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Generalized double tilted algebras

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Abstract. We introduce the class of generalized double tilted artin algebras and prove that it coincides with the class of artin algebras whose AR-quiver admits a faithful generalized standard almost directed component. A homological characterization of faithful generalized standard almost directed components is also established.

0. Introduction.

Tilted algebras have for a long time played a central role in the representation theory of artin algebras. Being a more general class than the class of hereditary algebras, they occur more frequently, and at the same time the module theory for tilted algebras is closely related to the module theory for hereditary algebras. The more general class of quasitilted algebras was introduced in [**9**]. This class contains also the canonical algebras, and has been the focus of much attention during the last years. A useful feature of the class of quasitilted algebras is the nice homological characterization as the algebras where each indecomposable module has projective or injective dimension at most one, and where in addition the global dimension is at most two. Dropping the last property we have the class of shod algebras, introduced and investigated in [**2**]. A shod algebra which is not quasitilted is said to be a strict shod algebra. Both for quasitilted algebras and for shod algebras there are several interesting descriptions and results (see [**2**], [**6**], [**7**], [**9**], [**13**], [**18**], [**20**], [**23**]), for example in terms of paths from indecomposable injective to indecomposable projective modules ([**9**], [**2**]).

As shown in [**18**] the strict shod algebras are closely related to tilted algebras. A tilted algebra is characterized in terms of the existence of an AR-component with a faithful section Δ such that $\mathrm{Hom}(M, D\,\mathrm{Tr}\,N) = 0$ for M and N on Δ (see [**17**] and [**20**]). The more general concepts of double section and double tilted algebras were introduced in [**18**], and the strict shod algebras were characterized in these terms. In particular, there are (left and right) tilted algebras naturally associated with a strict shod algebra, and the module theory for the strict shod algebras is closely related to the module theory for the associated tilted algebras.

In this paper we generalize the concepts of section and double section to what we call a multisection Δ, along with its left and right parts Δ_l and Δ_r. Modelled on defining the double tilted algebras on the basis of double sections in [**18**], we define generalized double tilted algebras on the basis of a multisection. We also characterize

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the generalized double tilted algebras as those having an AR-component with a faithful multisection Δ such that $\mathrm{Hom}(M, D\,\mathrm{Tr}\,N) = 0$ for M in Δ_r and N in Δ_l. On the other hand, this connects up with the work in [**20**], since we show that an algebra Λ is generalized double tilted if and only if there is an almost directed generalized standard faithful AR-component. We also describe the shape of all AR-components for such algebras.

For tilted algebras we have global dimension at most two, and for shod algebras it is at most three. For multisections we have the concept of n-section, where n may be infinite, and where a section is a 1-section and a double section a 2-section. Correspondingly we have n-double tilted algebras and we show that they have global dimension at most $n+1$.

We now describe the content of this paper section by section. In section 1 we recall some background material, including essential definitions. In section 2 we introduce the concept of multisection in a component of the AR-quiver of an artin algebra, along with its core, and the concept of generalized double tilted algebra is introduced and investigated in section 3. The results on global dimension are given in section 4, and in section 5 we give a homological characterization of faithful generalized standard almost directed components.

We refer also to the recent papers [**3**], [**4**] for generalizations of (strict) shod algebras in other, but related, directions. Recently, we have been informed by I. Assem and F. U. Coelho that they are also working on generalizations of shod algebras.

1. Preliminaries.

In this section we recall some definitions which will be used in this paper. We also refer to [**1**] and [**18**] for relevant background material.

By an algebra we mean a basic artin algebra Λ over a fixed commutative artin ring R, and by a Λ-module is meant a finitely generated left Λ-module. We denote the category of these modules by $\mathrm{mod}\,\Lambda$, and $\mathrm{ind}\,\Lambda$ denotes the full subcategory formed by the indecomposable modules. Then $\mathrm{rad}(\mathrm{mod}\,\Lambda)$ denotes the radical of $\mathrm{mod}\,\Lambda$, that is, the ideal in $\mathrm{mod}\,\Lambda$ generated by all noninvertible morphisms between modules in $\mathrm{ind}\,\Lambda$. The infinite radical $\mathrm{rad}^{\infty}(\mathrm{mod}\,\Lambda)$ of $\mathrm{mod}\,\Lambda$ is the intersection of all powers $\mathrm{rad}^n(\mathrm{mod}\,\Lambda)$, $n \geq 1$, of $\mathrm{rad}(\mathrm{mod}\,\Lambda)$. We denote by Γ_Λ the AR-quiver of Λ and by τ_Λ and τ_Λ^- the AR-translations $D\,\mathrm{Tr}$ and $\mathrm{Tr}\,D$, with D the ordinary duality and Tr the transpose. We shall not distinguish between an indecomposable Λ-module and the vertex of Γ_Λ corresponding to it. Following [**21**], a component $\mathscr{C}$ of Γ_Λ is said to be *generalized standard* if $\mathrm{rad}^{\infty}(X, Y) = 0$ for all X and Y in $\mathscr{C}$. Finally, a full subquiver Σ of Γ_Λ is said to be *directed* (respectively, *almost directed*) if all modules from Σ (respectively, all but finitely many modules from Σ) do not lie on oriented cycles in Γ_Λ.

Recall that a module M in Γ_Λ is called *left stable* (respectively, *right stable*) if $\tau_\Lambda^n M$ (respectively, $\tau_\Lambda^{-n} M$) is defined for all $n \in \boldsymbol{N}$. Moreover, M is called *stable* if it is both left stable and right stable. For a component $\mathscr{C}$ of Γ_Λ, we denote by $\mathscr{C}_l$ (respectively, $\mathscr{C}_r$) the *left stable part* (respectively, *right stable part*) of $\mathscr{C}$ obtained by deleting in $\mathscr{C}$ the τ-orbits of projective (respectively, injective) modules, and by $\mathscr{C}_s$ the *stable part* obtained by deleting in $\mathscr{C}$ the nonstable modules and arrows attached to them. By a left stable component (respectively, right stable component, stable component) of Γ_A we mean a connected component of $(\Gamma_A)_l$ (respectively, $(\Gamma_A)_r, (\Gamma_A)_s$). We refer to [**14**] and [**26**] (respectively, [**15**]) for the shapes of stable (respectively, left stable and right stable) components of Γ_A.

For M and N in $\operatorname{ind} A$ a path from M to N in $\operatorname{ind} A$ is given by a sequence of morphisms

$$M = X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_t \xrightarrow{f_t} X_{t+1} = N$$

where, for each i, X_i is in $\operatorname{ind} A$ and each f_i is a nonzero nonisomorphism, and $t \geq 1$. Then M is said to be a *predecessor* of N and N a *successor* of M in $\operatorname{ind} A$. If the morphisms f_i are in addition irreducible, then the path is a *path of irreducible morphisms*. If M is isomorphic to N, then the path is an *oriented cycle* in $\operatorname{ind} A$. Every indecomposable A-module M is also called its own (trivial) predecessor and successor.

Let $Z_1 \to Z_2 \to \cdots \to Z_n \to Z_{n+1}$ be a path of irreducible morphisms in $\operatorname{ind} A$. If $Z_{i-1} \simeq \tau Z_{i+1}$ for some i with $2 \leq i \leq n$, we say that Z_i is a *hook* of the path. The path is said to be *sectional* if it has no hook. It is well known (see [**1**, (VII.2.4)]) that the composition of irreducible morphisms forming a sectional path is nonzero.

We shall prove a technical fact playing a crucial role in the proof of the main result of Section 4.

Let A be a finite dimensional algebra over an algebraically closed field K and $\mathscr{C}$ a directed generalized standard component of Γ_A. For each indecomposable module X in $\mathscr{C}$ we fix irreducible morphisms $f_i^X : X \to E_i^X$, $1 \leq i \leq m_X$, where $E_1^X, \ldots, E_{m_X}^X$ are indecomposable modules from $\mathscr{C}$ (not necessarily nonisomorphic) such that

$$f = (f_1^X, \ldots, f_{m_X}^X)^t : X \to E_1^X \oplus \cdots \oplus E_{m_X}^X$$

is a minimal left almost split morphism in $\operatorname{mod} A$. Let M and N be indecomposable modules in $\mathscr{C}$ such that there is a path of irreducible morphisms in $\mathscr{C}$ from M to N and every such a path is sectional. Denote by $\mathscr{S}(M,N)$ the set of all (sectional) paths of irreducible morphisms of the forms

$$M = X_1 \xrightarrow{f_{i_1}^{X_1}} X_2 \xrightarrow{f_{i_2}^{X_2}} \cdots \longrightarrow X_r \xrightarrow{f_{i_r}^{X_r}} X_{r+1} = N$$

with $1 \leq i_s \leq m_{X_s}$, $1 \leq s \leq r$. Observe that the compositions $f_{i_r}^{X_r} \cdots f_{i_1}^{X_1}$ are nonzero (see [**1**, (VII.2.4)]). Moreover, since $\mathscr{C}$ is generalized standard, there is a natural number n such that $\operatorname{rad}^n(M,N) = \operatorname{rad}^\infty(M,N) = 0$ (see [**1**, (V.7.2)]), and consequently $\mathscr{S}(M,N)$ is finite. Denote by $\max(M,N)$ the maximum of length of paths in $\mathscr{S}(M,N)$. Finally, we note that $\operatorname{End}_A(X) \simeq K$ for any indecomposable module X in $\mathscr{C}$, because $\mathscr{C}$ is generalized standard and directed. Then the following fact holds.

Proposition 1.1. *Let $\mathscr{C}$ be a directed generalized standard component of Γ_A, M and N indecomposable modules in $\mathscr{C}$, $\varphi_1, \ldots, \varphi_t$ be the compositions of irreducible morphisms forming pairwise different paths $p_1, \ldots, p_t$ in $\mathscr{S}(M,N)$ and let $\xi_1, \ldots, \xi_t$ be automorphisms of N. Then the sum $\xi_1\varphi_1 + \cdots + \xi_t\varphi_t$ is nonzero.*

Proof. We prove the claim by induction on $\max(M,N)$. We may assume that $t \geq 2$, because the composition of irreducible morphisms on a sectional path is nonzero.

