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The bifurcation set of a complex polynomial function of two variables and the Newton polygons of singularities at infinity

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Abstract. A. Némethi and A. Zaharia have defined the explicit set for a complex polynomial function $f : \boldsymbol{C}^n \to \boldsymbol{C}$ and conjectured that the bifurcation set of the global fibration of f is given by the union of the set of critical values and the explicit set of f. They have proved only the case $n = 2$ and f is Newton non-degenerate. In the present paper we will prove this for the case $n = 2$, containing the Newton degenerate case, by using toric modifications and give an expression of the bifurcation set of f in the words of Newton polygons.

1. Introduction.

Let $f : \boldsymbol{C}^n \to \boldsymbol{C}$ be a polynomial function. It is well known that there exists a finite set $\Gamma \subset \boldsymbol{C}$ such that $f : \boldsymbol{C}^n \backslash f^{-1}(\Gamma) \to \boldsymbol{C} \backslash \Gamma$ is a locally trivial fibration. There are many different proofs about the finiteness of Γ, for instance [**V**], [**B1**], [**H-Lê**] and [**N**]. We denote by B_f the smallest set of Γ with the above property and call this the *bifurcation set*. For the set of critical values Σ_f of f we have $\Sigma_f \subseteq B_f$, but the equality does not hold in general. This is because the topology of the global fibration depends on not only the singularities in $\boldsymbol{C}^n$ but also the singularities at infinity. We can easily see the difference of Σ_f and B_f in the example $f(x, y) = x(xy - 1)$. In [**K**], A. G. Kouchnirenko has proved that if f is convenient and Newton non-degenerate then $B_f = \Sigma_f$, and in [**B1**] and [**B2**], S. A. Broughton has defined a certain class called *tame polynomials* and proved $B_f = \Sigma_f$ for them. After that it was expected to find more large classes with the property $B_f = \Sigma_f$. In the beginning of nineties, A. Némethi and A. Zaharia have defined a finite set S_f called the *explicit set* [**N-Z1**] and proved $B_f \subseteq \Sigma_f \cup S_f$. In particular for $n = 2$ they have proved that if $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ is a not convenient and Newton non-degenerate polynomial then $B_f = \Sigma_f \cup S_f$. The estimations of this and other classes are described in [**N-Z2**].

On the other hand we can obtain the information of the fibration by considering the singularities at infinity corresponding to the projective completion.

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H. V. Hà and D. T. Lê have proved that when $n = 2$, $c \in \boldsymbol{C}$ is an element of the bifurcation set of f if and only if the Euler characteristic of $f^{-1}(c)$ is different from the one of the general fibers [**H-Lê**]. For partial extensions of this result to the higher dimensional cases, see [**P**] and [**S-T**]. In the case when $n = 2$, V. T. Le and M. Oka have shown an estimation of the number of the critical values at infinity of f in the words of the Newton polygon of f [**Le-O**].

To explain the content of the present paper, we first give the definition of the explicit set of f. For a given f, let $\operatorname{grad} f(z)$ be the gradient vector defined by

$$\operatorname{grad} f(z) = {}^t\left(\overline{\frac{\partial f}{\partial z_1}}(z), \dots, \overline{\frac{\partial f}{\partial z_n}}(z)\right),$$

where $z = {}^t(z_1, \dots, z_n)$ and the overlines mean their complex conjugations. We set

$$M(f) = \{z \in \boldsymbol{C}^n \mid \text{there exists } \lambda \in \boldsymbol{C} \text{ such that } \operatorname{grad} f(z) = \lambda z\}.$$

Then the explicit set S_f of f is defined by

$$S_f = \left\{ c \in \boldsymbol{C} \;\middle|\; \begin{array}{l} \text{there exists a sequence } \{z^k\} \subseteq M(f) \text{ such that} \\ \lim_{k\to\infty} \|z^k\| = \infty \text{ and } \lim_{k\to\infty} f(z^k) = c \end{array} \right\}.$$

In the present paper we study the explicit set of a polynomial function $f : \boldsymbol{C}^2 \to \boldsymbol{C}$, which contains the case where f is Newton degenerate. If c is an element of the explicit set of f, by Curve Selection Lemma (see [**M**] and [**N-Z2**]), there exists a real analytic curve $p : (0, \varepsilon) \to M(f)$ such that $\lim_{t\to 0}\|p(t)\| = \infty$ and $\lim_{t\to 0} f(p(t)) = c$. We call this curve the *explicit path*. To study the existence of the explicit path we will define a certain inductive algorithm for making a tower of toric modifications with respect to $p(t)$. By using this algorithm we will show that the existence condition of the explicit path is equivalent to the inconstancy of the Milnor numbers of singularities at infinity after the toric modifications. Finally we will prove that:

Theorem 1.1. *Let* $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ *be a polynomial map. Then* $B_f = \Sigma_f \cup S_f$.

This result follows independently from the study of Lojasiewicz numbers in [**H**]. We will prove this from the viewpoint of Newton polygons and toric modifications. In the last section we will give an expression of the elements of the bifurcation set of f in the words of Newton polygons (Theorem 6.5), which is an extended result of the Némethi and Zaharia's expression in [**N-Z1**]. As a corollary we lead the Le and Oka's estimation of the number of the critical values at infinity of f [**Le-O**].

In Section 2 we study the relation of explicit paths on $M(f)$ and the Newton polygon of f and give a first stage of the proof of Theorem 1.1, that is, for the case where f is Newton non-degenerate or not convenient. In Section 3 we give the definitions of toric compactifications and toric modifications, and in Section 4, for a fixed explicit path on $M(f)$, we give an inductive algorithm for making a tower of toric modifications. After finite times inductive toric modifications, we obtain a certain transformed function f^{δ} of f around the limited point of the explicit path. In Section 5 we show two theorems about relations between the explicit path of f, the Newton polygon of f^{δ} and the Euler characteristics of $f^{-1}(0)$ and $f^{-1}(\varepsilon)$ for a generic ε, and complete the proof of Theorem 1.1. Finally in Section 6 we give an expression of the elements of the bifurcation set of f in the words of Newton polygons and estimate the number of the elements.

In this paper we will use the following notations: $\boldsymbol{C}^* = \boldsymbol{C}\backslash\{0\}$, $\boldsymbol{Z}_{\geq 0} = \{n \in \boldsymbol{Z} \mid n \geq 0\}$ and $\boldsymbol{R}_{\geq 0} = \{x \in \boldsymbol{R} \mid x \geq 0\}$.

2. Explicit paths of $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ and Newton polygons.

Let $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ be a polynomial function not depending only on one variable. Let S_f be the explicit set of f, which is defined in Section 1.

Definition 2.1. Let ε be a sufficiently small real positive number and let $p : (0, \varepsilon) \to M(f)$ be a real analytic curve on $M(f)$ with $\lim_{t\to 0}\|p(t)\| = \infty$. If $p(t)$ satisfies $\lim_{t\to 0} f(p(t)) = c$ for some $c \in \boldsymbol{C}$ with $|c| < \infty$, we call $p(t)$ *an explicit path*.

Now we fix some $c \in S_f$. For a convenience we assume $0 \in S_f$ by considering the polynomial $f(x, y) - c$.

Lemma 2.2. *If f has a factor x^2 (or y^2), the path $p(t) = {}^t(0, 1/t)$ (resp. $p(t) = {}^t(1/t, 0)$) is an explicit path with $f(p(t)) \equiv 0$. In this case $p(t)$ is on the singular locus of f and therefore $0 \in \Sigma_f \cap S_f$. If f has neither a factor x^2 nor y^2, both $x(t)$ and $y(t)$ are not constant zero.*

Proof. When f has either a factor x^2 or y^2, the lemma is obvious. Now we consider the case where f has neither a factor x^2 nor y^2. We assume that $x(t)$ is constant zero and $\lim_{t\to 0}|y(t)| = \infty$. When $f(0, y)$ is a polynomial of y, we have $\lim_{t\to 0}|f(0, y(t))| = \infty$. This contradicts $\lim_{t\to 0}|f(p(t))| < \infty$. When $f(0, y)$ is constant, since $\lim_{t\to 0} f(0, y(t)) = 0$, $f(0, y)$ is constant zero. Set $f(x, y) = x^{\gamma} g(x, y)$ where γ is a positive integer and $g(x, y)$ is a polynomial with $g(0, y) \not\equiv 0$. Since f does not have a factor x^2, we have $\gamma = 1$. Then $(\partial f/\partial x)(0, y) \equiv g(0, y) \not\equiv 0$ and $(\partial f/\partial y)(0, y) \equiv 0$. By substituting these for the equation $\operatorname{grad} f(p(t)) = \lambda(t) p(t)$ we have $\lambda(t) \not\equiv 0$, and then $y(t) \equiv 0$. This is a contradiction. We can prove $y(t) \not\equiv 0$ by the same way. $\square$

Let $p(t) = {}^t(x(t), y(t))$ be an explicit path. Now we suppose that both $x(t)$ and $y(t)$ are not constant zero. We can describe $p(t)$ as

$$(2.1) \qquad p(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} at^{k\alpha} + a_1 t^{k\alpha+1} + a_2 t^{k\alpha+2} + \text{higher terms} \\ bt^{k\beta} + b_1 t^{k\beta+1} + b_2 t^{k\beta+2} + \text{higher terms} \end{pmatrix},$$

where $a, b \in \boldsymbol{C}^*$, $k \in \boldsymbol{N}$ and $(\alpha, \beta) \neq (0,0)$ is a pair of coprime integers. Since $\lim_{t\to 0}\|p(t)\| = \infty$, either α or β is negative.

Let $f(x, y) = \sum_{(m,n)} a_{m,n}x^m y^n$ be a given polynomial where $m, n \geq 0$. We define the Newton polygon $\Delta(f)$ of f by the convex hull of the integral points $(m, n) \in \boldsymbol{R}^2$ such that $a_{m,n} \neq 0$. If $\Delta(f)$ intersects both positive axes we say f is *convenient*. Otherwise we say f is *not convenient*. Let $\Pi = {}^t(p, q) \neq (0,0)$ be a pair of coprime integers, called *a primitive covector*. For a given Π we consider the linear function $pX + qY$ where $(X, Y) \in \Delta(f)$ and denote its minimal value by $d(\Pi; f)$. We set

$$\Delta(\Pi; f) := \{(X, Y) \in \Delta(f) \mid pX + qY = d(\Pi; f)\}$$

and call this *a boundary face* (resp. *a boundary vertex*) if $\dim \Delta(\Pi; f) = 1$ (resp. $\dim \Delta(\Pi; f) = 0$). We define the partial sum $f_\Pi(x, y)$ by

$$f_\Pi(x, y) := \sum_{(m,n)\in\Delta(\Pi;f)} a_{m,n}x^m y^n$$

and call this *the boundary function* of the covector Π. In particular, if $\Delta(\Pi; f)$ is a face we call this *a face function*. If $f_\Pi(x, y) = 0$ has a non-zero multiple root we say f_Π is *degenerate*. Otherwise we say f_Π is *non-degenerate*. If $\Delta(f)$ possesses a boundary face whose face function is degenerate we say f is *Newton degenerate*, otherwise we say f is *Newton non-degenerate*.

Lemma 2.3. *Let $p(t)$ be an explicit path of f given by* (2.1) *such that either α or β is negative and* $\lim_{t\to 0} f(p(t)) = 0$. *Then the primitive covector $P = {}^t(\alpha, \beta)$ and the leading coefficients (a, b) satisfy one of the next conditions*:

(i) *$d(P; f) > 0$ and $\Delta(P; f)$ is a face*;

(ii) *$d(P; f) > 0$ and $\Delta(P; f)$ is a vertex on the axes*;

(iii) *$d(P; f) \leq 0$, $\Delta(P; f)$ is a face and (a, b) is a multiple root of $f_P(x, y) = 0$.*

Proof. Assume that $P = {}^t(\alpha, \beta)$ and (a, b) do not satisfy the above conditions. Then we have the following three cases:

(1) $d(P; f) > 0$ and $\Delta(P; f)$ is a vertex not on the axes;

(2) $d(P; f) \leq 0$ and $\Delta(P; f)$ is a vertex;

(3) $d(P; f) \leq 0$, $\Delta(P; f)$ is a face and (a, b) is not a multiple root of $f_P(x, y) = 0$.

Substituting $p(t)$ for f we have

$$f(p(t)) = f_P(a,b)t^{kd(P;f)} + \text{higher terms}, \tag{2.2}$$

and substituting it for $\operatorname{grad} f(p(t)) = \lambda(t)p(t)$ we have

$$\begin{pmatrix} At^{kd(P;f)-k\alpha} + \cdots \\ Bt^{kd(P;f)-k\beta} + \cdots \end{pmatrix} = \lambda(t)\begin{pmatrix} at^{k\alpha} + \cdots \\ bt^{k\beta} + \cdots \end{pmatrix}, \tag{2.3}$$

where $A, B \in \boldsymbol{C}$ are possibly zero. In the case (2), $\lim_{t\to 0} f(p(t)) = \infty$ if $d(P;f) < 0$ and $\lim_{t\to 0} f(p(t)) = f_P(a,b) \neq 0$ if $d(P;f) = 0$. Then these contradict $\lim_{t\to 0} f(p(t)) = 0$. We consider the cases (1) and (3). Assume that $\lambda(t) \equiv 0$. This implies that $(\partial f/\partial x)(x(t), y(t)) \equiv 0$ and $(\partial f/\partial y)(x(t), y(t)) \equiv 0$. However it is easy to see that this is impossible under the assumption of (1) or (3). Thus $\lambda(t) \not\equiv 0$. Put

$$\lambda(t) = \lambda_0 t^{\gamma} + \lambda_1 t^{\gamma+1} + \lambda_2 t^{\gamma+2} + \text{higher terms},$$

where $\gamma \in \boldsymbol{Z}$ and $\lambda_0 \in \boldsymbol{C}^*$. Comparing the valuations of (2.3) we have two equations

$$kd(P;f) - k\alpha = \gamma + k\alpha,$$

$$kd(P;f) - k\beta = \gamma + k\beta.$$

These equalities imply that $\alpha = \beta$. Since α and β are coprime and $|x(t)|^2 + |y(t)|^2 \to \infty$ as $t \to 0$ by the assumption, we have $\alpha = \beta = -1$, and hence the case (1) does not occur. Assume the case (3). f_P takes the form

$$f_P(x,y) = Cx^r y^s \prod_{i=1}^{\ell} (x + A_i y)^{\nu_i},$$

where $C, A_i \in \boldsymbol{C}^*$, $A_i \neq A_j$ if $i \neq j$, $r, s \in \boldsymbol{Z}_{\geq 0}$ and $\nu_i \in \boldsymbol{N}$. By the assumption, we may assume $a + A_{i_0} b = 0$ for some i_0 with $\nu_{i_0} = 1$ because (a,b) is not a multiple root. We can assume $a + A_\ell b = 0$, $\nu_\ell = 1$ and $a + A_i b \neq 0$ for $i = 1, \ldots, \ell - 1$. Putting $G(x,y) = Cx^r y^s \prod_{i=1}^{\ell-1} (x + A_i y)^{\nu_i}$, we have

$$\frac{\partial f_P}{\partial x}(p(t)) = G(a,b)t^{-k(r+s+\sum_{i=1}^{\ell-1} \nu_i)} + \text{higher terms},$$

$$\frac{\partial f_P}{\partial y}(p(t)) = A_\ell G(a,b)t^{-k(r+s+\sum_{i=1}^{\ell-1} \nu_i)} + \text{higher terms}.$$

From the leading coefficients of the equation $\operatorname{grad} f(p(t)) = \lambda(t)p(t)$, we have

$$\begin{pmatrix} \overline{G(a,b)} \\ \overline{A_\ell G(a,b)} \end{pmatrix} = \lambda_0 \begin{pmatrix} a \\ b \end{pmatrix}.$$

Then $a\overline{A_\ell} = b$. On the other hand $a + A_\ell b = 0$. Hence we have $|A_\ell|^2 + 1 = 0$, which is a contradiction. □

Lemma 2.4. *Suppose that $p(t)$ is in the case* (i) *or* (ii) *of Lemma* 2.3. *Then* $0 \in B_f$.

Proof. In these cases f is not convenient and satisfies $f(0,0) = 0$. Then from [**N-Z1**] Prop. 6 Step 2 we have $0 \in B_f$. □

Thus Theorem 1.1 is proved except for the case (iii) of Lemma 2.3.

3. Toric compactifications and toric modifications.

Let $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ be a polynomial function. First we define an admissible toric compactification with respect to the Newton polygon $\Delta(f)$. Let $Q_i = {}^t(p_i, q_i)$, $i = 1, 2, \dots, \eta$ be primitive covectors such that

(i) either p_i or q_i is negative;
(ii) $\Delta(Q_i; f)$ is a boundary face;
(iii) the indices are assigned in the counter-clockwise orientation.

Let $R_i = {}^t(r_i, s_i)$, $i = 1, 2, \dots, \mu$ be primitive covectors which satisfy the following:

(1) $R_1 = {}^t(1, 0)$, $R_2 = {}^t(0, 1)$;
(2) either r_i or s_i is negative for each R_i, $i = 3, \dots, \mu$;
(3) $\{Q_i\}$ is contained in $\{R_3, \dots, R_\mu\}$;
(4) the indices are assigned in the counter-clockwise orientation;
(5) the determinants of the matrices (R_i, R_{i+1}), $i = 1, \dots, \mu - 1$, and (R_μ, R_1) are 1.

For a convenience, we set $R_{\mu+1} = R_1$. For each $\mathrm{Cone}(R_i, R_{i+1})$, $i = 2, \dots, \mu$, an affine coordinate chart $(u_i, v_i) \in \boldsymbol{C}^2$ is defined by the coordinate transformation

$$x = u_i^{r_i} v_i^{r_{i+1}}, \quad y = u_i^{s_i} v_i^{s_{i+1}}.$$

Then the corresponding smooth toric variety X is obtained by gluing these coordinate charts, which can be described as

$$X = \boldsymbol{C}^{*2} \coprod_{i=1}^{\mu} E(R_i) = \boldsymbol{C}^2 \coprod_{i=3}^{\mu} E(R_i),$$

where $E(R_i)$ is the exceptional divisor corresponding to the covector R_i. Let $\pi : X \to \boldsymbol{C}^2$ be the associated proper mapping. This is called *the admissible toric compactification associated with* $\{R_1, \dots, R_\mu\}$.

Next we consider an admissible toric modification. The admissible toric modification is usually associated with a polynomial or a local analytic function.

