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Thread construction revisited

By Jin-ichi Itoh and Kazuyoshi Kiyohara

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Abstract. Staude's thread construction of ellipsoid is revisited from a new view-point concerning the length of geodesic segments. Thanks to the general nature of this view-point, one obtains similar thread construction on other stages, i.e., on "Liouville manifolds".

1. Introduction.

As is well known, ellipse (resp. hyperbola) in the Euclidean plane $\mathbb{R}^2$ is characterized as a locus of points such that the sum (resp. the difference) of the distances from two fixed points is constant. This property enables one to draw ellipse by means of thread and pins (and also to draw hyperbola with the help of a bit more complicated tools; cf. [**7**], [**8**]). Otto Staude would be the first mathematician who proved that a similar "thread construction" is possible for quadratic surfaces in $\mathbb{R}^3$ ([**9**]).

In this paper we shall explain Staude's construction in view of a simple inequality on the length of geodesics. Since the nature of the inequality is general enough, we obtain similar construction in more general setting, i.e., in Liouville manifolds. For example, hyperquadrics in the Euclidean space $\mathbb{R}^n$ and those in the hyperbolic spaces, intersections of two confocal hyperquadrics in one of them, etc.

This paper is organized as follows. In Section 2 we review the thread construction of Staude for tri-axial ellipsoids in $\mathbb{R}^3$ along with the description by Hilbert and Cohn-Vossen [**2**] and we give an explanation why it works well by means of our inequality. Through Sections 3 to 5 we state thread construction in Liouville manifolds. First, in Section 3, we briefly review the notion and the basic properties of Liouville manifold. The notion of focal submanifold is given, and four typical examples are illustrated. The behavior of geodesics on it is explained in Section 4. We prove there our main theorem (Theorem 4.1) concerning an inequality on the length of geodesics. The thread construction in this setting is then stated and explained in Section 5 in view of this theorem.

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2. Thread construction in $\mathbb{R}^3$.

First, we define two quadratic curves in $\mathbb{R}^3 = \{(u_1, u_2, u_3)\}$, called *focal curves*:

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$$C_2: \ u_2 = 0, \quad \frac{u_1^2}{a_1 - a_2} + \frac{u_3^2}{a_3 - a_2} = 1,$$

$$C_3: \ u_3 = 0, \quad \frac{u_1^2}{a_1 - a_3} + \frac{u_2^2}{a_2 - a_3} = 1,$$

where $a_1 > a_2 > a_3$ are fixed constants. We denote by $C_2^{\pm}$ the connected components of C_2 satisfying $\pm u_1 > 0$. Put

$$L = u_1\text{-axis} : u_2 = u_3 = 0,$$

$$C_2 \cap L = \{\pm s_2\}, \quad C_3 \cap L = \{\pm s_3\} \quad (s_j = \sqrt{a_1 - a_j}),$$

see Figure 1.

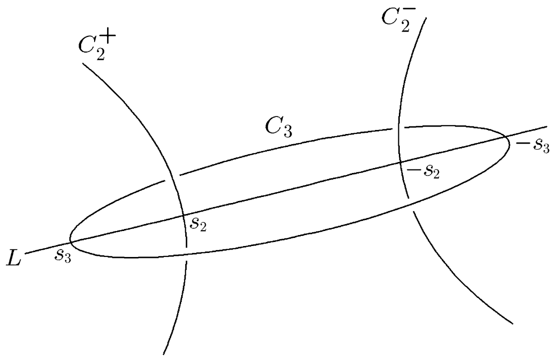

Figure 1.