Assume $\max(M,N)=1$. Then $\mathscr{S}(M,N)$ consists of irreducible morphisms $f_{i_1}^M, f_{i_2}^M, \dots, f_{i_q}^M$, $1 \le i_1 < i_2 < \cdots < i_q \le m_M$, such that the residue classes

$$f_{i_1}^M + \operatorname{rad}^2(M,N), f_{i_2}^M + \operatorname{rad}^2(M,N), \dots, f_{i_q}^M + \operatorname{rad}^2(M,N)$$

form a basis of $\operatorname{rad}(M,N)/\operatorname{rad}^2(M,N)$ over $K = \operatorname{End}_\Lambda(N)$. Since $\varphi_1, \dots, \varphi_t \in \mathscr{S}(M,N)$ are pairwise different, we obtain $\xi_1\varphi_1 + \cdots + \xi_t\varphi_t \ne 0$.

Assume $\max(M,N) \ge 2$. Observe that there is a minimal right almost split morphism of the form $A \oplus B \xrightarrow{(g,h)} N$, where $A = A_1 \oplus \cdots \oplus A_s$, the modules A_i are indecomposable, $g = (g_1, \dots, g_s)$ and the irreducible morphisms $g_1 : A_1 \to N, \dots, g_s : A_s \to N$ form a complete family of the final irreducible morphisms $f_{i_r}^{X_r} : X_r \to N$ on the paths of irreducible morphisms in $\mathscr{S}(M,N)$. Clearly, there are no paths of irreducible morphisms from M to indecomposable direct summands of B. We note also that $\operatorname{End}_\Lambda(N) \cong K \cong \operatorname{End}_\Lambda(A_i)$, $1 \le i \le s$. In particular, we have $\xi_i = \lambda_i \operatorname{id}_N$ with $\lambda_i \in K \backslash \{0\}$, for $i \in \{1, \dots, t\}$. For each $i \in \{1, \dots, t\}$, there is $j(i) \in \{1, \dots, s\}$ such that $\varphi_i = g_{j(i)}\varphi_i'$ for φ_i' being the composition of irreducible morphisms on a path from $\mathscr{S}(M, A_{j(i)})$. Then consider $\xi_i' = \lambda_i \operatorname{id}_{A_{j(i)}} \in \operatorname{End}_\Lambda(A_{j(i)})$. For $j \in \{1, \dots, s\}$, denote by $\psi_j : M \to A_j$ the sum of all morphisms $\xi_i'\varphi_i'$ with $j(i) = j$, and put $\psi = [\psi_1, \dots, \psi_s]^t : M \to A$. Then we have $\varphi = \xi_1\varphi_1 + \cdots + \xi_t\varphi_t = g\psi$. Note that all paths of irreducible morphisms from M to the modules $A_1, \dots, A_s$ are sectional and $\max(M, A_j) < \max(M,N)$ for any $j \in \{1, \dots, s\}$. For the inductive step, we may assume that the proposition holds for the compositions of irreducible morphisms of sectional paths from $\mathscr{S}(M, A_1), \dots, \mathscr{S}(M, A_s)$ and automorphisms of $A_1, \dots, A_s$. Then $\psi \ne 0$. Suppose $\varphi = g\psi = 0$. Then N is nonprojective and there is an almost split sequence of the form

$$0 \longrightarrow \tau_\Lambda N \xrightarrow{(u,v)^t} A \oplus B \xrightarrow{(g,h)} N \longrightarrow 0,$$

and a nonzero morphism $\psi' : M \to \tau_\Lambda N$ such that $\psi = u\psi'$. Since the component $\mathscr{C}$ is generalized standard, ψ' is a sum of compositions of irreducible morphisms between indecomposable modules in $\mathscr{C}$. Therefore, there is a nonsectional path of irreducible morphisms

$$M \to \cdots \to \tau_\Lambda N \to A_1 \to N$$

from M to N, a contradiction. Hence $\xi_1\varphi_1 + \cdots + \xi_t\varphi_t = g\psi \ne 0$. $\square$

2. Almost directed components.

In this section we introduce some concepts playing a fundamental role in our further investigations.

Let Λ be an artin algebra and let $\mathscr{C}$ be a component of Γ_Λ. Recall that following [**15**], [**20**], a full connected valued subquiver Δ of $\mathscr{C}$ is called a *section* of $\mathscr{C}$ if Δ is directed, convex in $\mathscr{C}$ and intersects each τ_Λ-orbit in $\mathscr{C}$ exactly once. We introduce the following more general concept. A full connected valued subquiver Δ of $\mathscr{C}$ is said to be a *multisection* in $\mathscr{C}$ if the following conditions are satisfied:

(1) Δ is almost directed.

(2) Δ is convex in $\mathscr{C}$.

(3) For each τ_Λ-orbit $\mathcal{O}$ in $\mathscr{C}$ we have $1 \le |\Delta \cap \mathcal{O}| < \infty$.

(4) $|\Delta \cap \mathcal{O}| = 1$ for all but finitely many τ_Λ-orbits $\mathcal{O}$ in $\mathscr{C}$.

(5) No proper full convex valued subquiver of Δ satisfies the conditions (1)–(4).

We also note that the double sections, defined in [**18**], are special types of multisections.

For a multisection Δ in a component $\mathscr{C}$ of Γ_Λ, we consider also the following full valued subquivers of $\mathscr{C}$:

$$\Delta_l' = \{X \in \Delta;\ \text{there is a nonsectional path } X \to \cdots \to P \text{ with } P \text{ projective}\},$$

$$\Delta_r' = \{X \in \Delta;\ \text{there is a nonsectional path } I \to \cdots \to X \text{ with } I \text{ injective}\},$$

$$\Delta_l'' = \{X \in \Delta_l';\ \tau_\Lambda^- X \notin \Delta_l'\},$$

$$\Delta_r'' = \{X \in \Delta_r';\ \tau_\Lambda X \notin \Delta_r'\},$$

$$\Delta_l = (\Delta \backslash \Delta_r') \cup \tau_\Lambda \Delta_r'',$$

$$\Delta_r = (\Delta \backslash \Delta_l') \cup \tau_\Lambda^- \Delta_l'',$$

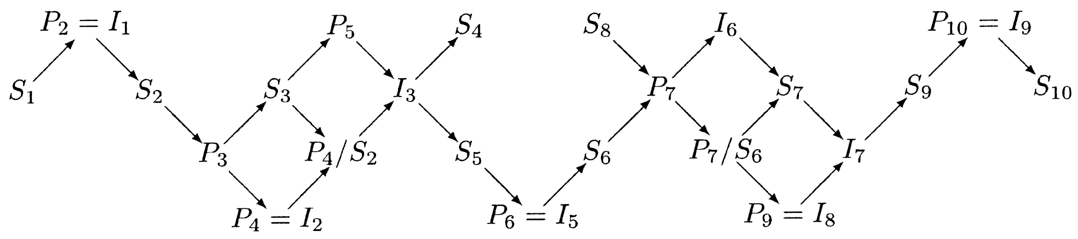
$$\Delta_c = \Delta_l' \cap \Delta_r' \ \text{(the core of } \Delta).$$

The following example shows that in general Δ_l and Δ_r are not subquivers of Δ.

Example 2.1. Let Λ be the bound quiver algebra KQ/I, where KQ is the paths algebra of the quiver

$$Q:\quad \begin{tikzcd} & & 4 \arrow[d, "\gamma"] & & & 8 & & \\ 1 & 2 \arrow[l, "\alpha"'] & 3 \arrow[l, "\beta"'] & 5 \arrow[l, "\sigma"'] & 6 \arrow[l, "\delta"'] & 7 \arrow[l, "\xi"'] \arrow[u, "\eta"'] & 9 \arrow[l, "\varrho"'] & 10 \arrow[l, "\nu"'] \end{tikzcd}$$

over a field K and I is the ideal in KQ generated by $\alpha\beta, \beta\sigma, \sigma\delta, \delta\xi, \xi\varrho$ and $\varrho\nu$. Then Γ_Λ is of the form



where S_i, P_i, I_i, $1 \le i \le 10$, denote the simple, projective, injective module given by the vertex i. Since the modules $P_2 = I_1$ and $P_{10} = I_9$ are projective-injective, the full translation subquiver Δ of Γ_Λ formed by all modules of Γ_Λ except the simple modules S_1, S_4, S_8 and S_{10} is the unique multisection of Γ_Λ. Moreover, we have $\Delta_l' = \Delta \backslash \{I_8, I_7, S_9, I_9\}$, $\Delta_r' = \Delta \backslash \{P_2, S_2, P_3, P_4\}$, and hence $\Delta_c = \{S_3, P_5, P_4/S_2, I_3, S_5, P_6, S_6, P_7, I_6, P_7/S_6, S_7\}$. Then $\Delta_l'' = \{I_1, I_2, P_5, I_3, I_5, I_6, P_7/S_6, S_7\}$, and $\Delta_r'' = \{S_3, P_4/S_2, P_5, P_6, P_7, I_6, P_9, P_{10}\}$. Therefore, Δ_l is a disjoint union of the subquivers $P_2 \to S_2 \to P_3 \to P_4$ and S_8, and Δ_r is a disjoint union of the subquivers S_4 and $I_8 \to I_7 \to S_9 \to I_9$. Finally, observe that S_4 and S_8 do not belong to Δ, and hence Δ_l and Δ_r are not subquivers of Δ.

The Example 2.9 below shows that we may have $\Delta_l' = \Delta_r' = \Delta = \Gamma_A$, and hence Δ_l and Δ_r empty. In fact, it is the case for all connected nonsimple selfinjective algebras of finite representation type.

The following property of multisections will be essential for our further considerations.

Lemma 2.2. *Let Δ be a multisection in a component $\mathscr{C}$ of Γ_A and $\mathscr{O}$ a τ_A-orbit in $\mathscr{C}$ such that $\Delta \cap \mathscr{O} = \{\tau_A^{m-1}X, \dots, X\}$ for some $m \geq 2$. Then there exist in Δ paths $I \to \cdots \to \tau_A^{m-1}X$ and $X \to \cdots \to P$ with I injective and P projective.*

Proof. Observe first that if there is in $\mathscr{C}$ a path $I \to \cdots \to \tau_A^{m-1}X$ with I injective then it lies entirely in Δ, because Δ contains a module from the τ_A-orbit of I and is convex in $\mathscr{C}$. Similarly, if $\mathscr{C}$ admits a path $X \to \cdots \to P$ with P projective then it lies entirely in Δ.