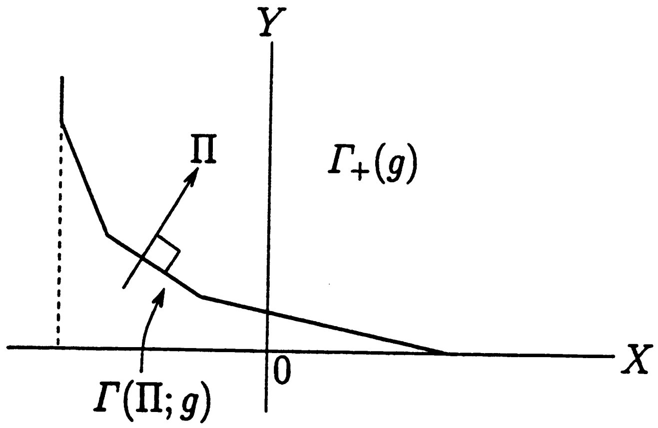

Figure 1. The Newton polygon $\Gamma_+(g)$ and the boundary face $\Gamma(\Pi; g)$ of Π.

But now we define it for some special functions. Let $U \subset \boldsymbol{C}^2$ be a small neighborhood of $(0,0)$ and let $g : U \to \boldsymbol{C}$ be a local function on U given by

$$g(x, y) = \sum_{(m,n)} a_{m,n} x^m y^n,$$

where $(m,n) \in \boldsymbol{Z}^2$ with $m > -M$ for some non-negative integer M and $n \geq 0$. We define the Newton polygon $\Gamma_+(g)$ of g as a germ by the convex closure of $\bigcup_{(m,n)}((m,n) + \boldsymbol{R}^2_{\geq 0})$ where the union is taken for all (m,n) such that $a_{m,n} \neq 0$. $\Gamma_+(g)$ is, for example, as shown in Figure 1. *The Newton boundary* $\Gamma(g)$ is the union of compact faces of $\Gamma_+(g)$. For a given primitive covector $\Pi = {}^t(p,q)$ with $p, q > 0$ we consider the linear function $pX + qY$ where $(X, Y) \in \Gamma_+(g)$ and denote its minimal value by $d(\Pi; g)$. We set

$$\Gamma(\Pi; g) := \{(X, Y) \in \Gamma(g) \mid pX + qY = d(\Pi; g)\}$$

and call this *a boundary face* (resp. *a boundary vertex*) if $\dim \Gamma(\Pi; g) = 1$ (resp. $\dim \Gamma(\Pi; g) = 0$). We define the partial sum $g_\Pi(x, y)$ by

$$g_\Pi(x, y) := \sum_{(m,n) \in \Gamma(\Pi; g)} a_{m,n} x^m y^n$$

and call this *the boundary function* of the covector Π. In particular, if $\Gamma(\Pi; g)$ is a face we call this *a face function*. If $g_\Pi(x, y) = 0$ has a non-zero multiple root we say g_Π is *degenerate*. Otherwise we say g_Π is *non-degenerate*.

Let g be a local function given as above and let $\hat{Q}_i = {}^t(\hat{p}_i, \hat{q}_i)$, $i = 1, 2, \ldots, \hat{\eta}$ be primitive covectors such that

(i) both $\hat{p}_i$ and $\hat{q}_i$ are positive;
(ii) $\Gamma(\hat{Q}_i; g)$ is a boundary face;
(iii) the indices are assigned in the counter-clockwise orientation.

Let $\hat{R}_i = {}^t(\hat{r}_i, \hat{s}_i)$, $i = 1, 2, \ldots, \hat{\mu}$ be primitive covectors which satisfy the following:

(1) $\hat{R}_1 = {}^t(1,0)$, $\hat{R}_{\hat{\mu}} = {}^t(0,1)$;
(2) both $\hat{r}_i$ and $\hat{s}_i$ are positive for each $\hat{R}_i$, $i = 2, \dots, \hat{\mu}-1$;
(3) $\{\hat{Q}_i\}$ is contained in $\{\hat{R}_2, \dots, \hat{R}_{\hat{\mu}-1}\}$;
(4) the indices are assigned in the counter-clockwise orientation;
(5) the determinants of the matrices $(\hat{R}_i, \hat{R}_{i+1})$, $i = 1, \dots, \hat{\mu}-1$, are 1.

For each $\mathrm{Cone}(\hat{R}_i, \hat{R}_{i+1})$, $i = 1, \dots, \hat{\mu}-1$, an affine coordinate chart $(u_i, v_i) \in \boldsymbol{C}^2$ is defined by the coordinate transformation

$$x = u_i^{\hat{r}_i} v_i^{\hat{r}_{i+1}}, \quad y = u_i^{\hat{s}_i} v_i^{\hat{s}_{i+1}}.$$

Then the corresponding variety Y is obtained by gluing these coordinate charts, which can be described as

$$Y = U \coprod_{i=2}^{\hat{\mu}-1} E(\hat{R}_i),$$

where $E(\hat{R}_i)$ is the exceptional divisor corresponding to the covector $\hat{R}_i$. Let $\pi : Y \to \boldsymbol{C}^2$ be the associated proper mapping. This is called *the admissible toric modification associated with* $\{\hat{R}_1, \dots, \hat{R}_{\hat{\mu}}\}$.

For further information about toric compactifications and toric modifications, see [**O2**].

4. A modification tower according to an explicit path.

Let $p(t)$ be an explicit path of f given by (2.1) such that either α or β is negative and $\lim_{t\to 0} f(p(t)) = 0$. Now we assume that $P = {}^t(\alpha, \beta)$ and (a, b) satisfy the condition (iii) of Lemma 2.3. Let $R_i = {}^t(r_i, s_i)$, $i = 1, \dots, \mu$ be primitive covectors which associate an admissible toric compactification $\pi : Y_1 \to \boldsymbol{C}^2$ with respect to $\Delta(f)$. In this case $\Delta(P; f)$ is a face, hence $P = R_{i_0}$ for some $i_0 \in \boldsymbol{Z}$ with $3 \le i_0 \le \mu$. We call the coordinates (u_{1,i_0}, v_{1,i_0}) associated with $\mathrm{Cone}(R_{i_0}, R_{i_0+1})$ *the supporting toric coordinates* of $p(t)$. On this coordinate chart, $p(t)$ is changed into the parametrization

$$\begin{aligned} q_{1,i_0}(t) = \begin{pmatrix} u_{1,i_0}(t) \\ v_{1,i_0}(t) \end{pmatrix} &= \begin{pmatrix} x(t)^{s_{i_0+1}} y(t)^{-r_{i_0+1}} \\ x(t)^{-\beta} y(t)^{\alpha} \end{pmatrix} \\ &= \begin{pmatrix} a^{s_{i_0+1}} b^{-r_{i_0+1}} t^{k\alpha s_{i_0+1} - k\beta r_{i_0+1}} + \text{higher terms} \\ a^{-\beta} b^{\alpha} t^{-k\alpha\beta + k\beta\alpha} + \text{higher terms} \end{pmatrix} \\ &= \begin{pmatrix} a^{s_{i_0+1}} b^{-r_{i_0+1}} t^{k} + \text{higher terms} \\ a^{-\beta} b^{\alpha} + \text{higher terms} \end{pmatrix}. \end{aligned}$$

From $k \ge 1$ we have $\lim_{t\to 0} q_{1,i_0}(t) = (0, a^{-\beta} b^{\alpha})$. Consider the coordinate change given by

$$x_1 = u_{1,i_0}, \quad y_1 = v_{1,i_0} - a^{-\beta}b^{\alpha}.$$

We denote by O_1 the origin of the coordinate chart (x_1, y_1). We call hereafter (x_1, y_1) *the translated coordinates* at O_1. Let $p_1(t) = {}^t(x_1(t), y_1(t))$ be the parametrization of $q_{1,i_0}(t)$ in the translated coordinates (x_1, y_1). Then $p_1(t)$ is a real analytic curve which satisfies $\lim_{t\to 0} p_1(t) = (0,0)$. The polynomial function f can be extended as a rational function on Y_1 and its restriction to the supporting toric coordinate chart (u_{1,i_0}, v_{1,i_0}), which we denote by g_{1,i_0}, is given by

$$g_{1,i_0}(u_{1,i_0}, v_{1,i_0}) = c_1 u_{1,i_0}^{d(P;f)} v_{1,i_0}^{d(R_{i_0+1};f)} \left\{ \prod_{j=1}^{\ell_{i_0}} (v_{1,i_0} + A_j)^{v_{i_0,j}} + u_{1,i_0} h_{1,i_0}(u_{1,i_0}, v_{1,i_0}) \right\},$$

where $c_1, A_j \in \boldsymbol{C}^*$, $A_j \neq A_{j'}$ if $j \neq j'$, $\ell_{i_0}, v_{i_0,j} \in \boldsymbol{N}$ and h_{1,i_0} is a polynomial of variables (u_{1,i_0}, v_{1,i_0}). Since (a,b) is a multiple root of $f_P(x,y) = 0$ we have $a^{-\beta}b^{\alpha} + A_{j_0} = 0$ for some $1 \le j_0 \le \ell_{i_0}$ with $v_{i_0,j_0} \ge 2$. On the translated coordinates (x_1, y_1), g_{1,i_0} is given by a rational function $f^1(x_1, y_1)$ which takes the form

$$f^1(x_1, y_1) = c_1 x_1^{d(P;f)} \{ y_1^{v_{i_0,j_0}} h_1(y_1) + x_1 h_1'(x_1, y_1) \}, \tag{4.1}$$

where h_1 is a local analytic function of one variable y_1 with $h_1(0) \neq 0$ and h_1' is of two variables (x_1, y_1).

Now we define the coordinate change inductively. Assume that we have constructed admissible toric modifications $\pi_i : Y_i \to Y_{i-1}$ with center O_{i-1} with respect to $\Gamma_+(f^{i-1})$ for $i = 2, \dots, \sigma$, where $f^{i-1}(x_{i-1}, y_{i-1})$ is the restriction of the pull-back $(\pi \circ \pi_2 \circ \cdots \circ \pi_{i-1})^* f$ to a neighborhood of O_{i-1}, considerd as a function on a translated coordinate chart (x_{i-1}, y_{i-1}).

Let U_σ be a neighborhood of O_σ in Y_σ with the translated coordinate chart (x_σ, y_σ). Let $f^\sigma : U_\sigma \to \boldsymbol{C}$ be the restriction of the pull-back $\pi_\sigma^* f^{\sigma-1}$ to U_σ. This takes the form

$$f^\sigma(x_\sigma, y_\sigma) = c_\sigma x_\sigma^{d_\sigma} \{ y_\sigma^{v_\sigma} h_\sigma(y_\sigma) + x_\sigma h_\sigma'(x_\sigma, y_\sigma) \}, \tag{4.2}$$

where $c_\sigma \in \boldsymbol{C}^*$, $d_\sigma, v_\sigma \in \boldsymbol{Z}$ with $d_\sigma \le 0$, $v_\sigma \ge 2$, h_σ is a local analytic function of one variable y_σ with $h_\sigma(0) \neq 0$ and h_σ' is of two variables (x_σ, y_σ). Let $p_\sigma(t) = {}^t(x_\sigma(t), y_\sigma(t))$ be the real analytic curve given by $(\pi \circ \pi_2 \circ \cdots \circ \pi_\sigma)^{-1} p(t)$, which is written in the translated coordinates (x_σ, y_σ) at O_σ. By the analyticity of $p(t)$ and by the properness of $\pi \circ \pi_2 \circ \cdots \circ \pi_\sigma : Y_\sigma \to \boldsymbol{C}^2$, $p_\sigma(t)$ is also real analytic at $t = 0$. Since $x_\sigma = 0$ defines the divisor which contracts to the center $O_{\sigma-1}$ by π_σ, $x_\sigma(t)$ is not constant zero. If $y_\sigma(t)$ is also not constant zero we can describe $p_\sigma(t)$ as

$$p_\sigma(t) = \begin{pmatrix} x_\sigma(t) \\ y_\sigma(t) \end{pmatrix} = \begin{pmatrix} a_\sigma t^{k_\sigma \alpha_\sigma} + a_{\sigma,1} t^{k_\sigma \alpha_\sigma + 1} + \text{higher terms} \\ b_\sigma t^{k_\sigma \beta_\sigma} + b_{\sigma,1} t^{k_\sigma \beta_\sigma + 1} + \text{higher terms} \end{pmatrix}, \tag{4.3}$$

where $a_\sigma, b_\sigma \in \boldsymbol{C}^*$, $k_\sigma \in \boldsymbol{N}$ and $(\alpha_\sigma, \beta_\sigma) \neq (0,0)$ is a pair of coprime integers. From $\lim_{t\to 0} p_\sigma(t) = (0,0)$, both α_σ and β_σ are positive.

Definition 4.1. If the primitive covector $P_\sigma = {}^t(\alpha_\sigma, \beta_\sigma)$ and the leading coefficients (a_σ, b_σ) satisfy that $d(P_\sigma; f^\sigma) \leq 0$ and (a_σ, b_σ) is a multiple root of $f^\sigma_{P_\sigma}(x_\sigma, y_\sigma) = 0$, then we say $p_\sigma(t)$ is *not terminated*. Otherwise we say $p_\sigma(t)$ is *terminated*.

Note that if $y_\sigma(t) \equiv 0$, $p_\sigma(t)$ is terminated. Now we assume that $p_\sigma(t)$ is not terminated and consider a toric modification with the center O_σ for constructing the next stage. In this case $p_\sigma(t)$ can be described as (4.3). Let $R_{\sigma,i} = {}^t(r_{\sigma,i}, s_{\sigma,i})$, $i = 1, \dots, \mu_\sigma$ be primitive covectors which associate an admissible toric modification with respect to $\Gamma_+(f^\sigma)$. In this case $\Gamma(P_\sigma; f^\sigma)$ is a face, hence $P_\sigma = R_{\sigma,i_\sigma}$ for some $i_\sigma \in \boldsymbol{Z}$ with $2 \leq i_\sigma \leq \mu_\sigma - 1$. The supporting toric coordinates $(u_{\sigma,i_\sigma}, v_{\sigma,i_\sigma})$ associated with $\mathrm{Cone}(R_{\sigma,i_\sigma}, R_{\sigma,i_\sigma+1})$ is defined by

$$x_\sigma = u_{\sigma+1,i_\sigma}^{\alpha_\sigma} v_{\sigma+1,i_\sigma}^{r_{\sigma,i_\sigma+1}}, \quad y_\sigma = u_{\sigma+1,i_\sigma}^{\beta_\sigma} v_{\sigma+1,i_\sigma}^{s_{\sigma,i_\sigma+1}}, \tag{4.4}$$

and $p_\sigma(t)$ is changed into the parametrization

$$\begin{aligned} q_{\sigma+1,i_\sigma}(t) &= \begin{pmatrix} u_{\sigma+1,i_\sigma}(t) \\ v_{\sigma+1,i_\sigma}(t) \end{pmatrix} = \begin{pmatrix} x_\sigma(t)^{s_{\sigma,i_\sigma+1}} y_\sigma(t)^{-r_{\sigma,i_\sigma+1}} \\ x_\sigma(t)^{-\beta_\sigma} y_\sigma(t)^{\alpha_\sigma} \end{pmatrix} \\ &= \begin{pmatrix} a_\sigma^{s_{\sigma,i_\sigma+1}} b_\sigma^{-r_{\sigma,i_\sigma+1}} t^{k_\sigma\alpha_\sigma s_{\sigma,i_\sigma+1} - k_\sigma\beta_\sigma r_{\sigma,i_\sigma+1}} + \text{higher terms} \\ a_\sigma^{-\beta_\sigma} b_\sigma^{\alpha_\sigma} t^{-k_\sigma\alpha_\sigma\beta_\sigma + k_\sigma\beta_\sigma\alpha_\sigma} + \text{higher terms} \end{pmatrix} \\ &= \begin{pmatrix} a_\sigma^{s_{\sigma,i_\sigma+1}} b_\sigma^{-r_{\sigma,i_\sigma+1}} t^{k_\sigma} + \text{higher terms} \\ a_\sigma^{-\beta_\sigma} b_\sigma^{\alpha_\sigma} + \text{higher terms} \end{pmatrix}. \end{aligned} \tag{4.5}$$

From $k_\sigma \geq 1$ we have $\lim_{t\to 0} q_{\sigma+1,i_\sigma}(t) = (0, a_\sigma^{-\beta_\sigma} b_\sigma^{\alpha_\sigma})$. Consider the coordinate change given by

$$x_{\sigma+1} = u_{\sigma+1,i_\sigma}, \quad y_{\sigma+1} = v_{\sigma+1,i_\sigma} - a_\sigma^{-\beta_\sigma} b_\sigma^{\alpha_\sigma}.$$

We denote by $O_{\sigma+1}$ the origin of the coordinate chart $(x_{\sigma+1}, y_{\sigma+1})$. The curve $q_{\sigma+1,i_\sigma}(t)$ is written as $p_{\sigma+1}(t) = (x_{\sigma+1}(t), y_{\sigma+1}(t))$ in the translated coordiantes $(x_{\sigma+1}, y_{\sigma+1})$. If $y_{\sigma+1}(t)$ are not constant zero, we can describe $p_{\sigma+1}(t)$ as

$$p_{\sigma+1}(t) = \begin{pmatrix} x_{\sigma+1}(t) \\ y_{\sigma+1}(t) \end{pmatrix} = \begin{pmatrix} a_{\sigma+1} t^{k_{\sigma+1}\alpha_{\sigma+1}} + a_{\sigma+1,1} t^{k_{\sigma+1}\alpha_{\sigma+1}+1} + \text{higher terms} \\ b_{\sigma+1} t^{k_{\sigma+1}\beta_{\sigma+1}} + b_{\sigma+1,1} t^{k_{\sigma+1}\beta_{\sigma+1}+1} + \text{higher terms} \end{pmatrix}.$$

Comparing this and (4.5), we have

$$k_{\sigma+1}\alpha_{\sigma+1} = k_\sigma. \tag{4.6}$$

The pull-back $g_{\sigma+1,i_\sigma} = \pi_{\sigma+1}^* f^\sigma$ of f^σ in the supporting toric coordinates $(u_{\sigma+1,i_\sigma}, v_{\sigma+1,i_\sigma})$ is given by

$$\begin{aligned} g_{\sigma+1,i_\sigma}&(u_{\sigma+1,i_\sigma}, v_{\sigma+1,i_\sigma}) \\ &= c_{\sigma+1} u_{\sigma+1,i_\sigma}^{d(P_\sigma;f^\sigma)} v_{\sigma+1,i_\sigma}^{d(R_{\sigma,i_\sigma+1};f^\sigma)} \left\{ \prod_{j=1}^{\ell_{i_\sigma}} (v_{\sigma+1,i_\sigma} + A_j)^{\nu_{i_\sigma,j}} + u_{\sigma+1,i_\sigma} h_{\sigma+1,i_\sigma} \right\}, \end{aligned}$$

where $c_{\sigma+1}, A_j \in \boldsymbol{C}^*$, $A_j \neq A_{j'}$ if $j \neq j'$, $\ell_{i_\sigma}, \nu_{i_\sigma,j} \in \boldsymbol{N}$ and $h_{\sigma+1,i_\sigma}$ is a local analytic function of variables $(u_{\sigma+1,i_\sigma}, v_{\sigma+1,i_\sigma})$. Since $p_\sigma(t)$ is not terminated, $d(P_\sigma; f^\sigma) < 0$ and $a_\sigma^{-\beta_\sigma} b_\sigma^{\alpha_\sigma} + A_{j_\sigma} = 0$ for some $1 \le j_\sigma \le \ell_{i_\sigma}$ with $\nu_{i_\sigma,j_\sigma} \ge 2$. Let $f^{\sigma+1}$ be the restriction of $g_{\sigma+1,i_\sigma}$ to the translated coordinates $(x_{\sigma+1}, y_{\sigma+1})$. This has the similar expression as (4.2):

$$f^{\sigma+1}(x_{\sigma+1}, y_{\sigma+1}) = c_{\sigma+1} x_{\sigma+1}^{d_{\sigma+1}} \{ y_{\sigma+1}^{\nu_{\sigma+1}} h_{\sigma+1}(y_{\sigma+1}) + x_{\sigma+1} h'_{\sigma+1}(x_{\sigma+1}, y_{\sigma+1}) \},$$

where $d_{\sigma+1} = d(P_\sigma; f^\sigma)$, $\nu_{\sigma+1} = \nu_{i_\sigma,j_\sigma}$, $h_{\sigma+1}$ is a local analytic function of one variable $y_{\sigma+1}$ with $h_{\sigma+1}(0) \neq 0$ and $h'_{\sigma+1}$ is of two variables $(x_{\sigma+1}, y_{\sigma+1})$.

