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Classification of links up to self pass-move

Dedicated to Professor Shin'ichi Suzuki for his 60th birthday

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Abstract. A pass-move and a #-move are local moves on oriented links defined by L. H. Kauffman and H. Murakami respectively. Two links are self pass-equivalent (resp. self #-equivalent) if one can be deformed into the other by pass-moves (resp. #-moves), where none of them can occur between distinct components of the link. These relations are equivalence relations on ordered oriented links and stronger than link-homotopy defined by J. Milnor. We give two complete classifications of links with arbitrarily many components up to self pass-equivalence and up to self #-equivalence respectively. So our classifications give subdivisions of link-homotopy classes.

1. Introduction.

We shall work in piecewise linear category. All links will be assumed to be ordered and oriented.

A *pass-move* [**5**] (resp. *#-move* [**7**]) is a local move on oriented links as illustrated in Figure 1.1(a) (resp. 1.1(b)). If the four strands in Figure 1.1(a) (resp. 1.1(b)) belong to the same component of a link, we call it a *self pass-move* (resp. *self #-move*) ([**1**], [**13**], [**14**], [**15**]). We note that pass-moves and #-moves are called #(II)-moves and #(I)-moves respectively in first author's prior papers [**13**], [**14**], [**15**], [**16**], etc. Two links are said to be *self pass-equivalent* (resp. *self #-equivalent*) if one can be deformed into the other by a finite sequence of self pass-moves (resp. self #-moves). Two links are said to be *link-homotopic* if one can be deformed into the other by finite sequence of *self crossing changes* ([**6**]). Since both self pass-move and self #-move are realized by self crossing changes, self pass-equivalence and self #-equivalence are stronger than link-homotopy. Link-homotopy classification is achieved by J. Milnor [**6**] for 3-component links, by J. Levine [**4**] for 4-component links, and by N. Habegger and X. S. Lin [**2**] for all links. In this paper we give two complete classifications of links with arbitrarily many components up to self pass-equivalence and up to self #-equivalence respectively. So our classifications give subdivisions of link-homotopy classes.

An n-component link $l = k_1 \cup \cdots \cup k_n$ is called a *proper link* if the linking number $\mathrm{lk}(l - k_i, k_i)$ is even for any $i (= 1, \ldots, n)$. For a proper link $l = k_1 \cup \cdots \cup k_n$, we call $\mathrm{Arf}(l) - \sum_{i=1}^{n} \mathrm{Arf}(k_i)$ $(\in \boldsymbol{Z}_2)$ the *reduced Arf invariant* [**13**] and denote it by $\overline{\mathrm{Arf}}(l)$, where Arf is the *Arf invariant* ([**11**]). (The Arf invariant is sometime called the *Robertello-Arf invariant*.)

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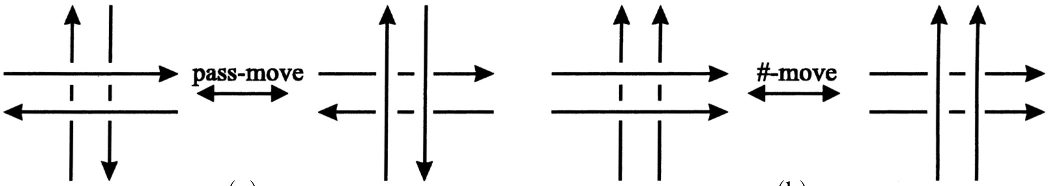
Key Words and Phrases. #-move, pass-move, link-homotopy, Arf invariant.



(a) (b)

Figure 1.1.

