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A generalization of Andreev's Theorem

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Abstract. Andreev's Theorem studies the existence of compact hyperbolic polyhedra of a given combinatorial type and given dihedral angles, all of them acute. In this paper we consider the same problem but without any restriction on the dihedral angles. We solve it for the *descendants of the tetrahedron*, i.e. those polyhedra that can be obtained from the tetrahedron by successively truncating vertices; for instance, the first of them is the triangular prism.

1. Introduction.

Andreev's Theorem arises in the context of hyperbolic reflection groups. For these groups, a fundamental domain is given by a hyperbolic polyhedron with dihedral angles of the form π/n with n an integer greater than 1. Then, a classification of these groups may be given by determining all these polyhedra. Andreev's theorem solves this problem (see [**1**], [**2**]): it determines the space of dihedral angles of (compact or finite volume) hyperbolic polyhedra with the restriction that all the dihedral angles are not greater than $\pi/2$. Another reason to describe polyhedra from their dihedral angles is that, for hyperbolic trivalent polyhedra (those with exactly three faces incident to each vertex), dihedral angles uniquely determine the polyhedra. This is mainly deduced from Cauchy's lemmas ([**4**], see also [**10**]).

We are interested on the generalization of Andreev's theorem: given the combinatorial type of a polyhedron and some real numbers $\alpha_{ij} \in (0, \pi)$ assigned to its edges, find necessary and sufficient conditions on the α_{ij} so that there exists a geometric (spherical or hyperbolic) polyhedron of the given combinatorial type and with dihedral angles the given α_{ij}.

Such a generalization is easily established for the tetrahedron (see [**11**] or [**9**], or also [**6**]). In this paper we generalize Andreev's Theorem for some kind of polyhedra, the so called *tetrahedron's descendants.* These are the polyhedra obtained from the tetrahedron by successively truncating vertices. For example the triangular prism is the first tetrahedron's descendant. We give the explicit description of the space of dihedral angles of the triangular prism (Theorems 3.1, 3.5), that appears quite complicated. For the remaining tetrahedron's descendants we explain an algorithm that provides the desired generalization of Andreev's Theorem (Theorem 4.3).

Our approach consists in using the *Gram matrices* of the polyhedra. The Gram matrix of a geometric polyhedron is a symmetric matrix whose entries are in correspondence with pairs of faces of the polyhedron. If two faces are adjacent, then the corresponding entry is equal to minus the cosine of the dihedral angle at the common edge. Thus, the

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Gram matrix contains all the information about the dihedral angles, and some extra information. In [**6**] we characterized the space of Gram matrices of hyperbolic polyhedra of a given combinatorial type (actually the result there was more general: for d-polytopes in any geometric space), as a subset of certain $\boldsymbol{R}^N$. To obtain the space of dihedral angles we must eliminate the entries of the Gram matrix not corresponding to dihedral angles. This corresponds to doing a certain projection from the space of Gram matrices. The conditions obtained in [**6**] describing the space of Gram matrices are polynomial equalities or inequalities in the entries of the Gram matrix, and therefore this space is a real semialgebraic set. By the Tarski-Seidenberg theorem (see for instance [**3**]), the projection of a real semialgebraic set is again real semialgebraic, although, in general, it is difficult to find its explicit description. In this paper we explicitly perform this projection for the triangular prism, so we obtain the space of (cosines of) dihedral angles, described by polynomial equalities and inequalities.

As a corollary of the proofs of our results we obtain an alternative proof of the fact that a tetrahedron's descendant is uniquely determined by its dihedral angles (which we knew a priori since tetrahedron's descendants are trivalent).

We remark that the method used here to describe the space of dihedral angles gets considerably more complicated for other simple examples of polyhedra, like the cube. We refer to [**6**] for a partial solution for the cube.

The paper is organized as follows: in Section 2 we review some basic definitions and state the characterization of the Gram matrices of polyhedra; we also give some geometric interpretations. Sections 3 and 4 contain the generalization of Andreev theorem for the triangular prism, and for the remaining tetrahedron's descendants, respectively.

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2. Polyhedra and Gram matrices.

An *affine polyhedron* is a bounded subset $P \subset \boldsymbol{R}^3$ defined as the intersection of finitely many closed halfspaces. The *combinatorial type* $\mathscr{P}$ of the polyhedron P is the lattice consisting of the set of vertices, edges and faces of P. We use roman capital letters $P, F, C, V, \dots$ for affine polyhedra and their faces and calligraphic letters $\mathscr{P}, \mathscr{F}, \mathscr{C}, \mathscr{V}, \dots$ for the corresponding elements of the combinatorial type of the polyhedron. An *oriented cycle* of $\mathscr{P}$ is an ordered family $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$ of three faces of $\mathscr{P}$ so that $\mathscr{C}_1, \mathscr{C}_2$ are adjacent (meeting along an edge of $\mathscr{P}$) and the three of them are incident to a vertex $\mathscr{V}$. A *maximal oriented cycle* is a family $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4$ so that $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$ is an oriented cycle incident to a vertex $\mathscr{V}$ and $\mathscr{C}_4$ does not contain $\mathscr{V}$. An oriented cycle or maximal oriented cycle induces an orientation on the boundary of P, which is a topological 2-sphere. We say that two oriented cycles (or maximal oriented cycles) have the *same orientation* if they induce the same orientation on this 2-sphere (see [**6**] for more details).

Let f be any non-degenerate quadratic form in the vector space $\boldsymbol{R}^4$. We call the pair $(\boldsymbol{R}^4, f)$ a *geometric space*. This includes two particular cases defining spherical and hyperbolic spaces. If f is the usual euclidean inner product, then $f^{-1}(1)$ is $\boldsymbol{S}^3$. To obtain a model of hyperbolic space, we take f of signature $(3, 1)$. More precisely, let (x_1, x_2, x_2, x_4) be the coordinates of the vector x with respect to the canonical basis of

$\boldsymbol{R}^4$ and let f be the quadratic form defined by $f(x) = x_1^2 + x_2^2 + x_3^2 - x_4^2$. Then, the hyperbolic space is the upper sheet of the hyperboloid $f^{-1}(-1)$:

$$\boldsymbol{H}^3 = \{x \in \boldsymbol{R}^4 \mid f(x) = -1,\, x_4 > 0\}.$$

A *polyhedral cone* in the vector space $\boldsymbol{R}^4$ is a subset of the form

$$\hat{P} = \bigcap_{i=1}^{n} \hat{H}_i^-,$$

where $\hat{H}_i^-$ is a closed halfspace bounded by the linear hyperplane $\hat{H}_i$, and such that the intersection of the hyperplanes $\hat{H}_i$ is the origin of $\boldsymbol{R}^4$. There exists an affine hyperplane A such that $\hat{P} \cap A$ is bounded, and therefore a polyhedron. The *vertices, edges* and *faces* of a polyhedral cone $\hat{P}$ are defined to be the cone over the vertices, edges and faces of the polyhedron $\hat{P} \cap A$, and the *combinatorial type* of $\hat{P}$ is defined to be that of $\hat{P} \cap A$.

We denote by H_i^- the intersection of $\hat{H}_i^-$ with either $\boldsymbol{S}^3$ or $\boldsymbol{H}^3$. Then, the intersection of the polyhedral cone $\hat{P} = \cap \hat{H}_i^-$ with either $\boldsymbol{S}^3$ or $\boldsymbol{H}^3$ will be, respectively, a *spherical* or *hyperbolic polyhedron* $P = \cap H_i^-$. When P is hyperbolic, a vertex is called *finite* if the corresponding vectorial ray in $\hat{P}$ intersects $\boldsymbol{H}^3$.