For a general point $p \in \mathbb{R}^3$, i.e., a point not lying on coordinate planes $N_i : u_i = 0$ $(i = 2, 3)$, there are three (confocal) quadrics passing through $p = (u_1(p), u_2(p), u_3(p))$:

$$Q_i(p): \ \frac{u_1^2}{a_1 - \lambda_i} + \frac{u_2^2}{a_2 - \lambda_i} + \frac{u_3^2}{a_3 - \lambda_i} = 1, \quad (i = 1, 2, 3),$$
$$a_1 \geq \lambda_1 > a_2 > \lambda_2 > a_3 > \lambda_3.$$

Here, $Q_1(p)$ is a connected component of 2-sheeted hyperboloid, $Q_2(p)$ is a 1-sheeted hyperboloid, and $Q_3(p)$ is an ellipsoid. In case $u_1(p) = 0$, we suppose $Q_1(p)$ is the plane $u_1 = 0$. Put

$$Q_i(p) \cap L = \{\pm r_i(p)\}, \ r_i(p) > 0 \ (i = 2, 3), \quad Q_1(p) \cap L = \{r_1(p)\}.$$

Then

$$r_i(p)^2 = a_1 - \lambda_i$$

and

$$r_3(p) \geq s_3 \geq r_2(p) \geq s_2 \geq r_1(p) \geq -s_2. \tag{2.1}$$

The system of functions $(\lambda_1, \lambda_2, \lambda_3)$ thus obtained is nothing but the elliptic coordinate system. Since the functions λ_i are continuously extended to the whole space $\mathbb{R}^3$, we obtain

PROPOSITION 2.1. (1) *The correspondence $p \mapsto (r_1(p), r_2(p), r_3(p))$ is a local diffeomorphism around a general point $p \in \mathbb{R}^3$.*
(2) *The functions r_i are continuously extended to the whole $\mathbb{R}^3$ and satisfy the inequality* (2.1).

In particular, there are the following correspondences:

$$p \in N_i \iff r_i(p) = s_i \quad \text{or} \quad r_{i-1}(p) = s_i \quad (i = 2, 3).$$
$$p \in C_3 \iff r_3(p) = s_3 \quad \text{and} \quad r_2(p) = s_3.$$
$$p \in \pm C_2 \iff r_2(p) = s_2 \quad \text{and} \quad r_1(p) = \pm s_2.$$

We then have the following theorem. The proof will be given in Section 4 under more general setting.

THEOREM 2.2. *If $p(t)$ is a geodesic (a straight line) in $\mathbb{R}^3$, then the length of any segment of it is equal to or greater than the sum of the distances that the corresponding three points $r_i(p(t))$ $(1 \leq i \leq 3)$ on the line L moved out. Moreover, the equality holds if and only if the geodesic $p(t)$ passes through both two focal curves C_2 and C_3.*

REMARK 2.3. (1) The above intersection of a geodesic with C_2 includes the case where they intersects "at infinity", i.e., the case where the geodesic (a straight line) is parallel to one of the asymptotes to the hyperbola C_2.

(2) For almost all points, there are just four straight lines which pass the given point and which pass both two focal curves. (See Section 4 for the detailed explanation.)

Now, let us explain the thread construction due to Staude ([**9**, Section 13], Hilbert and Cohn-Vossen [**2**]). Taking a general point $p_0 \in \mathbb{R}^3$, we shall consider four broken line segments. First let us consider the shortest broken line segment $p(t)$ which starts at $p_0 = p(0)$, passes a point $p(t_1)$ on C_2^+ and reaches the point $s_3 \in L$ at $t = t_2$, where t is the length parameter (see Figure 2).

Then we have:

PROPOSITION 2.4. (1) *The movement of each $r_i(p(t))$ on each interval $(0, t_1)$ and (t_1, t_2) is monotone.*
(2) *$r_1(p(t))$ moves from $r_1(p_0)$ to s_2 for $0 \leq t \leq t_1$, and stay there for $t_1 \leq t \leq t_2$.*
(3) *$r_2(p(t))$ moves from $r_2(p_0)$ to s_2 for $0 \leq t \leq t_1$, and then reverses the direction and moves up to s_3 for $t_1 \leq t \leq t_2$.*
(4) *$r_3(p(t))$ moves monotonously from $r_3(p)$ to s_3 for $0 \leq t \leq t_2$.*