THEOREM 1.1. *Let $l = k_1 \cup \cdots \cup k_n$ and $l' = k_1' \cup \cdots \cup k_n'$ be n-component links. Then the following* (i) *and* (ii) *hold.*

(i) *l and l' are self pass-equivalent if and only if they are link-homotopic,* $\mathrm{Arf}(k_i) = \mathrm{Arf}(k_i')$ *for any i $(i = 1, \dots, n)$, and* $\mathrm{Arf}(k_{i_1} \cup \cdots \cup k_{i_p}) = \mathrm{Arf}(k_{i_1}' \cup \cdots \cup k_{i_p}')$ *for any proper links $k_{i_1} \cup \cdots \cup k_{i_p} \subseteq l$ and $k_{i_1}' \cup \cdots \cup k_{i_p}' \subseteq l'$.*

(ii) *l and l' are self #-equivalent if and only if they are link-homotopic and* $\overline{\mathrm{Arf}}(k_{i_1} \cup \cdots \cup k_{i_p}) = \overline{\mathrm{Arf}}(k_{i_1}' \cup \cdots \cup k_{i_p}')$ *for any proper links $k_{i_1} \cup \cdots \cup k_{i_p} \subseteq l$ and $k_{i_1}' \cup \cdots \cup k_{i_p}' \subseteq l'$.*

For two-component links, both self pass-equivalence classification and self #-equivalence classification have been done by the first author ([**15**]). His proof can be applied only to two-component links. So we need different approach to proving Theorem 1.1.

A link $l = k_1 \cup \cdots \cup k_n$ is said to be ***Z***$_2$*-algebraically split* if $\mathrm{lk}(k_i, k_j)$ is even for any i, j $(1 \leq i < j \leq n)$. We note that if $l = k_1 \cup \cdots \cup k_n$ is $\boldsymbol{Z}_2$-algebraically split link, then l and $k_i \cup k_j$ $(1 \leq i < j \leq n)$ are proper.

THEOREM 1.2. *Let $l = k_1 \cup \cdots \cup k_n$ and $l' = k_1' \cup \cdots \cup k_n'$ be n-component* ***Z***$_2$*-algebraically split links. If l and l' are link-homotopic, then*

$$\overline{\mathrm{Arf}}(l) + \sum_{1 \leq i < j \leq n} \overline{\mathrm{Arf}}(k_i \cup k_j) = \overline{\mathrm{Arf}}(l') + \sum_{1 \leq i < j \leq n} \overline{\mathrm{Arf}}(k_i' \cup k_j') \ \ (\in \boldsymbol{Z}_2).$$

By combining Theorems 1.1 and 1.2, we have the following corollary.

COROLLARY 1.3. *Let $l = k_1 \cup \cdots \cup k_n$ and $l' = k_1' \cup \cdots \cup k_n'$ be n-component* ***Z***$_2$*-algebraically split links. Then the following* (i) *and* (ii) *hold.*

(i) *l and l' are self pass-equivalent if and only if they are link-homotopic,* $\mathrm{Arf}(k_i) = \mathrm{Arf}(k_i')$ *for any i, and* $\mathrm{Arf}(k_i \cup k_j) = \mathrm{Arf}(k_i' \cup k_j')$ *for any i, j $(1 \leq i < j \leq n)$.*

(ii) *l and l' are self #-equivalent if and only if they are link-homotopic and* $\overline{\mathrm{Arf}}(k_i \cup k_j) = \overline{\mathrm{Arf}}(k_i' \cup k_j')$ *for any i, j $(1 \leq i < j \leq n)$.*

2. Preliminaries.

In this section, we collect several results in order to prove Theorems 1.1 and 1.2.

Let $l = k_1 \cup \cdots \cup k_n$ and $l' = k_1' \cup \cdots \cup k_n'$ be n-component links. Let D^4 be the unit 4-ball, $L_\#$ a link in ∂D^4 as illustrated in Figure 2.1, and $C_\#$ the cone with the center of D^4 and $L_\#$. Let $\mathscr{A} = A_1 \cup \cdots \cup A_n$ be a disjoint union of n annuli $A_1, \dots, A_n$. Suppose that there is a continuous map $f : \mathscr{A} \to S^3 \times [0, 1]$ with $f(\partial \mathscr{A}) \subset \partial(S^3 \times [0, 1])$ such that