Let $\hat{P} = \cap_{i=1}^n \hat{H}_i^-$ be a polyhedral cone such that no $\hat{H}_i$ is lightlike (that is, f restricted to $\hat{H}_i$ is non-degenerate). The *outward normal vector* to the halfspace $\hat{H}_i^-$ (or to H_i^-) is the unique vector e_i with $f(e_i) = \pm 1$ and satisfying

$$\hat{H}_i^- = \{x \in \boldsymbol{R}^4 \mid f(x, e_i) \leq 0\}$$

(we are using the notation f for both the quadratic form and its associated bilinear form). The *Gram matrix* of $\hat{P}$ (or of P) is defined to be the matrix of inner products of the outward normal vectors to the halfspaces $\hat{H}_i^-$, that is

$$G(\hat{P}) = G(P) = (f(e_i, e_j))_{i,j=1,\ldots,n}.$$

In [**6**], we characterized the Gram matrices of polyhedral cones in $(\boldsymbol{R}^4, f)$ of a given combinatorial type. We state here the results.

Notation. Given a matrix G, we will denote by $G\left[\begin{smallmatrix} i_1 \ldots i_r \\ j_1 \ldots j_s \end{smallmatrix}\right]$ the submatrix of G obtained by taking the rows $i_1 \ldots i_r$ and the columns $j_1 \ldots j_s$; for square submatrices we will denote by $G\left(\begin{smallmatrix} i_1 \ldots i_r \\ j_1 \ldots j_r \end{smallmatrix}\right)$ the determinant of that submatrix. For short, we will write $G[i_1 \ldots i_r] = G\left[\begin{smallmatrix} i_1 \ldots i_r \\ i_1 \ldots i_r \end{smallmatrix}\right]$ and $G_{i_1 \ldots i_r} = G\left(\begin{smallmatrix} i_1 \ldots i_r \\ i_1 \ldots i_r \end{smallmatrix}\right)$.

Theorem 2.1 (Theorem 4.1 *in* [**6**], *for* $d = 3$). *Let $\mathscr{P}$ be the combinatorial type of a polyhedron with n ordered faces $\mathscr{C}_1, \ldots, \mathscr{C}_n$; let $(\boldsymbol{R}^4, f)$ be a geometric space and let G be a real symmetric matrix of order n and rank 4. Suppose that the signature of G equals the signature of f and that $g_{ii} = \pm 1$ for all $i = 1, \ldots, n$. Then there exists a polyhedral cone $\hat{P} \subset (\boldsymbol{R}^4, f)$ with the combinatorial type of $\mathscr{P}$ and with Gram matrix $G(\hat{P})$ equal to G if and only if the following conditions hold*:

(R) (*Rank*) *Given any vertex of* $\mathscr{P}$ *and all the faces* $\mathscr{C}_{i_1}, \dots, \mathscr{C}_{i_m}$ *incident to it, the submatrix* $G[i_1 \dots i_m]$ *has rank less than or equal to* 3.

(P_4) (*Principal minors of order* 4) *If* $\mathscr{C}_{i_1}, \dots, \mathscr{C}_{i_4}$ *is a maximal oriented cycle of* $\mathscr{P}$, *then* $G_{i_1 \dots i_4} \det f > 0$.

(M_4) (*Mixed minors of order* 4) *If* $\mathscr{C}_{i_1}, \dots, \mathscr{C}_{i_4}$ *and* $\mathscr{C}_{j_1}, \dots, \mathscr{C}_{j_4}$ *are two maximal oriented cycles with the same orientation, then* $G\binom{i_1 \dots i_4}{j_1 \dots j_4} \det f > 0$.

Moreover, if these conditions hold, the polyhedral cone $\hat{P}$ *is unique up to an orthogonal transformation of* $(\boldsymbol{R}^4, f)$.

Furthermore, in the hyperbolic case, the polyhedron $P = \hat{P} \cap \boldsymbol{H}^3$ *is compact if and only if the following conditions hold*:

(P_3) (*Principal minors of order* 3) *If* $\mathscr{C}_i, \mathscr{C}_j, \mathscr{C}_k$ *are faces of* $\mathscr{P}$ *incident to a vertex, then* $G_{ijk} > 0$.

(M_3) (*Mixed minors of order* 3) *If* $\mathscr{C}_{i_1}, \mathscr{C}_{i_2}, \mathscr{C}_{i_3}$ *and* $\mathscr{C}_{j_1}, \mathscr{C}_{j_2}, \mathscr{C}_{j_3}$ *are oriented cycles of* $\mathscr{P}$ *with the same orientation, then* $G\binom{i_1 i_2 i_3}{j_1 j_2 j_3} > 0$.

(Here, if the signature of f is (s_+, s_-), then $\det f = (-1)^{s_-}$.)

We remark that conditions (M_4) and (M_3) contain, respectively, conditions (P_4) and (P_3). We state them separately because they have different geometric meaning (see below).

2.1. Geometric interpretation of the minors of the Gram matrix.

The quadratic form f in $\boldsymbol{R}^4$ determines a non degenerate quadratic form $\bigwedge^m f$ in $\bigwedge^m \boldsymbol{R}^4$ defined by

$$\Big(\bigwedge^m f\Big)(u_1 \wedge \ldots \wedge u_m, v_1 \wedge \ldots \wedge v_m) = \det(f(u_i, v_j))_{i,j=1,\ldots,m}.$$

We can obtain some geometric interpretation of the conditions in Theorem 2.1 by studying the geometric spaces $(\bigwedge^4 \boldsymbol{R}^4, \bigwedge^4 f)$ and $(\bigwedge^3 \boldsymbol{R}^4, \bigwedge^3 f)$.

(a) Since $\bigwedge^4 \boldsymbol{R}^4$ has dimension one and $\bigwedge^4 f$ is not degenerate, the inner product of two vectors is zero if and only if one of the vectors is zero. Expressed in terms of determinants of submatrices of the Gram matrix, we have that

$$G\binom{i_1 \dots i_4}{j_1 \dots j_4} = 0 \quad \text{if and only if} \quad G_{i_1 \dots i_4} = 0 \ \text{ or } G_{j_1 \dots j_4} = 0.$$

In the spherical case $\bigwedge^4 f$ is positive definite, while in the hyperbolic case $\bigwedge^4 f$ is negative definite. In both cases, the condition $G\binom{i_1 \dots i_4}{j_1 \dots j_4} \det f > 0$ means that the vectors $e_{i_1} \wedge \cdots \wedge e_{i_4}$ and $e_{j_1} \wedge \cdots \wedge e_{j_4}$ are in the same vectorial ray of $\bigwedge^4 \boldsymbol{R}^4$. Therefore, condition ($M_4$) of Theorem 2.1 means that the vectors in $\bigwedge^4 \boldsymbol{R}^4$ corresponding to maximal oriented cycles of $\mathscr{P}$ with the same orientation are all in the same vectorial ray. As an immediate consequence, this condition can be reduced: if there are N maximal oriented cycles in $\mathscr{P}$, it is enough to consider $N-1$ convenient maximal cycles.