Suppose $\tau_A^{m-1}X$ is not a successor of an injective module in $\mathscr{C}$. Let Ω be the family of all modules in Δ which are predecessors of $\tau_A^{m-1}X$ in $\mathscr{C}$ (equivalently, in Δ). Then it follows from our assumption that Ω consists of noninjective modules. Denote by Σ the full valued subquiver of $\mathscr{C}$ given by all modules from $\Delta \backslash \Omega$ and $\tau_A^- \Omega$. Clearly, Σ is a full connected valued subquiver of $\mathscr{C}$ satisfying the conditions (1)–(4) of a multisection. Moreover, every module of $\tau_A^- \Omega$ is a predecessor of $\tau_A^{m-2}X$ (because the paths from modules in Ω to $\tau_A^{m-1}X$ consist of noninjective modules) and hence belongs to Δ, again by the convexity of Δ in $\mathscr{C}$. Therefore, Σ is a convex valued subquiver of Δ such that $\mathscr{O} \cap \Sigma = \{\tau_A^{m-2}X, \dots, X\}$, which contradicts the minimality of Δ. This shows that $\tau_A^{m-1}X$ is a successor of an injective module in $\mathscr{C}$. Similarly, we prove that X is a predecessor of a projective module in $\mathscr{C}$. □

As a consequence we obtain the following characterization of sections.

Corollary 2.3. *Let Δ be a multisection of a component $\mathscr{C}$ of Γ_A. The following statements are equivalent*:

(i) *Δ is a section of $\mathscr{C}$.*
(ii) *$\Delta = \Delta_l$.*
(iii) *$\Delta = \Delta_r$.*

Proof. Assume that Δ is a section of $\mathscr{C}$. We claim that $\Delta_r' = \varnothing = \Delta_l'$, and consequently $\Delta_l = \Delta = \Delta_r$. Indeed, suppose Z is a module in Δ_r'. Then there exists a path $I \to \cdots \to \tau_A X \to Y \to X \to \cdots \to Z$ in $\mathscr{C}$ with I injective. Since Δ is a section, I and Z belong to Δ, then $\{\tau_A X, X\}$ is contained in Δ, by the convexity of Δ in $\mathscr{C}$. This contradicts the fact that Δ intersects each τ_A-orbit in $\mathscr{C}$ exactly once. Therefore, $\Delta_r' = \varnothing$. Similarly, we prove that $\Delta_l' = \varnothing$.

Conversely, assume that $\Delta_l = \Delta$ (respectively, $\Delta_r = \Delta$) holds. In order to prove that Δ is a section of $\mathscr{C}$, it is enough to show that $|\Delta \cap \mathscr{O}| = 1$ for every τ_A-orbit $\mathscr{O}$ in $\mathscr{C}$. Suppose $\mathscr{O}$ is a τ_A-orbit in $\mathscr{C}$ such that $\Delta \cap \mathscr{O} = \{\tau_A^{m-1}X, \dots, X\}$ for some $m \geq 2$. It follows from Lemma 2.2 that there exist in $\mathscr{C}$ paths $I \to \cdots \to \tau_A^{m-1}X$ and $X \to \cdots \to P$ with I injective and P projective. Hence we obtain a path in $\mathscr{C}$ of the form

$$I \to \cdots \to \tau_A^{m-1}X \to Y_{m-1} \to \tau_A^{m-2}X \to \cdots \to \tau_A X \to Y_1 \to X \to \cdots \to P.$$

This implies $\{\tau_A^{m-2}X, \ldots, X\} \subseteq \Delta_r'$ and $\{\tau_A^{m-1}X, \ldots, \tau_A X\} \subseteq \Delta_l'$. But then $\tau_A^{m-2}X \in \Delta \backslash \Delta_l$ and $\tau_A X \in \Delta \backslash \Delta_r$, a contradiction. □

The following proposition describes basic properties of multisections.

PROPOSITION 2.4. *Let $\mathscr{C}$ be a component of Γ_A with a multisection Δ. Then we have the following statements.*

(i) *Every cycle in $\mathscr{C}$ lies in Δ_c.*

(ii) *Δ_c is finite.*

(iii) *Every indecomposable module M in $\mathscr{C}$ is in Δ_c or is a predecessor of Δ_l or a successor of Δ_r in $\mathscr{C}$.*

PROOF. (i) Let $X = X_1 \to X_2 \to \cdots \to X_m \to X_{m+1} = X$ be an oriented cycle in $\mathscr{C}$. We note first that the cycle $X = X_1 \to \cdots \to X_m = X$ is not sectional (see [**1**, (VII.2.6)]). Then for any natural numbers r, s such that the modules $\tau_A^r X_i$ and $\tau_A^{-s} X_i$ are nonzero for all $i \in \{1, 2, \ldots, m\}$, all modules $\tau_A^t X_i$, $i \in \{1, 2, \ldots, m\}$, $-s \le t \le r$, lie on one oriented cycle in $\mathscr{C}$. Suppose first that the modules $X_1, \ldots, X_m$ are stable, and let Γ be a stable component of Γ_A containing these modules. Then it follows from Zhang's theorem [**26**] (see also [**14**, (2.7)]) that Γ consists of τ_A-periodic modules, and consequently is either a stable tube $\boldsymbol{Z}\boldsymbol{A}_\infty/(\tau^r)$, $r \ge 1$, or of the form $\boldsymbol{Z}\Omega/G$ for a Dynkin quiver Ω and an admissible group G of automorphisms of $\boldsymbol{Z}\Omega$ (see [**8**] or [**1**, (VII.4.1)]). Since the multisection Δ intersects each τ_A-orbit in $\mathscr{C}$ and is convex in $\mathscr{C}$, we conclude that Γ is contained in Δ. Moreover, since Δ is almost directed, we obtain that Γ is a finite quiver of the form $\boldsymbol{Z}\Omega/G$. Further, $\Gamma \ne \mathscr{C}$, because otherwise $\mathscr{C}$ is the Auslander-Reiten quiver of a connected part of A and is stable. Therefore, there is an indecomposable module Z in Γ which is a direct predecessor of a projective module or a direct successor of an injective module in $\mathscr{C}$. It is well known that any left stable (respectively, right stable) component of Γ_A containing a τ_A-periodic module is in fact stable. Hence, $\mathscr{C}$ contains a finite nonperiodic τ_A-orbit connected in $\mathscr{C}$ to a (periodic) τ_A-orbit of Γ. Then for any indecomposable module Y in Γ there are nonsectional paths $I \to \cdots \to Y$ and $Y \to \cdots \to P$ with I injective and P projective, and hence Y belongs to Δ_c. Consequently, Γ is contained in Δ_c. In particular, the modules $X_1, \ldots, X_m$ lie in Δ_c. Assume now that one of the modules $X_1, \ldots, X_m$ is not stable. We claim that there exist $i, j \in \{1, \ldots, m\}$ and nonnegative integers p and q such that $\tau_A^p X_i$ is projective, $\tau_A^{-q} X_j$ is injective, and all modules $\tau_A^s X_k$, $-q \le s \le p$, $1 \le k \le m$, are nonzero. Without loss of generality, we may assume that one of the modules $X_1, \ldots, X_m$ is not right stable. Let q be the minimal natural number such that $\tau_A^{-q} X_j$ is injective for some $j \in \{1, \ldots, m\}$. Suppose the modules $X_1, \ldots, X_m$ are left stable and let $\mathscr{D}$ be the left stable component of Γ_A containing these modules. Since X_j is not stable, $\mathscr{D}$ is not stable and so does not contain a τ_A-periodic module. Then it follows from [**15**, (2.3)] that there exists an infinite sectional path

$$\cdots \to \tau_A^{2r} Y_1 \to \tau_A^r Y_s \to \cdots \to \tau_A^r Y_1 \to Y_s \to \cdots \to Y_1$$

with $r > s$ such that $Y_1, \ldots, Y_s$ is a complete set of representatives of the τ_A-orbits in $\mathscr{D}$, the module Y_s injective, and the modules Y_s and $\tau_A^m Y_1, \ldots, \tau_A^m Y_s$, $m \ge r$, lie on common oriented cycles in $\mathscr{D}$. Since Δ contains a module from the τ_A-orbit of Y_s, the convexity of Δ implies that all modules $\tau_A^m Y_j$, $m \geq r$, $1 \leq j \leq s$, belong to Δ. But then Δ contains infinitely many modules lying on oriented cycles in $\mathscr{C}$, a contradiction. Therefore, one of the modules $X_1, \dots, X_m$ is not left stable, and so there exists a minimal natural number p such that $\tau_A^p X_i$ is projective for some $i \in \{1, \dots, m\}$. Since all modules $\tau_A^t X_i$, $i \in \{1, 2, \dots, m\}$, $-q \leq t \leq p$, lie on a common oriented cycle, and this cycle is not sectional, we conclude that for any module X_k, $k \in \{1, 2, \dots, m\}$, there exist nonsectional paths $\tau_A^{-q} X_j \to \cdots \to X_k$ and $X_k \to \cdots \to \tau_A^p X_i$ with $\tau_A^{-q} X_j$ injective and $\tau_A^p X_i$ projective. Therefore, all modules $X = X_1, X_2, \dots, X_m$ lie in Δ_c.

(ii) Since by (i) every cycle in $\mathscr{C}$ lies in $\Delta_c \subseteq \Delta$ and Δ is almost directed we conclude that $\mathscr{C}$ is also almost directed. Suppose that Δ_c is infinite. Clearly then Δ and $\mathscr{C}$ are also infinite. Invoking now the conditions (3) and (4) for a multisection, we conclude that the stable part $\mathscr{C}_s$ of $\mathscr{C}$ contains a connected component Γ of the form $\boldsymbol{Z}\Sigma$, for a directed connected full valued subquiver Σ of Δ containing infinitely many modules from Δ_c. Observe also that $\mathscr{C}$ can be obtained from the connected components of $\mathscr{C}_s$ by gluing along the nonstable τ_A-orbit of $\mathscr{C}$. Then the section Σ of Δ contains at most finitely many modules which are simultaneously the targets of paths in $\mathscr{C}$ with injective sources and the sources of paths in $\mathscr{C}$ with projective targets. Therefore, Σ contains at most finitely many modules from Δ_c, a contradiction. This shows that Δ_c is finite.