(i) $(\partial(S^3 \times [0,1]), f(\partial A_i)) = (S^3 \times \{0\}, k_i) \cup (-S^3 \times \{1\}, -k_i')$ $(i = 1, \dots, n)$, and
(ii) there are finite points $p_1, \dots, p_m$ in $f(\mathscr{A}) \cap (S^3 \times (0,1))$ such that
- the inverse image $f^{-1}(p_j)$ of each p_j is a set of 4 points and belongs to a single annulus,
- $f : \mathscr{A} - \bigcup_j f^{-1}(p_j) \to S^3 \times [0,1]$ is a locally flat embedding, and
- each p_j has a small neighborhood $N(p_j)$ in $S^3 \times [0,1]$ such that $(N(p_j), N(p_j) \cap \mathscr{A})$ is homeomorphic to $(D^4, C_\#)$,

where $-X$ denotes X with the opposite orientation. Then $f(\mathscr{A})$ is called a *pass-annuli* between l and l'.

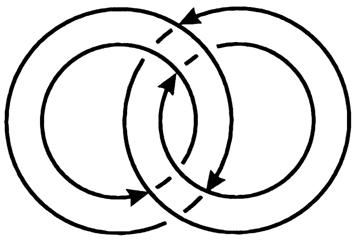


Figure 2.1.

The following is proved by the first author in [**14**].

LEMMA 2.1. *Two links l and l' are self pass-equivalent if and only if there is a pass-annuli between them.*

It is known that a pass-move is realized by a finite sequence of #-moves ([**8**]). Thus we have the following.

LEMMA 2.2. *If two links l and l' are self pass-equivalent, then they are self #-equivalent.*

A *Γ-move* [**5**] denotes a local move on oriented links as illustrated in Figure 2.2.

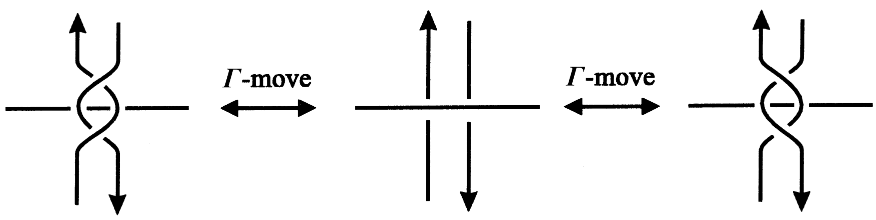


Figure 2.2.

The following is known [**5**].

LEMMA 2.3. *A Γ-move is realized by a single pass-move.*

Let $l = k_1 \cup \cdots \cup k_n$ and $l' = k_1' \cup \cdots \cup k_n'$ be n-component links such that there is a 3-ball B^3 in S^3 with $B^3 \cap (l \cup l') = l$. Let $b_1, \dots, b_n$ be mutually disjoint disks in S^3 such that $b_i \cap l = \partial b_i \cap k_i$ and $b_i \cap l' = \partial b_i \cap k_i'$ are arcs for each i. Then the link

$l \cup l' \cup (\bigcup_{i=1}^{n} \partial b_i) - (\bigcup \operatorname{int}(b_i \cap (l \cup l')))$ is called a *band sum* (or a *product fusion* [**12**]) of l and l' and denoted by $(k_1 \#_{b_1} k_1') \cup \cdots \cup (k_n \#_{b_n} k_n')$. Note that a band sum of l and l' is $\boldsymbol{Z}_2$-algebraically split if $\operatorname{lk}(k_i, k_j) \equiv \operatorname{lk}(k_i', k_j') \pmod 2$ $(1 \leq i < j \leq n)$.

The following is proved by the first author in [**12**].

Lemma 2.4. *Two links l and l' are link-homotopic if and only if there is a band sum of l and $-\overline{l'}$ that is link-homotopic to a trivial link, where $(S^3, -\overline{l'}) \cong (-S^3, -l')$.*

By the definition of the Arf invariant via 4-dimensional topology ([**11**]), we have the following.