(b) If f has signature $(3,1)$, $\bigwedge^3 \boldsymbol{R}^4$ has signature $(1,3)$. Then, now $(\bigwedge^3 f)^{-1}(1)$ is a hyperboloid of two sheets. Condition (P_3) in Theorem 2.1 imposes the condition that some vectors of $\bigwedge^3 \boldsymbol{R}^4$ (those corresponding to vertices of the polyhedron) be in this hyperboloid. And condition (M_3) imposes the condition that all these vectors be in the same sheet of the hyperboloid. As a consequence, condition (M_3) can also be reduced. By the same argument, we can prove: if two minors $G_{i_1i_2i_3}$ and $G_{j_1j_2j_3}$ of G are positive and there exists another non negative minor $G_{k_1k_2k_3}$, then $G\binom{i_1i_2i_3}{j_1j_2j_3} > 0$ if and only if $G\binom{i_1i_2i_3}{k_1k_2k_3}G\binom{j_1j_2j_3}{k_1k_2k_3} > 0$.

Finally, we remark that, given an orientation of $\boldsymbol{R}^4$, the *Hodge star operators* are isomorphisms $*$ between $\bigwedge^m \boldsymbol{R}^4$ and $\bigwedge^{4-m} \boldsymbol{R}^4$. For $m = 4$, $*(u_1 \wedge \cdots \wedge u_4)$ is equal to the determinant of the matrix of coordinates of the vectors $u_1, \ldots, u_4$ with respect to any orthonormal and positively oriented basis. For $m = 3$, the $*$ operator is a direct generalization of the vector product in $\boldsymbol{R}^3$, and we recall here some properties (see [**8**] or [**6**]).

Lemma 2.2. *Let $u, u_1, u_2, u_3,\ v, v_1, v_2, v_3 \in \boldsymbol{R}^4$; then:*

(a) $f(*(u_1 \wedge u_2 \wedge u_3),\, v) = *(u_1 \wedge u_2 \wedge u_3 \wedge v)$
(b) $(\bigwedge^3 f)(u_1 \wedge u_2 \wedge u_3,\, v_1 \wedge v_2 \wedge v_3) = (\det f)\ \ *(u_1 \wedge u_2 \wedge u_3 \wedge *(v_1 \wedge v_2 \wedge v_3))$
(c) $(\bigwedge^3 f)(u_1 \wedge u_2 \wedge u_3,\, v_1 \wedge v_2 \wedge v_3) = (\det f)\ f(*(u_1 \wedge u_2 \wedge u_3), *(v_1 \wedge v_2 \wedge v_3))$.

If C_i, C_j, C_k is an oriented cycle of P, we define $v_{ijk} = *(e_i \wedge e_j \wedge e_k)$. In [**6**] we proved that: if we consider an oriented cycle incident to each vertex of $\mathscr{P}$ so that all these cycles have the same orientation, then either the set of vectors v_{ijk} or the set of their opposite vectors determine the vertices of P.

Using this fact and Lemma 2.2, we can directly obtain some geometric relations for polyhedra from minors of their Gram matrices:

Lemma 2.3. *Let $P \in \boldsymbol{H}^3$ be a polyhedron and let G be its Gram matrix.*

(a) *Let $C_{i_1}, C_{i_2}, C_{i_3}$ and $C_{j_1}, C_{j_2}, C_{j_3}$ be oriented cycles with the same orientation incident to finite vertices $V_{i_1i_2i_3}, V_{j_1j_2j_3}$. Then the distance between these vertices is given by*

$$-\cosh\big(\mathrm{d}(V_{i_1i_2i_3}, V_{j_1j_2j_3})\big) = \frac{-G\binom{i_1i_2i_3}{j_1j_2j_3}}{\sqrt{G_{i_1i_2i_3}}\sqrt{G_{j_1j_2j_3}}}.$$

(b) *Let V_{ijk} be a finite vertex of P and let C_l be a face not containing this vertex. Then the distance between them is given by*

$$\sinh^2\big(\mathrm{d}(V_{ijk}, C_l)\big) = \frac{-G_{ijkl}}{G_{ijk}}.$$

Sylvester identities.

In the proofs in this paper we will use some matrix relations obtained from the so called *Sylvester identity* (see [**7**]). We state here this identity.

Let G be a square matrix of order n. Let us fix an order p submatrix of G, for example the square submatrix $G[1\ 2 \ldots p]$. We construct the matrix $B = G\left\|\begin{smallmatrix}1 & 2 \ldots p\\ 1 & 2 \ldots p\end{smallmatrix}\right\|$ by bordering the submatrix $G[1\,2\dots p]$ with one row and one column among the remaining rows and columns of G and collecting their determinants; that is, B is a square matrix of order $n-p$ whose entries are of the form

$$b_{ij} = G\binom{1\;2\dots p\;i}{1\;2\dots p\;j},$$

where $p+1 \leq i \leq n$ and $p+1 \leq j \leq n$.

With the above notation, the *Sylvester identity* for B states that

$$\det G\left\|\begin{smallmatrix}1\;2\dots p\\1\;2\dots p\end{smallmatrix}\right\| = G\binom{1\;2\dots p}{1\;2\dots p}^{n-p-1}\det G.$$

3. The triangular prism.

Let $\mathscr{P}$ be the combinatorial type of a triangular prism, with the faces labeled as in Figure 1. If the faces $\mathscr{C}_i, \mathscr{C}_j$ are adjacent, we denote by $\mathscr{E}_{ij}$ the common edge. Let $\alpha_{ij} \in (0,\pi)$ be arbitrary numbers associated to the edges $\mathscr{E}_{ij}$. In this section we answer the following question: is there a compact hyperbolic (resp. spherical) triangular prism with dihedral angles the given numbers?

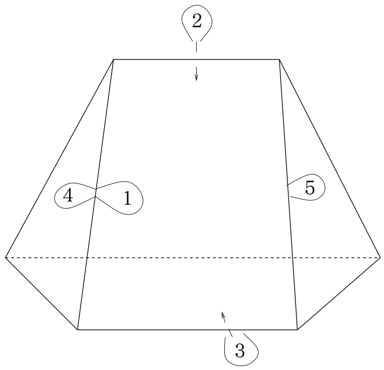

Figure 1. Triangular prism.

The method we will use to work out this problem is simple: we collect the given numbers to construct the matrix

$$G = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{12} & 1 & a_{23} & a_{24} & a_{25} \\ a_{13} & a_{23} & 1 & a_{34} & a_{35} \\ a_{14} & a_{24} & a_{34} & 1 & u \\ a_{15} & a_{25} & a_{35} & u & 1 \end{pmatrix}, \tag{1}$$

with $a_{ij} = -\cos\alpha_{ij}$, and u an unknown. By imposing the condition that the rank of G be equal to 4, we obtain at most two solutions for u. For each of these solutions we check whether the resulting matrix satisfies the conditions in Theorem 2.1 for the triangular prism. In practice, nevertheless, this process is long and quite technical.

We order the numbers a_{ij} according to a fixed ordering of the edges of $\mathscr{P}$. Then we regard (a_{ij}) as a point in $\boldsymbol{R}^9$, and we denote by $\mathscr{A}^{CH}$ the subset of those points (a_{ij}) corresponding to dihedral angles of compact hyperbolic triangular prisms; we refer to it as the *space of dihedral angles of compact hyperbolic triangular prisms* (regardless that, actually, the numbers a_{ij} are minus the cosines of the dihedral angles, rather than the dihedral angles themselves). Answering the question above is, then, describing the subset $\mathscr{A}^{CH}$. We do that in the Theorem 3.1.