PROOF. The only nontrivial statement would be (4). For this it is enough to note

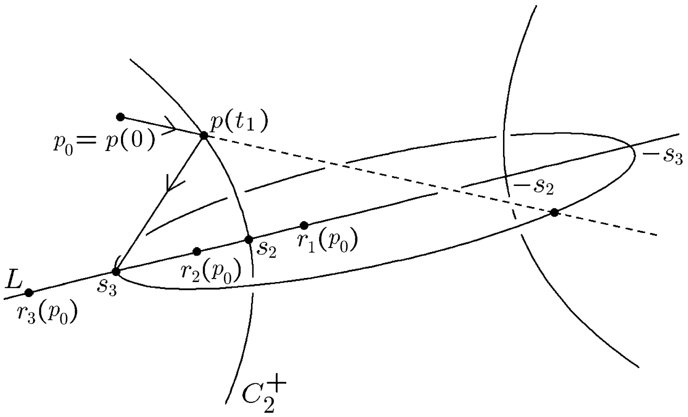

Figure 2.

the fact that the turning point $p(t_1)$ should be in the same side as p_0 with respect to the plane N_3; otherwise, taking the reflection point with respect to the plane N_3 as the new turning point, one would obtain a shorter broken line. □

As a consequence of Theorem 2.2, we have the following proposition.

Proposition 2.5. (1) *Each straight line composing the broken line* $\{p(t)\}$ *passes through both focal curves* C_2 *and* C_3.
(2) *The length of the broken line segment* $\{p(t)\}$ *is equal to*

$$r_3(p_0) + r_2(p_0) - r_1(p_0) - s_2.$$

Proof. By Theorem 2.2 the length of $\{p(t)\}$ is equal to or greater than the sum of the distances that the three points $r_i(p(t))$ moved out, which is

$$\begin{aligned} &\{s_2 - r_1(p_0)\} + \{(r_2(p_0) - s_2) + (s_3 - s_2)\} + \{r_3(p_0) - s_3\} \\ &\quad = r_3(p_0) + r_2(p_0) - r_1(p_0) - s_2, \end{aligned}$$

(see Figure 3). Note that this sum of the distances is common to other broken line segments which joins p_0 and s_3 and whose turning (broken) point is on C_2^+ and in the same side as p_0 with respect to the plane $u_3 = 0$.

Therefore, to prove the proposition, it is enough to show that there is only one point $p_1 \in C_2^+$ such that $u_3(p_0)$ and $u_3(p_1)$ have the same sign and that the line joining p_0 and p_1 passes through C_3. Since the projection image of C_3 from the point p_0 to the

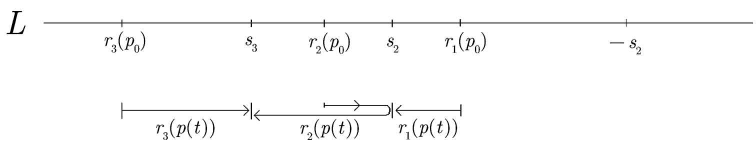

Figure 3.

plane N_2 is an ellipse passing through $\pm s_3$, it follows that this image intersects C_2^+ at two points; one is on the part $u_3 > 0$ and the other one is on the part $u_3 < 0$. Thus there is only one point $p_1 \in C_2^+$ such that $u_3(p_0)$ and $u_3(p_1)$ have the same sign and that the line joining p_0 and p_1 passes through C_3. □

In a similar way, one can obtain the following shortest broken line segments $\bar{p}(t)$, $q(t)$, and $\bar{q}(t)$ with the length parameter t:

- $\bar{p}(t)$ starts at $p_0 = \bar{p}(0)$, turns the direction at a point $\bar{p}(\bar{t}_1)$ on C_3 and reaches the point $s_2 \in L \cap C_2^+$ at $t = \bar{t}_2$ (cf. Figure 4, Figure 5).
  The length of $\{\bar{p}(t)\}$ is, by Theorem 2.2, equal to
  $$r_3(p_0) - r_1(p_0) + (s_3 - r_2(p_0)).$$
- $q(t)$ starts at $p_0 = q(0)$, turns the direction at a point $q(t_1')$ on C_2^- and reaches the point $-s_3 \in L$ at $t = t_2'$ (cf. Figure 6, Figure 7).
  The length of $\{q(t)\}$ is equal to
  $$r_1(p_0) - (-r_3(p_0)) + (-s_2 - (-r_2(p_0))).$$
- $\bar{q}(t)$ starts at $p_0 = \bar{q}(0)$, turns the direction at a point $\bar{q}(\bar{t}_1')$ on C_3 and reaches the point $-s_2 \in L \cap C_2^-$ at $t = \bar{t}_2'$ (cf. Figure 8, Figure 9).
  The length of $\{\bar{q}(t)\}$ is equal to
  $$r_1(p_0) - (-r_3(p_0)) + (-r_2(p_0) - (-s_3)).$$

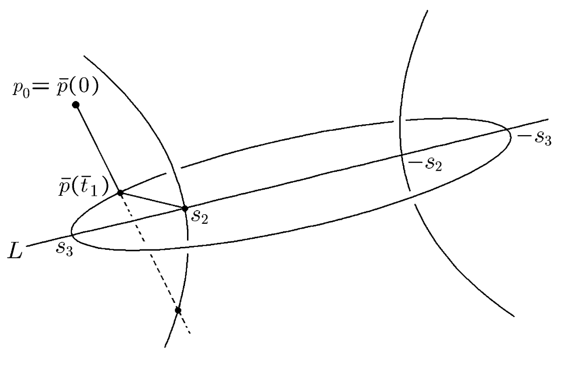

Figure 4.

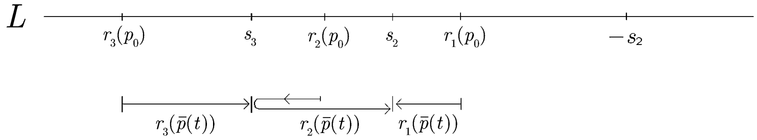

Figure 5.

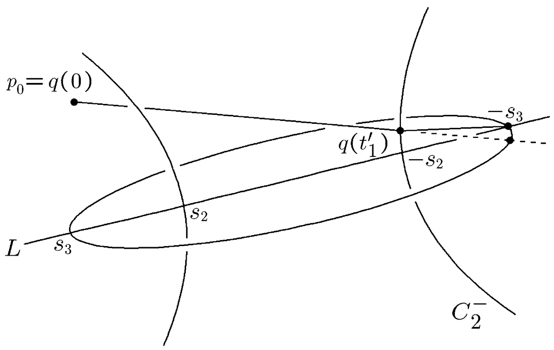

Figure 6.

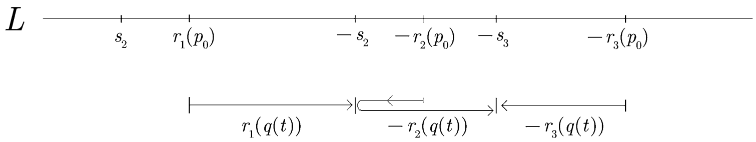

Figure 7.

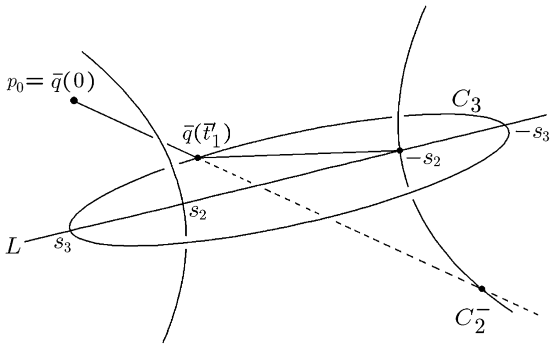

Figure 8.