Lemma 2.5. *Let l and l' be proper links and L a band sum of l and $-\overline{l'}$. Then L is proper and* $\operatorname{Arf}(L) = \operatorname{Arf}(l) + \operatorname{Arf}(l')$ $(\in \boldsymbol{Z}_2)$.

The following lemma forms an interesting contrast to the lemma above.

Lemma 2.6. *Let $l = k_1 \cup k_2$ and $l' = k_1' \cup k_2'$ be 2-component links with* $\operatorname{lk}(k_1, k_2)$ *and* $\operatorname{lk}(k_1', k_2')$ *odd. Let $L = (k_1 \#_{b_1} (-\overline{k_1'})) \cup (k_2 \#_{b_2} (-\overline{k_2'}))$ be a band sum and L' a band sum obtained from L by adding a single full-twist to b_2; see Figure 2.3. Then L and L' are proper and link-homotopic, and* $\operatorname{Arf}(L) \neq \operatorname{Arf}(L')$.

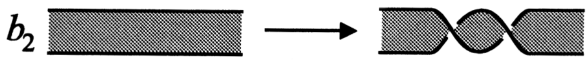


Figure 2.3.

Proof. Clearly L and L' are proper and link-homotopic. So we shall show $\operatorname{Arf}(L) \neq \operatorname{Arf}(L')$.

Let a_i be the ith coefficient of the Conway polynomial. Then we have

$$a_3(L) - a_3(L') = a_2((k_1 \#_{b_1} (-\overline{k_1'})) \cup k_2 \cup (-\overline{k_2'})).$$

It is known that the third coefficient of the Conway polynomial of a two-component proper link is mod 2 congruent to the sum of the Arf invariants of the link and the components [**9**]. This and Lemma 2.5 imply $\operatorname{Arf}(L) - \operatorname{Arf}(L') = a_3(L) - a_3(L')$ $(\in \boldsymbol{Z}_2)$. By [**3**],

$$\begin{aligned} &a_2((k_1 \#_{b_1} (-\overline{k_1'})) \cup k_2 \cup (-\overline{k_2'})) \\ &\quad = \operatorname{lk}(k_1 \#_{b_1} (-\overline{k_1'}), k_2) \operatorname{lk}(k_2, -\overline{k_2'}) + \operatorname{lk}(k_2, -\overline{k_2'}) \operatorname{lk}(-\overline{k_2'}, k_1 \#_{b_1} (-\overline{k_1'})) \\ &\qquad + \operatorname{lk}(-\overline{k_2'}, k_1 \#_{b_1} (-\overline{k_1'})) \operatorname{lk}(k_1 \#_{b_1} (-\overline{k_1'}), k_2). \end{aligned}$$

Thus we have $\operatorname{Arf}(L) - \operatorname{Arf}(L') = 1$ $(\in \boldsymbol{Z}_2)$. □

A Δ-move [**8**] is a local move on links as illustrated in Figure 2.4. If at least two of the three strands in Figure 2.4 belong to the same component of a link, we call it a *quasi self Δ-move* ([**10**]). Two links are said to be *quasi self Δ-equivalent* if one can be deformed into the other by a finite sequence of quasi self Δ-moves.

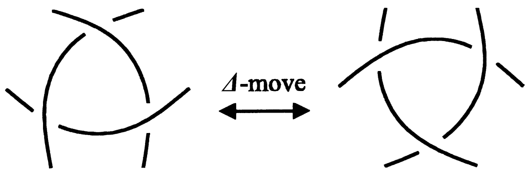


Figure 2.4.

The following is proved by Y. Nakanishi and the first author in [**10**].

Lemma 2.7. *Two links are link-homotopic if and only if they are quasi self Δ-equivalent.*

3. Proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.2. Since l is link-homotopic to l', by Lemma 2.7, l is quasi self Δ-equivalent to l'. It is sufficient to consider the case that l' is obtained from l by a single quasi self Δ-move.