Similarly, we denote by $\mathscr{A}^S$ the space of dihedral angles of spherical triangular prisms, and we describe this subset in Theorem 3.5

THEOREM 3.1. *Let $\mathscr{P}$ be a triangular prism with labeled faces. Let $\alpha_{12}, \ldots, \alpha_{35} \in (0, \pi)$ and $a_{ij} = -\cos\alpha_{ij}$. Then, there exists a compact hyperbolic triangular prism $P \subset \boldsymbol{H}^3$ with dihedral angles α_{ij} at the edges $\mathscr{E}_{ij}$ if and only if $(a_{12}, \ldots, a_{35})$ is in the subset $\mathscr{A}^{CH} = \mathscr{S}_0^H \cap \mathscr{S}_0^C \cap (\mathscr{R}_+ \cup \mathscr{R}_-)$, where*

$$\mathscr{S}_0^H = \left\{ \begin{matrix} -1 < a_{ij} < 1 \\ G_{1234} < 0,\ G_{1235} < 0 \end{matrix} \right\},$$

$$\mathscr{S}_0^C = \left\{ \begin{matrix} G_{124} > 0,\ G_{134} > 0,\ G_{234} > 0,\ G_{125} > 0,\ G_{235} > 0,\ G_{135} > 0 \\ G\binom{124}{234} > 0,\ G\binom{124}{143} > 0,\ G\binom{152}{135} > 0,\ G\binom{152}{253} > 0 \end{matrix} \right\},$$

$$\mathscr{R}_+ = \{G_{123} \geq 0\} \cap \mathscr{S}_{12} \cap \mathscr{S}_{13} \cap \mathscr{S}_{23},$$

$$\mathscr{R}_- = \{G_{123} < 0\} \cap (\mathscr{S}_{12} \cup \mathscr{S}'_{12}) \cap (\mathscr{S}_{13} \cup \mathscr{S}'_{13}) \cap (\mathscr{S}_{23} \cup \mathscr{S}'_{23});$$

the subsets $\mathscr{S}_{12}, \mathscr{S}'_{12}, \ldots$ are

$$\mathscr{S}_{12} = \left\{ \begin{matrix} G\binom{123}{124} G\binom{123}{152} > 0 \\ G\binom{123}{124}(G_{124}G_{1235} - G_{125}G_{1234}) < 0 \end{matrix} \right\},$$

$$\mathscr{S}'_{12} = \left\{ \begin{matrix} G\binom{123}{124} \leq 0,\ G\binom{123}{152} \geq 0 \\ G\binom{123}{124}^2 + G\binom{123}{152}^2 > 0 \end{matrix} \right\};$$

and the remaining subsets are similar: first, we rewrite the expression $G_{124}G_{1235} - G_{125}G_{1234}$ in $\mathscr{S}_{12}$ as $G_{124}G_{1523} - G_{152}G_{1243}$; then $\mathscr{S}_{13}$ is obtained from $\mathscr{S}_{12}$ by changing the indices 124 *to* 143 *and* 152 *to* 135; *and $\mathscr{S}_{23}$, $\mathscr{S}'_{13}$, etc. are obtained in an analogous way.*

It is clear that the subset $\mathscr{S}_0^H$ consists just of the obvious restrictions on the a_{ij} and the restrictions coming from condition (P_4) of Theorem 2.1. Also, the subset $\mathscr{S}_0^C$ corresponds to the conditions (P_3) and (M_3) of that theorem. Then, what is left is to analyse how the conditions (R) and (M_4) are translated into some subset of $\boldsymbol{R}^9$. This is the main part of the proof.

3.1. Proof of the necessary conditions.

Suppose that $P \subset \boldsymbol{H}^3$ is a compact hyperbolic triangular prism, let $G = G(P)$ be its Gram matrix, and let $A = (a_{ij}) \in \boldsymbol{R}^9$, with a_{ij} equal to minus the cosines of the dihedral angles of P. Then, G satisfies the conclusions of Theorem 2.1. From conditions (P_4), (P_3) and (M_3) of that theorem, we immediately see that the point (a_{ij}) belongs to $\mathscr{S}_0^H$ and to $\mathscr{S}_0^C$ (we just need to check the orientation of the cycles in Figure 1).

Next, we study the consequences obtained from condition (M_4) in Theorem 2.1, that is, the signs of some particular mixed minors of G of order 4. In the remaining of the proof we will show that the condition $G\binom{1243}{1245} < 0$ implies that A belongs to the subset $\{G_{123} \geq 0\} \cap \mathscr{S}_{12}$ or to the subset $\{G_{123} < 0\} \cap (\mathscr{S}_{12} \cup \mathscr{S}'_{12})$. Studying in an analogous way the implications of the conditions $G\binom{1432}{1435} < 0$ and $G\binom{2341}{2345} < 0$ we complete the proof.

Since the minor $G\binom{1243}{1245}$ contains the entry $u = G(4,5)$, we first determine the value of u. By (R), we have $\det G = 0$, and therefore, u is a root of the polynomial

$$\det G = -G_{123}u^2 - 2G\tbinom{1234}{1235}_{|u=0} u + \det G_{|u=0}.$$

We split the analysis in two subcases.

Case A: $G_{123} \neq 0$; then we have

$$u = \frac{G\binom{1234}{1235}_{|u=0} \pm \sqrt{G_{1234}G_{1235}}}{-G_{123}}$$

(the factorization of the discriminant comes from the Sylvester identity $G_{123}G_{12345} = G\binom{1234}{1235}^2 - G_{1234}G_{1235}$). Notice that, by condition (P_4) of Theorem 2.1, $G_{1234}G_{1235} > 0$, and therefore u is a well defined real number. Moreover, we can also determine the sign without ambiguity: since the cycles $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_4$ and $\mathscr{C}_1, \mathscr{C}_5, \mathscr{C}_2$ of $\mathscr{P}$ have the same orientation, condition (M_4) in Theorem 2.1 implies that $G\binom{1234}{1235} > 0$. From this inequality we obtain that $-G_{123}u < G\binom{1234}{1235}_{|u=0}$ (expanding $G\binom{1234}{1235}$ as a polynomial in u), and this finally implies that

$$u = \frac{G\binom{1234}{1235}_{|u=0} - \sqrt{G_{1234}G_{1235}}}{-G_{123}}.$$

Expanding $G\binom{1243}{1245}$ as a polynomial in u and substituting u by the value obtained above (and using a suitable Sylvester identity), we get that

$$G\binom{1243}{1245} = \frac{G\binom{123}{152}G_{1234} + G\binom{123}{124}\sqrt{G_{1234}G_{1235}}}{-G_{123}}. \tag{2}$$

We still consider two subcases, according to G_{123} being negative or positive. Then, Lemma 3.2 below shows that, in the first case, $A \in \mathscr{S}_{12} \cup \mathscr{S}'_{12}$, and in the second case, $A \in \mathscr{S}_{12}$ (for the second part, notice that the conditions $G\binom{1243}{1523} < 0$ and $G\binom{124}{152} > 0$ are included in conditions (M_4) and (M_3), respectively).

Case B: $G_{123} = 0$. We first notice that, in this case, the value of u is equal to $\frac{\det G_{|u=0}}{2G\binom{1234}{1235}_{|u=0}}$, where the denominator is different from zero: indeed, $G\binom{1234}{1235}_{|u=0} = G\binom{1234}{1235}$, and $G\binom{1234}{1235} \neq 0$ because G_{1234}, G_{1235} are both different from zero (see Section 2.1(a)). Then, Lemma 3.3 below shows that $A \in \mathscr{S}_{12}$.