(iii) Let M be an indecomposable module in $\mathscr{C}$ and $\mathscr{O}$ the τ_A-orbit of M. Since Δ is a multisection in $\mathscr{C}$, we have $m = |\Delta \cap \mathscr{O}| \geq 1$. Let $\Delta \cap \mathscr{O} = \{\tau_A^{m-1} X, \dots, X\}$. Assume first that $m \geq 2$. Then it follows from Lemma 2.2 that there are paths $I \to \cdots \to \tau_A^{m-1} X$ and $X \to \cdots \to P$ with I injective and P projective. Hence $\{\tau_A^{m-2} X, \dots, X\} \subseteq \Delta_r'$, $\{\tau_A^{m-1} X, \dots, \tau_A X\} \subseteq \Delta_l'$, and $\{\tau_A^{m-2} X, \dots, \tau_A X\} \subseteq \Delta_c$. Observe that $\tau_A^m X \notin \Delta_r'$ and $\tau_A^- X \notin \Delta_l'$, because $\tau_A^m X$ and $\tau_A^- X$ do not belong to Δ. Moreover, if $\tau_A^{m-1} X \notin \Delta_r'$ (respectively, $X \notin \Delta_l'$) then $\tau_A^{m-1} X \in \Delta_l$ (respectively, $X \in \Delta_r$). Further, if $\tau_A^{m-1} X \in \Delta_r'$ then $\tau_A^{m-1} X \in \Delta_c$, and $\tau_A^m X \in \Delta_l$ provided $\tau_A^{m-1} X$ is nonprojective. Similarly, if $X \in \Delta_l'$ then $X \in \Delta_c$, and $\tau_A^- X \in \Delta_r$ provided X is noninjective.

Assume now $m = 1$ and $X \notin \Delta_c = \Delta_l' \cap \Delta_r'$. Observe that if $X \notin \Delta_l'$ and $X \notin \Delta_r'$ then $X \in \Delta_l \cap \Delta_r$. Suppose $X \in \Delta_l'$ and $X \notin \Delta_r'$. Since $\tau_A^- X \notin \Delta$, we have $X \in \Delta_l$, and $\tau_A^- X \in \Delta_r$, provided X is noninjective (and $X \in \Delta_r$ otherwise). Similarly, if $X \in \Delta_r'$ and $X \notin \Delta_l'$, then $X \in \Delta_r$, and $\tau_A X \in \Delta_l$ provided X is nonprojective (and $X \in \Delta_l$ otherwise). Summing up, we conclude that the module M lies in Δ_c or is a predecessor of a module from Δ_l or a successor of a module from Δ_r. □

Theorem 2.5. *Let $\mathscr{C}$ be a component of Γ_A. Then $\mathscr{C}$ is almost directed if and only if $\mathscr{C}$ admits a multisection Δ.*

Proof. The sufficiency part follows from Proposition 2.4. We shall prove the necessity part. Assume that $\mathscr{C}$ is almost directed. In order to prove that $\mathscr{C}$ admits a multisection it is enough to show that there exists a full connected valued subquiver Σ of $\mathscr{C}$ satisfying the conditions (1)–(4). Indeed, then any minimal full convex valued subquiver Δ of Σ satisfying the conditions (1)–(4) is a multisection in $\mathscr{C}$. If $\mathscr{C}$ is finite we may take $\Sigma = \mathscr{C}$. Therefore, assume that $\mathscr{C}$ is infinite.

Since $\mathscr{C}$ is infinite, then the left stable part $\mathscr{C}_l$ of $\mathscr{C}$ or the right stable part $\mathscr{C}_r$ of $\mathscr{C}$ is infinite. Assume $\mathscr{C}_l$ is infinite and let $\mathscr{D}$ be an infinite component of $\mathscr{C}_l$. We claim that $\mathscr{D}$ is directed. Suppose to the contrary that $\mathscr{D}$ contains an oriented cycle. If $\mathscr{D}$ is stable, then applying [**26**] we conclude that $\mathscr{D}$ is a stable tube and consequently contains infinitely many modules lying on oriented cycles in $\mathscr{D}$, a contradiction because $\mathscr{C}$ is almost directed. Thus assume (without loss of generality) that $\mathscr{D}$ contains an injective module. Then it follows from [**15**, (2.3)] that there exists an infinite sectional path

$$\cdots \to \tau_A^{2t} X_1 \to \tau_A^t X_s \to \cdots \to \tau_A^t X_1 \to X_s \to \cdots \to X_1$$

with $t > s$ such that $X_1, X_2, \ldots, X_s$ is a complete set of representatives of the τ_A-orbits in $\mathscr{D}$, and all modules $\tau_A^m X_1, \ldots, \tau_A^m X_s$, $m \geq t$, lie on oriented cycles in $\mathscr{D}$. But then again there are infinitely many modules in $\mathscr{D}$, and hence in $\mathscr{C}$, lying on oriented cycles. Therefore, $\mathscr{D}$ is directed. Applying now [**15**, (3.4)] we obtain that there exists a connected valued quiver Ω without oriented cycles such that $\mathscr{D}$ is isomorphic to a full translation subquiver of $\boldsymbol{Z}\Omega$ which is closed under predecessors. In particular, $\mathscr{D}$ admits a section isomorphic to Ω. Similarly, we prove that if $\mathscr{E}$ is an infinite component of $\mathscr{C}_r$ then $\mathscr{E}$ is directed and admits a section Θ.

Let $\mathscr{D}_1, \ldots, \mathscr{D}_p$ be the family of all infinite components of $\mathscr{C}_l$ and $\mathscr{E}_1, \ldots, \mathscr{E}_q$ the family of all infinite components of $\mathscr{C}_r$. It follows from our discussion above that $\mathscr{D}_1, \ldots, \mathscr{D}_p$, $\mathscr{E}_1, \ldots, \mathscr{E}_q$ are directed translation quivers and admit respectively sections $\Omega_1, \ldots, \Omega_p$, $\Theta_1, \ldots, \Theta_q$. Further, let $\mathscr{O}_1, \ldots, \mathscr{O}_m$ be all finite τ_A-orbits in $\mathscr{C}$. Choose a finite family $\mathscr{X}$ of modules in $\mathscr{C}$ intersecting each of the quivers $\Omega_1, \ldots, \Omega_p$, $\Theta_1, \ldots, \Theta_q$, $\mathscr{O}_1, \ldots, \mathscr{O}_m$ exactly once. Since $\mathscr{C}$ is a connected quiver, for any modules X and X' from $\mathscr{X}$, there is in $\mathscr{C}$ a finite walk $X = X_1 — X_2 — \cdots — X_{t-1} — X_t = X'$ connecting X and X'. Hence there exists a finite full valued subquiver $\mathscr{Y}$ in $\mathscr{C}$ such that $\mathscr{X} \subseteq \mathscr{Y}$ and any two modules Y and Y' in $\mathscr{Y}$ can be connected by a walk consisting entirely of modules from $\mathscr{Y}$. Observe that $\mathscr{C}$ can be obtained from the directed translation quivers $\mathscr{D}_1, \ldots, \mathscr{D}_p$ and $\mathscr{E}_1, \ldots, \mathscr{E}_q$ by gluing along (finitely many) finite τ_A-orbits. Moreover, $\mathscr{C}$ is locally finite, that is, any module in $\mathscr{C}$ is a source or target of at most finitely many arrows. Therefore, each of the directed quivers $\Omega_1, \ldots, \Omega_p$ (respectively, $\Theta_1, \ldots, \Theta_q$) contains at most finitely many modules which are sources or targets of paths in $\mathscr{C}$ with targets or sources in $\mathscr{Y}$. Let Σ be the convex hull of the full valued subquivers $\Omega_1, \ldots, \Omega_p$, $\Theta_1, \ldots, \Theta_q$, $\mathscr{O}_1, \ldots, \mathscr{O}_m$, and $\mathscr{Y}$ in $\mathscr{C}$. Then Σ satisfies the required conditions (1)–(4). This finishes the proof. □

As a consequence we obtain the following.

Corollary 2.6. *Let $\mathscr{C}$ be a component of Γ_A. Then $\mathscr{C}$ is directed if and only if $\mathscr{C}$ admits a directed multisection Δ.*

Recall that a family $\mathscr{X}$ of modules in $\operatorname{mod} A$ is called *faithful* if the intersection of the (right) annihilators $\operatorname{ann}_A(X) = \{a \in A; Xa = 0\}$ of all modules X in $\mathscr{X}$ is zero. It is well known (see [**19**, (2.4)]) that $\mathscr{X}$ is faithful if and only if there exist a monomorphism $A_A \to M$ and an epimorphism $N \to D(A)_A$ for some finite direct sums M and N of modules from $\mathscr{X}$. We have the following fact.

Lemma 2.7. *Let $\mathscr{C}$ be a component of Γ_A with a multisection Δ. Then $\mathscr{C}$ is faithful if and only if Δ is faithful.*

Proof. It is a straightforward extension of arguments applied in the proof of [**20**, Lemma 3], invoking the property (2.4)(iii) of a multisection. □

Our next aim is to introduce a numerical invariant $w(\Delta)$ of a multisection Δ in a component $\mathscr{C}$ of Γ_Λ, called the *width* of Δ. Let p be a path in Δ. We write the path p as an ordered sequence of modules associated with the vertices. A subpath $(M, Z^{(1)}, \tau_\Lambda^{-1}M, Z^{(2)}, \tau_\Lambda^{-2}M, \dots, Z^{(n)}, \tau_\Lambda^{-n}M)$ is called a *hook path* of length n (if $n \geq 1$), and it is a *maximal hook path* if it is not contained in any hook path of larger length. Associated with the path p is the following sequence of maximal hook paths (if there are any hook paths). Start with a maximal hook path $(M, Z^{(1)}, \tau_\Lambda^{-1}M, Z^{(2)}, \tau_\Lambda^{-2}M, \dots, Z^{(n)}, \tau_\Lambda^{-n}M)$, where M is the first module on p which is the start of a hook subpath of p. Then take the maximal hook subpath of p with the start at the first possible successor of $\tau_\Lambda^{-n}M$ on p, etc. Denote by $i(p)$ the sum of the length of these hook subpaths of p. In particular we have $i(p) = 0$ if and only if p is sectional. We define the *width* $w(\Delta)$ of Δ to be the maximum of $i(p) + 1$ for all paths p in Δ. Observe that if Δ is a section [**15**] (respectively, double section [**18**]) in $\mathscr{C}$ then $w(\Delta) = 1$ (respectively, $w(\Delta) = 2$). Observe also that Δ is directed if and only if $w(\Delta) < \infty$. We say that a multisection Δ in $\mathscr{C}$ is an *n-section* if $n = w(\Delta)$.