Suppose that the three strands of the Δ-move that is applied to the deformation from l into l' belong to one component of l. Without loss of generality we may assume that the component is k_1. Note that k_i and k_i' are ambient isotopic for any $i(\neq 1)$, and that $k_i \cup k_j$ and $k_i' \cup k_j'$ are ambient isotopic for any $i < j$ $(i \neq 1)$. Since a Δ-move changes the value of the Arf invariant ([**8**]), we have $\mathrm{Arf}(l) \neq \mathrm{Arf}(l')$, $\mathrm{Arf}(k_1) \neq \mathrm{Arf}(k_1')$ and $\mathrm{Arf}(k_1 \cup k_j) \neq \mathrm{Arf}(k_1' \cup k_j')$. Thus we have $\overline{\mathrm{Arf}}(l) = \overline{\mathrm{Arf}}(l')$ and $\overline{\mathrm{Arf}}(k_1 \cup k_j) = \overline{\mathrm{Arf}}(k_1' \cup k_j')$. So we have the conclusion.

We now consider the other case, i.e., the three strands of the Δ-move belong to exactly two components of l. Without loss of generality we may assume that the two components are k_1 and k_2. Note that k_i and k_i' are ambient isotopic for any i, and that $k_i \cup k_j$ and $k_i' \cup k_j'$ are ambient isotopic for any $i < j$ $((i, j) \neq (1, 2))$. Since $\mathrm{Arf}(l) \neq \mathrm{Arf}(l')$ and $\mathrm{Arf}(k_1 \cup k_2) \neq \mathrm{Arf}(k_1' \cup k_2')$, $\mathrm{Arf}(l) + \mathrm{Arf}(k_1 \cup k_2) = \mathrm{Arf}(l') + \mathrm{Arf}(k_1' \cup k_2')$ $(\in \boldsymbol{Z}_2)$. This completes the proof. $\square$

Lemma 3.1. *Let $l = k_1 \cup \cdots \cup k_n$ and $l' = k_1' \cup \cdots \cup k_n'$ be n-component $\boldsymbol{Z}_2$-algebraically split links. If l and l' are link-homotopic,* $\mathrm{Arf}(k_i) = \mathrm{Arf}(k_i')$ $(i = 1, \ldots, n)$ *and* $\mathrm{Arf}(k_i \cup k_j) = \mathrm{Arf}(k_i' \cup k_j')$ $(1 \leq i < j \leq n)$, *then l and l' are self pass-equivalent.*

Proof. Since l is link-homotopic to l', by Lemma 2.7, l is quasi self Δ-equivalent to l'. Let u be the minimum number of quasi self Δ-moves which are needed to deform l into l'. By Theorem 1.2, $\mathrm{Arf}(l) = \mathrm{Arf}(l')$. Since a Δ-move changes the value of the Arf invariant, u is even. It is sufficient to consider the case $u = 2$. Therefore, there is a continuous map $f : \mathscr{A} = A_1 \cup \cdots \cup A_n \to S^3 \times [0, 1]$ from a disjoint union of n annuli $A_1, \ldots, A_n$ with $f(\partial\mathscr{A}) \subset \partial(S^3 \times [0, 1])$ such that

(i) $(\partial(S^3 \times [0, 1]), f(\partial A_i)) = (S^3 \times \{0\}, k_i) \cup (-S^3 \times \{1\}, -k_i')$ $(i = 1, \ldots, n)$, and

(ii) there are two points p_1, p_2 in $f(\mathscr{A}) \cap (S^3 \times (0,1))$ such that
- the inverse image $f^{-1}(p_t)$ of each p_t is a set of 3 points and belongs to at most two annuli,
- $f : \mathscr{A} - f^{-1}(p_1) \cup f^{-1}(p_2) \to S^3 \times [0,1]$ is a locally flat, level-preserving embedding, and
- each p_t has a small neighborhood $N(p_t)$ in $S^3 \times [0,1]$ such that $(N(p_t), N(p_t) \cap f(\mathscr{A}))$ is homeomorphic to (D^4, C_Δ), where C_Δ is the cone with the center of the unit 4-ball D^4 and the Borromean rings in ∂D^4.