Lemma 3.2. *Let G be a matrix as in (1). Suppose that $G_{123} \neq 0$ and that G_{1234}, G_{1235} are negative, and let $u = \frac{G\binom{1234}{1235}_{|u=0} - \sqrt{G_{1234}G_{1235}}}{-G_{123}}$.*

(a) *Suppose that $G_{123} < 0$; then $G\binom{1243}{1245} < 0$ if and only if $A \in \mathscr{S}_{12} \cup \mathscr{S}'_{12}$.*
(b) *Suppose that $G_{123} > 0$; then $G\binom{1243}{1245} < 0$ if and only if $A \in \mathscr{S}_{12} \cup \mathscr{S}''_{12}$, where*

$$\mathscr{S}''_{12} = \left\{ \begin{matrix} G\binom{123}{124} \geq 0, \ G\binom{123}{152} \leq 0 \\ G\binom{123}{124}^2 + G\binom{123}{152}^2 > 0 \end{matrix} \right\}.$$

Furthermore, if $G_{12} > 0$, $G\binom{1243}{1523} < 0$ and $G\binom{124}{152} > 0$, then $G\binom{1243}{1245} < 0$ implies that $A \in \mathscr{S}_{12}$.

Proof. (a) From the expression (2) for $G\binom{1243}{1245}$, in the case $G_{123} < 0$ we have that $G\binom{1243}{1245} < 0$ is equivalent to

$$G\binom{123}{124}\sqrt{G_{1234}G_{1235}} < -G\binom{123}{152}G_{1234}, \tag{3}$$

so we will prove that (3) is equivalent to $A \in \mathscr{S}_{12} \cup \mathscr{S}'_{12}$. If $A \in \mathscr{S}'_{12}$, then (3) is readily satisfied. If $A \in \mathscr{S}_{12}$, then we still have to consider the cases where $G\binom{123}{124}$ and $G\binom{123}{152}$ are both positive or both negative. Taking into account that G_{1234} and G_{123} are negative, and using some Sylvester inequalities, we easily obtain the result.

The proof of the converse is also straightforward by studying the different possibilities of signs for $G\binom{123}{124}$ and $G\binom{123}{152}$.

(b) The first part is proved in a completely analogous way as part (a). Then, it is enough to prove that, with the conditions we have added, $G\binom{1243}{1245} < 0$ implies that $G\binom{123}{124}G\binom{123}{152} > 0$. This is clearly obtained from the equality

$$G\binom{123}{152}G\binom{124}{123} = -G_{12}G\binom{1243}{1523} + G_{123}G\binom{124}{152},$$

which is a modification (by changing files and columns) of a Sylvester identity. □

Lemma 3.3. *Suppose that $G_{12} > 0$, $G_{123} = 0$, $G_{1234} < 0, G_{1235} < 0$ and $G\binom{1234}{1235} > 0$ and let $u = \frac{\det G_{|u=0}}{2G\binom{1234}{1235}_{|u=0}}$. Then $G\binom{1243}{1245}$ is negative if and only if $A \in \mathscr{S}_{12}$.*

Proof. From a Sylvester identity and from $G_{123} = 0$, we get that $G_{12}G\binom{1234}{1235} = G\binom{123}{124}G\binom{123}{152}$. By the hypothesis on the signs, we have that $G\binom{123}{124}G\binom{123}{152} > 0$. To finish, we will prove that $G\binom{1243}{1245} < 0$ is equivalent to $G\binom{123}{124}(G_{124}G_{1235} - G_{125}G_{1234}) < 0$. Let us denote $F_1 = G_{124}G_{1235} - G_{125}G_{1234}$.

Substituting u by its value in $G\binom{1243}{1245}$ and using that $G\binom{1234}{1235}_{|u=0} > 0$, we have that

$G\binom{1243}{1245} < 0$ is equivalent to

$$-G\binom{123}{124} \det G_{|u=0} + 2G\binom{1234}{1235}_{|u=0} G\binom{1243}{1245}_{|u=0} < 0. \tag{4}$$

Let us call r to the last expression, which is the resultant, when $G_{123} = 0$, of the polynomials $\det G$ and $G\binom{1243}{1245}$ with respect to u. Calling R_1 to the resultant of $\det G$ and $G\binom{1243}{1245}$ with respect to u, we have that (see Lemma 3.4 below)

$$R_1 = -G_{123} G\binom{1243}{1245}^2_{|u=0} - G\binom{123}{124} r.$$

On the other hand, we show in Lemma 3.4 below that R_1 factorizes as $R_1 = G_{1234} F_1$. Then, under our hypothesis $G_{123} = 0$ and $G_{1234} < 0$, we have that $r < 0$ is equivalent to $G\binom{123}{124} F_1 < 0$. Therefore we have proved the desired result. □

Lemma 3.4.

(a) *Let* $F_1 = G_{124} G_{1235} - G_{125} G_{1234}$. *Then the following identity holds*:

$$F_1 = G\binom{123}{125} G\binom{1234}{1245} + G\binom{123}{124} G\binom{1235}{1245}.$$

(*In particular, the right-hand expression does not depend on* u.)

(b) *Let* R_1 *be the resultant of* $\det G$ *and* $G\binom{1243}{1245}$ *with respect to* u. *Then* $R_1 = G_{1234} F_1$.

Proof. (a) To prove the first identity we use some standard Sylvester identities:

$$\begin{aligned} & G\binom{123}{125} G\binom{1234}{1245} + G\binom{123}{124} G\binom{1235}{1245} \\ &\quad = G\binom{123}{125} \frac{G\binom{123}{124} G\binom{124}{125} - G\binom{123}{125} G_{124}}{G_{12}} + G\binom{123}{124} \frac{G\binom{123}{124} G_{125} - G\binom{123}{125} G\binom{125}{124}}{G_{12}} \\ &\quad = \frac{-G\binom{123}{125}^2 G_{124} + G\binom{123}{124}^2 G_{125}}{G_{12}} = G_{124} G_{1235} - G_{125} G_{1234} = F_1. \end{aligned}$$

(b) We compute the resultant R_1 as the determinant of the Sylvester matrix

$$\begin{pmatrix} -G_{123} & -2G\binom{1234}{1235}_{|u=0} & \det G_{|u=0} \\ -G\binom{123}{124} & G\binom{1243}{1245}_{|u=0} & 0 \\ 0 & -G\binom{123}{124} & G\binom{1243}{1245}_{|u=0} \end{pmatrix}.$$

Then

$$\begin{aligned} R_1 = \quad & -G_{123} G\binom{1243}{1245}^2_{|u=0} + G\binom{123}{124} \Big(-2G\binom{1234}{1235}_{|u=0} G\binom{1243}{1245}_{|u=0} + G\binom{123}{124} \det G_{|u=0} \Big) \\ =^{(i)} \quad & -G_{123} \big(G_{1234} G_{1245_{|u=0}} - G_{124} \det G_{|u=0} \big) \\ & + G\binom{123}{124} \Big(2G_{1234} G\binom{1235}{1245}_{|u=0} - G\binom{123}{124} \det G_{|u=0} \Big) \end{aligned}$$

$$\begin{aligned}
&= \quad G_{1234}\Big(- G_{123}G_{1245_{|u=0}} + 2G\tbinom{123}{124}G\tbinom{1235}{1245}_{|u=0}\Big) \\
&\qquad + \det G_{|u=0}\Big(G_{123}G_{124} - G\tbinom{123}{124}^2\Big) \\
&=^{(ii)} G_{1234}\Big(G\tbinom{123}{125}G\tbinom{1234}{1245}_{|u=0} + G\tbinom{123}{124}G\tbinom{1235}{1245}_{|u=0} - G_{12}\det G_{|u=0}\Big) \\
&\qquad + \det G_{|u=0}G_{12}G_{1234} \\
&= \quad G_{1234}\Big(G\tbinom{123}{125}G\tbinom{1234}{1245}_{|u=0} + G\tbinom{123}{124}G\tbinom{1235}{1245}_{|u=0}\Big) \\
&=^{(iii)} G_{1234}F_1,
\end{aligned}$$

where we have used several Sylvester identities, some of them easily recognizable; the others are: for (i) we have used the identity

$$-G\tbinom{123}{124}G_{12345} = -G_{1234}G\tbinom{1235}{1245} + G\tbinom{1234}{1245}G\tbinom{1235}{1234},$$

which is obtained by applying the Sylvester identity to the submatrix $G\left[{123 \atop 124}\right]$ of the matrix $G[12345]$; for (ii) we have used the identity

$$G_{12}G_{12345} = G_{123}G_{1245} - G\tbinom{123}{124}G\tbinom{1235}{1245} + G\tbinom{123}{125}G\tbinom{1234}{1245},$$

which is obtained by applying the Sylvester identity to $G[12]$ as a submatrix of G. Finally, (iii) is obtained from (a). $\square$