Definition 4.2. Let f and $p(t)$ be as above. If $p_\delta(t)$ is terminated after $\delta - 1$ times inductive toric modifications, we say $p(t)$ *has the depth* δ with respect to the modification tower $\pi_j : Y_j \to Y_{j-1}$, $j = 2, \dots, \delta$.

Lemma 4.3. *The depth of* $p(t)$ *is finite.*

Proof. Assume that there exists $p(t)$ such that the depth is infinite. Then for any $\sigma \in \boldsymbol{N}$, f^σ is described as (4.2) and satisfies $d_\sigma, \nu_\sigma \in \boldsymbol{Z}$ with $d_\sigma \le 0$ and $\nu_\sigma \ge 2$. Since $\nu_{\sigma+1}$ is the multiplicity of a multiple factor of a face function of $\Gamma_+(f^\sigma)$ we have $\nu_\sigma \ge \nu_{\sigma+1}$. We can assume that there exists some σ_0 such that $\nu_\sigma = \nu_{\sigma+1}$ for any $\sigma \ge \sigma_0$, otherwise ν_σ decreases and after finite steps $\Gamma_+(f^{\sigma+\text{finite}})$ does not have any degenerate faces. When $\nu_\sigma = \nu_{\sigma+1}$ for any $\sigma \ge \sigma_0$, $\Gamma_+(f^\sigma)$ must have only one boundary face $\Gamma(Q_{\sigma,1}; f^\sigma)$ with $Q_{\sigma,1} = {}^t(1, \beta_{\sigma,1})$ and $\beta_{\sigma,1} > 0$. Hence the primitive covector $P_\sigma = {}^t(\alpha_\sigma, \beta_\sigma)$ must be given by $P_\sigma = Q_{\sigma,1}$ for any $\sigma \ge \sigma_0$. From Figure 2 we have $d(Q_{\sigma+1,1}; f^{\sigma+1}) > d(Q_{\sigma,1}; f^\sigma)$. This means $d(Q_{\sigma+\text{finite},1}; f^{\sigma+\text{finite}}) > 0$ after finite steps, hence $p_{\sigma+\text{finite}}(t)$ is terminated. This contradicts the assumption. □

Hereafter we assume that $p(t)$ has the depth δ with respect to a tower of toric modifications $\pi_i : Y_i \to Y_{i-1}$, $i = 2, \dots, \delta$. We will use the same notations as above, the expression (4.2) for $1 \le \sigma \le \delta$ and (4.3) for $1 \le \sigma \le \delta - 1$. Remark that $p_\delta(t) = {}^t(x_\delta(t), y_\delta(t))$ may satisfy $y_\delta(t) \equiv 0$.

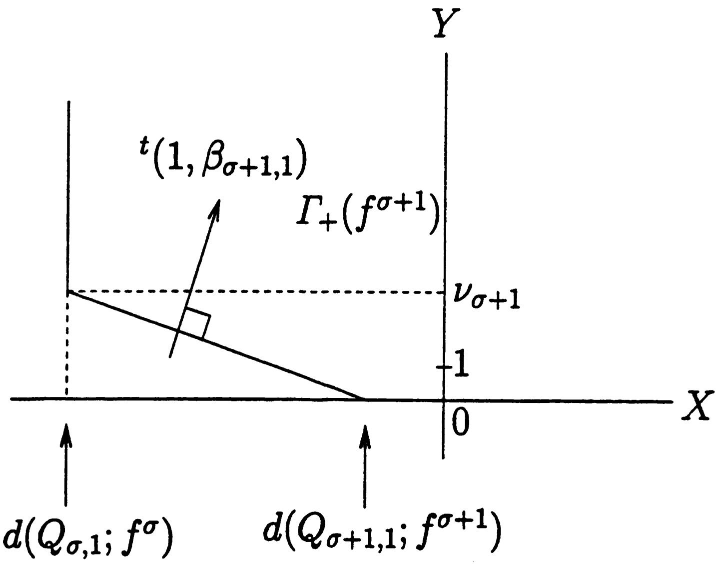

Figure 2. The Newton polygon $\Gamma_+(f^{\sigma+1})$ in the case when $\nu_\sigma = \nu_{\sigma+1}$.

5. Proof of the main theorem.

In this section we show two theorems about relations between the explicit path, the Newton polygon of f^δ and the Euler characteristics of $f^{-1}(0)$ and $f^{-1}(\varepsilon)$ for a generic $\varepsilon \in \boldsymbol{C}^*$. As a result of these theorems we will complete the proof of Theorem 1.1.

Let $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ be a polynomial function which is convenient or satisfies $f(0,0) \neq 0$. Note that we already proved Theorem 1.1 in the case when f is a not convenient function with $f(0,0) = 0$, see Lemma 2.4. Let $p(t)$ be an explicit path of f given by (2.1) such that $P = {}^t(\alpha, \beta)$ and (a, b) satisfy the condition (iii) of Lemma 2.3. We assume $p(t)$ is not on the singular locus of f. Suppose that the depth of $p(t)$ is δ. Let f^σ, $1 \leq \sigma \leq \delta$ be the pull-backed functions according to the toric compactification and the inductive toric modifications with respect to $p(t)$. Each f^σ has a description as (4.2) with $d_\sigma \leq 0$ and $\nu_\sigma \geq 2$. Set $f_z^\sigma = f^\sigma - z$ and $F_z^\sigma = x_\sigma^{-d_\sigma} f_z^\sigma$ for $z \in \boldsymbol{C}$. Note that for any $z \in \boldsymbol{C}$, $\Gamma_+(F_z^\sigma)$ is contained in $\boldsymbol{R}^2_{\geq 0}$. Since $x_\sigma = 0$ defines the divisor which collapses to a point of $Y_1 \backslash \boldsymbol{C}^2$, f^σ is regular on $U_\sigma \backslash \{x_\sigma = 0\}$ where U_σ is a small neighborhood of O_σ.

Lemma 5.1. *Suppose that f^σ is regular on $U_\sigma \backslash \{x_\sigma = 0\}$. Then F_0^σ has only an isolated singularity at O_σ. If $d_\sigma < 0$ then, for a generic $\varepsilon \in \boldsymbol{C}^*$, F_ε^σ also has only an isolated singularity at O_σ.*

Proof. Assume that F_0^σ has a non-isolated singularity at O_σ. Let $\varphi(s) \in U_\sigma$ be the singular locus with a parameter $s \in [0,1]$. Then $F_0^\sigma(\varphi(s)) \equiv 0$ and $(\partial F_0^\sigma / \partial x_\sigma)(\varphi(s)) \equiv (\partial F_0^\sigma / \partial y_\sigma)(\varphi(s)) \equiv 0$. From $\nu_\sigma \geq 2$ we have $(\partial F_0^\sigma / \partial y_\sigma) \cdot (0, y_\sigma) \not\equiv 0$, hence $\varphi(s) \not\subset \{x_\sigma = 0\}$. Then from $F_0^\sigma = x_\sigma^{-d_\sigma} f^\sigma$ we have $f^\sigma(\varphi(s)) \equiv 0$ and $(\partial f^\sigma / \partial x_\sigma)(\varphi(s)) \equiv (\partial f^\sigma / \partial y_\sigma)(\varphi(s)) \equiv 0$. But since f^σ is regular on $U_\sigma \backslash$

$\{x_\sigma = 0\}$, these lead a contradiction. If $d_\sigma < 0$, $F_\varepsilon^\sigma = 0$ passes through the origin O_σ. Since ε is generic, we can assume that f_ε^σ is regular on $U_\sigma \backslash \{x_\sigma = 0\}$. Hence by the same way as the case F_0^σ, we can say that F_ε^σ has only an isolated singularity at O_σ. □

F_0^σ satisfies $F_0^\sigma(0,0) = 0$. Also for a generic $\varepsilon \in \boldsymbol{C}^*$, F_ε^σ with $d_\sigma < 0$ satisfies $F_\varepsilon^\sigma(0,0) = 0$. For each analytic function F_z^σ with $F_z^\sigma(0,0) = 0$, we set $\Gamma_-(F_z^\sigma) := \{sT \mid T \in \Gamma(F_z^\sigma), 0 \le s \le 1\}$, which is the cone over the Newton boundary with vertex at the origin O. For an integral polyhedron $\Delta \subset \boldsymbol{R}^2_{\ge 0}$, the Newton number $\nu(\Delta)$ is defined by

$$\nu(\Delta) = 2\,\mathrm{Vol}(\Delta) - |\Delta \cap \{X\text{-axis}\}| - |\Delta \cap \{Y\text{-axis}\}| + \imath,$$

where the second and third terms are the length of the segments and $\imath$ is defined by $\imath = 1$ if $O \in \Delta$ and otherwise $\imath = 0$, see [**O1**]. For such analytic functions F_z^σ with $F_z^\sigma(0,0) = 0$, we define $\nu(F_z^\sigma) := \nu(\Gamma_-(F_z^\sigma))$.

Theorem 5.2. *Let $p(t)$ be an explicit path of f with depth $\delta \ge 1$ such that $\lim_{t\to 0} f(p(t)) = 0$ and $p(t)$ is not on the singular locus of f. Then*
(i) *$d_\delta = 0$, or*
(ii) *$d_\delta < 0$ and $\nu(F_0^\delta) > \nu(F_\varepsilon^\delta)$ for a generic $\varepsilon \in \boldsymbol{C}^*$.*

First we prepare a few lemmas to prove this theorem. Let $p(t)$ be an explicit path of f with depth δ given by (2.1) such that either α or β is negative and $\lim_{t\to 0} f(p(t)) = 0$. We set $x_0 = x$, $y_0 = y$, $p_0(t) = p(t)$, $k_0 = k$, $P_0 = P$, $\alpha_0 = \alpha$, $\beta_0 = \beta$ and $f^0 = f$. For $0 \le \sigma \le \delta - 1$, $p_\sigma(t)$ is not terminated. We assume that $p_\sigma(t)$ is described as (4.3) for $1 \le \sigma \le \delta - 1$. Let $R_{\sigma,j} = {}^t(r_{\sigma,j}, s_{\sigma,j})$, $j = 1, \dots, \mu_\sigma$ be primitive covectors which associate the admissible toric compactification $\pi : Y_1 \to \boldsymbol{C}^2$ for $\sigma = 0$ or the admissible toric modifications $\pi_{\sigma+1} : Y_{\sigma+1} \to Y_\sigma$ for $1 \le \sigma \le \delta - 1$. For the primitive covector $P_\sigma = {}^t(\alpha_\sigma, \beta_\sigma)$, $\Gamma(P_\sigma; f^\sigma)$ is a face, hence $P_\sigma = R_{\sigma,i_\sigma}$ for some $i_\sigma \in \boldsymbol{Z}$. From (4.4) we have

$$\begin{pmatrix} \partial f^\sigma/\partial x_\sigma \\ \partial f^\sigma/\partial y_\sigma \end{pmatrix} = \begin{pmatrix} s_{\sigma,i_\sigma+1} x_\sigma^{s_{\sigma,i_\sigma+1}-1} y_\sigma^{-r_{\sigma,i_\sigma+1}} & -\beta_\sigma x_\sigma^{-\beta_\sigma-1} y_\sigma^{\alpha_\sigma} \\ -r_{\sigma,i_\sigma+1} x_\sigma^{s_{\sigma,i_\sigma+1}} y_\sigma^{-r_{\sigma,i_\sigma+1}-1} & \alpha_\sigma x_\sigma^{-\beta_\sigma} y_\sigma^{\alpha_\sigma-1} \end{pmatrix} \begin{pmatrix} \partial g_{\sigma+1,i_\sigma}/\partial u_{\sigma+1,i_\sigma} \\ \partial g_{\sigma+1,i_\sigma}/\partial v_{\sigma+1,i_\sigma} \end{pmatrix}.$$

By substituting $p_\sigma(t)$ given by (4.3) for the above 2×2 matrix, we obtain

$$M_\sigma := \begin{pmatrix} A_{11,\sigma} t^{k_\sigma(1-\alpha_\sigma)} + \cdots & A_{12,\sigma} t^{-k_\sigma \alpha_\sigma} + \cdots \\ A_{21,\sigma} t^{k_\sigma(1-\beta_\sigma)} + \cdots & A_{22,\sigma} t^{-k_\sigma \beta_\sigma} + \cdots \end{pmatrix},$$

where

$$A_{11,\sigma} = s_{\sigma,i_\sigma+1} a_\sigma^{s_{\sigma,i_\sigma+1}-1} b_\sigma^{-r_{\sigma,i_\sigma+1}}, \quad A_{12,\sigma} = -\beta_\sigma a_\sigma^{-\beta_\sigma-1} b_\sigma^{\alpha_\sigma},$$

$$A_{21,\sigma} = -r_{\sigma,i_\sigma+1} a_\sigma^{s_{\sigma,i_\sigma+1}} b_\sigma^{-r_{\sigma,i_\sigma+1}-1}, \quad A_{22,\sigma} = \alpha_\sigma a_\sigma^{-\beta_\sigma} b_\sigma^{\alpha_\sigma-1}.$$

Lemma 5.3. *Let f^{σ} be as above for $0 \leq \sigma \leq \delta - 1$. If*

$$\begin{pmatrix} \mathrm{val}((\partial f^{\sigma+1}/\partial x_{\sigma+1})(p_{\sigma+1}(t))) \\ \mathrm{val}((\partial f^{\sigma+1}/\partial y_{\sigma+1})(p_{\sigma+1}(t))) \end{pmatrix} = \begin{pmatrix} -k_{\sigma+1}\alpha_{\sigma+1} + C_{\sigma+1} \\ -k_{\sigma+1}\beta_{\sigma+1} + C_{\sigma+1} \end{pmatrix}$$

for some integer $C_{\sigma+1}$, then the equality

$$\begin{pmatrix} \mathrm{val}((\partial f^{\sigma}/\partial x_{\sigma})(p_{\sigma}(t))) \\ \mathrm{val}((\partial f^{\sigma}/\partial y_{\sigma})(p_{\sigma}(t))) \end{pmatrix} = \begin{pmatrix} -k_{\sigma}\alpha_{\sigma} + C_{\sigma} \\ -k_{\sigma}\beta_{\sigma} + C_{\sigma} \end{pmatrix}$$

holds for some integer C_{σ}.

Proof. The following equality follows from the above computation and the obvious equalities $(\partial f^{\sigma+1}/\partial x_{\sigma+1})(p_{\sigma+1}(t)) = (\partial g_{\sigma+1,i_{\sigma}}/\partial u_{\sigma+1,i_{\sigma}})(q_{\sigma+1,i_{\sigma}}(t))$ and $(\partial f^{\sigma+1}/\partial y_{\sigma+1})(p_{\sigma+1}(t)) = (\partial g_{\sigma+1,i_{\sigma}}/\partial v_{\sigma+1,i_{\sigma}})(q_{\sigma+1,i_{\sigma}}(t))$:

$$\begin{pmatrix} (\partial f^{\sigma}/\partial x_{\sigma})(p_{\sigma}(t)) \\ (\partial f^{\sigma}/\partial y_{\sigma})(p_{\sigma}(t)) \end{pmatrix} = M_{\sigma}\begin{pmatrix} (\partial f^{\sigma+1}/\partial x_{\sigma+1})(p_{\sigma+1}(t)) \\ (\partial f^{\sigma+1}/\partial y_{\sigma+1})(p_{\sigma+1}(t)) \end{pmatrix},$$

where M_{σ} is the 2×2 matrix defined as above. From the valuations of this equation we have

$$\begin{aligned} &\begin{pmatrix} \mathrm{val}((\partial f^{\sigma}/\partial x_{\sigma})(p_{\sigma}(t))) \\ \mathrm{val}((\partial f^{\sigma}/\partial y_{\sigma})(p_{\sigma}(t))) \end{pmatrix} \\ &\quad = \begin{pmatrix} \min\{k_{\sigma}(1-\alpha_{\sigma}) - k_{\sigma+1}\alpha_{\sigma+1} + C_{\sigma+1}, -k_{\sigma}\alpha_{\sigma} - k_{\sigma+1}\beta_{\sigma+1} + C_{\sigma+1}\} \\ \min\{k_{\sigma}(1-\beta_{\sigma}) - k_{\sigma+1}\alpha_{\sigma+1} + C_{\sigma+1}, -k_{\sigma}\beta_{\sigma} - k_{\sigma+1}\beta_{\sigma+1} + C_{\sigma+1}\} \end{pmatrix}. \end{aligned}$$

By using (4.6) and $k_{\sigma+1}\beta_{\sigma+1} > 0$, we get

$$\begin{pmatrix} \mathrm{val}((\partial f^{\sigma}/\partial x_{\sigma})(p_{\sigma}(t))) \\ \mathrm{val}((\partial f^{\sigma}/\partial y_{\sigma})(p_{\sigma}(t))) \end{pmatrix} = \begin{pmatrix} -k_{\sigma}\alpha_{\sigma} - k_{\sigma+1}\beta_{\sigma+1} + C_{\sigma+1} \\ -k_{\sigma}\beta_{\sigma} - k_{\sigma+1}\beta_{\sigma+1} + C_{\sigma+1} \end{pmatrix}.$$

Putting $C_{\sigma} = -k_{\sigma+1}\beta_{\sigma+1} + C_{\sigma+1}$, we complete the proof of the assertion. □

Lemma 5.4. *Assume that $p(t)$ is not on the singular locus of f and the depth $\delta \geq 1$. Then the following properties hold*:

(i) *both $(\partial f^1/\partial x_1)(p_1(t))$ and $(\partial f^1/\partial y_1)(p_1(t))$ are not constant zero.*

(ii) *$p_1(t)$ satisfies the following inequality*:

$$\mathrm{val}\left(\frac{\partial f^1}{\partial x_1}(p_1(t))\right) < \mathrm{val}\left(\frac{\partial f^1}{\partial y_1}(p_1(t))\right) + \min\{0, k_1(\beta_1 - \alpha_1)\}.$$

Proof. Assume that the assertion (i) does not hold. If $(\partial f^1/\partial x_1) \cdot (p_1(t))$ and $(\partial f^1/\partial y_1)(p_1(t))$ are both constant zero, $p(t)$ is on the singular locus of f, which contradicts the assumption. Then we have (i-1n) or (i-2n) below.