The following examples show that for any $n \in N \cup \{\infty\}$ there exists an AR-component $\mathscr{C}$ having a multisection Δ with $w(\Delta) = n$.

Example 2.8. Let Λ the bound quiver algebra KQ/I, where KQ is the path algebra of the quiver

$$Q : 1 \to 2 \to \cdots \to n \to n+1 \to n+2$$

over a field K and I the ideal in KQ generated by all paths in Q of length 2. Then the AR-quiver Γ_Λ of Λ is of the form

$$\begin{tikzcd}[column sep=small]
 & P_{n+1} \arrow[dr] & & P_n \arrow[dr] & & P_1 \arrow[dr] & \\
S_{n+2} \arrow[ur] & & S_{n+1} \arrow[ur] & & \cdots \arrow[ur] & & S_1
\end{tikzcd}$$

where S_i and P_i, $1 \leq i \leq n+2$, denote the simple and indecomposable projective module at the vertex i, respectively. Then the full subquiver Δ of Γ_Λ formed by all vertices of Γ_Λ except S_{n+2} and S_1 is a multisection with $w(\Delta) = n$. We also note that $\operatorname{gl.dim} \Lambda = n+1$.

Example 2.9. Let K be a field and $\Lambda = K[X]/(X^2)$. Then Γ_Λ is of the form

$$S \rightleftarrows \Lambda$$

where $S = K[X]/(X)$. Moreover, $\Delta = \Gamma_\Lambda$ is a multisection in Γ_Λ, $w(\Delta) = \infty$ and $\operatorname{gl.dim} \Lambda = \infty$.

Our next example shows that an almost directed AR-component admits usually many multisections.

Example 2.10. Let Λ be the bound quiver algebra KQ/I, where KQ is the path algebra of the quiver

$$Q : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xleftarrow{\gamma} 4 \xleftarrow{\sigma} 5 \xleftarrow{\delta} 6 \xleftarrow{\varepsilon} 7 \xrightarrow{\eta} 8$$

over a field K and I is the ideal in KQ generated by $\gamma\sigma, \sigma\delta$ and $\delta\varepsilon$. Then Γ_A is of the form

$$\begin{tikzcd}[column sep=small]
 & P_4 \arrow[dr] & & S_2 \arrow[dr] & & S_1 & & & & & & \\
S_3 \arrow[ur] \arrow[dr] & & M \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] & & & & S_8 \arrow[dr] & & I_6 \arrow[dr] & \\
 & P_2 \arrow[ur] \arrow[dr] & & I_3 \arrow[ur] \arrow[dr] & & & & & & P_7 \arrow[ur] \arrow[dr] & & S_7 \\
 & & P_1 \arrow[ur] & & S_4 \arrow[dr] & & S_5 \arrow[dr] & & S_6 \arrow[ur] & & I_8 \arrow[ur] & \\
 & & & & & P_5 \arrow[ur] & & P_6 \arrow[ur] & & & &
\end{tikzcd}$$

where $S_i, P_i, I_i,\ 1 \le i \le 8$, denote the simple, projective, injective module given by the vertex i. Then the following families of modules form all multisections of Γ_A:

$$\Delta^{(1)} = \mathscr{X} \cup \{P_4, M, I_6\}, \quad \Delta^{(2)} = \mathscr{X} \cup \{P_4, M, S_8\},$$

$$\Delta^{(3)} = \mathscr{X} \cup \{S_2, M, I_6\}, \quad \Delta^{(4)} = \mathscr{X} \cup \{S_2, M, S_8\},$$

$$\Delta^{(5)} = \mathscr{X} \cup \{S_2, I_2, I_6\}, \quad \Delta^{(6)} = \mathscr{X} \cup \{S_2, I_2, S_8\},$$

$$\Delta^{(7)} = \mathscr{X} \cup \{S_1, I_2, I_6\}, \quad \Delta^{(8)} = \mathscr{X} \cup \{S_1, I_2, S_8\},$$

where $\mathscr{X} = \{I_3, S_4, P_5, S_5, P_6, S_6, P_7\}$. Observe that, for each $i \in \{1, 2, \ldots, 8\}$, we have $w(\Delta^{(i)}) = 3$ and $\Delta_c^{(i)} = \{S_5\}$.

The following general fact shows that the width and the core of a multisection of an almost directed component $\mathscr{C}$ are in fact invariants of $\mathscr{C}$.

Proposition 2.11. *Let $\mathscr{C}$ be a component of Γ_A and Δ, Σ multisections in $\mathscr{C}$. Then $w(\Delta) = w(\Sigma)$ and $\Delta_c = \Sigma_c$.*

Proof. Observe that every module X lying on a path $I \to \cdots \to X \to \cdots \to P$ in $\mathscr{C}$ with I injective and P projective lies in $\Delta \cap \Sigma$. Consequently $\Delta_c = \Sigma_c$. Moreover, it follows from Lemma 2.2 that, if $p = (M, Z^{(1)}, \tau_A^- M, Z^{(2)}, \tau_A^{-2} M, \ldots, Z^{(n)}, \tau_A^{-n} M)$ is a hook path in Δ (respectively, in Σ), then M is a successor of an injective module and $\tau_A^{-n} M$ is a predecessor of a projective module in $\mathscr{C}$, and hence p is also a hook path in Σ (respectively, in Δ). Then the equality $w(\Delta) = w(\Sigma)$ also easily follows. $\square$

3. Generalized double tilted algebras.

We introduce here the class of generalized double tilted algebras, containing the classes of tilted algebras [**10**], double tilted algebras [**18**], and all algebras of finite representation type.

A connected artin algebra A is said to be a *generalized double tilted algebra* if the following conditions are satisfied:

(1) Γ_A admits a component $\mathscr{C}$ with a faithful multisection Δ.

(2) There exists a tilted factor algebra A_l of A (not necessarily connected) such that Δ_l is a disjoint union of sections of connecting components of the connected parts of A_l, and the category of all predecessors of Δ_l in $\operatorname{ind} A$ coincides with the category of all predecessors of Δ_l in $\operatorname{ind} A_l$.

(3) There exists a tilted factor algebra A_r of A (not necessarily connected) such that Δ_r is a disjoint union of sections of connecting components of the connected parts of A_r, and the category of all successors of Δ_r in $\operatorname{ind} A$ coincides with the category of all successors of Δ_r in $\operatorname{ind} A_r$.

In the above notation, A_l and A_r are said to be the *left tilted algebra* and the *right tilted algebra* of A, respectively. Moreover, $\mathscr{C}$ is said to be a *connecting component* of A. If Δ is a section of $\mathscr{C}$ then $A_l = A = A_r$ and hence A is a tilted algebra. Further, if $w(\Delta) \geq 2$, then $\mathscr{C}$ is the unique connecting component of A. We say that a connected artin algebra A is an *n-double tilted algebra* if Γ_A admits a component $\mathscr{C}$ with an n-section Δ satisfying the above conditions (1)–(3). It follows from Proposition 2.11 and properties of tilted algebras that $n = w(\Delta)$ does not depend on the choice of multisection Δ and connecting component $\mathscr{C}$. Observe that 1-double tilted algebras and 2-double tilted algebras are exactly tilted algebras [**10**] and double tilted algebras [**18**], respectively. Finally, we note that every connected artin algebra of finite representation type is a generalized double tilted algebra.

We have the following characterization of generalized double tilted algebras.

Theorem 3.1. *Let A be a basic connected artin algebra. The following conditions are equivalent*:

(i) *A is generalized double tilted.*

(ii) *Γ_A admits a faithful generalized standard almost directed component.*

(iii) *Γ_A admits a component $\mathscr{C}$ with a faithful multisection Δ such that $\operatorname{Hom}_A(X, \tau_A Y) = 0$ for all modules X from Δ_r and Y from Δ_l.*

Proof. It is a straightforward extension (invoking Theorem 2.5) of arguments applied in the proofs of [**20**, Theorem 3] and [**18**, Theorem 7.3]), where similar characterizations of tilted and double tilted algebras have been established. □

We obtain the following consequences.

Proposition 3.2. *Let A be a basic connected artin algebra. Then A is n-double tilted, for some $n \geq 2$, if and only if Γ_A admits a faithful generalized standard almost directed component with a nonsectional path from an injective module to projective module.*

Proposition 3.3. *Let A be a basic connected artin algebra. Then A is n-double tilted, for some $n \geq 3$, if and only if Γ_A admits a faithful generalized standard component with a multisection Δ such that $\Delta_c \neq \varnothing$.*

We have also the following results on the structure of the module category of a generalized double tilted algebra of infinite representation type.

Theorem 3.4. *Let A be a basic connected generalized double tilted algebra of infinite representation type which is not tilted, and let $\mathscr{C}$ be the connecting component of Γ_A.*

Then there are a hereditary artin algebra ${}_{\infty}H$ and a tilting ${}_{\infty}H$-module ${}_{\infty}T$ without nonzero preinjective direct summands, and a hereditary artin algebra H_{∞} and a tilting H_{∞}-module T_{∞} without nonzero preprojective directed summands such that the following statements hold:

(i) *The tilted algebras* ${}_{\infty}\Lambda = \mathrm{End}_{{}_{\infty}H}({}_{\infty}T)^{op}$ *and* $\Lambda_{\infty} = \mathrm{End}_{H_{\infty}}(T_{\infty})^{op}$ *are factor algebras of* Λ.

(ii) *The torsion-free part* $\mathscr{Y}({}_{\infty}T)$ *of* $\mathrm{mod}\,{}_{\infty}\Lambda$ *is a full exact subcategory of* $\mathrm{mod}\,\Lambda$ *which is closed under predecessors of indecomposable modules.*

(iii) *The torsion part* $\mathscr{X}(T_{\infty})$ *of* $\mathrm{mod}\,\Lambda_{\infty}$ *is a full exact subcategory of* $\mathrm{mod}\,\Lambda$ *which is closed under successors of indecomposable modules.*

(iv) ${}_{\infty}\Lambda$ *is a factor algebra of* ${}_{l}\Lambda$ *and* ${}_{\infty}\mathscr{C} = \mathscr{Y}({}_{\infty}T) \cap \mathscr{C}$ *is the torsion-free part of the family* $\mathscr{C}_{{}_{\infty}T}$ *of the connecting components of* ${}_{\infty}\Lambda$ *and is also a full translation subquiver of* $\mathscr{C}$ *which is closed under predecessors in* $\mathscr{C}$.