3.2. Proof of the sufficient conditions.

Let now $A = (a_{12}, \ldots, a_{35})$ be a point in $\mathscr{A}^{CH}$. We begin by constructing a matrix G as (1), where u has either the value given in Lemma 3.2 or in Lemma 3.3, depending on G_{123} being different from or equal to zero. In both cases the hypothesis $A \in \mathscr{S}_0^H$ implies that the value given to u is a well defined real number.

Next, we will check that the matrix G satisfies all the conditions in Theorem 2.1; this will guarantee the existence of the desired prism.

(I) First, by the choice of u, we have that $\det G = 0$. Since $A \in \mathscr{S}_0^H$, we have the signs of minors $G_{12} > 0$ and $G_{1234} < 0$, which imply that the signature of G is $(3, 1)$.

(II) Conditions (P_4) and (M_4) are equivalent to the matrix $\bigwedge^4 G$ of minors of order 4 of G having the following signs (see Figure 1 to check the orientations of the maximal cycles).

	1234	1235	1245	1345	2345
1234	$-$	$+$	$+$	$-$	$+$
1235	$+$	$-$	$-$	$+$	$-$
1245	$+$	$-$	$-$	$+$	$-$
1345	$-$	$+$	$+$	$-$	$+$
2345	$+$	$-$	$-$	$+$	$-$

By the hypothesis $A \in \mathscr{S}_0^H$, we have $G_{1234} < 0$ and $G_{1235} < 0$; on the other hand, since $\det G = 0$, the matrix $\bigwedge^4 G$ has rank 1, and therefore we can find out the signs of all its entries from the signs of some of them. For instance, it is enough to prove that

$$\begin{aligned} G\tbinom{1243}{1245} &< 0 \quad \bigl(\text{equivalent to } \; G\tbinom{1234}{1245} > 0\bigr) \\ G\tbinom{2341}{2345} &< 0 \quad \bigl(\text{equivalent to } \; G\tbinom{1234}{2345} > 0\bigr) \\ G\tbinom{1435}{1432} &< 0 \quad \bigl(\text{equivalent to } \; G\tbinom{1345}{1234} < 0\bigr) \\ G\tbinom{1523}{1524} &< 0 \quad \bigl(\text{equivalent to } \; G\tbinom{1235}{1245} < 0\bigr). \end{aligned}$$

By the symmetries of the prism, it is enough to show one of the above inequalities. We will prove that $G\binom{1243}{1245} < 0$. If $G_{123} \neq 0$, then this is obtained from Lemma 3.2. If $G_{123} = 0$, to apply Lemma 3.3, we first need to have $G\binom{1234}{1235} > 0$. But this holds because, since $G_{123} = 0$, then $G\binom{1234}{1235} = \frac{G\binom{123}{152}G\binom{124}{123}}{G_{12}}$ and $A \in \mathscr{S}_{12}$.

(III) Since $A \in \mathscr{S}_0^C$, all principal minors corresponding to vertices of P are positive. This implies condition (P_3).

(IV) To prove (M_3), by the geometric interpretation of Section 2.1(b), it is enough to show that the mixed minors $G\binom{124}{234}, G\binom{124}{143}, G\binom{152}{135}$, $G\binom{152}{253}$ and $G\binom{124}{152}$ are all positive. This is true for the first four minors since $A \in \mathscr{S}_0^C$. We study the sign of the remaining minor $G\binom{124}{152}$. We distinguish two cases, depending whether A is in $\mathscr{R}_+$ or in $\mathscr{R}_-$.

Case A: $A \in \mathscr{R}_+$. We use the identity

$$2G\tbinom{123}{124}G\tbinom{124}{125}G\tbinom{125}{123} = -G_{12}G_{123}G_{1245} + G_{125}G\tbinom{123}{124}^2 + G_{124}G\tbinom{123}{125}^2,$$

which is obtained applying the Sylvester identity to the submatrix $G[12]$ of G and using that $\det G = 0$. We already know that G_{12}, G_{125}, G_{124} are positive, $G_{1245} \leq 0$, and $G_{123} \geq 0$; since $A \in \mathscr{S}_{12}$, we have that $G\binom{123}{124}$ and $G\binom{123}{125}$ are not zero. Therefore, the right-hand side of the previous expression is strictly positive. On the other hand, also since $A \in \mathscr{S}_{12}$, we have $G\binom{123}{124}G\binom{123}{152} > 0$, and therefore $G\binom{124}{152}$ must be positive.

Case B: $A \in \mathscr{R}_-$. We expand $G\binom{124}{125} = -G\binom{124}{152}$ as a polynomial in u and substitute u by its value:

$$G\tbinom{124}{125} = \frac{G_{12}G\tbinom{1234}{1235}_{|u=0} - G_{12}\sqrt{G_{1234}G_{1235}} - G_{123}G\tbinom{124}{125}_{|u=0}}{-G_{123}}.$$

Since $-G_{123} > 0$, we must show that the numerator in the previous expression is negative. Using a Sylvester identity, this is equivalent to

$$-G\tbinom{123}{125}G\tbinom{124}{123} < G_{12}\sqrt{G_{1234}G_{1235}}. \tag{5}$$

The right-hand side of the last expression is positive. We study the sign of the left-hand side in the cases $A \in \mathscr{S}_{12}$ and $A \in \mathscr{S}'_{12}$.

B(i) If $A \in \mathscr{S}'_{12}$, then $G\binom{124}{123} \leq 0$ and $-G\binom{123}{125} \geq 0$, so that (5) is automatically satisfied.

B(ii) If $A \in \mathscr{S}_{12}$, then $-G\binom{123}{125}G\binom{124}{123} > 0$, so that (5) is equivalent to

$$G\binom{123}{125}^2 G\binom{124}{123}^2 < G_{12}^2 G_{1234} G_{1235}.$$

So we must prove that the expression $F_2 = G_{12}^2 G_{1234} G_{1235} - G\binom{123}{125}^2 G\binom{124}{123}^2$ is positive. First, we use Sylvester identities to change mixed minors into principal minors, and we get

$$F_2 = G_{123}(-G_{123}G_{124}G_{125} + G_{12}G_{124}G_{1235} + G_{12}G_{125}G_{1234}) = G_{123}R_2,$$

where we have denoted by R_2 the expression into brackets.

On the other hand, under the hypothesis that $G_{123} < 0$ and $G_{124}, G_{125} > 0$, we have that

$$R_2 < R_2 - G_{123}G_{124}G_{125} = -G_{124}G\binom{123}{125}^2 - G_{125}G\binom{123}{124}^2,$$

where again we have used Sylvester identities. Then, we obtain that $R_2 < 0$, and therefore, $F_2 > 0$.