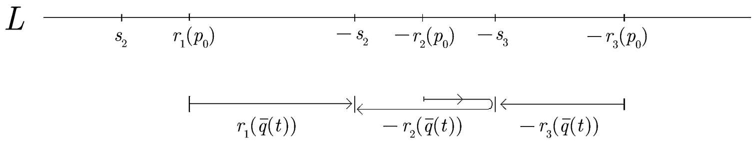

Figure 9.

Thus:

$$\text{Length of } \{p(t)\} = r_3(p_0) - r_1(p_0) + (r_2(p_0) - s_2)$$
$$\text{Length of } \{\bar{p}(t)\} = r_3(p_0) - r_1(p_0) + (s_3 - r_2(p_0))$$
$$\text{Length of } \{q(t)\} = r_1(p_0) - (-r_3(p_0)) + (-s_2 - (-r_2(p_0)))$$
$$\text{Length of } \{\bar{q}(t)\} = r_1(p_0) - (-r_3(p_0)) + (-r_2(p_0) - (-s_3)).$$

Therefore, the sum of the lengths of $\{p(t)\}$ and $\{\bar{q}(t)\}$ is:

$$2r_3(p_0) + s_3 - s_2,$$

which is constant when p_0 is on the ellipsoid

$$\frac{u_1^2}{a_1 - \lambda} + \frac{u_2^2}{a_2 - \lambda} + \frac{u_3^2}{a_3 - \lambda} = 1$$

for a fixed $\lambda < a_3$ $(r_3(p_0) = \sqrt{a_1 - \lambda})$.

Similarly, the difference of the lengths of the segments $\{q(t)\}$ and $\{p(t)\}$ is equal to

$$2r_1(p_0),$$

which is constant when p_0 is on the two-sheeted hyperboloid

$$\frac{u_1^2}{a_1 - \lambda} + \frac{u_2^2}{a_2 - \lambda} + \frac{u_3^2}{a_3 - \lambda} = 1$$

for a fixed $a_1 > \lambda > a_2$ $\quad (r_1(p_0) = \sqrt{a_1 - \lambda})$.

Also, the difference of the lengths of the segments $\{p(t)\}$ and $\{\bar{p}(t)\}$ is equal to

$$2r_2(p_0) - s_2 - s_3,$$

which is constant when p is on the one-sheeted hyperboloid

$$\frac{u_1^2}{a_1 - \lambda} + \frac{u_2^2}{a_2 - \lambda} + \frac{u_3^2}{a_3 - \lambda} = 1$$

for a fixed $a_2 > \lambda > a_3$ $\quad (r_2(p_0) = \sqrt{a_1 - \lambda})$.

Remark 2.6. Actually, Staude's paper [**9**] contains more general thread configurations, i.e., the case where the focal curves C_2 and C_3 are replaced by confocal hyperboloids of one sheet and confocal ellipsoids respectively. We do not treat this case here. See [**1**] for such configurations and detailed historical remarks.

3. Liouville manifolds.

Liouville manifold is, roughly speaking, a class of Riemannian manifold whose geodesic flow is integrated in the same way as that of ellipsoid. In particular, its geodesic flow is completely integrable in the sense of Hamiltonian mechanics. (For the precise definition, see [**6**].) Here, we need a certain restricted version (a subclass of "Liouville manifold of rank one, type (C)" in [**6**]) and we shall explain it now.

3.1. Construction.

Liouville manifold treated in the present paper is defined with n constants $a_1 > a_2 > \cdots > a_n$ and a positive C^∞ function $A(\lambda)$ defined on the interval $-\infty < \lambda \leq a_1$.

To make the constructed manifold being complete, we assume the following condition on the growth rate of $A(\lambda)$ as $\lambda \to -\infty$:

$$\int_{-\infty}^{c} \frac{A(\lambda)}{\sqrt{-\lambda}} d\lambda = \infty, \tag{3.1}$$

where c is any negative number.