(i-1n) If $(\partial f^1/\partial x_1)(p_1(t)) \equiv 0$ and $(\partial f^1/\partial y_1)(p_1(t)) \not\equiv 0$, we have

$$\begin{pmatrix} (\partial f/\partial x)(p(t)) \\ (\partial f/\partial y)(p(t)) \end{pmatrix} = \begin{pmatrix} A_{11,0}t^{k(1-\alpha)} + \cdots & A_{12,0}t^{-k\alpha} + \cdots \\ A_{21,0}t^{k(1-\beta)} + \cdots & A_{22,0}t^{-k\beta} + \cdots \end{pmatrix} \begin{pmatrix} 0 \\ (\partial f^1/\partial y_1)(p_1(t)) \end{pmatrix}.$$

Then

$$\begin{pmatrix} \mathrm{val}((\partial f/\partial x)(p(t))) \\ \mathrm{val}((\partial f/\partial y)(p(t))) \end{pmatrix} = \begin{pmatrix} -k\alpha + \mathrm{val}((\partial f^1/\partial y_1)(p_1(t))) \\ -k\beta + \mathrm{val}((\partial f^1/\partial y_1)(p_1(t))) \end{pmatrix}.$$

(i-2n) If $(\partial f^1/\partial x_1)(p_1(t)) \not\equiv 0$ and $(\partial f^1/\partial y_1)(p_1(t)) \equiv 0$, we have

$$\begin{pmatrix} (\partial f/\partial x)(p(t)) \\ (\partial f/\partial y)(p(t)) \end{pmatrix} = \begin{pmatrix} A_{11,0}t^{k(1-\alpha)} + \cdots & A_{12,0}t^{-k\alpha} + \cdots \\ A_{21,0}t^{k(1-\beta)} + \cdots & A_{22,0}t^{-k\beta} + \cdots \end{pmatrix} \begin{pmatrix} (\partial f^1/\partial x_1)(p_1(t)) \\ 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} \mathrm{val}((\partial f/\partial x)(p(t))) \\ \mathrm{val}((\partial f/\partial y)(p(t))) \end{pmatrix} = \begin{pmatrix} k(1-\alpha) + \mathrm{val}((\partial f^1/\partial x_1)(p_1(t))) \\ k(1-\beta) + \mathrm{val}((\partial f^1/\partial x_1)(p_1(t))) \end{pmatrix}.$$

(iin) Assume that the assertion (i) of Lemma 5.4 holds and the assertion (ii) does not hold. Since

$$\begin{pmatrix} (\partial f/\partial x)(p(t)) \\ (\partial f/\partial y)(p(t)) \end{pmatrix} = \begin{pmatrix} A_{11,0}t^{k(1-\alpha)} + \cdots & A_{12,0}t^{-k\alpha} + \cdots \\ A_{21,0}t^{k(1-\beta)} + \cdots & A_{22,0}t^{-k\beta} + \cdots \end{pmatrix} \begin{pmatrix} (\partial f^1/\partial x_1)(p_1(t)) \\ (\partial f^1/\partial y_1)(p_1(t)) \end{pmatrix},$$

if the inequality $\mathrm{val}((\partial f^1/\partial x_1)(p_1(t))) \geq \mathrm{val}((\partial f^1/\partial y_1)(p_1(t)))$ holds, we have

$$\begin{pmatrix} \mathrm{val}((\partial f/\partial x)(p(t))) \\ \mathrm{val}((\partial f/\partial y)(p(t))) \end{pmatrix} = \begin{pmatrix} -k\alpha + \mathrm{val}((\partial f^1/\partial y_1)(p_1(t))) \\ -k\beta + \mathrm{val}((\partial f^1/\partial y_1)(p_1(t))) \end{pmatrix}.$$

If $\mathrm{val}((\partial f^1/\partial x_1)(p_1(t))) \geq \mathrm{val}((\partial f^1/\partial y_1)(p_1(t))) + k_1(\beta_1 - \alpha_1)$, we also have the same equality by using (4.6).

For all cases (i-1n), (i-2n) and (iin), note that $(\mathrm{val}((\partial f/\partial x)(p(t))), \mathrm{val}((\partial f/\partial y)(p(t)))) = (-k\alpha + C, -k\beta + C)$ for a suitable $C \in \boldsymbol{Z}$. Thus comparing the valuations of the equality $\mathrm{grad}\, f(p(t)) = \lambda(t)p(t)$, we have

$$-k\alpha + C = \mathrm{val}(\lambda(t)) + k\alpha,$$

$$-k\beta + C = \mathrm{val}(\lambda(t)) + k\beta.$$

This implies $\alpha = \beta$. Since α and β are coprime and $\|p(t)\| \to \infty$ as $t \to 0$ by the assumption, we have $\alpha = \beta = -1$. Therefore we can assume that the first coordinate change corresponding to the toric compactification is defined by $x = u_{1,i}^{-1}v_{1,i}^{-\mu}$ and $y = u_{1,i}^{-1}v_{1,i}^{-\mu-1}$ for some positive integer μ. Then the conjugation of $\mathrm{grad}\, f(p(t))$ is given by

$$\begin{aligned}
&\begin{pmatrix} (\partial f/\partial x)(p(t)) \\ (\partial f/\partial y)(p(t)) \end{pmatrix} \\
&= \begin{pmatrix} -(\mu+1)x(t)^{-\mu-2}y(t)^{\mu} & y(t)^{-1} \\ \mu x(t)^{-\mu-1}y(t)^{\mu-1} & -x(t)y(t)^{-2} \end{pmatrix} \begin{pmatrix} (\partial f^1/\partial x_1)(p_1(t)) \\ (\partial f^1/\partial y_1)(p_1(t)) \end{pmatrix} \\
&= \begin{pmatrix} -(\mu+1)a^{-\mu-2}b^{\mu}t^{2k}+\cdots & b^{-1}t^k+\cdots \\ \mu a^{-\mu-1}b^{\mu-1}t^{2k}+\cdots & -ab^{-2}t^k+\cdots \end{pmatrix} \begin{pmatrix} (\partial f^1/\partial x_1)(p_1(t)) \\ (\partial f^1/\partial y_1)(p_1(t)) \end{pmatrix},
\end{aligned}$$

where $p(t) = {}^t(x(t), y(t))$ and a and b are the leading coefficients of $x(t)$ and $y(t)$ respectively. In the cases (i-1n) and (iin) we have

$$\begin{pmatrix} (\partial f/\partial x)(p(t)) \\ (\partial f/\partial y)(p(t)) \end{pmatrix} = \begin{pmatrix} b^{-1}B_1t^{k+q} + \text{higher terms} \\ -ab^{-2}B_1t^{k+q} + \text{higher terms} \end{pmatrix},$$

where B_1 is the leading coefficient, and q is the valuation, of $(\partial f^1/\partial y_1)(p_1(t))$. Comparing the leading coefficients of $\operatorname{grad} f(p(t)) = \lambda(t)p(t)$, we have

$$\begin{aligned}
\overline{b^{-1}B_1} &= \lambda_0 a, \\
-\overline{ab^{-2}B_1} &= \lambda_0 b,
\end{aligned}$$

where $\lambda_0 \in \boldsymbol{C}^*$ is the leading coefficient of $\lambda(t)$. Hence we have $|a|^2 + |b|^2 = 0$, which is a contradiction. In the case (i-2n) we have

$$\begin{pmatrix} (\partial f/\partial x)(p(t)) \\ (\partial f/\partial y)(p(t)) \end{pmatrix} = \begin{pmatrix} -(\mu+1)a^{-\mu-2}b^{\mu}A_1t^{2k+p} + \text{higher terms} \\ \mu a^{-\mu-1}b^{\mu-1}A_1t^{2k+p} + \text{higher terms} \end{pmatrix},$$

where A_1 is the leading coefficient, and p is the valuation, of $(\partial f^1/\partial x_1)(p_1(t))$. Comparing the leading coefficients of $\operatorname{grad} f(p(t)) = \lambda(t)p(t)$, we have

$$\begin{aligned}
-(\mu+1)\overline{a^{-\mu-2}b^{\mu}A_1} &= \lambda_0 a, \\
\mu\overline{a^{-\mu-1}b^{\mu-1}A_1} &= \lambda_0 b.
\end{aligned}$$

Then $\mu|a|^2 + (\mu+1)|b|^2 = 0$, which is a contradiction. This completes the proof of the assertions (i) and (ii) of Lemma 5.4. $\square$

Now we consider the local functions f^{δ} and f^{δ}_{ε} for a generic $\varepsilon \in \boldsymbol{C}^*$ on the translated coordinate chart (x_{δ}, y_{δ}), which is the terminated stage of a tower of toric modifications. Let $Q_{\delta,i},\ i = 1, \dots, \eta_{\delta}$ be primitive covectors corresponding to the faces of the Newton boundary $\Gamma(f^{\delta})$.

Definition 5.5. If the primitive covector $Q_{\delta,\eta_{\delta}} = {}^t(1, \beta_{\delta,\eta_{\delta}})$ satisfies $d(Q_{\delta,\eta_{\delta}}; f^{\delta}) > 0$ and the face function $f^{\delta}_{Q_{\delta,\eta_{\delta}}}(x_{\delta}, y_{\delta})$ takes the form

$$f^{\delta}_{Q_{\delta,\eta_{\delta}}}(x_{\delta}, y_{\delta}) = A_{\delta}x_{\delta}^{e_{\delta}}(x_{\delta}^{\beta_{\delta,\eta_{\delta}}} + B_{\delta}y_{\delta}), \tag{5.1}$$

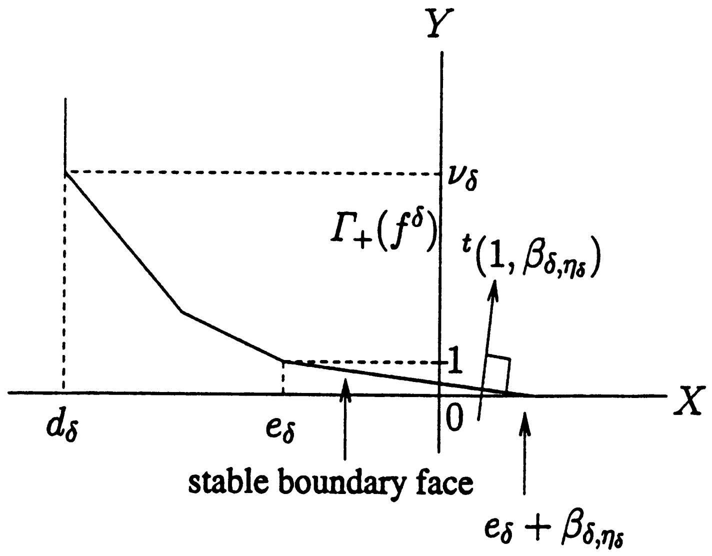

Figure 3. A stable boundary face.

where $A_\delta, B_\delta \in \boldsymbol{C}^*$ and $e_\delta \in \boldsymbol{Z}$, then we call $\Gamma(Q_{\delta,\eta_\delta}; f^\delta)$ *a stable boundary face* (see Figure 3).

In particular we have $\nu(F_0^\delta) > \nu(F_\varepsilon^\delta)$ for a generic $\varepsilon \in \boldsymbol{C}^*$ if $\Gamma(f^\delta)$ has a boundary face $\Gamma(Q_{\delta,\imath}; f^\delta)$ which is not stable and satisfies $d(Q_{\delta,\imath}; f^\delta) > 0$ for some $1 \le \imath \le \eta_\delta$.

Lemma 5.6. *We assume that $p(t)$ is not on the singular locus of f. Suppose that each $Q_{\delta,i}$, $i = 1, \dots, \eta_\delta$ satisfies that $d(Q_{\delta,i}; f^\delta) \le 0$ or $\Delta(Q_{\delta,i}; f^\delta)$ is a stable face. Then for $p_\delta(t) = {}^t(x_\delta(t), y_\delta(t))$ both $x_\delta(t)$ and $y_\delta(t)$ are not constant zero.*

Proof. Since $x_\delta = 0$ defines the divisor which collapses to a point of $Y_1 \backslash \boldsymbol{C}^2$, it is obvious that $x_\delta(t) \not\equiv 0$. Thus we assume $y_\delta(t) \equiv 0$. If both $(\partial f^\delta/\partial x_\delta)(x_\delta(t), 0)$ and $(\partial f^\delta/\partial y_\delta)(x_\delta(t), 0)$ are constant zero, $y_\delta = 0$ is on the singular locus of f. Now we suppose that $(\partial f^\delta/\partial x_\delta)(x_\delta(t), 0) \equiv 0$ and $(\partial f^\delta/\partial y_\delta)(x_\delta(t), 0) \not\equiv 0$. By Lemma 5.4 (i) we have $\delta \ge 2$ and

$$\begin{pmatrix} (\partial f^{\delta-1}/\partial x_{\delta-1})(p_{\delta-1}(t)) \\ (\partial f^{\delta-1}/\partial y_{\delta-1})(p_{\delta-1}(t)) \end{pmatrix}$$
$$= \begin{pmatrix} A_{11,\sigma-1} t^{k_{\delta-1}(1-\alpha_{\delta-1})} + \cdots & A_{12,\sigma-1} t^{-k_{\delta-1}\alpha_{\delta-1}} + \cdots \\ A_{21,\sigma-1} t^{k_{\delta-1}(1-\beta_{\delta-1})} + \cdots & A_{22,\sigma-1} t^{-k_{\delta-1}\beta_{\delta-1}} + \cdots \end{pmatrix} \begin{pmatrix} 0 \\ (\partial f^\delta/\partial y_\delta)(p_\delta(t)) \end{pmatrix}.$$

By considering the valuations, we have

$$\begin{pmatrix} \mathrm{val}((\partial f^{\delta-1}/\partial x_{\delta-1})(p_{\delta-1}(t))) \\ \mathrm{val}((\partial f^{\delta-1}/\partial y_{\delta-1})(p_{\delta-1}(t))) \end{pmatrix} = \begin{pmatrix} -k_{\delta-1}\alpha_{\delta-1} + \mathrm{val}((\partial f^\delta/\partial y_\delta)(p_\delta(t))) \\ -k_{\delta-1}\beta_{\delta-1} + \mathrm{val}((\partial f^\delta/\partial y_\delta)(p_\delta(t))) \end{pmatrix}.$$

Then from Lemma 5.3 and 5.4 (ii) we have a contradiction. Next we suppose that $(\partial f^{\delta}/\partial x_{\delta})(x_{\delta}(t),0) \not\equiv 0$ and $(\partial f^{\delta}/\partial y_{\delta})(x_{\delta}(t),0) \equiv 0$. By Lemma 5.4 (i) we have $\delta \geq 2$ and

$$\begin{pmatrix} (\partial f^{\delta-1}/\partial x_{\delta-1})(p_{\delta-1}(t)) \\ (\partial f^{\delta-1}/\partial y_{\delta-1})(p_{\delta-1}(t)) \end{pmatrix}$$
$$= \begin{pmatrix} A_{11,\sigma-1}t^{k_{\delta-1}(1-\alpha_{\delta-1})}+\cdots & A_{12,\sigma-1}t^{-k_{\delta-1}\alpha_{\delta-1}}+\cdots \\ A_{21,\sigma-1}t^{k_{\delta-1}(1-\beta_{\delta-1})}+\cdots & A_{22,\sigma-1}t^{-k_{\delta-1}\beta_{\delta-1}}+\cdots \end{pmatrix} \begin{pmatrix} (\partial f^{\delta}/\partial x_{\delta})(p_{\delta}(t)) \\ 0 \end{pmatrix}.$$

By considering the valuations, we have

$$\begin{pmatrix} \mathrm{val}((\partial f^{\delta-1}/\partial x_{\delta-1})(p_{\delta-1}(t))) \\ \mathrm{val}((\partial f^{\delta-1}/\partial y_{\delta-1})(p_{\delta-1}(t))) \end{pmatrix} = \begin{pmatrix} k_{\delta-1}(1-\alpha_{\delta-1}) + \mathrm{val}((\partial f^{\delta}/\partial x_{\delta})(p_{\delta}(t))) \\ k_{\delta-1}(1-\beta_{\delta-1}) + \mathrm{val}((\partial f^{\delta}/\partial x_{\delta})(p_{\delta}(t))) \end{pmatrix}.$$

Then from Lemma 5.3 and 5.4 (ii) we have a contradiction. Suppose that $(\partial f^{\delta}/\partial x_{\delta})(x_{\delta}(t),0) \not\equiv 0$ and $(\partial f^{\delta}/\partial y_{\delta})(x_{\delta}(t),0) \not\equiv 0$. Then we can assume that $\Gamma_{+}(f^{\delta})$ intersects the X-axis. As $\lim_{t\to 0} f(x_{\delta}(t),0) = 0$ by the assumption, $\Gamma_{+}(f^{\delta})$ intersects the positive X-axis. By the assumption in the lemma, $\Gamma(Q_{\delta,\eta_{\delta}}; f^{\delta})$ is a stable face and $d(Q_{\delta,i}; f^{\delta}) \leq 0$ for $1 \leq i \leq \eta_{\delta}-1$. Suppose that the face function $f^{\delta}_{Q_{\delta,\eta_{\delta}}}$ is described as (5.1). Since $\deg_{x_{\delta}}((\partial f^{\delta}/\partial x_{\delta})(x_{\delta},0)) = e_{\delta}+\beta_{\delta,i}-1$ and $\deg_{x_{\delta}}((\partial f^{\delta}/\partial y_{\delta})(x_{\delta},0)) = e_{\delta}$, we have $\mathrm{val}((\partial f^{\delta}/\partial x_{\delta})(p_{\delta}(t))) \geq \mathrm{val}((\partial f^{\delta}/\partial y_{\delta})(p_{\delta}(t)))$. We can assume $\delta \geq 2$ by Lemma 5.4 (ii) and then the following equality holds:

$$\begin{pmatrix} (\partial f^{\delta-1}/\partial x_{\delta-1})(p_{\delta-1}(t)) \\ (\partial f^{\delta-1}/\partial y_{\delta-1})(p_{\delta-1}(t)) \end{pmatrix}$$
$$= \begin{pmatrix} A_{11,\delta-1}t^{k_{\delta-1}(1-\alpha_{\delta-1})}+\cdots & A_{12,\delta-1}t^{-k_{\delta-1}\alpha_{\delta-1}}+\cdots \\ A_{21,\delta-1}t^{k_{\delta-1}(1-\beta_{\delta-1})}+\cdots & A_{22,\delta-1}t^{-k_{\delta-1}\beta_{\delta-1}}+\cdots \end{pmatrix} \begin{pmatrix} (\partial f^{\delta}/\partial x_{\delta})(p_{\delta}(t)) \\ (\partial f^{\delta}/\partial y_{\delta})(p_{\delta}(t)) \end{pmatrix}.$$