This completes the proof of the sufficient conditions and hence of Theorem 3.1.

3.3. Spherical triangular prisms.

In the spherical case the result is as follows.

THEOREM 3.4. *The space of (cosines) of dihedral angles of spherical triangular prisms is the subset* $\mathscr{A}^S = \mathscr{S}_0^S \cap (\mathscr{S}_{12}'' \cup \mathscr{S}_{12}''') \cap (\mathscr{S}_{13}'' \cup \mathscr{S}_{13}''') \cup (\mathscr{S}_{23}'' \cup \mathscr{S}_{23}''')$ *where we have denoted by* $\mathscr{S}_0^S$, $\mathscr{S}_{12}''$, *etc., the following sets*:

$$\mathscr{S}_0^S = \left\{ \begin{array}{l} a_{ij} \in (-1,1) \\ G_{124} > 0, G_{1234} > 0, G_{1235} > 0 \end{array} \right\},$$

$$\mathscr{S}_{12}'' = \left\{ \begin{array}{l} G\binom{123}{124} \geq 0,\ G\binom{123}{152} \leq 0 \\ G\binom{123}{124}^2 + G\binom{123}{152}^2 > 0 \end{array} \right\},$$

$$\mathscr{S}_{12}''' = \left\{ \begin{array}{l} G\binom{123}{124}G\binom{123}{152} > 0 \\ G\binom{123}{124}(G_{124}G_{1235} - G_{125}G_{1234}) > 0 \end{array} \right\};$$

and the other sets in an analogous way.

PROOF. The proof is similar to the one of the Theorem 3.1. The entry u of the Gram matrix has now the value $u = \frac{G\binom{1234}{1235}_{|u=0} + \sqrt{G_{1234}G_{1235}}}{-G_{123}}$, and we must use an appropriate modification of Lemma 3.2. $\square$

3.4. Geometric comments.

In the previous theorems the necessary condition can also be proved by geometric means: suppose, for instance, that P is a compact hyperbolic triangular prism; we will prove that the point A corresponding to its dihedral angles is in $\mathscr{A}^{CH}$. Conditions in $\mathscr{S}_0^H \cap \mathscr{S}_0^C$ are clear from Theorem 2.1.

Suppose that $G_{123} \geq 0$. This means that the planes containing the faces C_1, C_2, C_3 intersect in a finite or infinite point O of $\boldsymbol{H}^3$. There are two possibilities: either the point O is "closer" to the face C_4 than to the face C_5 or it is "farther".

Suppose we are in the first case. By looking at Figure 1, we can check that the cycles $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$ and $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_4$ (and $\mathscr{C}_1, \mathscr{C}_5, \mathscr{C}_2$) have the same orientation, and then $G\binom{123}{124}$ and $G\binom{123}{152}$ are positive, by Theorem 2.1 applied to the tetrahedra of faces $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4$ and $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_5$. Also, it is geometrically clear that the distance from the vertex V_{124} to the face C_3 is smaller than the distance from V_{152} to the same face. By Lemma 2.3(b), we have that $\frac{-G_{1243}}{G_{124}} < \frac{-G_{1523}}{G_{152}}$, and this is equivalent to $G_{124}G_{1523} - G_{152}G_{1243} < 0$. Therefore, $A \in \mathscr{S}_{12}$.

In the second case, $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$ has different orientation than $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_4$ (and $\mathscr{C}_1, \mathscr{C}_5, \mathscr{C}_2$). Then, now $G\binom{123}{124}$ and $G\binom{123}{152}$ are negative; on the other hand, now we have $G_{124}G_{1523} - G_{152}G_{1243} > 0$, so again $A \in \mathscr{S}_{12}$.

If $G_{123} < 0$, now the three planes above have a common orthogonal plane Π. If this plane intersects the closed segment $V_{124}V_{152}$ then $A \in \mathscr{S}'_{12}$; otherwise we still have $A \in \mathscr{S}_{12}$.

We can do the same study for the other edges $\mathscr{E}_{23}$ and $\mathscr{E}_{13}$, to get the complete result.

3.5. Example.

We show a three dimensional slice of the space of dihedral angles (*not* the cosines) of compact hyperbolic triangular prisms: we consider $\alpha_{12} = \alpha_{23} = \alpha_{31} = \alpha$, $\alpha_{41} = \alpha_{42} = \alpha_{43} = \beta$, and $\alpha_{51} = \alpha_{52} = \alpha_{53} = \gamma$, and we analyze the conditions in Theorem 3.1. The inequalities $G_{1234} < 0$ and $G_{1235} < 0$ are equivalent to

$$\cos\alpha > \frac{1 - 3\cos^2\beta}{2} \qquad \text{and} \qquad \cos\alpha > \frac{1 - 3\cos^2\gamma}{2},$$

respectively. By the symmetry of this example of triangular prism, the subsets $\mathscr{S}_{12}, \mathscr{S}_{13}, \mathscr{S}_{23}$ are equal, and so are $\mathscr{S}'_{12}, \mathscr{S}'_{13}, \mathscr{S}'_{23}$.

We can easily check that

$$\mathscr{S}_{12} = \left\{\beta > \frac{\pi}{2},\, \gamma < \frac{\pi}{2},\, \beta + \gamma < \pi\right\} \cup \left\{\beta < \frac{\pi}{2},\, \gamma > \frac{\pi}{2},\, \beta + \gamma < \pi\right\}$$

$$\mathscr{S}'_{12} = \left\{0 \leq \beta,\, \gamma \leq \frac{\pi}{2},\, (\beta, \gamma) \neq \left(\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}.$$

The condition that G_{123} be negative, zero or positive is equivalent to 3α being respectively less than, equal to or greater than π. Similarly, $G_{124} > 0$ and $G_{125} > 0$ (the remaining minors in $\mathscr{S}_0^C$ are equal to these two) are equivalent to $\alpha + 2\beta > \pi$ and $\alpha + 2\gamma > \pi$. Finally, in the particular case we are studying, the conditions about the mixed minors in $\mathscr{S}_0^C$ are automatically implied by $G_{1234} < 0$ and $G_{1235} < 0$.

In Figure 2 we show the subset of $\boldsymbol{R}^3$ obtained. It is apparent in this figure that the space of dihedral angles is not convex. This illustrates the general result proved in [**5**]

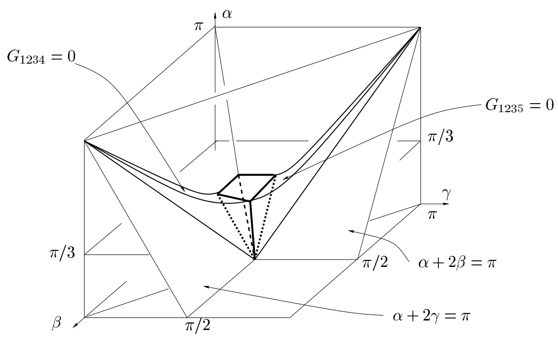

Figure 2. Three dimensional slice of the space of dihedral angles of compact hyperbolic triangular prisms; the Andreev region is the inverted pyramid marked in boldface.

that the space of dihedral angles of hyperbolic polyhedra of any fixed combinatorial type is not convex. We can also see in Figure 2 that the region corresponding to $\alpha, \beta, \gamma \leq \pi/2$ is convex, as proved in Andreev's Theorem.