A singular point p_t is called *type* (i) if $f^{-1}(p_t) \subset A_i$, and *type* (i,j) $(i<j)$ if $f^{-1}(p_t) \subset A_i \cup A_j$. Note that if p_t is type (i) (resp. type (i,j)), then $\partial(N(p_t) \cap f(\mathscr{A})) \subset f(A_i)$ (resp. $\subset f(A_i \cup A_j)$). For each i (resp. i,j), let u_i (resp. $u_{i,j}$) be the number of the singular points of type (i) (resp. type (i,j)). We note that a number of Δ-moves which are needed to deform k_i into k_i' (resp. $k_i \cup k_j$ into $k_i' \cup k_j'$) is equal to u_i (resp. $u_{i,j} + u_i + u_j$). By the hypothesis of this lemma, we have u_i and $u_{i,j} + u_i + u_j$ are even. Hence u_i and $u_{i,j}$ are even. This implies that both p_1 and p_2 are the same type.

Suppose that p_1 and p_2 are type (i,j). Without loss of generality we may assume that $(i,j) = (1,2)$ and two components of the Borromean rings $\partial(N(p_1) \cap f(\mathscr{A}))$ belong to $f(A_2)$. Let α be an arc in $f(A_1) \cap (S^3 \times (0,1))$ that connects two singular points p_1 and p_2 of type $(1,2)$, and let $(S^3, L) = (\partial N(\alpha), \partial(N(\alpha) \cap f(A_1 \cup A_2)))$. Then L is a 5-component link as illustrated in either Figure 3.1(a) or (b). In the case that L is as Figure 3.1(a), we can deform L into a trivial link by applying Γ-moves to the sublink $L \cap f(A_2)$; see Figure 3.2. In the case that L is as Figure 3.1(b), we can deform L into the link as in Figure 3.2(a) by two Γ-moves, one is applied to $L \cap f(A_1)$ and the other to $L \cap f(A_2)$; see Figure 3.3. It follows from this and Figure 3.2 that L can be deformed into a trivial link by Γ-moves, one is applied to $L \cap f(A_1)$ and the others to $L \cap f(A_2)$.

Suppose that p_1 and p_2 are type (i). Let α be an arc in $f(A_i) \cap (S^3 \times (0,1))$ that connects two singular points p_1 and p_2 of type (i), and let $(S^3, L) = (\partial N(\alpha), \partial(N(\alpha) \cap f(A_i)))$. By the argument similar to that in the above, L can be deformed into a trivial link by applying Γ-moves to $L \cap f(A_i)$.

Therefore, by Lemma 2.3, we can constract pass-annuli in $S^3 \times [0,1]$ between l and l'. Lemma 2.1 completes the proof. □

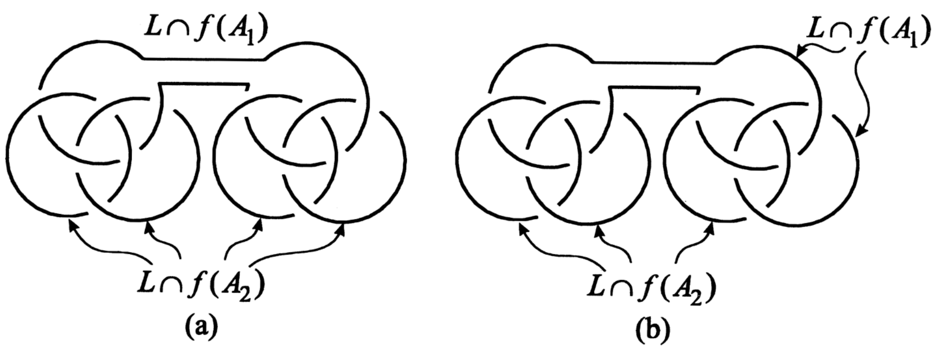


Figure 3.1.