(v) Λ_{∞} *is a factor algebra of* Λ_r *and* $\mathscr{C}_{\infty} = \mathscr{X}(T_{\infty}) \cap \mathscr{C}$ *in the torsion part of the family* $\mathscr{C}_{T_{\infty}}$ *of the connecting components of* Λ_{∞} *and is also a full translation subquiver of* $\mathscr{C}$ *which is closed under successors in* $\mathscr{C}$.

(vi) $\mathscr{Y}({}_{\infty}T)$ *and* $\mathscr{X}(T_{\infty})$ *have no common nonzero modules.*

(vii) *The family of indecomposable* Λ*-modules which are neither in* $\mathscr{Y}({}_{\infty}T)$ *nor in* $\mathscr{X}(T_{\infty})$ *is finite and coincides with the family of all indecomposable modules in* $\mathscr{C}$ *which are neither in* ${}_{\infty}\mathscr{C}$ *nor in* $\mathscr{C}_{\infty}$.

Proof. It is a direct extension of arguments applied in the proof of [**20**, Theorem 1], where the structure of the module categories of algebras having faithful generalized standard directed components has been established. □

We note that if the left tilted algebra Λ_l (respectively, the right tilted algebra Λ_r) of Λ has a finite torsion-free part $\mathscr{Y}(T_l)$ (respectively finite torsion part $\mathscr{X}(T_r)$) then ${}_{\infty}\Lambda$ (respectively Λ_{∞}) is zero. The known structure of AR-components of tilted algebras (see [**10**], [**11**], [**12**], [**16**], [**19**], [**25**]) and the above theorem lead to the following description of the AR-quivers of nontilted generalized double tilted algebras of infinite representation type.

Corollary 3.5. *Let* Λ *be a basic connected generalized double tilted algebra of infinite representation type, and assume that* Λ *is not tilted. Then we have the following.*

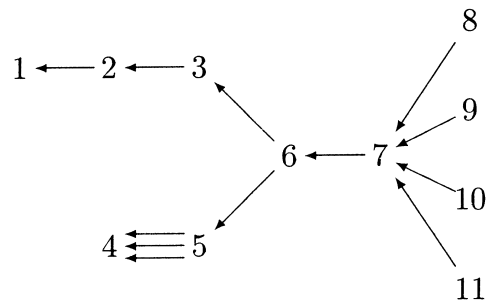
(i) *Every component of* Γ_{Λ} *different from the connecting component either lies entirely in* $\mathscr{Y}({}_{\infty}T)$ *or lies entirely in* $\mathscr{X}(T_{\infty})$.

(ii) *Every component of* Γ_{Λ} *contained in* $\mathscr{Y}({}_{\infty}T)$ *is either preprojective, a stable tube* $\boldsymbol{Z}\boldsymbol{A}_{\infty}/(\tau^m)$, *for* $m \geq 1$, *of the form* $\boldsymbol{Z}\boldsymbol{A}_{\infty}$, *or can be obtained from a stable tube or a component of type* $\boldsymbol{Z}\boldsymbol{A}_{\infty}$ *by a finite number of ray insertions* (*in the sense of* [**19**]).

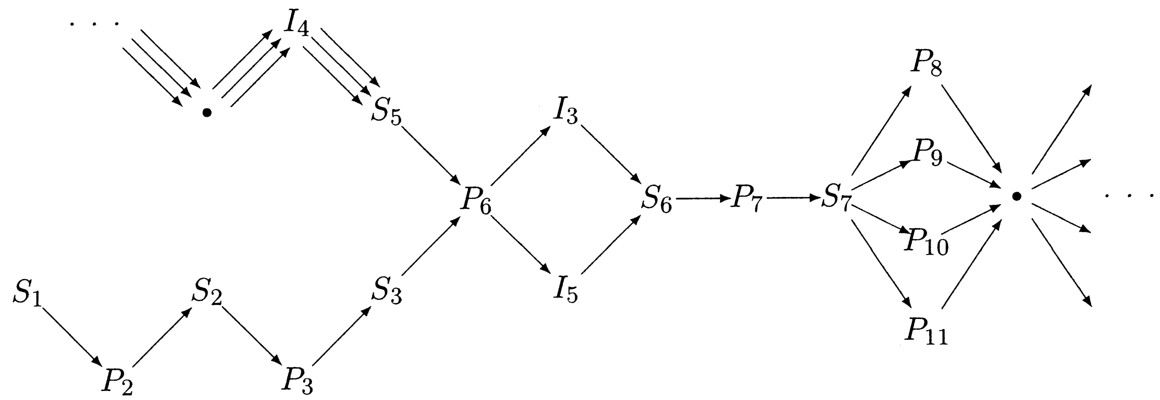
(iii) *Every component of* Γ_{Λ} *contained in* $\mathscr{X}(T_{\infty})$ *is either preinjective, a stable tube* $\boldsymbol{Z}\boldsymbol{A}_{\infty}/(\tau^m)$, *for* $m \geq 1$, *of the form* $\boldsymbol{Z}\boldsymbol{A}_{\infty}$, *or can be obtained from a stable tube or a component of type* $\boldsymbol{Z}\boldsymbol{A}_{\infty}$ *by a finite number of coray insertions* (*in the sense of* [**19**]).

We illustrate the above considerations with the following example.

Example 3.6. Let Q be the quiver



KQ the path algebra of Q over a field K, I the ideal in KQ generated by all paths in Q of length 2, and $\Lambda = KQ/I$. Then Γ_Λ admits a component $\mathscr{C}$ of the form



Observe that the full convex subquiver Δ of $\mathscr{C}$ given by the modules P_2, S_2, P_3, S_3, I_4, S_5, P_6, I_3, I_5, S_6, P_7, S_7, P_8, P_9, P_{10}, P_{11} is the unique multisection in $\mathscr{C}$ and $w(\Delta) = 4 = i(p) + 1$, for p being the path

$$S_2 \to P_3 \to S_3 \to P_6 \to I_5 \to S_6 \to P_7 \to S_7.$$

Further, Δ_l' (respectively, Δ_r') is the subquiver of Δ given by the modules P_2, S_2, P_3, S_3, I_4, S_5, P_6, I_3, I_5, S_6 (respectively, S_3, P_6, I_3, I_5, S_6, P_7, S_7, P_8, P_9, P_{10}, P_{11}), and hence $\Delta_c = \Delta_l' \cap \Delta_r'$ is the subquiver given by P_3, S_3, P_6, I_3, I_5, S_6. Moreover, Δ_l'' (respectively, Δ_r'') is given by the modules P_2, P_3, I_4, I_3, I_5, S_6 (respectively, P_3, S_3, P_6, I_3, P_7, P_8, P_9, P_{10}, P_{11}). Hence $\Delta_l = (\Delta \backslash \Delta_r') \cup \tau_\Lambda \Delta_r''$ is given by the modules P_2, S_2, I_4, S_5 and $\Delta_r = (\Delta \backslash \Delta_l') \cup \tau_\Lambda^- \Delta_l''$ is given by the modules P_7, S_7, P_8, P_9, P_{10}, P_{11}. Therefore, the left tilted algebra Λ_l of Λ is the product $H_1 \times H_2$, where H_1 is the path algebra of the full subquiver of Q given by the vertices 1 and 2 and H_2 is the path algebra of the full subquiver of Q given by the vertices 4 and 5, and the right tilted algebra Λ_r of Λ is the bound quiver algebra $K\Sigma/J$ with Σ being the full subquiver of Q given by the vertices 6, 7, 8, 9, 10, 11 and J the ideal in $K\Sigma$ generated by all paths in Σ of length 2. Finally, ${}_\infty\Lambda = H_2$ and Λ_∞ is the path algebra $K\Omega$ of the full subquiver Ω of Q given by the vertices 7, 8, 9, 10, 11. Thus the AR-quiver Γ_Λ of Λ consists of a preprojective component, an infinite family of components of type $\boldsymbol{Z}A_\infty$, the connecting component $\mathscr{C}$, a preinjective component and an infinite family of stable tubes (3 of them of the form $\boldsymbol{Z}A_\infty/(\tau^2)$ and the remaining ones of the form $\boldsymbol{Z}A_\infty/(\tau)$).

4. Global dimension.

Let Λ be a generalized double tilted algebra over an algebraically closed field K, $\mathscr{C}$ a connecting component of Γ_Λ and Δ a faithful multisection in $\mathscr{C}$. In particular, we have

$\mathrm{Hom}_\Lambda(D(\Lambda), \tau_\Lambda X) = 0$ for any predecessor X of Δ_l in $\mathrm{ind}\,\Lambda$ and $\mathrm{Hom}_\Lambda(\tau_\Lambda^- Y, \Lambda) = 0$ for any successor Y of Δ_r in $\mathrm{ind}\,\Lambda$. Then every predecessor of Δ_l in $\mathrm{ind}\,\Lambda$ has projective dimension at most one, and every successor of Δ_r in $\mathrm{ind}\,\Lambda$ has injective dimension at most one (see [**19**, (2.4)]). In particular, for all but finitely many indecomposable Λ-modules X we have $\mathrm{pd}_\Lambda X \le 1$ or $\mathrm{id}_\Lambda X \le 1$. On the other hand, as we have seen in Section 2, there are generalized double tilted algebras of arbitrary global dimension. The aim of this section is to establish a bound for the global dimension of a generalized double tilted algebra in terms of the width of a multisection. We need some notation.

For an indecomposable module N in the multisection Δ, we denote by $i(N)$ the maximum of the numbers $i(p)$, for all paths in Δ ending at N, as defined in Section 2. We note that if Δ is a section then $i(N) = 0$ for all modules N in Δ. We have the following easy observation.

Lemma 4.1. *Let N be a module in Δ with $i(N) > 0$. Then there exists a path p in Δ starting with an injective module and ending at N such that $i(N) = i(p)$.*

Proof. Assume $i(N) > 0$ and choose a path p' in Δ ending at N such that $i(p') = i(N)$. Let $M, Z^{(1)}, \tau_\Lambda^{-1} M, \ldots, Z^{(t)}, \tau_\Lambda^{-t} M$ be the first maximal hook subpath of p'. Then by Lemma 2.2 there is some path in Δ from an injective module to M, and composing it with the path p' we obtain the required path p, because $i(p) \ge i(p')$ and $i(p') = i(N)$ is maximal. □

Theorem 4.2. *Let Λ be an n-double tilted algebra. Then* $\mathrm{gl.dim}\,\Lambda \le n+1$.