Since $\mathrm{val}((\partial f^{\delta}/\partial x_{\delta})(p_{\delta}(t))) \geq \mathrm{val}((\partial f^{\delta}/\partial y_{\delta})(p_{\delta}(t)))$, we have

$$\begin{pmatrix} \mathrm{val}((\partial f^{\delta-1}/\partial x_{\delta-1})(p_{\delta-1}(t))) \\ \mathrm{val}((\partial f^{\delta-1}/\partial y_{\delta-1})(p_{\delta-1}(t))) \end{pmatrix} = \begin{pmatrix} -k_{\delta-1}\alpha_{\delta-1} + \mathrm{val}((\partial f^{\delta}/\partial y_{\delta})(p_{\delta}(t))) \\ -k_{\delta-1}\beta_{\delta-1} + \mathrm{val}((\partial f^{\delta}/\partial y_{\delta})(p_{\delta}(t))) \end{pmatrix}.$$

Then from Lemma 5.3 and 5.4 (ii) we have a contradiction. □

Lemma 5.7. *Suppose $p_{\delta}(t)$ is given by* (4.3) *such that both α_{δ} and β_{δ} are positive. Then the primitive covector $P_{\delta} = {}^{t}(\alpha_{\delta}, \beta_{\delta})$ satisfies one of the next conditions*:

(i) *$d(P_{\delta}; f^{\delta}) > 0$ and $\Gamma(P_{\delta}; f^{\delta})$ is a face which is not a stable face*;

(ii) *$d(P_{\delta}; f^{\delta}) > 0$, $\Gamma(P_{\delta}; f^{\delta})$ is a vertex on the X-axis and $\Gamma(f^{\delta})$ does not have stable faces.*

Proof. Assume that $P_\delta = {}^t(\alpha_\delta, \beta_\delta)$ does not satisfy the above conditions. Let (a_δ, b_δ) be the leading coefficients of $p_\delta(t)$ as before. Because $p_\delta(t)$ is terminated, we have five cases:

(1) $d(P_\delta; f^\delta) \le 0$ and $\Gamma(P_\delta; f^\delta)$ is a vertex;
(2) $d(P_\delta; f^\delta) \le 0$, $\Gamma(P_\delta; f^\delta)$ is a face and (a_δ, b_δ) is not a multiple root of $f^\delta_{P_\delta}(x_\delta, y_\delta) = 0$;
(3) $d(P_\delta; f^\delta) > 0$ and $\Gamma(P_\delta; f^\delta)$ is a vertex not on the X-axis.
(4) $d(P_\delta; f^\delta) > 0$, $\Gamma(P_\delta; f^\delta)$ is a vertex on the X-axis and $\Gamma(f^\delta)$ has a stable boundary face;
(5) $d(P_\delta; f^\delta) > 0$ and $\Gamma(P_\delta; f^\delta)$ is a stable boundary face.

Substituting $p_\delta(t)$ for f^δ we have

$$f^\delta(p_\delta(t)) = f^\delta_{P_\delta}(a_\delta, b_\delta) t^{k_\delta d(P_\delta; f^\delta)} + \text{higher terms},$$

and for the derivatives of f^δ we have

$$(5.2) \qquad \begin{pmatrix} (\partial f^\delta/\partial x_\delta)(p_\delta(t)) \\ (\partial f^\delta/\partial y_\delta)(p_\delta(t)) \end{pmatrix} = \begin{pmatrix} A t^{k_\delta d(P_\delta; f^\delta) - k_\delta \alpha_\delta} + \text{higher terms} \\ B t^{k_\delta d(P_\delta; f^\delta) - k_\delta \beta_\delta} + \text{higher terms} \end{pmatrix},$$

where A and $B \in \boldsymbol{C}$ are possibly zero. We consider the case (1). Obviously that $\lim_{t\to 0} f^\delta(p_\delta(t)) = \infty$ if $d(P_\delta; f^\delta) < 0$ and $\lim_{t\to 0} f^\delta(p_\delta(t)) = f^\delta_{P_\delta}(a_\delta, b_\delta) \ne 0$ if $d(P_\delta; f^\delta) = 0$. Both cases contradict the assumption $\lim_{t\to 0} f^\delta(p_\delta(t)) = 0$. We prove the non-existence of the case (2). In this case because $d(P_\delta; f^\delta) \le 0$ and $\lim_{t\to 0} f^\delta(p_\delta(t)) = 0$, we have $f^\delta_{P_\delta}(a_\delta, b_\delta) = 0$. Since (a_δ, b_δ) is not a multiple root, $(\partial f^\delta_{P_\delta}/\partial x_\delta)(a_\delta, b_\delta) \ne 0$ and $(\partial f^\delta_{P_\delta}/\partial y_\delta)(a_\delta, b_\delta) \ne 0$. Hence $A, B \ne 0$ and then we have

$$\begin{pmatrix} \operatorname{val}((\partial f^\delta/\partial x_\delta)(p_\delta(t))) \\ \operatorname{val}((\partial f^\delta/\partial y_\delta)(p_\delta(t))) \end{pmatrix} = \begin{pmatrix} -k_\delta \alpha_\delta + k_\delta d(P_\delta; f^\delta) \\ -k_\delta \beta_\delta + k_\delta d(P_\delta; f^\delta) \end{pmatrix}.$$

From Lemma 5.3 and 5.4 (ii) we have a contradiction. Next we consider the cases (3), (4) and (5). In the case (3), since $(\partial f^\delta_{P_\delta}/\partial y_\delta)(a_\delta, b_\delta) \ne 0$, A and B in (5.2) satisfy $A \in \boldsymbol{C}$ and $B \in \boldsymbol{C}^*$. Then the following inequality holds:

$$\operatorname{val}\left(\frac{\partial f^\delta}{\partial x_\delta}(p_\delta(t))\right) - \operatorname{val}\left(\frac{\partial f^\delta}{\partial y_\delta}(p_\delta(t))\right) \ge k_\delta(\beta_\delta - \alpha_\delta).$$

In the case (4) we suppose that the face function on the stable face is described as (5.1). Then $\Gamma(P_\delta; \partial f^\delta/\partial x_\delta)$ is the vertex $(e_\delta + \beta_\delta - 1, 0)$ and $\Gamma(P_\delta; \partial f^\delta/\partial y_\delta)$ is the vertex $(e_\delta, 0)$. Hence we have the following inequality:

$$\operatorname{val}\left(\frac{\partial f^\delta}{\partial x_\delta}(p_\delta(t))\right) = k_\delta \alpha_\delta(e_\delta + \beta_\delta - 1) \ge k_\delta \alpha_\delta e_\delta = \operatorname{val}\left(\frac{\partial f^\delta}{\partial y_\delta}(p_\delta(t))\right).$$

In the case (5) we also suppose that the face function on the stable face is described as (5.1). Then $\Gamma(P_\delta; \partial f^\delta/\partial x_\delta)$ is a face which contains $(e_\delta + \beta - 1, 0)$ and $\Gamma(P_\delta; \partial f^\delta/\partial y_\delta)$ is the vertex $(e_\delta, 0)$. Hence we have the following inequality:

$$\mathrm{val}\left(\frac{\partial f^\delta}{\partial x_\delta}(p_\delta(t))\right) \geq k_\delta \alpha_\delta(e_\delta + \beta_\delta - 1) \geq k_\delta \alpha_\delta e_\delta = \mathrm{val}\left(\frac{\partial f^\delta}{\partial y_\delta}(p_\delta(t))\right).$$

In all cases (3), (4) and (5), we can assume $\delta \geq 2$ by Lemma 5.4 (ii). Then by the same argument as (iin) in the proof of Lemma 5.4, we have

$$\begin{pmatrix} \mathrm{val}((\partial f^{\delta-1}/\partial x_{\delta-1})(p_{\delta-1}(t))) \\ \mathrm{val}((\partial f^{\delta-1}/\partial y_{\delta-1})(p_{\delta-1}(t))) \end{pmatrix} = \begin{pmatrix} -k_{\delta-1}\alpha_{\delta-1} + C \\ -k_{\delta-1}\beta_{\delta-1} + C \end{pmatrix}$$

for some integer C. Therefore from Lemma 5.3 and 5.4 (ii) we have a contradiction. $\square$

Proof of Theorem 5.2. We suppose that $p(t)$ is not on the singular locus of f. Since $p_\delta(t)$ is terminated, $d_\delta \leq 0$. We assume that f is not in the case (i), namely we assume $d_\delta < 0$. From Lemma 5.1, F_0^δ and F_ε^δ have only isolated singularities at O_σ. We can assume that $\Gamma_+(f^\delta)$ has a boundary face $\Gamma(P_{\delta,\imath}; f^\delta)$ with $d(P_{\delta,\imath}; f^\delta) > 0$ which is not stable, otherwise $p_\delta(t)$ is described as (4.3) by Lemma 5.6 and there does not exist such an explicit path $p(t)$ by Lemma 5.7. Comparing $\Gamma_+(F_0^\delta)$ and $\Gamma_+(F_\varepsilon^\delta)$ we have the inequality $\nu(F_0^\delta) > \nu(F_\varepsilon^\delta)$. $\square$

Let $\chi(\mathscr{M})$ denote the Euler characteristic of a manifold $\mathscr{M}$. As seen in the proof of Lemma 2.4, if f is not convenient and satisfies $f(0,0) = 0$ then $0 \in B_f$. Assume the results in Section 2. Then Theorem 1.1 follows from this fact, Theorem 5.2 and the next theorem.

Theorem 5.8. *Let f be a polynomial which is convenient or satisfies $f(0,0) \neq 0$, and we assume $0 \notin \Sigma_f$. Let $p(t)$ be an explicit path of f with depth $\delta \geq 1$ such that $\lim_{t\to 0} f(p(t)) = 0$. We set a generic $\varepsilon \in \boldsymbol{C}^*$. Suppose one of the following*:

(i) $d_\delta = 0$, *or*

(ii) $d_\delta < 0$ *and* $\nu(F_0^\delta) > \nu(F_\varepsilon^\delta)$.

Then $\chi(f^{-1}(0)) > \chi(f^{-1}(\varepsilon))$.

Before proving Theorem 5.8, we prepare a useful lemma.

Lemma 5.9. *Let $h : \boldsymbol{C}^2 \to \boldsymbol{C}$ be a local analytic function with $h(0,0) = 0$ and $h(0,y) \not\equiv 0$. Assume that h has only an isolated singularity at the origin O. Let $g = x^\gamma h$ where γ is a non-negative integer. Denote by $\mathscr{F}(g)$ (resp. $\mathscr{F}(x^N g)$) the Milnor fiber of g (resp. $x^N g$) at O where N is a positive integer. Let $I(\phi, x; O)$ be the algebraic intersection number of a divisor $\phi = 0$ and $x = 0$ at O.*

(I) *Assume that* $\gamma = 0$. *Then for any* $N \in \boldsymbol{N}$,

$$\chi(\mathscr{F}(x^N g)) = \chi(\mathscr{F}(g)) - (N+1)I(h, x; 0) + N.$$

(II) *Assume that* $\gamma > 0$. *Then for any* $N \in \boldsymbol{N}$,

$$\chi(\mathscr{F}(x^N g)) = \chi(\mathscr{F}(g)) - NI(h, x; 0) + N.$$

Before proving this lemma we recall the definition of the complexities of resolutions of plane curve singularities (see [**Lê-O**]). Let $\pi : Y \to U$ be a resolution map so that the pull-back $\pi^* g$ has normal crossing singularities at any non-empty intersection of two divisors. To each divisor E_i, $i = 1, 2, \dots, s$ we give a vertex v_i. If two divisors E_i and E_j have an intersection, we join two vertices by an edge. Then we obtain a graph $\mathscr{G}(\pi)$. Let $\delta(v_i)$ be the number of edges meeting at the vertex v_i in $\mathscr{G}(\pi)$. The complexity of π is defined by

$$\rho(\pi) = 1 + \sum_{i=1}^{s} \max(\delta(v_i) - 2, 0).$$

Proof of Lemma 5.9. Let $\tilde{R}_j = {}^t(\tilde{r}_j, \tilde{s}_j)$, $j = 1, \dots, \tilde{\mu}$ be primitive covectors which associate an admissible toric modification $\pi : Y \to \boldsymbol{C}^2$ with respect to $\Gamma_+(g)$ and let $\tilde{Q}_i = {}^t(\tilde{\alpha}_i, \tilde{\beta}_i)$, $i = 1, \dots, \tilde{\eta}$ be primitive covectors which correspond to the faces of the Newton boundary $\Gamma(g)$. For $i = 1, \dots, \tilde{\eta}$, each face function $g_{\tilde{Q}_i}(x, y)$ has the factorization

$$g_{\tilde{Q}_i}(x, y) = C_i x^{A_i} y^{B_i} \prod_{j=1}^{\tilde{\ell}_i} (x^{\tilde{\beta}_i} + c_{i,j} y^{\tilde{\alpha}_i})^{\tilde{v}_{i,j}},$$

where $C_i, c_{i,j} \in \boldsymbol{C}^*$, $c_{i,j} \neq c_{i,j'}$ if $j \neq j'$, $A_i \in \boldsymbol{Z}$, $B_i \in \boldsymbol{Z}_{\geq 0}$ and $\tilde{\ell}_i, \tilde{v}_{i,j} \in \boldsymbol{N}$. We set

$$\tilde{v}_i = \sum_{j=1}^{\tilde{\ell}_i} I(V_h, E(\tilde{Q}_i); z_{i,j}) = \sum_{j=1}^{\tilde{\ell}_i} \tilde{v}_{i,j}. \tag{5.3}$$

When $\tilde{Q}_1 = \tilde{R}_2$, by adding a primitive covector $\tilde{R}$ between $\tilde{R}_1$ and $\tilde{R}_2$ such that $\det(\tilde{R}_1, \tilde{R}) = \det(\tilde{R}, \tilde{R}_2) = 1$, we can assume that $\tilde{Q}_1 \neq \tilde{R}_2$. Also we can assume that $\tilde{Q}_{\tilde{\eta}} \neq \tilde{R}_{\tilde{\mu}-1}$. Let $E(\tilde{R}_j)$ (resp. $E(\tilde{Q}_i)$) be the exceptional divisors of the resolution of g corresponding to $\tilde{R}_j$ (resp. $\tilde{Q}_i$) and let $m(\tilde{R}_j; \psi)$ (resp. $m(\tilde{Q}_i; \psi)$) be the multiplicity of the pull-back $\pi^*\psi$ along $\tilde{R}_j$ (resp. $\tilde{Q}_i$) where ψ is a germ of an analytic function at the origin O. Let V_h be the strict transform of $h = 0$ and let V_h' be the union of V_h and the strict transform of $x = 0$ which is given by the divisor $E(\tilde{R}_1)$. For $1 \leq i \leq \tilde{\mu}$, we set

$$E^*(\tilde{R}_i) = E(\tilde{R}_i) \setminus \left(V_h \cup \bigcup_{1 \leq j \leq \tilde{\mu}, j \neq i} E(\tilde{R}_j) \right),$$

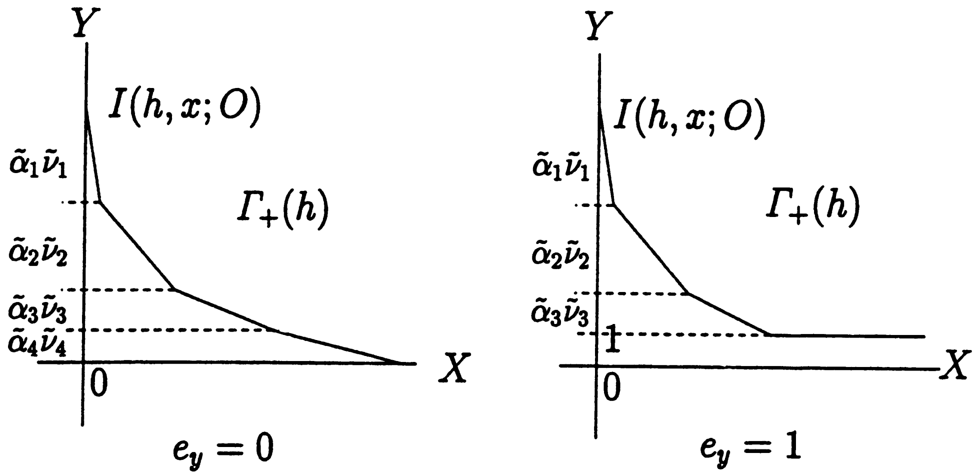

Figure 4. Newton polygons $\Gamma_+(h)$ in the cases $e_y = 0$ and 1.

$$E'^*(\tilde{R}_i) = E(\tilde{R}_i) \setminus \left(V'_h \cup \bigcup_{1 \le j \le \tilde{\mu}, j \ne i} E(\tilde{R}_j) \right).$$

First we prove the case (II). We define a non-negative integer e_y by $h = y^{e_y} h_0(x, y)$ where h_0 is a local analytic function with $h_0(x, 0) \not\equiv 0$. Since h has only an isolated singularity, $e_y = 0$ or 1. From Figure 4 we have

$$I(h, x; O) = \deg_y h(0, y) = \sum_{i=1}^{\tilde{\eta}} \tilde{\alpha}_i \tilde{\nu}_i + e_y. \tag{5.4}$$

We prove this case by using on induction on the complexity $\rho(\phi)$ where ϕ is a resolution map of g such that the pull-back $\phi^* g$ has only normal crossing singularities.