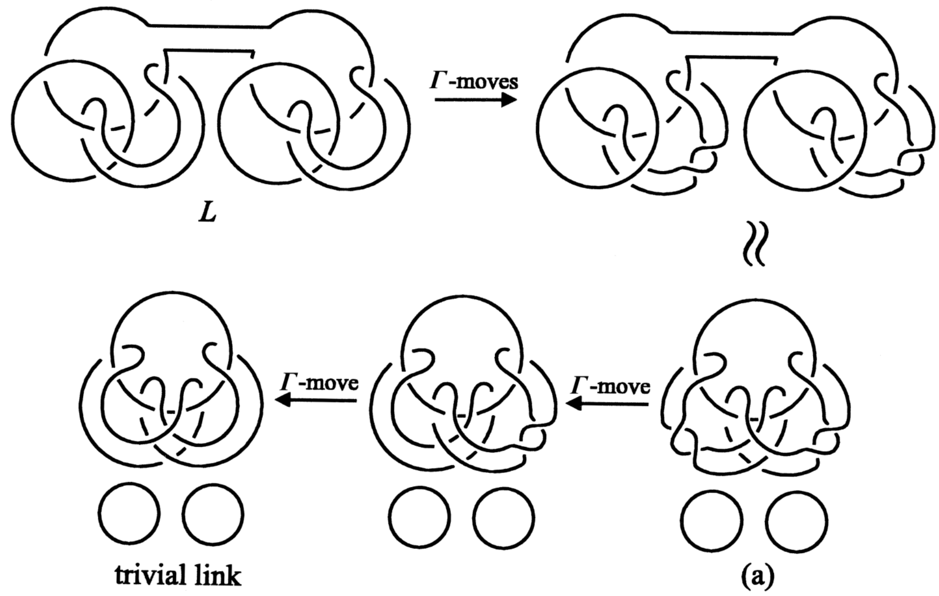


Figure 3.2.

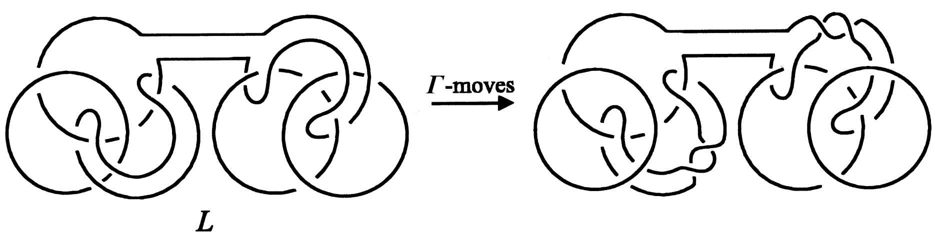


Figure 3.3.

PROOF OF THEOREM 1.1. Since a self pass-move (resp. a self #-move) is realized by link-homotopy and it preserves Arf (resp. $\overline{\mathrm{Arf}}$) [**15**, Proposition], we have the 'only if' part of (i) (resp. (ii)). We shall prove the 'if' parts.

(i) For a link $l = k_1 \cup \cdots k_n$, let G_l^o (resp. G_l^e) be a graph with the vertex set $\{k_1, \ldots, k_n\}$ and the edge set $\{k_i k_j \,|\, \mathrm{lk}(k_i, k_j) \text{ is odd}\}$ (resp. $\{k_i k_j \,|\, \mathrm{lk}(k_i, k_j) \text{ is even}\}$). Note that $G_l^o \cup G_l^e$ is the complete graph with n vertices. For a band sum $L = K_1 \cup \cdots \cup K_n (= (k_1 \#_{b_1} (-\overline{k_1'})) \cup \cdots \cup (k_n \#_{b_n} (-\overline{k_n'})))$ of l and $-\overline{l'}$, let A_L be a graph with the vertex set $\{K_1, \ldots, K_n\}$ and the edge set $\{K_i K_j \,|\, \mathrm{Arf}(K_i \cup K_j) = 0\}$. (Note that L is a $\boldsymbol{Z}_2$-algebraically split link since l and l' are link-homotopic.)