First, we define n positive numbers $\alpha_1, \dots, \alpha_n$ (α_n may be ∞) by the formula

$$\alpha_i = 2\int_{a_{i+1}}^{a_i} \frac{A(\lambda)\,d\lambda}{\sqrt{(-1)^i \prod_{j=1}^n (\lambda - a_j)}} \qquad (1 \le i \le n-1),$$

$$\alpha_n = \frac{1}{2}\int_{-\infty}^{a_n} \frac{A(\lambda)\,d\lambda}{\sqrt{(-1)^n \prod_{j=1}^n (\lambda - a_j)}}.$$

Also, we define n functions $f_i(x_i)$ $(1 \le i \le n)$;

$$f_i : \mathbb{R}/\alpha_i\mathbb{Z} = \{x_i\} \to [a_{i+1}, a_i] \qquad (i \le n-1),$$

$$f_n : (-\alpha_n, \alpha_n) \to (-\infty, a_n]$$

as the inverse functions of the integrals of the 1-forms

$$\frac{A(\lambda)\,d\lambda}{2\sqrt{(-1)^i \prod_{j=1}^n (\lambda - a_j)}},$$

i.e., as the functions satisfying

$$\begin{gathered}
\left(\frac{df_i}{dx_i}\right)^2 = \frac{(-1)^i 4 \prod_{j=1}^n (f_i - a_j)}{A(f_i)^2}, \\
f_i(0) = a_{i+1}, \quad f_i\left(\frac{\alpha_i}{4}\right) = a_i \quad (i \le n-1), \\
f_i(-x_i) = f_i(x_i) = f_i\left(\frac{\alpha_i}{2} - x_i\right) \quad (i \le n-1), \\
f_n(0) = a_n, \quad \lim_{x_n \to \alpha_n} f_n(x_n) = -\infty, \quad f_n(-x_n) = f_n(x_n).
\end{gathered} \tag{3.2}$$

Let us construct a generalized cylinder

$$R = \prod_{i=1}^{n-1} (\mathbb{R}/\alpha_i\mathbb{Z}) \times (-\alpha_n, \alpha_n) = \{(x_1, \dots, x_n)\}.$$

Let $\tau_1, \dots, \tau_{n-1}$ be the involutions on R given by

$$\tau_i(x_1,\dots,x_n)=\left(x_1,\dots,-x_i,\frac{\alpha_{i+1}}{2}-x_{i+1},\dots,x_n\right)\quad(i\le n-2),$$
$$\tau_{n-1}(x_1,\dots,x_n)=(x_1,\dots,-x_{n-1},-x_n),$$

and let G be the group of transformations of R generated by τ_i's. Then G is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ and the quotient space $M=R/G$ is, with the natural differentiable structure, diffeomorphic to $\mathbb{R}^n$.

Now, put

$$b_{ij}(x_i)=\begin{cases}(-1)^i\prod_{\substack{1\le k\le n-1\\ k\ne j}}(f_i(x_i)-a_{k+1}) & (1\le j\le n-1)\\ (-1)^{i+1}\prod_{1\le k\le n-1}(f_i(x_i)-a_{k+1}) & (j=n)\end{cases}$$

and define functions $F_1,\dots,F_n$ on the cotangent bundle by

$$\sum_{j=1}^n b_{ij}(x_i)F_j=\xi_i^2\qquad(1\le i\le n),\tag{3.3}$$

where $(\xi_1,\dots,\xi_n)$ are fiber coordinates associated with $(x_1,\dots,x_n)$. Although F_i have singularities as functions on T^*R, they define well-defined, smooth functions on T^*M via the quotient map $R\to M$. Moreover, it turns out that F_n is positive definite on each fiber. Thus it defines a Riemannian metric g on M;

$$g=\sum_{i=1}^n(-1)^{n-i}\prod_{j\ne i}(f_j(x_j)-f_i(x_i))\,dx_i^2,$$

with which M becomes a complete Riemannian manifold. The metric g is also expressed as

$$g=\sum_{i=1}^n\frac{\prod_{j\ne i}(f_i-f_j)}{-4\prod_{k=1}^n(f_i-a_k)}\,A(f_i)^2\,df_i^2.\tag{3.4}$$

