$ *forcing* $\mathfrak{e}_2^{\text{const}} = \lambda = \mathfrak{c}$ *and* $\mathfrak{e}^{\text{const}} = \kappa$.
(b) (Kamo, [**Ka1**]) $\mathfrak{v}_2^{\text{const}} < \mathfrak{v}^{\text{const}}$ *is consistent; in fact, given* κ *regular uncountable and* $\lambda = \lambda^{\omega} > \kappa$, *there is a p.o.* $\boldsymbol{P}$ *forcing* $\mathfrak{v}_2^{\text{const}} = \kappa$ *and* $\mathfrak{v}^{\text{const}} = \lambda = \mathfrak{c}$.

PROOF. This proof is similar to the one of Theorem 3.6.
(a) Let $\langle \boldsymbol{P}_\alpha, \dot{\boldsymbol{Q}}_\alpha; \alpha < \lambda \rangle$ be a finite support iteration of ccc forcing such that
- for even α, $\Vdash_\alpha \dot{\boldsymbol{Q}}_\alpha = \dot{\boldsymbol{P}}^2$,
- for odd α, $\Vdash_\alpha \dot{\boldsymbol{Q}}_\alpha$ is a subforcing of $\dot{\boldsymbol{P}}^\omega$ of size $< \kappa$.

Guarantee that we take care of all small subforcings of $\boldsymbol{P}^\omega$ by a book-keeping argument. Then the only thing we need to prove is $\mathfrak{e}^{\text{const}} \leq \kappa$: argue by induction that a strongly evading family of size κ (which is added after the first κ stages of the iteration) is preserved along the iteration. For the even successor step, this follows from the crucial lemma, for the odd successor step, use the well-known analog of 4.1 for forcing notions of size $< \kappa$, and for the limit step, use a standard argument.

(b) First add λ many Cohen reals. Then make a finite support iteration $\langle \boldsymbol{P}_\alpha, \dot{\boldsymbol{Q}}_\alpha; \alpha < \kappa \rangle$ of $\boldsymbol{P}^2$. $\mathfrak{v}^{\text{const}} = \mathfrak{c} = \lambda$ follows from Lemma 4.1 using standard arguments. □

5. Problems.

Apart from Conjecture 3.1 mentioned at the beginning of Section 3, the following are open.

QUESTION 5.1 (Kamo [**Ka2**]). *Is* $\mathfrak{v}^{\text{const}} < \mathsf{non}(\mathcal{N})$ *consistent? If yes, is even* $\mathfrak{v}^{\text{const}} < \min\{\mathfrak{d}, \mathsf{non}(\mathcal{N})\}$ *consistent? If no, what about* $\mathfrak{v}_2^{\text{const}}$? *Dually, is* $\mathfrak{e}^{\text{const}} > \mathsf{cov}(\mathcal{N})$ *consistent?*

In view of Theorem 1.5, the following is of interest as well.

QUESTION 5.2 (Kamo [**Ka1**], [**Ka2**]). *Is* $\mathfrak{v}_2^{\text{const}} < \mathsf{non}(\mathcal{M})$ *consistent? Dually, is* $\mathfrak{e}_2^{\text{const}} > \mathsf{cov}(\mathcal{M})$ *consistent?*

Recall that $\mathfrak{v}^{\text{const}} \geq \mathsf{non}(\mathcal{M})$ is a theorem of *ZFC* [**Ka1**]. In case both questions have a positive answer, we may even ask

QUESTION 5.3. *Is* $\mathfrak{v}_2^{\text{const}}$ *consistently smaller than the splitting number* $\mathfrak{s}$? *Dually, is* $\mathfrak{e}_2^{\text{const}}$ *consistently larger than the reaping number* $\mathfrak{r}$?

To appreciate the connection, recall that $\mathfrak{s} \leq \mathsf{non}(\mathscr{M}), \mathsf{non}(\mathscr{N})$ in *ZFC*. Apart from Question 5.3, there is no connection between the prediction and evasion numbers on one hand and $\mathfrak{s}$ and $\mathfrak{r}$ on the other hand: $\mathfrak{v}_2^{\mathrm{const}}$ is consistently larger than $\mathfrak{r}$ (either use the model for $\mathfrak{v}_2^{\mathrm{const}} > \mathsf{cof}(\mathscr{N})$ of [**Ka1**] and note the forcing involved is P-point preserving, or make a short iteration of σ-centered forcing over a model of *MA* and use arguments of [**BSh**] to see $\mathfrak{v}_2^{\mathrm{const}}$ stays large), $\mathfrak{v}^{\mathrm{const}}$ is consistently smaller than $\mathfrak{r}$ (this holds in the model for Theorem 3.6 (b) because the iterands of the short iteration are Suslin ccc forcing notions [**BJ**] so that $\mathfrak{r}$ stays large) and $\mathfrak{v}_2^{\mathrm{const}}$ is consistently larger than $\mathfrak{s}$ (e.g. in the Cohen real model). Dual statements hold for $\mathfrak{e}_2^{\mathrm{const}}$ and $\mathfrak{e}^{\mathrm{const}}$, as well.

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