CLAIM. *There is a band sum L* of l and $-\overline{l'}$ *such that L is link-homotopic to a trivial link and A_L is the complete graph with n vertices.*

PROOF. Let T be a maximal subgraph of G_l^o that does not contain a cycle. Since T does not contain a cycle, by Lemmas 2.4 and 2.6, there is a band sum L of l and l' such that L is link-homotopic to a trivial link and $T \subset h(A_L)$, where $h : A_L \to G_l^o \cup G_l^e$ the natural map defined by $h(K_i) = k_i$ and $h(K_i K_j) = k_i k_j$. By Lemma 2.5, we have $G_l^e \subset h(A_L)$. Since h is injective and $G_l^o \cup G_l^e$ is the complete graph, it is sufficient to prove that h is surjective. Let E be the set of edges which are not contained in $h(A_L)$, and $H^o = h(A_L) \cap G_l^o$. Suppose $E \neq \varnothing$. Then there is an edge $e \in E$ such that there is a cycle C in $H^o \cup e$ containning e whose any *chord* are not contained in G_l^o, where a chord denotes an edge connecting two nonadjacent edges of C. (In fact, for each $e_i \in E$, consider the minimum length l_i of cycles in $H^o \cup e_i$ containning e_i and choose an edge e and a cycle C in $H^o \cup e$ containning e so that the length of C is equal to $\min\{l_i \mid e_i \in E\}$.) Without loss of generality we may assume that $C = k_1k_2\cdots k_ck_1$ and $e = k_1k_2$. Set $l_c = k_1 \cup \cdots \cup k_c$ and $L_c = K_1 \cup \cdots \cup K_c$. Since C has no chords in G_l^o, all chords are in G_l^e. Thus we have $k_ik_j \subset H^o \cup G_l^e (= h(A_L))$ for any i, j $(1 \le i < j \le c)$ except for $(i, j) = (1, 2)$. This implies $\mathrm{Arf}(K_i \cup K_j) = 0$ for any i, j $(1 \le i < j \le c, (i, j) \ne (1, 2))$. The fact that C has no chords in G_l^o implies l_c is a propre link. By the hypothesis about the Arf invariants and Lemma 2.5, we have $\mathrm{Arf}(L_c) = 2\,\mathrm{Arf}(l_c) = 0$ $(\in \boldsymbol{Z}_2)$ and $\mathrm{Arf}(K_i) = 2\,\mathrm{Arf}(k_i) = 0$ $(\in \boldsymbol{Z}_2)$ $(i = 1, \ldots, c)$. Since L_c is link-homotopic to a trivial link, by Theorem 1.2, $\mathrm{Arf}(K_1 \cup K_2) = 0$. This contradicts $e = k_1k_2 \in E$. □

By Claim, there is a band sum $L = K_1 \cup \cdots \cup K_n$ of l and $-\overline{l'}$ such that L is link-homotopic to a trivial link, $\mathrm{Arf}(K_i) = 0$ $(i = 1, \ldots, n)$ and $\mathrm{Arf}(K_i \cup K_j) = 0$ $(1 \le i < j \le n)$. By Lemma 3.1, L is self pass-equivalent to a trivial link. Since L is a band sum of l and $-\overline{l'}$, we can constract a pass-annuli between l and l'. Lemma 2.1 completes the proof.

(ii) Since a #-move changes the value of the Arf invariant [7], by applying self #-moves, we may assume that $\mathrm{Arf}(k_i) = \mathrm{Arf}(k_i')$ for any i. Theorem 1.1(i) and Lemma 2.2 complete the proof. □

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