4. Descendants of the tetrahedron.

Let P be an affine polyhedron and let V be a vertex of P. Consider a plane H that intersects each edge incident to V in an interior point, and let H^- be the closed halfspace determined by H not containing V. Then we say that the polyhedron $P \cap H^-$ is obtained from P by *truncating* V.

A *tetrahedron's descendant* is a polyhedron P that can be obtained from a tetrahedron by successively truncating vertices. That is, there exists a chain of polyhedra $P_4, P_5, \ldots, P_n = P$ where P_4 is a tetrahedron and so that P_k is obtained from P_{k-1} by truncating a vertex. We call $P_4, P_5, \ldots, P_n$ a *generating path* for P. We label the faces of P as $C_1, \ldots, C_n$, so that $C_1, \ldots, C_4$ are the faces of P_4 (labeled in some arbitrary way), C_5 is the new face appearing in P_5 in the truncating process, etc.

We remark that a tetrahedron's descendant is trivalent and that adjacent faces in P_r are still adjacent in P_{r+1}. We will use the following lemma.

Lemma 4.1. *Let P be a tetrahedron's descendant, and $P_1, \ldots, P_n$ a generating path. Then, each $r = 5, \ldots, n$ determines uniquely four indices i_r, j_r, k_r, s_r smaller than r so that the faces $C_{i_r}, C_{j_r}, C_{k_r}, C_r$ are pairwise adjacent, and so are the faces $C_{i_r}, C_{j_r}, C_{k_r}, C_{s_r}$.*

Proof. Clearly, $C_{i_r}, C_{j_r}, C_{k_r}$ are the three faces incident to the vertex V of P_{r-1} that is truncated to obtain P_r. If V was a vertex of the original tetrahedron, then C_{s_r} is the fourth face of that tetrahedron. Otherwise, V has appeared in the trucation process: one of the faces incident to V, say C_{k_r}, is the truncating face, and the other two were incident to the vertex that is truncated. In this case, C_{s_r} is the third face incident to this vertex. $\square$

If $\mathscr{P}$ is the combinatorial type of a tetrahedron's descendant with n faces and $\alpha_{ij} \in (0, \pi)$ are numbers assigned to its edges, we construct the $n \times n$ symmetric matrix $G =$

$G(\alpha, \bar{x})$ whose entry (i, j) is equal to: $-\cos\alpha_{ij}$ if the faces $\mathscr{C}_i, \mathscr{C}_j$ of $\mathscr{P}$ are adjacent; an unknown x_{ij}, otherwise.

PROPOSITION 4.2. *Let $\mathscr{P}$ be the combinatorial type of a tetrahedron's descendant and let $\alpha_{ij} \in (0, \pi)$ be numbers assigned to its edges. With the notation of* Lemma 4.1, *suppose that for each $r \geq 4$, it holds $G_{i_r j_r k_r r} < 0$ (where for $r = 4$ we set $i_4 = 1, j_4 = 2, k_4 = 3$). Then there exists at most one value $\bar{x}_0$ for the vector of unknowns such that*

(i) *the matrix $G(\alpha, \bar{x}_0)$ has rank 4 and signature $(3, 1)$;*
(ii) *for each $r > 4$ then $G\binom{i_r j_r k_r r}{i_r j_r k_r s_r} > 0$.*

PROOF. (i) Consider $r = 5, \ldots, n$. By Lemma 4.1, the submatrix $G[i_r j_r k_r s_r r]$ has exactly one unknown, namely $x_{s_r r}$. In the same way as we did for the triangular prism (see Section 3.2), we assign a value to $x_{s_r r}$ with the condition that the determinant of $G[i_r j_r k_r s_r r]$ be equal to zero. Explicitly:

- if $G_{i_r j_r k_r} = 0$, the unique possible value is:

$$x_{s_r r} = \frac{G_{i_r j_r k_r l_r s_r}{}_{|x_{sr}=0}}{2G\binom{i_r j_r k_r r}{i_r j_r k_r s_r}_{|x_{s_r r}=0}};$$

- if $G_{i_r j_r k_r} \neq 0$, there are two possible values for $x_{s_r r}$, but only one of them satisfies the further condition $G\binom{i_r j_r k_r r}{i_r j_r k_r s_r} > 0$, given in (ii). This value is

$$x_{s_r r} = \frac{G\binom{i_r j_r k_r s}{i_r j_r k_r r}_{|x_{s_r r}=0} - \sqrt{G_{i_r j_r k_r s} G_{i_r j_r k_r r}}}{-G_{i_r j_r k_r}}.$$

(In both cases the value assigned is a well defined real number because $G_{i_r j_r k_r s_r}$ and $G_{i_r j_r k_r r}$ are strictly negative.)

There are still some unknowns in G. Assuming we have assigned a value to all the unknowns of the submatrix $G[1 \ldots r-1]$, then for each $t < r$, $t \neq i_r, j_r, k_r, s_r$, we assign to the unknown x_{tr} the unique value so that $G\binom{i_r j_r k_r s_r\, r}{i_r j_r k_r s_r\, t} = 0$, that is

$$x_{tr} = \frac{-G\binom{i_r j_r k_r s_r\, r}{i_r j_r k_r s_r\, t}_{|x_{tr}=0}}{G_{i_r j_r k_r s_r}}. \qquad \square$$

Once we have the matrix $G(\alpha, \bar{x}_0)$ constructed above, to decide the existence of a hyperbolic polyhedron with the combinatorial type of $\mathscr{P}$ and with dihedral angles α_{ij}, we just need to check whether this matrix satisfies or not all the conditions in Theorem 2.1. Thus, we have

THEOREM 4.3. *Let $\mathscr{P}$ be the combinatorial type of a tetrahedron's descendant and let $\alpha_i \in (0, \pi)$ be numbers assigned to its edges. Then there exists $P \subset \boldsymbol{H}^3$ with the same combinatorial type as $\mathscr{P}$ and with dihedral angles α_{ij} if and only if*

(i) *for each $r \geq 4$, $G_{i_r j_r k_r r} < 0$;*

(ii) *the matrix G constructed in* Proposition 4.2 *satisfies the conditions in* Theorem 2.1 *referred to $\mathscr{P}$.*

Moreover, the polyhedron P is unique up to hyperbolic isometry.

PROOF. We only need to prove that if P is a hyperbolic polyhedron realizing $\mathscr{P}$ and with dihedral angles α_{ij}, then its Gram matrix $G(P)$ coincides with the matrix $G(\alpha, \bar{x}_0)$ constructed in Proposition 4.2, and this will follow if $G(P)$ satisfies conditions (i) and (ii) of that proposition.

Clearly, $G(P)$ satisfies (i), but we remark that (ii) is not automatically deduced from condition (M_4) of Theorem 2.1 because, in general, neither the faces $\mathscr{C}_{i_r}, \mathscr{C}_{j_r}, \mathscr{C}_{k_r}, \mathscr{C}_r$ nor $\mathscr{C}_{i_r}, \mathscr{C}_{j_r}, \mathscr{C}_{k_r}, \mathscr{C}_{s_r}$ are maximal oriented cycles. Therefore, we need the following extra argument.

For the given polyhedron $P = H_1^- \cap \cdots \cap H_n^-$, we consider the auxiliary polyhedron $P' = H_{i_r}^- \cap H_{j_r}^- \cap H_{k_r}^- \cap H_{s_r}^- \cap H_r^-$; then, P' is a (not necessarily compact) triangular prism, with s_r, r the indices of the non-adjacent faces. Therefore, by Theorem 2.1 applied to P', it holds $G\binom{i_r j_r s_r k_r}{i_r r\, j_r k_r} < 0$; hence, changing rows and columns we have the result. □

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