Let $\mathcal{F}$ be the vector space spanned by $F_1,\dots,F_n$. From the formula (3.3) one can easily see that $\mathcal{F}$ is commutative with respect to the Poisson bracket:

$$\{F_i,F_j\}=0\quad\text{for any } i,j.$$

Since the Hamiltonian of the geodesic flow of the Riemannian manifold M (with the metric g) is $F_n/2$, the geodesic flow is completely integrable by means of the first integrals in $\mathcal{F}$. The pair $(M,\mathcal{F})$ thus obtained is the Liouville manifold which we use here.

3.2. Examples.

Here we shall describe four typical examples.

(I) If the function $A(\lambda)$ is identically 1, then the resulting Riemannian manifold M

is isometric to the flat $\mathbb{R}^n$. In this case the functions $f_i(x_i)$ are nothing but the elliptic coordinates λ_i defined by the identity in λ

$$\sum_{i=1}^n \frac{u_i^2}{a_i-\lambda} - 1 = \frac{-\prod_{j=1}^n(\lambda_j-\lambda)}{\prod_{i=1}^n(a_i-\lambda)}, \quad (u_1,\dots,u_n)\in\mathbb{R}^n$$

and the inequality

$$a_1 \geq \lambda_1 \geq a_2 \geq \lambda_2 \geq \cdots \geq a_n \geq \lambda_n > -\infty.$$

The Euclidean metric g_0 is given by

$$g_0 = \frac{1}{4}\sum_{i=1}^n \frac{\prod_{k\neq i}(\lambda_i-\lambda_k)}{-\prod_{k=1}^n(\lambda_i-a_k)}\, d\lambda_i^2.$$

(II) The two-sheeted hyperboloid

$$\sum_{i=0}^n \frac{u_i^2}{a_i} = 1 \qquad (a_0 > 0 > a_1 > \cdots > a_n), \quad u_0 > 0,$$

is isometric to the Liouville manifold constructed with the constants $a_1,\dots,a_n$ and the function

$$A(\lambda) = \sqrt{\frac{-\lambda}{a_0-\lambda}} \qquad \text{on} \quad (-\infty, a_1].$$

In fact, by using the elliptic coordinates $\lambda_1,\dots,\lambda_n$ $(a_i \geq \lambda_i \geq a_{i+1})$ on the hyperboloid defined by the following identity in λ:

$$\sum_{i=0}^n \frac{u_i^2}{a_i-\lambda} - 1 = \frac{\prod_{j=1}^n \lambda(\lambda_j-\lambda)}{\prod_{i=0}^n(a_i-\lambda)},$$

the metric is expressed as

$$g = \frac{1}{4}\sum_{i=1}^n \frac{\lambda_i\prod_{k\neq i}(\lambda_i-\lambda_k)}{-\prod_{k=0}^n(\lambda_i-a_k)}\, d\lambda_i^2.$$

Comparing this with the formula (3.4), one obtains the above $A(\lambda)$ by putting $f_i = \lambda_i$.

(III) Let us consider the hyperbolic space H^n with constant curvature -1 realized in the Minkowski space $\{(u_0,\dots,u_n)\}$ with the flat Lorentz metric $-du_0^2 + \sum_{i=1}^n du_i^2$;

$$H^n: \ -u_0^2 + \sum_{i=1}^n u_i^2 = -1, \qquad u_0 > 0.$$

For given constants $a_0 > a_1 > \cdots > a_n$, the elliptic coordinates $\lambda_1, \ldots, \lambda_n$ are defined on H by the identity in λ;

$$-\frac{u_0^2}{a_0 - \lambda} + \sum_{i=1}^{n} \frac{u_i^2}{a_i - \lambda} = \frac{-\prod_{j=1}^{n}(\lambda_j - \lambda)}{\prod_{i=0}^{n}(a_i - \lambda)}$$

and the inequalities

$$a_0 > a_1 \geq \lambda_1 \geq a_2 \geq \cdots \geq a_n \geq \lambda_n > -\infty.$$