(i) Suppose $\rho(\phi) = 1$, namely g is Newton non-degenerate. We consider an admissible toric modification $\pi : Y \to \boldsymbol{C}^2$ with respect to $\Gamma_+(g)$ as above. Comparing the Newton polygons $\Gamma_+(g)$ and $\Gamma_+(x^N g)$ and the pull-backs $\pi^* g$ and $\pi^*(x^N g)$, we have

$$\begin{aligned} m(\tilde{R}_j; x^N g) &= m(\tilde{R}_j; g) + \tilde{r}_j N, \\ \chi(E'^*(\tilde{R}_j)) &= \chi(E^*(\tilde{R}_j)), \quad \text{for } j = 1, \dots, \tilde{\mu}, \\ \chi(E^*(\tilde{R}_{\tilde{\mu}-1})) &= 1 - e_y, \\ \tilde{r}_{\tilde{\mu}-1} &= 1. \end{aligned}$$

Since $\{\tilde{Q}_1, \dots, \tilde{Q}_{\tilde{\eta}}\} \subset \{\tilde{R}_1, \dots, \tilde{R}_{\tilde{\mu}}\}$, we also have

$$\begin{aligned} m(\tilde{Q}_i; x^N g) &= m(\tilde{Q}_i; g) + \tilde{\alpha}_i N, \\ \chi(E'^*(\tilde{Q}_i)) &= \chi(E^*(\tilde{Q}_i)) = -\tilde{\ell}_i, \quad \text{for } i = 1, \dots, \tilde{\eta}. \end{aligned}$$

The non-degeneracy implies $\tilde{v}_i = \tilde{\ell}_i$. Using a theorem of N. A'Campo (see [**AC**]), we have

$$
\begin{aligned}
\chi(\mathscr{F}(x^N g)) &= \sum_{i=1}^{\tilde{\eta}} m(\tilde{Q}_i; x^N g)\chi(E'^*(\tilde{Q}_i)) + m(\tilde{R}_{\tilde{\mu}-1}; x^N g)\chi(E'^*(\tilde{R}_{\tilde{\mu}-1})) \\
&= \sum_{i=1}^{\tilde{\eta}} (m(\tilde{Q}_i; g) + \tilde{\alpha}_i N)\chi(E^*(\tilde{Q}_i)) + (m(\tilde{R}_{\tilde{\mu}-1}; g) + \tilde{r}_{\tilde{\mu}-1} N)\chi(E^*(\tilde{R}_{\tilde{\mu}-1})) \\
&= \sum_{i=1}^{\tilde{\eta}} m(\tilde{Q}_i; g)\chi(E^*(\tilde{Q}_i)) + m(\tilde{R}_{\tilde{\mu}-1}; g)\chi(E^*(\tilde{R}_{\tilde{\mu}-1})) \\
&\quad + N\left\{\sum_{i=1}^{\tilde{\eta}} \tilde{\alpha}_i(-\tilde{\ell}_i) + 1 - e_y\right\} \\
&\overset{(5.4)}{=} \chi(\mathscr{F}(g)) - NI(h, x; O) + N.
\end{aligned}
$$

(ii) We assume that the assertion (II) holds for any $h(x, y)$ whose resolution complexity is less than or equal to n and prove the case $\rho(\phi) = n + 1$. Let g be an analytic function given by $g = x^\gamma h$ where $\gamma > 0$ and the resolution complexity of $h(x, y)$ is $n + 1$. We consider an admissible toric modification $\pi : Y \to \boldsymbol{C}^2$ with respect to $\Gamma_+(g)$ as before. Under the above notation, the number of the (topological) intersection points of the strict transform V_h and the divisor $E_{\tilde{Q}_i}$ is given by $\tilde{\ell}_i$. Let $z_{i,j}$, $j = 1, 2, \ldots, \tilde{\ell}_i$ be the intersection points, $(x_{i,j}, y_{i,j})$ the translated coordinates at $z_{i,j}$ and let $g_{z_{i,j}}$ be the local function of the pull-back $\pi^* g$ on the coordinate chart $(x_{i,j}, y_{i,j})$. The pull-back $\pi^* x^N$ is given by $x_{i,j}^{\tilde{\alpha}_i N}(y_{i,j} + c_{i,j})^{\tilde{r}_{\xi(i)+1} N}$ where $\tilde{Q}_i = \tilde{R}_{\xi(i)}$. As $(y_{i,j} + c_{i,j})^{\tilde{r}_{\xi(i)+1} N}$ is a unit at $z_{i,j}$, the pull-back $\pi^*(x^N g)$ can be replaced by $x_{i,j}^{\tilde{\alpha}_i N} g_{z_{i,j}}$. Let $\mathscr{F}(g_{z_{i,j}})$ (resp. $\mathscr{F}(x_{i,j}^{\tilde{\alpha}_i N} g_{z_{i,j}})$) be the Milnor fiber of $g_{z_{i,j}}$ (resp. $x_{i,j}^{\tilde{\alpha}_i N} g_{z_{i,j}}$) at $z_{i,j}$. The complexity of $g_{z_{i,j}}$ is at most n, so by using the assumption of the induction we have

$$
\chi(\mathscr{F}(x_{i,j}^{\tilde{\alpha}_i N} g_{z_{i,j}})) = \chi(\mathscr{F}(g_{z_{i,j}})) - \tilde{\alpha}_i N \tilde{v}_{i,j} + \tilde{\alpha}_i N. \tag{5.5}
$$

Then $\chi(\mathscr{F}(x^N g))$ can be modified as follows:

$$
\begin{aligned}
\chi(\mathscr{F}(x^N g)) &= \sum_{i=1}^{\tilde{\eta}} \left\{ m(\tilde{Q}_i; x^N g)\chi(E'^*(\tilde{Q}_i)) + \sum_{j=1}^{\tilde{\ell}_i} \chi(\mathscr{F}(x_{i,j}^{\tilde{\alpha}_i N} g_{z_{i,j}})) \right\} \\
&\quad + m(\tilde{R}_{\tilde{\mu}-1}; x^N g)\chi(E'^*(\tilde{R}_{\tilde{\mu}-1}))
\end{aligned}
$$

$$\overset{(5.5)}{=} \sum_{i=1}^{\tilde{\eta}} \Bigg\{ (m(\tilde{Q}_i; g) + \tilde{\alpha}_i N)\chi(E^*(\tilde{Q}_i))$$

$$+ \sum_{j=1}^{\tilde{\ell}_i} \{\chi(\mathscr{F}(g_{z_{i,j}})) - \tilde{\alpha}_i N \tilde{v}_{i,j} + \tilde{\alpha}_i N\} \Bigg\}$$

$$+ (m(\tilde{R}_{\tilde{\mu}-1}; g) + \tilde{r}_{\tilde{\mu}-1} N)\chi(E^*(\tilde{R}_{\tilde{\mu}-1}))$$

$$= \sum_{i=1}^{\tilde{\eta}} \Bigg\{ m(\tilde{Q}_i; g)\chi(E^*(\tilde{Q}_i)) + \sum_{j=1}^{\tilde{\ell}_i} \chi(\mathscr{F}(g_{z_{i,j}})) \Bigg\} + m(\tilde{R}_{\tilde{\mu}-1}; g)\chi(E^*(\tilde{R}_{\tilde{\mu}-1}))$$

$$+ \sum_{i=1}^{\tilde{\eta}} \Bigg\{ \tilde{\alpha}_i N(-\tilde{\ell}_i) + \sum_{j=1}^{\tilde{\ell}_i} \tilde{\alpha}_i N(-\tilde{v}_{i,j} + 1) \Bigg\} + N(1 - e_y)$$

$$\overset{(5.3),(5.4)}{=} \chi(\mathscr{F}(g)) - NI(h, x; O) + N.$$

Thus the proof of the assertion (II) is completed. Next we prove the case (I). Assume that $\gamma = 0$ (thus $g = h$). From the Newton polygon $\Gamma_+(g)$ we have the relations

$$m(\tilde{R}_2; g) = I(g, x; O), \quad \chi(E'^*(\tilde{R}_2)) = 0, \quad \chi(E^*(\tilde{R}_2)) = 1.$$

Thus

$$\chi(\mathscr{F}(x^N g)) = \sum_{i=1}^{\tilde{\eta}} \Bigg\{ m(\tilde{Q}_i; x^N g)\chi(E'^*(\tilde{Q}_i)) + \sum_{j=1}^{\tilde{\ell}_i} \chi(\mathscr{F}(x_{i,j}^{\tilde{\alpha}_i N} g_{z_{i,j}})) \Bigg\}$$

$$+ m(\tilde{R}_2; x^N g)\chi(E'^*(\tilde{R}_2)) + m(\tilde{R}_{\tilde{\mu}-1}; x^N g)\chi(E'^*(\tilde{R}_{\tilde{\mu}-1}))$$

$$\overset{(II)}{=} \sum_{i=1}^{\tilde{\eta}} \Bigg\{ (m(\tilde{Q}_i; g) + \tilde{\alpha}_i N)\chi(E^*(\tilde{Q}_i)) + \sum_{j=1}^{\tilde{\ell}_i} \{\chi(\mathscr{F}(g_{z_{i,j}})) - \tilde{\alpha}_i N \tilde{v}_{i,j} + \tilde{\alpha}_i N\} \Bigg\}$$

$$+ (m(\tilde{R}_{\tilde{\mu}-1}; g) + \tilde{r}_{\tilde{\mu}-1} N)\chi(E^*(\tilde{R}_{\tilde{\mu}-1}))$$

$$= \sum_{i=1}^{\tilde{\eta}} \Bigg\{ m(\tilde{Q}_i; g)\chi(E^*(\tilde{Q}_i)) + \sum_{j=1}^{\tilde{\ell}_i} \chi(\mathscr{F}(g_{z_{i,j}})) \Bigg\} + m(\tilde{R}_2; g)\chi(E^*(\tilde{R}_2))$$

$$+ m(\tilde{R}_{\tilde{\mu}-1}; g)\chi(E^*(\tilde{R}_{\tilde{\mu}-1})) + \sum_{i=1}^{\tilde{\eta}} \Bigg\{ \tilde{\alpha}_i N(-\tilde{\ell}_i) + \sum_{j=1}^{\tilde{\ell}_i} \tilde{\alpha}_i N(-\tilde{v}_{i,j} + 1) \Bigg\}$$

$$- m(\tilde{R}_2; g) + \tilde{r}_{\tilde{\mu}-1} N \chi(E^*(\tilde{R}_{\tilde{\mu}-1}))$$

$$= \chi(\mathscr{F}(g)) + \sum_{i=1}^{\tilde\eta} N\left\{-\sum_{j=1}^{\tilde\ell_i}\tilde\alpha_i\tilde\nu_{i,j}\right\} - I(g,x;O) + N(1-e_y)$$

$$\overset{(5.3),(5.4)}{=} \chi(\mathscr{F}(g)) - (N+1)I(g,x;0) + N.$$

Thus the proof of the assertion (I) is completed. □

Let $g(x,y)$, $z_{i,j}$ and $g_{z_{i,j}}$ be as in the case (I) of Lemma 5.9. Under the same notations as in the proof of Lemma 5.9, observe the following equality which we have used in the inductive argument:

$$\chi(\mathscr{F}(g)) = \sum_{i=1}^{\tilde\mu} m(\tilde R_i;g)\chi(E^*(\tilde R_i)) + \sum_{i=1}^{\tilde\eta}\sum_{j=1}^{\tilde\ell_i}\chi(\mathscr{F}(g_{z_{i,j}})), \tag{5.6}$$

where $\chi(E^*(\tilde R_i)) = 0$ unless $\tilde R_i$ is either $\tilde Q_j$ or $\tilde R_{\tilde\mu-1}$.

Proof of Theorem 5.8. Let $R_j = {}^t(r_j,s_j)$, $j=1,\dots,\mu$ be the primitive covectors which associate an admissible toric compactification $\pi: Y_1 \to \boldsymbol{C}^2$ with respect to $\Delta(f_\varepsilon)$ as in Section 3. Let $Q_i = {}^t(p_i,q_i)$, $i=1,\dots,\eta$ be the covectors among $\{R_3,\dots,R_\mu\}$ such that $\Delta(Q_i;f_\varepsilon)$ is a boundary face of $\Delta(f_\varepsilon)$. Set $\Delta := \Delta(f_\varepsilon)$ and suppose that $g(x,y)$ is a polynomial such that $\Delta(g) = \Delta$ and $V_g := g^{-1}(0) \subset \boldsymbol{C}^2$ has only isolated singularities. Let $\overline{V}_g$ be the closure of V_g in Y_1 and $\{\xi_1,\dots,\xi_k\}$ be the singular points of $\overline{V}_g$. Denote by $\mu(\overline{V}_g,\xi_n)$ the Milnor number of the local defining function of $\overline{V}_g$ at ξ_n. Then there exists an integer θ_Δ which depends only on Δ so that

$$\chi(\overline{V}_g) = \theta_\Delta + \sum_{n=1}^{k}\mu(\overline{V}_g,\xi_n), \tag{5.7}$$

see [**O2**].

Now we consider our situation. For simplicity, we put $X_0 := f^{-1}(0)$ and $X_\varepsilon := f^{-1}(\varepsilon)$ and denote their closures in Y_1 by $\overline{X}_0$ and $\overline{X}_\varepsilon$ respectively. Suppose that the factorization of the face function is given as

$$f_{Q_i}(x,y) = C_i x^{A_i} y^{B_i} \prod_{j=1}^{\ell_i}(x^{q_i} + c_{i,j}y^{p_i})^{\nu_{i,j}},$$

where $C_i, c_{i,j} \in \boldsymbol{C}^*$, $c_{i,j} \neq c_{i,j'}$ if $j \neq j'$, $A_i \in \boldsymbol{Z}$, $B_i \in \boldsymbol{Z}_{\geq 0}$ and $\ell_i, \nu_{i,j} \in \boldsymbol{N}$. Note that by the assumption, there is no boundary face $\Delta(Q_i;f)$ of $\Delta(f)$ such that $d(Q_i;f) > 0$. Let $\{z_{i,j}\}_{j=1,\dots,\ell_i}$ be the intersections of $\overline{X}_0$ and the divisor $E(Q_i)$ where $z_{i,j}$ corresponds to the factor $(x^{q_i} + c_{i,j}y^{p_i})^{\nu_{i,j}}$. $z_{i,j}$ can be a singular point of $\overline{X}_0$ if $\nu_{i,j} \geq 2$ and $z_{i,j} \in \overline{X}_\varepsilon$ if and only if $d(Q_i;f) < 0$.

Let $I_{i,0}$ (resp. $I_{i,\varepsilon}$) be the number of (topological) intersection points of $\overline{X}_0$ and $E(Q_i)$ (resp. of $\overline{X}_\varepsilon$ and $E(Q_i)$). Under the above notations, we have $I_{i,0} = I_{i,\varepsilon} = \ell_i$ for Q_i with $d(Q_i; f) < 0$. Put $I = \sum^- I_{i,0}$ where $\sum^-$ is the sum for Q_i with $d(Q_i; f) < 0$. Also put $N_0 = \sum^0 I_{i,0}$ and $N_\varepsilon = \sum^0 I_{i,\varepsilon}$ where $\sum^0$ is the sum for Q_i with $d(Q_i; f) = 0$. By the additivity of the Euler characteristics, we have

$$\begin{aligned} \chi(\overline{X}_0) &= \chi(X_0) + I + N_0, \\ \chi(\overline{X}_\varepsilon) &= \chi(X_\varepsilon) + I + N_\varepsilon. \end{aligned} \tag{5.8}$$

By the assumption, $f(x, y)$ is conveinent or satisfies $f(0,0) \neq 0$. If $f(x, y)$ is convenient, there are no Q_i with $d(Q_i; f) = 0$ and thus $N_0 = N_\varepsilon = 0$. If $f(0,0) \neq 0$, since ε is generic, f_ε is non-degenerate on any face $\Delta(Q_i; f_\varepsilon)$ with $d(Q_i; f) = 0$. Thus N_ε coincides with the sum of the algebraic intersection numbers of $E(Q_i)$ and $\overline{X}_\varepsilon$ for Q_i with $d(Q_i; f) = 0$. Namely

$$N_\varepsilon = \sum\nolimits^0 \sum_{j=1}^{\ell_i} v_{i,j}.$$

Thus by the above consideration, we get

$$\begin{aligned} \chi(X_0) - \chi(X_\varepsilon) &\overset{(5.8)}{=} \chi(\overline{X}_0) - \chi(\overline{X}_\varepsilon) + N_\varepsilon - N_0 \\ &\overset{(5.7)}{=} \sum\nolimits^- \sum_{j=1}^{\ell_i} (\mu(\overline{X}_0, z_{i,j}) - \mu(\overline{X}_\varepsilon, z_{i,j})) + \sum\nolimits^0 \sum_{j=1}^{\ell_i} (\mu(\overline{X}_0, z_{i,j}) + v_{i,j} - 1). \end{aligned}$$

Note that $\mu(\overline{X}_0, z_{i,j}) - \mu(\overline{X}_\varepsilon, z_{i,j}) \geq 0$ when $d(Q_i; f) < 0$. Note also that $\mu(\overline{X}_0, z_{i,j}) + v_{i,j} - 1 \geq 0$ when $d(Q_i; f) = 0$, and the strict inequality holds if $v_{i,j} \geq 2$. Thus we get the inequality

$$\chi(X_0) \geq \chi(X_\varepsilon).$$

If we have some $z_{i,j}$ with $d(Q_i; f) = 0$ and $v_{i,j} \geq 2$, we have $\chi(X_0) > \chi(X_\varepsilon)$. If we have some $z_{i,j}$ with $d(Q_i; f) < 0$ such that $\mu(\overline{X}_0, z_{i,j}) > \mu(\overline{X}_\varepsilon, z_{i,j})$, we have also $\chi(X_0) > \chi(X_\varepsilon)$. Recall that $P = {}^t(\alpha, \beta)$ is the primitive covector associated with the valuations of $p(t)$ given by (2.1). We have seen that $P = Q_{i_0}$ for some $1 \leq i_0 \leq \eta$. When $\delta = 1$ and $d_\delta = 0$, we must have $d(P; f) = 0$ and $f_P(x, y)$ is degenerate by Lemma 2.3. Thus the assertion follows from the above argument. Recall also that O_i, $i = 1, \dots, \delta - 1$ are the centers of the inductive toric modifications $\pi_{i+1} : Y_{i+1} \to Y_i$. We may assume that $d(P; f) < 0$ and $O_1 = z_{i_0, j_0}$ for some j_0. To complete the proof we will show that

$$\mu(\overline{X}_0, z_{i_0, j_0}) > \mu(\overline{X}_\varepsilon, z_{i_0, j_0}). \tag{5.9}$$

Let $\mathscr{F}_0^{\sigma}$ (resp. $\mathscr{F}_\varepsilon^{\sigma}$) be the Milnor fiber of F_0^{σ} (resp. F_ε^{σ}) at O_σ. The inequality (5.9) is equivalent to

$$\chi(\mathscr{F}_0^{1}) < \chi(\mathscr{F}_\varepsilon^{1}).$$

We prove this inequality by induction. The induction starts by Assertion 5.10, and Assertion 5.11 guarantees the inductive step

$$\chi(\mathscr{F}_0^{\sigma}) < \chi(\mathscr{F}_\varepsilon^{\sigma}) \Rightarrow \chi(\mathscr{F}_0^{\sigma-1}) < \chi(\mathscr{F}_\varepsilon^{\sigma-1}).$$

This completes the proof of Theorem 5.8, assuming Assertion 5.10 and 5.11 below. □

Assertion 5.10. (I) *Suppose* $\delta \geq 2$. *If* $d_\delta = 0$, *then* $\chi(\mathscr{F}_0^{\delta-1}) < \chi(\mathscr{F}_\varepsilon^{\delta-1})$.
(II) *Suppose* $\delta \geq 1$. *If* $d_\delta < 0$ *and* $\nu(F_0^{\delta}) > \nu(F_\varepsilon^{\delta})$, *then* $\chi(\mathscr{F}_0^{\delta}) < \chi(\mathscr{F}_\varepsilon^{\delta})$.