Proof. Let Δ be an n-section in a connecting component $\mathscr{C}$ of Γ_Λ. We may assume that $n = \omega(\Delta) < \infty$, or equivalently (Corollary 2.6) that $\mathscr{C}$ is directed. We first prove that for any indecomposable module N in Δ we have $\mathrm{pd}_\Lambda N \le i(N) + 1$. Let N be a module from Δ. Assume first $i(N) = 0$. Then any path in Δ ending at N is sectional. Since Δ_l is a disjoint union of sections of connecting components of the connected parts of the left tilted algebra Λ_l of Λ and the injective Λ_l-modules are successors of Δ_l in $\mathrm{ind}\,\Lambda_l$, we infer that every injective predecessor of a module from $\mathscr{C}$ in $\mathrm{ind}\,\Lambda$ lies in $\mathscr{C}$. This implies that $\mathrm{Hom}_\Lambda(D(\Lambda), \tau_\Lambda N) = 0$, and consequently $\mathrm{pd}_\Lambda N \le 1$ (see [**19**, (2.4)]), because otherwise there would be a path in Δ of the form $I \to \tau_\Lambda N \to X \to N$, contradicting $i(N) = 0$. Assume $i(N) > 0$. For each indecomposable module X in $\mathscr{C}$, we fix irreducible morphisms $f_i^X : X \to E_i^X$, $1 \le i \le m_X$, where $E_1^X, \ldots, E_{m_X}^X$ are indecomposable modules from $\mathscr{C}$ (not necessarily nonisomorphic) such that

$$f = (f_1^X, \ldots, f_{m_X}^X)^t : X \to E_1^X \oplus \cdots \oplus E_{m_X}^X$$

is a minimal left almost split morphism in $\mathrm{mod}\,\Lambda$. Denote by $\mathscr{F}$ the family of all chosen irreducible morphisms f_i^X, $X \in \mathscr{C}$, $1 \le i \le m_X$. Consider an exact sequence

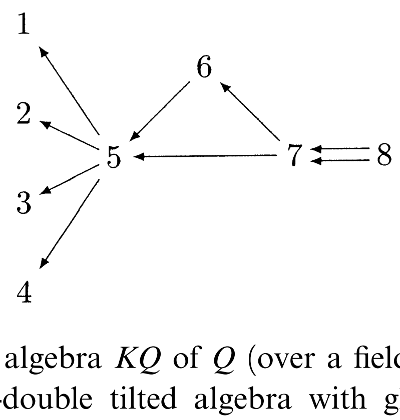
$$0 \to \Omega N \xrightarrow{u} P_N \xrightarrow{v} N \to 0$$

where $v : P_N \to N$ is a projective cover of N, and let $P_N = P_1 \oplus \cdots \oplus P_t$ for some indecomposable modules P_i. Let $v_j : P_j \to N$, $1 \le j \le t$, be the restrictions of v to the summands P_j. Since $\mathscr{C}$ is a generalized standard component, we may choose $v : P_N \to N$ such that, if P_j belongs to $\mathscr{C}$, then the morphism v_j is a scalar multiplication (by a nonzero element of $K = \mathrm{End}_\Lambda(N)$) of a composition of irreducible morphisms from the family $\mathscr{F}$. Let L be an indecomposable direct summand of ΩN. Assume L belongs to Δ. We claim that then there is a nonsectional path of irreducible morphisms from L to N. Without loss of generality we may assume that $\{1, \dots, r\}$ is the set of all $j \in \{1, \dots, t\}$ such that the composition $u_j : L \to P_j$ of the restriction of u to L with the canonical projection $P_N \to P_j$ is nonzero. Since the connecting component $\mathscr{C}$ of the generalized double tilted algebra Λ is convex in $\mathrm{ind}\,\Lambda$ (see also Theorem 3.4) the modules $P_1, \dots, P_r$ belong to $\mathscr{C}$ and, by our choice of v, the morphisms $v_1, \dots, v_r$ are nonzero scalar multiplications of irreducible morphisms on pairwise different paths of irreducible morphisms from $\mathscr{F}$. Since $\mathscr{C}$ is a generalized standard component, invoking the universal property of left almost split morphisms, we conclude that the morphisms $u_j : L \to P_j$, $1 \le j \le r$, are linear combinations of compositions of irreducible morphisms from $\mathscr{F}$. Then $v_1 u_1 + \cdots + v_r u_r$ is a linear combination of compositions of irreducible morphisms on pairwise different paths from L to N consisting of irreducible morphisms from $\mathscr{F}$. Since $v_1 u_1 + \cdots + v_r u_r = 0$, applying Proposition 1.1, we conclude that there is a nonsectional path of irreducible morphisms from L to N, and consequently $i(L) < i(N)$. Then by the inductive assumption we have $\mathrm{pd}_\Lambda L \le i(L) + 1$. If L is not in Δ, then L is a predecessor of Δ_l in $\mathrm{ind}\,\Lambda$, as a predecessor of the module $N \in \Delta$ in $\mathrm{ind}\,\Lambda$, and hence $\mathrm{pd}_\Lambda L \le 1$. Therefore, we obtain $\mathrm{pd}_\Lambda N \le i(N) + 1$.

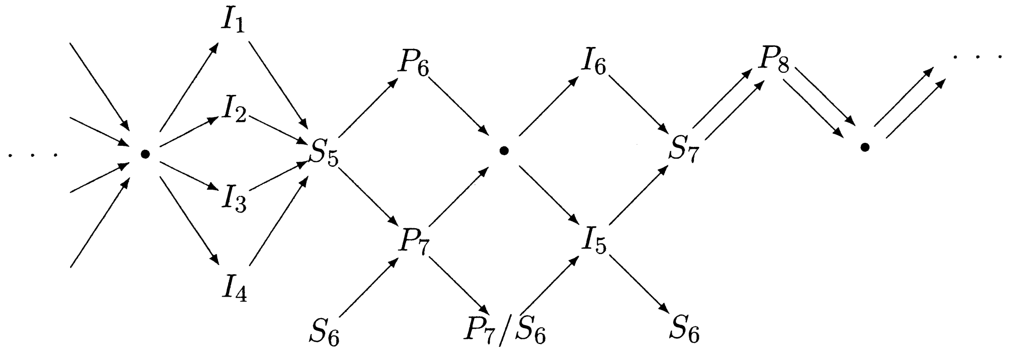
Let X be an arbitrary indecomposable Λ-module, and let Y be an indecomposable direct summand of ΩX. Then Y is a predecessor of an indecomposable projective module in $\mathrm{ind}\,\Lambda$. Since the multisection Δ is faithful, every indecomposable projective Λ-module is a predecessor of Δ in $\mathrm{ind}\,\Lambda$ and hence lies in Δ or is a predecessor of Δ_l in $\mathrm{ind}\,\Lambda$. Hence Y lies in Δ or is a predecessor of Δ_l in $\mathrm{ind}\,\Lambda$. In the first case we have $\mathrm{pd}_\Lambda X \le \mathrm{pd}_\Lambda Y + 1 \le i(Y) + 2 \le w(\Delta) + 1$, and in the second case $\mathrm{pd}_\Lambda X \le \mathrm{pd}_\Lambda Y + 1 \le 2$. In any case we obtain that $\mathrm{gl.dim}\,\Lambda \le w(\Delta) + 1 = n + 1$. □

Note that 1-double tilted algebras are tilted algebras, where we know that the global dimension is at most 2 [**10**]. The 2-double tilted algebras are the strict shod algebras [**18**] which have global dimension 3. Examples 2.8 and 2.9 show that there are n-double tilted algebras Λ (for any $n \in \boldsymbol{N} \cup \{\infty\}$) with $\mathrm{gl.dim}\,\Lambda = n + 1$. But in general it is not the case.

Example 4.3. Let $\Lambda = KQ/I$, where Q is the quiver



and I is the ideal in the path algebra KQ of Q (over a field K) generated by all paths of length 2. Then Λ is an ∞-double tilted algebra with $\mathrm{gl.dim}\,\Lambda = 4$ and Γ_Λ admits a connecting component of the form

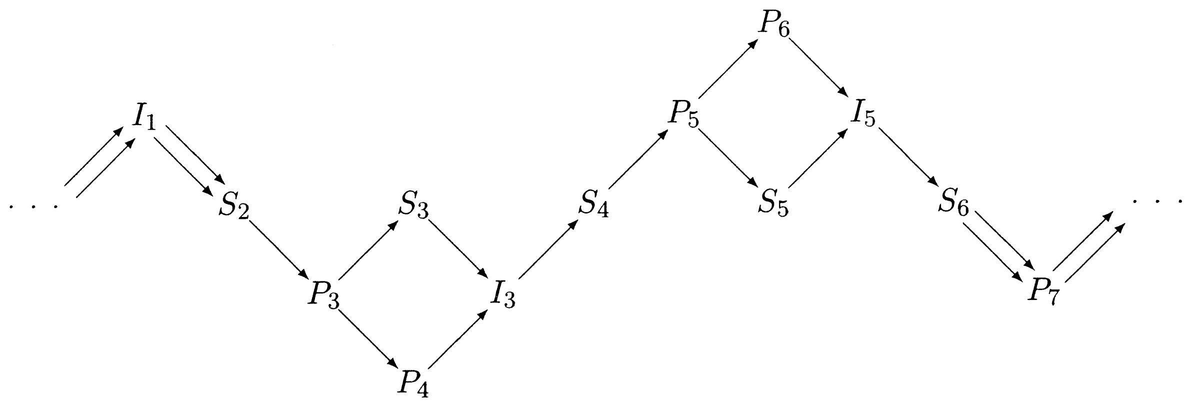


Moreover, both tilted algebras Λ_l and Λ_r are of infinite representation type.

EXAMPLE 4.4. Let $\Lambda = KQ/I$, where Q is the quiver

$$1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2 \xleftarrow{\gamma} 3 \xleftarrow{\sigma} 4 \xleftarrow{\xi} 5 \xleftarrow{\eta} 6 \underset{\nu}{\overset{\omega}{\leftleftarrows}} 7$$

and I the ideal in the path algebra KQ of Q (over a field K) generated by the paths $\alpha\gamma, \beta\gamma, \sigma\xi, \eta\omega$ and $\eta\nu$. Then Λ is a 4-double tilted algebra, $\mathrm{gl.dim}\,\Lambda = 2$, and Γ_Λ admits a connecting component of the form



5. Standard almost directed components.

The aim of this section is to prove a homological characterization of faithful generalized standard almost directed components.