Proof. First we consider the case (I). Thus we assume that $\delta \geq 2$ and $d(P_{\delta-1}; f^{\delta-1}) = 0$, where $f^{\delta-1}$ is the restriction of the pull-back of f to the translated coordinate chart centered at $O_{\delta-1}$. Recall that $F_0^{\delta-1} = x_{\delta-1}^{-d_{\delta-1}} f^{\delta-1}$. To compare $\chi(F_0^{\delta-1})$ and $\chi(F_\varepsilon^{\delta-1})$, we use an admissible toric modification $\pi_\delta : Y_\delta \to Y_{\delta-1}$ with respect to $\Gamma_+(F_0^{\delta-1})$. Let $R_{\delta-1,j}$, $j = 1, \ldots, \mu_{\delta-1}$ be the primitive covectors which associate π_δ. By assigning the indices as in Section 3, we can assume that $R_{\delta-1,1} = {}^t(1,0)$, $R_{\delta-1,\mu_{\delta-1}} = {}^t(0,1)$ and $P_{\delta-1} = R_{\delta-1,\iota_{\delta-1}}$ where $P_{\delta-1}$ is the primitive covector given by the valuations of $p_{\delta-1}(t)$. If $\Gamma(R_{\delta-1,i}; F_0^{\delta-1})$ is a face, the factorization of the face function is given by

$$(F_0^{\delta-1})_{R_{\delta-1,i}}(x_{\delta-1}, y_{\delta-1}) = C_i x_{\delta-1}^{A_i} y_{\delta-1}^{B_i} \prod_{j=1}^{\ell_i} (x_{\delta-1}^{s_{\delta-1,i}} + c_{i,j} y_{\delta-1}^{r_{\delta-1,i}})^{\nu_{i,j}},$$

where $C_i, c_{i,j} \in \boldsymbol{C}^2$, $c_{i,j} \neq c_{i,j'}$ if $j \neq j'$, $A_i, B_i \in \boldsymbol{Z}_{\geq 0}$ and $\ell_i, \nu_{i,j} \in \boldsymbol{N}$. Put $\nu_i = \sum_{j=1}^{\ell_i} \nu_{i,j}$. Let $E(R_{\delta-1,i})$ be the exceptional divisor corresponding to $R_{\delta-1,i}$ and we set

$$E^*(R_{\delta-1,i}; F_0^{\delta-1}) = E(R_{\delta-1,i}) \setminus \left(V_{F_0^{\delta-1}} \bigcup_{1 \leq j \leq \mu_{\delta-1}, j \neq i} E(R_{\delta-1,j}) \right),$$

where $V_{F_0^{\delta-1}}$ is the strict transform of $F_0^{\delta-1}$. Let $m(R_{\delta-1,i}; F_0^{\delta-1})$ be the multiplicity of the pull-back $\pi_\delta^* F_0^{\delta-1}$ along $R_{\delta-1,i}$. Let $\{z_{i,j,0}\}_{j=1,\ldots,\ell_i}$ be the intersection points of $V_{F_0^{\delta-1}}$ and $E(R_{\delta-1,i})$ and we denote the restriction of the pull-back $\pi_\delta^* F_0^{\delta-1}$ to a neighborhood of $z_{i,j,0}$ by $F_{i,j,0}$. We can consider that $z_{i,j}$ corresponds to the factor $(x_{\delta-1}^{s_{\delta-1,i}} + c_{i,j} y_{\delta-1}^{r_{\delta-1,i}})^{\nu_{i,j}}$. We denote $m_i = m(R_{\delta-1,i}; F_0^{\delta-1})$ and define an analytic function $f_{i,j}$ by $F_{i,j,0} = x_{i,j}^{m_i} f_{i,j}$ where $(x_{i,j}, y_{i,j})$ is the translated coordinates at $z_{i,j}$. Note that $\nu_{i,j} = I(x_{i,j}, f_{i,j}; z_{i,j})$. By using (5.6), we have $\chi(\mathscr{F}_0^{\delta-1}) = J_0 + K_0$ where

$$J_0 = \sum\nolimits^{-}\left(m_i\chi(E^*(R_{\delta-1,i};F_0^{\delta-1})) + \sum_{j=1}^{\ell_i}\chi(\mathscr{F}(F_{i,j,0}))\right),$$

$$K_0 = \sum\nolimits^{\geq 0}\left(m_i\chi(E^*(R_{\delta-1,i};F_0^{\delta-1})) + \sum_{j=1}^{\ell_i}\chi(\mathscr{F}(F_{i,j,0}))\right).$$

Here $\sum^{\geq 0}$ is the sum for $R_{\delta-1,i}$ with $d(R_{\delta-1,i};f^{\delta-1}) \geq 0$ and $\mathscr{F}(\psi)$ is the Milnor fiber of ψ at the origin.

Next we consider the pull-back $\pi_\delta^* F_\varepsilon^{\delta-1}$ of $F_\varepsilon^{\delta-1}$ by the same admissible toric modification $\pi_\delta : Y_\delta \to Y_{\delta-1}$. We define

$$E^*(R_{\delta-1,i};F_\varepsilon^{\delta-1}) = E(R_{\delta-1,i})\backslash\left(V_{F_\varepsilon^{\delta-1}} \bigcup_{1\leq j\leq \mu_{\delta-1}, j\neq i} E(R_{\delta-1,j})\right),$$

where $V_{F_\varepsilon^{\delta-1}}$ is the strict transform of $F_\varepsilon^{\delta-1}$. Let $m(R_{\delta-1,i};F_\varepsilon^{\delta-1})$ be the multiplicity of the pull-back $\pi_\delta^* F_\varepsilon^{\delta-1}$ along $R_{\delta-1,i}$. Let $\{z_{i,j,\varepsilon}\}_{j=1,\dots,\ell_i'}$ be the intersection points of $V_{F_\varepsilon^{\delta-1}}$ and $E(R_{\delta-1,i})$ and we denote the restriction of the pull-back $\pi_\delta^* F_\varepsilon^{\delta-1}$ to a neighborhood of $z_{i,j,\varepsilon}$ by $F_{i,j,\varepsilon}$. Note that in $\Gamma(F_\varepsilon^{\delta-1})$, the face $\Gamma(P_{\delta-1};F_0^{\delta-1})$ changes into a bigger face $\Gamma(P_{\delta-1};F_\varepsilon^{\delta-1})$ which touches the X-axis. The intersection $\Gamma(P_{\delta-1};F_\varepsilon^{\delta-1}) \cap \{X\text{-axis}\}$ is exactly the point $D := (-d_{\delta-1}, 0)$. The other faces corresponding to some $R_{\delta-1,j}$ with $j > i_{\delta-1}$ (they are in the right side of $\Gamma(Q_{\delta-1};F_0^{\delta-1})$) disappear in $\Gamma(F_\varepsilon^{\delta-1})$, see Figure 5. Comparing $\Gamma_+(F_0^{\delta-1})$ and $\Gamma_+(F_\varepsilon^{\delta-1})$, we can say that if $d(R_{\delta-1,i};f) < 0$ then $\ell_i = \ell_i'$ and $z_{i,j,0} = z_{i,j,\varepsilon}$ for all $j = 1,\dots,\ell_i$. Then from the equation (5.6), $\chi(\mathscr{F}_\varepsilon^{\delta-1})$ can be described as $\chi(\mathscr{F}_\varepsilon^{\delta-1}) = J_\varepsilon + K_\varepsilon$ where

$$J_\varepsilon = \sum\nolimits^{-}\left(m_i\chi(E^*(R_{\delta-1,i};F_0^{\delta-1})) + \sum_{j=1}^{\ell_i}\chi(\mathscr{F}(F_{i,j,\varepsilon}))\right),$$

$$K_\varepsilon = m(P_{\delta-1};F_0^{\delta-1})\chi(E^*(P_{\delta-1};F_\varepsilon^{\delta-1})) + m(R_{\delta-1,\mu_{\delta-1}-1};F_\varepsilon^{\delta-1}).$$

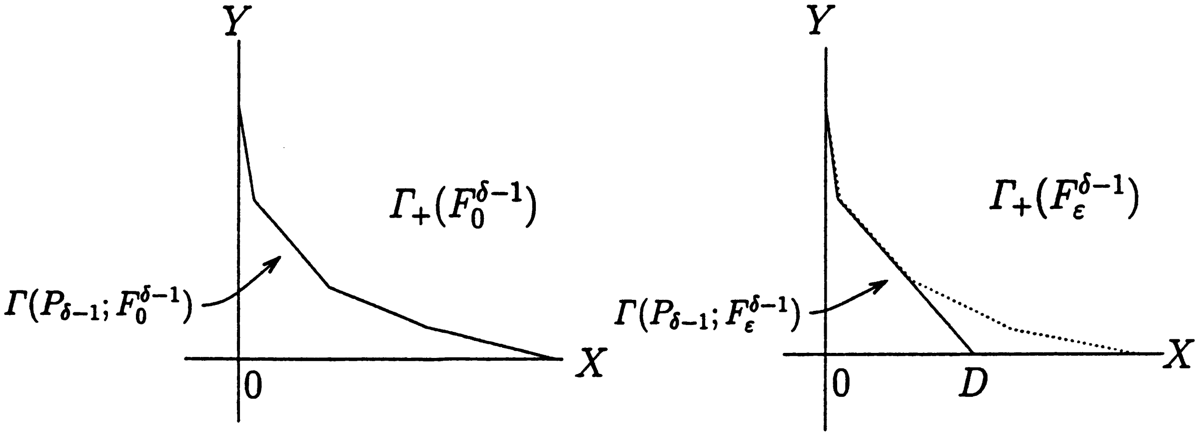

Figure 5. Newton polygons $\Gamma_+(F_0^{\delta-1})$ and $\Gamma_+(F_\varepsilon^{\delta-1})$.

Here $F_{i,j,\varepsilon}$ is the restriction of $\pi_\delta^* F_\varepsilon^{\delta-1}$ to a neighborhood of $z_{i,j,\varepsilon}$. By the upper semi-continuity of the Milnor numbers (= lower semi-continuity of the Euler characteristics) and Lemma 5.9, we have $J_0 \le J_\varepsilon$. Hence the expected strict inequality $\chi(\mathscr{F}_0^{\delta-1}) < \chi(\mathscr{F}_\varepsilon^{\delta-1})$ follows from

$$K_0 < K_\varepsilon. \tag{5.10}$$

For a given face $\Xi \subset \boldsymbol{R}^2_{\ge 0}$, let $\operatorname{Cone}(\Xi, O)$ be the cone over Ξ with vertex at the origin:

$$\operatorname{Cone}(\Xi, O) := \{sT \mid T \in \Xi, 0 \le s \le 1\}.$$

Since $\Delta(P_{\delta-1}; F_\varepsilon^{\delta-1})$ is non-degenerate,

$$m(P_{\delta-1}; F_0^{\delta-1})\chi(E^*(P_{\delta-1}; F_\varepsilon^{\delta-1})) = -2\operatorname{Vol}(\operatorname{Cone}(\Gamma(P_{\delta-1}; F_\varepsilon^{\delta-1}), O)),$$

see [**O2**]. Also it is easy to see that $m(R_{\delta-1,\mu_{\delta-1}-1}; F_\varepsilon^{\delta-1})$ is the length of the segment OD. Thus we have

$$K_\varepsilon = -\nu(\operatorname{Cone}(\Gamma(P_{\delta-1}; F_\varepsilon^{\delta-1}), O)) + 1,$$

where $\nu(\Delta)$ is the Newton number of an integral polyhedron $\Delta \subset \boldsymbol{R}^2_{\ge 0}$. To show the inequality (5.10), we first divide $\Gamma(F_0^{\delta-1})$ into two parts, Γ_L and Γ_R where Γ_L is the left upper part of $\Gamma(F_0^{\delta-1})$ relative to the face $\Gamma(P_{\delta-1}; F_0^{\delta-1})$ and Γ_R is the union of $\Gamma(P_{\delta-1}; F_0^{\delta-1})$ and its right lower part. Thus $\Gamma_-(F_0^{\delta-1}) = \operatorname{Cone}(\Gamma_L, O) \cup \operatorname{Cone}(\Gamma_R, O)$, see Figure 6. The inequality (5.10) follows from the following inequalities:

$$K_0 < -\nu(\operatorname{Cone}(\Gamma_R, O)) + 1, \tag{5.11}$$

$$-\nu(\operatorname{Cone}(\Gamma_R, O)) + 1 \le -\nu(\operatorname{Cone}(\Gamma(P_{\delta-1}; F_\varepsilon^{\delta-1}), O)) + 1 = K_\varepsilon. \tag{5.12}$$

The last inequality is immediate from the monotonity of the Newton numbers. From Lemma 5.9 and the inequality $\chi(\mathscr{F}(f_{i,j})) \le 1$,

$$\chi(\mathscr{F}(F_{i,j,0})) \le -(m_i + 1)(\nu_{i,j} - 1). \tag{5.13}$$

Then, when $\Gamma(R_{\delta-1,i}; F_0^{\delta-1})$ is a boundary face, we have the inequality:

$$\begin{aligned} m_i\chi(E^*(R_{\delta-1,i}; F_0^{\delta-1})) &+ \sum_{j=1}^{\ell_i} \chi(\mathscr{F}(F_{i,j,0})) \\ &\le -m_i\ell_i - \sum_{j=1}^{\ell_i}(m_i + 1)(\nu_{i,j} - 1) \\ &= -(m_i + 1)\nu_i + \ell_i = -m_i\nu_i - (\nu_i - \ell_i) \end{aligned}$$

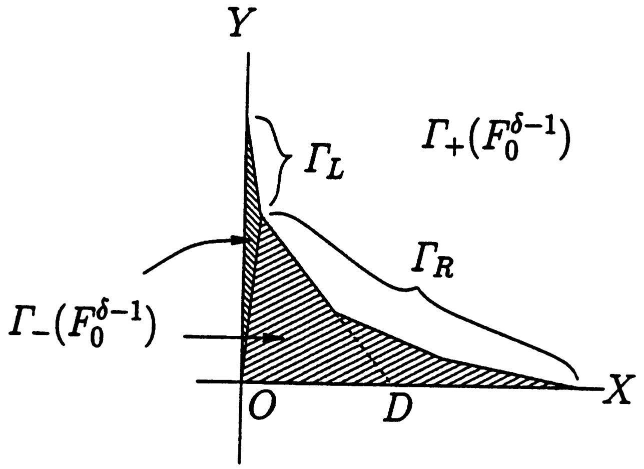

Figure 6. Partition of $\Gamma_-(F_0^{\delta-1})$ into $\mathrm{Cone}(\Gamma_L, O)$ and $\mathrm{Cone}(\Gamma_R, O)$.

$$\leq -m_i v_i = -2\,\mathrm{Vol}(\mathrm{Cone}(\Gamma(R_{\delta-1,i}; F_0^{\delta-1}), O)).$$

The strict inequality holds if there exists some j with $v_{i,j} \geq 2$. Therefore we have the strict inequality for $i = i_{\delta-1}$. Thus we get

$$\begin{aligned} K_0 &< -2\,\mathrm{Vol}(\mathrm{Cone}(\Gamma_R, O)) + |\mathrm{Cone}(\Gamma_R, O) \cap \{X\text{-axis}\}| \\ &= -v(\mathrm{Cone}(\Gamma_R, O)) + 1 \leq K_\varepsilon. \end{aligned}$$

This proves (5.11).

Now we consider the case (II), that is, $\delta \geq 1$, $d_\delta < 0$ and $v(F_0^\delta) > v(F_\varepsilon^\delta)$. The proof is exactly parallel to the previous case. Let $\pi_{\delta+1} : Y_{\delta+1} \to Y_\delta$ be an admissible toric modification with respect to $\Gamma_+(F_0^\delta)$. Let $R_{\delta,j}$, $j = 1, \ldots, \mu_\delta$ be the primitive covectors which associate $\pi_{\delta+1}$. By assigning the indices as in Section 3, we can assume that $R_{\delta,1} = {}^t(1,0)$, $R_{\delta,\mu_\delta} = {}^t(0,1)$ and $P_\delta = R_{\delta,i_\delta}$ where P_δ is the primitive covector given by the valuations of $p_\delta(t)$. Let Γ_L be the union of the boundary faces $\Gamma(R_{\delta,i}; F_0^\delta)$ such that $d(R_{\delta,i}; f^\delta) < 0$ and let Γ_R be the closure of $\Gamma(F_0^\delta) \backslash \Gamma_L$. Note that in $\Gamma(F_\varepsilon^\delta)$, there exists a unique face Ξ_ε which is not contained in the faces in Γ_L. This face corresponds to the face of $\Gamma(f_\varepsilon^\delta)$ which ends at the origin. We may assume that $\Gamma(R_{\delta,i'_\delta}; F_\varepsilon^\delta) = \Xi_\varepsilon$ and $\Gamma(R_{\delta,j}; F_0^\delta) \subset \Gamma_L$ if and only if $j < i'_\delta$. By the assumption, we have the strict inequality

$$v(\mathrm{Cone}(\Xi_\varepsilon, O)) < v(\mathrm{Cone}(\Gamma_R, O)).$$

From the equation (5.6), $\chi(\mathscr{F}_0^\delta)$ can be described as $\chi(\mathscr{F}_0^\delta) = J_0 + K_0$ where J_0 (resp. K_0) is the sum of

$$m(R_{\delta,i};F_0^\delta)\chi(E^*(R_{\delta,i};F_0^\delta))+\sum_{j=1}^{\ell_i}\chi(\mathscr{F}(F_{i,j,0}))$$

over $i<i'_\delta$ (resp. over $i\geq i'_\delta$) where $\{z_{i,1},\dots,z_{i,\ell_i}\}=E(R_{\delta,i})\cap V_{F_0^\delta}$ and $F_{i,j,0}$ is the restriction of $\pi_\delta^*F_0^\delta$ to a neighborhood of $z_{i,j}$. For $\chi(\mathscr{F}_\varepsilon^\delta)$, we have a similar partition $\chi(\mathscr{F}_\varepsilon^\delta)=J_\varepsilon+K_\varepsilon$ where J_ε is the sum of

$$m(R_{\delta,i};F_\varepsilon^\delta)\chi(E^*(R_{\delta,i};F_\varepsilon^\delta))+\sum_{j=1}^{\ell_i}\chi(\mathscr{F}(F_{i,j,\varepsilon}))$$

over $i<i'_\delta$ and, by the case of equality in (5.13), K_ε is given by

$$K_\varepsilon=-2\,\mathrm{Vol}(\mathrm{Cone}(\Gamma_\varepsilon,O))+|\mathrm{Cone}(\Gamma_\varepsilon,O)\cap\{X\text{-axis}\}|=-\nu(\mathrm{Cone}(\Gamma_\varepsilon,O))+1.$$

By the upper semi-continuity of the Milnor numbers, we have $J_0\leq J_\varepsilon$. On the other hand,

$$K_0\leq-\nu(\mathrm{Cone}(\Gamma_R,O))+1<-\nu(\mathrm{Cone}(\Gamma_\varepsilon,O))+1=K_\varepsilon.$$

Thus we conclude that $\chi(\mathscr{F}_0^\delta)<\chi(\mathscr{F}_\varepsilon^\delta)$. □

ASSERTION 5.11. *Let σ be an integer with $1\leq\sigma\leq\delta-1$. If $F_0^{\sigma+1}$ and $F_\varepsilon^{\sigma+1}$ have only isolated singularities at $O_{\sigma+1}$ and their Milnor fibers satisfy $\chi(\mathscr{F}_0^{\sigma+1})<\chi(\mathscr{F}_\varepsilon^{\sigma+1})$, then $\chi(\mathscr{F}_0^\sigma)<\chi(\mathscr{F}_\varepsilon^\sigma)$.*

PROOF. The proof of Assertion 5.11 is completely parallel to that of Assertion 5.10. Let $\pi_{\sigma+1}:Y_{\sigma+1}\to Y_\sigma$ be an admissible toric modification with respect to $\Gamma_+(F_0^\sigma)$ which is associated with covectors $R_{\sigma,i}$, $i=1,\dots,\mu_\sigma$. Suppose $P_\sigma=R_{\sigma,i_\sigma}$. Let Γ_L be the union of the boundary faces $\Gamma(R_{\sigma,i};F_0^\sigma)$ such that $d(R_{\sigma,i};f^\sigma)<0$ and let Γ_R be the closure of $\Gamma(F_0^\sigma)\backslash\Gamma_L$. We may assume that $\Gamma(R_{\sigma,j};F_0^\sigma)\subset\Gamma_L$ if and only if $j<i'_\sigma$ and also assume $i_\sigma<i'_\sigma$. Let $\{z_{i,j}\}_{j=1,\dots,\ell_i}$ be the intersection points of the strict transform of $F_0^\sigma=0$ and $E(R_{\sigma,i})$ as before. By using (5.6), the Euler characteristics $\chi(\mathscr{F}_0^\sigma)$ and $\chi(\mathscr{F}_\varepsilon^\sigma)$ can be decomposed into $\chi(\mathscr{F}_0^\sigma)=J_0+K_0$ and $\chi(\mathscr{F}_\varepsilon^\sigma)=J_\varepsilon+K_\varepsilon$ respectively. The inequality $K_0\leq K_\varepsilon$ follows by the monotonity of the Newton numbers as before. For $i<i'_\sigma$, the intersection points of the strict transform of $F_\varepsilon^\sigma=0$ and the divisor $E(R_{\sigma,i})$ coincide with $\{z_{i,j}\}_{j=1,\dots,\ell_i}$. We denote the restriction of the pull-back $\pi_{\sigma+1}^*F_0^\sigma$ (resp. $\pi_{\sigma+1}^*F_\varepsilon^\sigma$) to a neighborhood of $z_{i,j}$ by $F_{i,j,0}$ (resp. $F_{i,j,\varepsilon}$). By the assumption $\chi(\mathscr{F}_0^{\sigma+1})<\chi(\mathscr{F}_\varepsilon^{\sigma+1})$ and Lemma 5.9, we have $\chi(\mathscr{F}(F_{i_\sigma,j_\sigma,0}))<\chi(\mathscr{F}(F_{i_\sigma,j_\sigma,\varepsilon}))$ for some j_σ with $z_{i_\sigma,j_\sigma}=O_{\sigma+1}$. Thus we get the strict inequality $J_0<J_\varepsilon$. □

6. Bifurcation set in the words of Newton polygons.

In this section we present the elements of the bifurcation set of a polynomial f by using inductive toric modifications and estimate the number of the elements.

Example 6.1 (Kouchnirenko, [**K**]). If f is a convenient and Newton non-degenerate polynomial, then $S_f = \emptyset$. Namely $B_f = \Sigma_f$. This is obvious from Lemma 2.2 and 2.3.

Assume that f has both variables x and y. A face Δ of $\Delta(f)$ is called *a bad face* if the supporting line is different from coordinate axes and it passes through the origin. There exist at most two bad faces. Let $\mathfrak{B}$ be the set of bad faces. For $\Delta \in \mathfrak{B}$, let $f_\Delta(x, y)$ be the face function on Δ and define a set $\Sigma(\Delta) \subset \boldsymbol{C}$ by

$$\Sigma(\Delta) = \{f_\Delta(x_0, y_0) \in \boldsymbol{C} \mid (x_0, y_0) \in (\boldsymbol{C}^*)^2 \text{ and } \operatorname{grad} f_\Delta(x_0, y_0) = 0\}.$$

Example 6.2. (Némethi & Zaharia, [**N-Z1**]). Let f be a not convenient and Newton non-degenerate polynomial, not depending only on one variable, such that $f(0,0) = 0$. Then

$$B_f = \Sigma_f \cup \{0\} \bigcup_{\Delta \in \mathfrak{B}} \Sigma(\Delta).$$

We give a proof of this assertion by using the previous results.

Proof. By the definition of the bifurcation set we have $B_f \supseteq \Sigma_f$. Also we have $B_f \supseteq \Sigma_f \cup \{0\}$ from Lemma 2.4. Let $c \in S_f$. Suppose that the primitive covector $P = {}^t(\alpha, \beta)$, given by (2.1), satisfies $d(P; f - c) > 0$. If $c \neq 0$, there does not exist such primitive covectors. If $c = 0$, by Lemma 2.4, we have $0 \in B_f$. Next we suppose $d(P; f - c) < 0$. In this case the boundary face $\Delta(P; f - c)$ is independent of c. Since there does not exist degenerate boundary faces and by Lemma 2.3, P does not associate any explicit paths. Finally we suppose $d(P; f - c) = 0$. If $c = f_P(a, b) \in \Sigma(\Delta(P; f))$, then $d(P; f - c) = 0$, $f_P(x, y) - c$ is degenerate and (a, b) is a multiple root of $f_P(x, y) - c = 0$. Considering an admissible toric compactification $\pi : Y_1 \to \boldsymbol{C}^2$ with respect to $\Delta(f - c)$, $f^1(x_1, y_1) - c$ satisfies $d_1 = d(P; f - c) = 0$ where f^1 is the restriction of the pull-back $\pi^* f$ to a neighborhood of the point corresponding to the face $\Delta(P; f - c)$ and the multiple root (a, b). Then from Theorem 5.8 we have $c \in B_f$. On the other hand if $c \notin \Sigma_f \cup \{0\} \bigcup_{\Delta \in \mathfrak{B}} \Sigma(\Delta)$, from Lemma 2.3 and 2.4, $c \notin \Sigma_f \cup S_f$. Then we have $c \notin B_f$. $\square$

Let $\mathscr{S}$ be the set of a tower of toric modifications

$$\rho : Y_\sigma \xrightarrow{\pi_\sigma} Y_{\sigma-1} \to \cdots \xrightarrow{\pi_2} Y_1 \supset \boldsymbol{C}^2,$$

where $\pi_i : Y_i \to Y_{i-1}$ is an admissible toric modification with center O_{i-1} using the translated coordinates (x_{i-1}, y_{i-1}). We assume that the pull-back f^i of f to

Y_i has a non-ordinary singularity at O_i. We use the same notations as in Section 4 and 5. Set $F_\varepsilon^\sigma = x_\sigma^{-d_\sigma}(f^\sigma - \varepsilon)$ as before.

PROPOSITION 6.3. *Fix $\rho \in \mathscr{S}$ as above. Suppose that f^σ satisfies $d_\sigma < 0$ and $v_\sigma \geq 2$. If $v(F_{c_\sigma}^\sigma) > v(F_\varepsilon^\sigma)$ for a generic $\varepsilon \in \boldsymbol{C}^*$, then $c_\sigma \in B_f$.*

We remark here that there exists a unique such c_σ for each f^σ if it exists.

PROOF. From Theorem 5.8 if $v(F_{c_\sigma}^\sigma) > v(F_\varepsilon^\sigma)$ for a generic $\varepsilon \in \boldsymbol{C}^*$, then the Euler characteristic of $f^{-1}(\varepsilon)$ is less than of $f^{-1}(c_\sigma)$. □

Now we extend the definition of bad faces to $\Gamma_+(f^\sigma)$ as follows: A face Δ of $\Gamma_+(f^\sigma)$ is called *a bad face* if the covector $\tilde{Q}_{\sigma,\imath} = {}^t(\tilde{p}_{\sigma,\imath}, \tilde{q}_{\sigma,\imath})$ of the face Δ satisfies $\tilde{p}_{\sigma,\imath}, \tilde{q}_{\sigma,\imath} > 0$ and the supporting line passes through the origin. There exists at most one bad face for each $\Gamma_+(f^\sigma)$.

PROPOSITION 6.4. *Suppose that there is a bad face Δ of $\Gamma_+(f^\sigma)$. Then $\Sigma(\Delta) \subset B_f$.*

PROOF. Suppose $c_\sigma \in \Sigma(\Delta)$. We may assume $c_\sigma = f_\Delta^\sigma(a_\sigma, b_\sigma)$ for $(a_\sigma, b_\sigma) \in \boldsymbol{C}^{*2}$ where f_Δ is the face function on Δ. Then (a_σ, b_σ) is a multiple root of $f_\Delta^\sigma(x, y) - c_\sigma = 0$. Consider an admissible toric modification $\pi_{\sigma+1} : Y_{\sigma+1} \to Y_\sigma$ with respect to $\Gamma(f^\sigma - c_\sigma)$ and let $f^{\sigma+1} - c_\sigma$ be the restriction of the pull-back $\pi_{\sigma+1}^*(f^\sigma - c_\sigma)$ to a neighborhood of the point corresponding to the face Δ and the multiple root (a_σ, b_σ). Then $f^{\sigma+1} - c_\sigma$ is described as (4.2) and satisfies $d_{\sigma+1} = 0$. Hence by Theorem 5.8 we have $c_\sigma \in B_f$. □

THEOREM 6.5. *Let $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ be a polynomial not depending only on one variable. We define $C = \varnothing$ if f is convenient and $C = \{f(0,0)\}$ if f is not convenient. Also we define $C_\sigma = \{c_\sigma\}$ if there exists c_σ such that $v(F_{c_\sigma}^\sigma) > v(F_\varepsilon^\sigma)$ for a generic $\varepsilon \in \boldsymbol{C}$ and otherwise we define $C_\sigma = \varnothing$.* Then

$$B_f = \Sigma_f \cup C \cup \bigcup_{\rho \in \mathscr{S}} \left(C_\sigma \bigcup_{\Delta \in \mathfrak{B}_\sigma} \Sigma(\Delta) \right), \tag{6.1}$$

where $\mathfrak{B}_\sigma$ is the set of bad faces of $\Delta(f)$ or $\Gamma_+(f^\sigma)$. In the above notation ρ runs all modification towers with respect to possible explicit paths of f such that $d_\sigma < 0$.

This theorem generalizes the result of Némethi and Zaharia in Eample 6.2. We remark that by the condition $d_\delta < 0$, the algorithm for constructing towers of toric modifications is independent of the constant term of f. It is enough to check non-zero multiple factors on boundary faces $\Gamma(R_{\sigma,i}; f^\sigma)$ with $d(R_{\sigma,i}; f^\sigma) < 0$ on each stage. If it exists, we have to consider the next stage for each non-zero multiple factor. On each stage we should check the inequality $v(F^{\sigma}_{c_\sigma}) > v(F^{\sigma}_{\varepsilon})$ if $\sigma \geq 1$, and also check the existence of a bad face of $\Delta(f)$ or $\Gamma_+(f^\sigma)$.

Proof. We prove the inclusion $\subseteq$ by deriving a contradiction. Let c be an element of B_f such that c is not contained in the set of the right hand side of (6.1), that is, $c \notin \Sigma_f$ and there are no explicit paths $p(t)$ such that $\lim_{t\to 0} f(p(t)) = c$. Then, since $B_f \subseteq \Sigma_f \cup S_f$, $c \notin B_f$. This is a contradiction. We can easily check the opposite inclusion $\supseteq$ by using Proposition 6.3 and 6.4. □

The next corollary is a result of Le and Oka in [**Le-O**]. We define $\xi(f) = 1$ if $\Delta(f)$ has a boundary face with $d(Q_i; f) > 0$ or f has either a factor x^2 or y^2. Otherwise we define $\xi(f) = 0$. Note that $\xi(f)$ is equivalent to $\varepsilon(f)$ in [**Le-O**].

Corollary 6.6 (Le & Oka, [**Le-O**]). *Let f be a polynomial function with $f(0,0) = 0$. Then*

$$\sharp S_f \leq \sum\nolimits^{-} \sum_{j=1}^{\ell_i} (v_{i,j} - 1) + \sum\nolimits^{0} \sum_{j=1}^{\ell_i} v_{i,j} + \xi(f),$$

where $\sum^{-}$ (resp. $\sum^{0}$) is the sum for Q_i such that $\Delta(Q_i; f)$ is a face with $d(Q_i; f) < 0$ (resp. $d(Q_i; f) = 0$) and ℓ_i and $v_{i,j}$ are given by the form of the face function $f_{Q_i} = C_i x^{A_i} y^{B_i} \prod_{j=1}^{\ell_i} (x^{b_i} + c_{i,j} y^{a_i})^{v_{i,j}}$.

Proof. Suppose that $d(Q_{i_0}; f) < 0$. If $f_{Q_{i_0}}$ is non-degenerate, there does not exist explicit path. If $f_{Q_{i_0}}$ is degenerate, we consider an admissible toric compactification $\pi : Y_1 \to \boldsymbol{C}^2$ with respect to $\Delta(f)$ and construct the following branched tower of toric modifications: Let $\pi_{\sigma+1} : Y_{\sigma+1} \to Y_\sigma$ be an admissible toric modification in the tower. Let f^σ be the restriction of the pull-back of f in Y_σ to a neighborhood of O_σ. f_σ takes the form (4.2). The branches at $\pi_{\sigma+1}$ correspond to the multiple factors of the face functions of $\Gamma_+(f^\sigma)$. Let $\{v_{\sigma,j}\}_{j=1,\dots,s_\sigma}$ be the multiplicities of the multiple factors. By considering $\Gamma(f^\sigma)$, we can see that

$$\sum_{j=1}^{s_\sigma} v_{\sigma,j} \leq v_\sigma.$$

We have three cases to obtain an element of S_f:

(1) When $d_{\sigma_0} < 0$ and $v(F^{\sigma_0}_{c_{\sigma_0}}) > v(F^{\sigma_0}_{\varepsilon})$ for a generic $\varepsilon \in \boldsymbol{C}$, from Proposition 6.3 we have $c_{\sigma_0} \in S_f$. By considering $\pi_{\sigma_0} : Y_{\sigma_0} \to Y_{\sigma_0 - 1}$, we have the inequality

$$\sum_{j=1}^{s_{\sigma_0}-1} v_{\sigma_0-1,j} < v_{\sigma_0-1}. \tag{6.2}$$

(2) When $\Gamma(f^{\sigma_0})$ has a face $\Gamma(Q_{\sigma_0,\imath}; f^{\sigma_0})$ with $d(Q_{\sigma_0,\imath}; f^{\sigma_0}) = 0$, from Proposition 6.4 the number of the elements of explicit set corresponding to this face is at most $v_{\sigma_0,\imath} - 1$. Hence we also have the inequality (6.2).

(3) When $p(t)$ is on the singular locus of f, let $p_{\sigma_0}(t) = {}^t(x_{\sigma_0}(t), y_{\sigma_0}(t))$ and $c_{\sigma_0} = \lim_{t\to 0} f^{\sigma_0}(p_{\sigma_0}(t))$. If both $x_{\sigma_0}(t)$ and $y_{\sigma_0}(t)$ are not constant zero, $v_{\sigma_0} \geq 2$ by Lemma 5.7. Hence by considering $\pi_{\sigma_0} : Y_{\sigma_0} \to Y_{\sigma_0 - 1}$, we also have the inequality (6.2). If either $x_{\sigma_0}(t)$ or $y_{\sigma_0}(t)$ is not constant zero, since $x_{\sigma_0} = 0$ defines the divisor which collapses to a point in $Y_1 \backslash \boldsymbol{C}^2$, we can assume that $x_{\sigma_0}(t) \not\equiv 0$ and $y_{\sigma_0}(t) \equiv 0$. Suppose $f^{\sigma_0} - c_{\sigma_0}$ takes the form $f^{\sigma_0}(x_{\sigma_0}, y_{\sigma_0}) - c_{\sigma_0} = y_{\sigma_0}^{e_{\sigma_0}} g(x_{\sigma_0}, y_{\sigma_0})$ where $e_{\sigma_0} \in \boldsymbol{Z}_{\geq 0}$ and $g(x_{\sigma_0}, y_{\sigma_0})$ is a local function with $g(x_{\sigma_0}, 0) \not\equiv 0$. If $e_{\sigma_0} = 0$ then $f^{\sigma_0}(x_{\sigma_0}, 0) - c_{\sigma_0} = g(x_{\sigma_0}, 0) \not\equiv 0$. This contradicts $f^{\sigma_0}(x_{\sigma_0}(t), 0) \equiv c_{\sigma_0}$. If $e_{\sigma_0} = 1$ then $(\partial f^{\sigma_0}/\partial x_{\sigma_0})(x_{\sigma_0}, 0) = g(x_{\sigma_0}, 0) \not\equiv 0$. This contradicts the assumption that $p_{\sigma_0}(t)$ is on the singular locus of f. Hence $f^{\sigma_0}(x_{\sigma_0}, y_{\sigma_0}) - c_{\sigma_0}$ has a factor $y_{\sigma_0}^2$. By considering $\pi_{\sigma_0} : Y_{\sigma_0} \to Y_{\sigma_0 - 1}$, we also have the inequality (6.2).

Let $z_{i_0,j}$ be the point in Y_1 corresponding to each factor $(x^{b_{i_0}} + c_{i_0,j} y^{a_{i_0}})^{v_{i_0,j}}$ and let $\sharp c_{i_0,j}$ be the number of elements of S_f which correspond to branches of the tower of toric modifications starting from $z_{i_0,j}$. Then the strict inequality $\sharp c_{i_0,j} < v_{i_0,j}$ follows from the above inequalities in the cases (1), (2) and (3).

Suppose $d(Q_{i_1}; f) = 0$. The face function $f_{Q_{i_1}}$ takes the form $f_{Q_{i_1}} = C_{i_1} x^{A_{i_1}} \cdot y^{B_{i_1}} \prod_{j=1}^{\ell_{i_1}} (x^{\beta_{i_1}} + c_{i_1,j} y^{\alpha_{i_1}})^{v_{i_1,j}}$. Then the number of elements of S_f corresponding to the face $\Delta(Q_{i_1}; f)$ is at most $\sum_{j=1}^{\ell_{i_1}} v_{i_1,j}$.

Suppose $d(Q_{i_2}; f) > 0$. In this case there may exist an explicit path $p(t)$ with $\lim_{t\to 0} f(p(t)) = 0$. If f has either a factor x^2 or y^2, there exists an explicit path $p(t)$ with $\lim_{t\to 0} f(p(t)) \equiv 0$. If f has a boundary face $\Delta(Q_{i_2}; f)$ which satisfies $d(Q_{i_2}; f) > 0$, also there may exist an explicit path $p(t)$ with $\lim_{t\to 0} f(p(t)) = 0$. For these cases we need to define $\xi(f)$. □

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