Let Λ be an artin algebra. Following [**9**], we define two subcategories $\mathscr{L}_\Lambda$ and $\mathscr{R}_\Lambda$ of $\operatorname{ind}\Lambda$ as follows. The category $\mathscr{L}_\Lambda$ is formed by all modules X in $\operatorname{ind}\Lambda$ such that for every predecessor Y of X in $\operatorname{ind}\Lambda$ we have $\mathrm{pd}_\Lambda Y \leq 1$. Dually, $\mathscr{R}_\Lambda$ is formed by all modules X in $\operatorname{ind}\Lambda$ such that for every successor Y of X in $\operatorname{ind}\Lambda$ we have $\mathrm{id}_\Lambda Y \leq 1$. It is known [**2**, (2.1)] that $\operatorname{ind}\Lambda = \mathscr{L}_\Lambda \cup \mathscr{R}_\Lambda$ if and only if for every indecomposable Λ-module X we have $\mathrm{pd}_\Lambda X \leq 1$ or $\mathrm{id}_\Lambda X \leq 1$ (Λ is a shod algebra). Moreover, if Λ is basic connected, then Λ is shod if and only if Λ is tilted, double tilted or quasitilted of canonical type (see [**10**], [**18**], [**6**], [**7**]). In general, for a generalized double tilted algebra Λ we have $\operatorname{ind}\Lambda \backslash (\mathscr{L}_\Lambda \cup \mathscr{R}_\Lambda)$ finite but not necessarily empty, as the examples presented in the previous sections show. In fact it has been proved in [**24**] that $\mathscr{L}_\Lambda \cup \mathscr{R}_\Lambda$ is cofinite in $\operatorname{ind}\Lambda$ if and only if Λ is quasitilted or generalized double tilted.

Let Λ be an artin algebra and $\mathscr{C}$ a component of Γ_Λ. We define two full translation subquivers $\mathscr{L}_\mathscr{C}$ and $\mathscr{R}_\mathscr{C}$ in $\mathscr{C}$ as follows. The quiver $\mathscr{L}_\mathscr{C}$ is formed by all modules X in $\mathscr{C}$ such that for any path $Y = Y_1 \to Y_2 \to \cdots \to Y_t = X$ in $\operatorname{ind}\Lambda$ with $Y_1, Y_2, \dots, Y_t$ from $\mathscr{C}$ we have $\operatorname{pd}_\Lambda Y \le 1$. Dually, $\mathscr{R}_\mathscr{C}$ is formed by all modules X in $\mathscr{C}$ such that for every path $X = Z_1 \to Z_2 \to \cdots \to Z_m = Z$ in $\operatorname{ind}\Lambda$ with $Z_1, Z_2, \dots, Z_m$ from $\mathscr{C}$ we have $\operatorname{id}_\Lambda Z \le 1$. Observe that $\mathscr{L}_\mathscr{C}$ is closed under predecessors in $\mathscr{C}$ and $\mathscr{R}_\mathscr{C}$ is closed under successors in $\mathscr{C}$. Moreover, if Δ is a multisection of $\mathscr{C}$, then $\Delta_c \subseteq \mathscr{C}\backslash(\mathscr{L}_\mathscr{C} \cup \mathscr{R}_\mathscr{C})$.

Theorem 5.1. *Let Λ be a basic connected artin algebra, $\mathscr{C}$ a faithful component of Γ_Λ with a multisection Δ, and assume that $\mathscr{C}$ contains both a projective module and an injective module. Then $\mathscr{C}$ is generalized standard if and only if there is a decomposition $\mathscr{C} = \mathscr{L}_\mathscr{C} \cup \Delta_c \cup \mathscr{R}_\mathscr{C}$.*

Proof. Assume $\mathscr{C}$ is generalized standard. Then, by Theorem 3.1, Λ is a generalized double tilted algebra. Moreover, by Proposition 2.4, every indecomposable module in $\mathscr{C}$ belongs to Δ_c, or is a predecessor of Δ_l or a successor of Δ_r in $\mathscr{C}$. Since Λ is generalized double tilted, all predecessors of Δ_l in $\operatorname{ind}\Lambda$ belong to $\mathscr{L}_\Lambda$ and all successors of Δ_r in $\operatorname{ind}\Lambda$ belong to $\mathscr{R}_\Lambda$. Further, we have $\mathscr{L}_\Lambda \cap \mathscr{C} \subseteq \mathscr{L}_\mathscr{C}$ and $\mathscr{R}_\Lambda \cap \mathscr{C} \subseteq \mathscr{R}_\mathscr{C}$. Therefore, the required decomposition $\mathscr{C} = \mathscr{L}_\mathscr{C} \cup \Delta_c \cup \mathscr{R}_\mathscr{C}$ holds.

Assume now that $\mathscr{C} = \mathscr{L}_\mathscr{C} \cup \Delta_c \cup \mathscr{R}_\mathscr{C}$. Since $\mathscr{C}$ is a faithful component of Γ_Λ, applying Lemma 2.7 we infer the multisection Δ is faithful. Hence, in order to prove that $\mathscr{C}$ is generalized standard, it is enough to show that $\operatorname{Hom}_\Lambda(X, \tau_\Lambda Y) = 0$ for all modules X from Δ_r and Y from Δ_l (see Theorem 3.1). Suppose there exist $X \in \Delta_r$ and $Y \in \Delta_l$ such that $\operatorname{Hom}_\Lambda(X, \tau_\Lambda Y) \ne 0$. Let $\mathscr{D}$ be the full translation subquiver of $\mathscr{C}$ formed by all predecessors of Δ_l in $\mathscr{C}$ and $\mathscr{E}$ the full translation subquiver of $\mathscr{C}$ formed by all successors of Δ_r in $\mathscr{C}$. Since, by Proposition 2.4, all oriented cycles of $\mathscr{C}$ are entirely contained in Δ_c, the translation quivers $\mathscr{D}$ and $\mathscr{E}$ are directed. In particular, there is no path in $\mathscr{C}$ from X to $\tau_\Lambda Y$. Since $\operatorname{Hom}_\Lambda(X, \tau_\Lambda Y) \ne 0$ then there exists an infinite path

$$X = X_0 \to X_1 \to \cdots \to X_i \to X_{i+1} \to \cdots$$

in $\mathscr{C}$ such that $\operatorname{Hom}_\Lambda(X_i, \tau_\Lambda Y) \ne 0$ for all $i \ge 0$. We claim that there are $i \ge 0$ and a path in $\mathscr{C}$ of the form

$$I \to \tau_\Lambda Z \to V \to Z \to \cdots \to X_i$$

with I injective. It follows from our assumption on $\mathscr{C}$ that $\mathscr{C}$ contains at least one injective module. Further, since Δ intersects any τ_Λ-orbit of $\mathscr{C}$, every indecomposable injective module from $\mathscr{C}$ lies in Δ or in $\mathscr{E}$. Moreover, every indecomposable projective module from $\mathscr{C}$ lies in Δ or in $\mathscr{D}$. Invoking now the fact that $\mathscr{E}$ is directed we conclude that all but finitely many modules X_i, $i \ge 0$, belong to one connected component Γ of the right stable part $\mathscr{E}_r$ of $\mathscr{E}$. Moreover, the connected components of $\mathscr{E}_r$ are glued in $\mathscr{C}$ along the τ_Λ-orbits of injective modules, and by assumption $\mathscr{C}$ contains at least one injective module. Hence there is a path in $\mathscr{C}$ of the form

$$I \to \tau_\Lambda Z \to V \to Z \to \cdots \to U$$

with I injective and U in Γ. If Γ has only finitely many orbits then clearly there is in Γ a path $U \to \cdots \to X_i$, for some $i \geq 0$, and consequently a required path

$$I \to \tau_\Lambda Z \to V \to Z \to \cdots \to X_i.$$

Assume Γ has infinitely many orbits. Since Γ is directed then there are $j \geq 0$, $m \geq 0$ and an infinite path in Γ of the form

$$U \to \cdots \to \tau_\Lambda^{-m} X_j \to \tau_\Lambda^{-m} X_{j+1} \to \tau_\Lambda^{-m} X_{j+2} \to \cdots.$$

Applying now [**5**, Lemma 1.5] (see also [**22**, Lemma 4]) we infer that there are $i \geq j$ and a path in $\operatorname{ind} \Lambda$ of the form $\tau_\Lambda^{-m} X_i \to \cdots \to X_i$, consisting of modules from Γ, and we obtain a required path

$$I \to \tau_\Lambda Z \to V \to Z \to \cdots \to U \to \cdots \to \tau_\Lambda^{-m} X_i \to \cdots \to X_i.$$

Since $\operatorname{Hom}_\Lambda(X_i, \tau_\Lambda Y) \neq 0$ and there is no path in $\mathscr{C}$ from X_i to $\tau_\Lambda Y$, there exists an infinite path

$$\cdots \to Y_{t+1} \to Y_t \to \cdots \to Y_1 \to Y_0 = \tau_\Lambda Y$$

in $\mathscr{C}$ such that $\operatorname{Hom}_\Lambda(X_i, Y_t) \neq 0$ for all $t \geq 0$. Invoking our assumption that $\mathscr{C}$ contains at least one projective module and applying dual arguments to those above, we conclude that there are $t \geq 0$ and a path in $\operatorname{ind} \Lambda$ of the form

$$Y_t \to \cdots \to N \to W \to \tau_\Lambda^{-} N \to P$$

with P projective and consisting of modules from $\mathscr{C}$. Observe that $\operatorname{pd}_\Lambda Z \geq 2$ and $\operatorname{id}_\Lambda N \geq 2$, because $\operatorname{Hom}_\Lambda(I, \tau_\Lambda Z) \neq 0$ and $\operatorname{Hom}_\Lambda(\tau_\Lambda^{-} N, P) \neq 0$ (see [**19**, (2.4)]). Therefore, X_i and Y_t belong to $\mathscr{C} \backslash (\mathscr{L}_\mathscr{C} \cup \Delta_c \cup \mathscr{R}_\mathscr{C})$, which contradicts the assumption. This finishes the proof. $\square$

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