The metric g is described as

$$g = \frac{1}{4} \sum_{i=1}^{n} \frac{\prod_{j \neq i}(\lambda_i - \lambda_j)}{\prod_{k=0}^{n}(\lambda_i - a_k)} \, d\lambda_i^2.$$

It therefore turns out that H^n is isometric to the Liouville manifold constructed with the constants $a_1, \ldots, a_n$ and the function

$$A(\lambda) = \frac{1}{\sqrt{a_0 - \lambda}} \qquad \text{on} \quad (-\infty, a_1]$$

by putting $f_i = \lambda_i$.

(IV) The elliptic paraboloid

$$2u_0 + \sum_{i=1}^{n} \frac{u_i^2}{a_i} = 0 \qquad (0 > a_1 > \cdots > a_n)$$

has elliptic (parabolic) coordinates $\lambda_1, \ldots, \lambda_n$ defined by

$$2u_0 + \lambda + \sum_{i=1}^{n} \frac{u_i^2}{a_i - \lambda} = \frac{\lambda \prod_{j=1}^{n}(\lambda_j - \lambda)}{\prod_{i=1}^{n}(a_i - \lambda)}$$

and the inequalities

$$0 > a_1 \geq \lambda_1 \geq a_2 \geq \cdots \geq a_n \geq \lambda_n > -\infty.$$

With these the metric is described as

$$g = \frac{1}{4} \sum_{i=1}^{n} \frac{\lambda_i \prod_{j \neq i}(\lambda_i - \lambda_j)}{\prod_{k=1}^{n}(\lambda_i - a_k)} \, d\lambda_i^2.$$

Thus one knows that the elliptic paraboloid is constructed with the constants $a_1, \ldots, a_n$ and the function

$$A(\lambda) = \sqrt{-\lambda} \qquad \text{on} \quad (-\infty, a_1].$$

3.3. Special submanifolds.

We first introduce two kinds of submanifolds N_i and C_i of M $(2 \le i \le n)$:

$$N_i = \{x \in M \mid f_{i-1}(x_{i-1}) = a_i \quad \text{or} \quad f_i(x_i) = a_i\},$$
$$C_i = \{x \in M \mid f_{i-1}(x_{i-1}) = f_i(x_i) = a_i\}.$$

We shall call C_i *focal submanifolds* of M. Note that C_2 has two connected components $C_2^{\pm}$ and they are distinguished by the value of x_1; $x_1 = 0$ on C_2^+ and $x_1 = \alpha_1/2$ on C_2^-. We also define submanifolds $Q_i(p)$ $(1 \le i \le n, p \in M)$ of M by

$$Q_i(p) = \{x \in M \mid x_i = x_i(p)\},$$

which we simply call *coordinate hypersurfaces*. Actually, it is a manifold without boundary only when $x_i(p) \ne a_i, a_{i+1}$. When $x_i(p)$ is equal to a_i (resp. a_{i+1}), then $Q_i(p)$ represents a closed region of N_i (resp. N_{i+1}) whose boundary is C_i (resp. C_{i+1}). The following lemmas can be verified by comparing them with the case of flat $\mathbb{R}^n$.

Lemma 3.1. (1) *N_i is totally geodesic and diffeomorphic to $\mathbb{R}^{n-1}$.*
(2) *$C_i \subset N_i$ and C_i is diffeomorphic to $S^{i-2} \times \mathbb{R}^{n-i}$.*

Lemma 3.2. *$Q_i(p)$ $(2 \le i \le n)$ is diffeomorphic to $S^{i-1} \times \mathbb{R}^{n-i}$ and $Q_1(p)$ is diffeomorphic to $\mathbb{R}^{n-1}$, provided $x_i(p) \ne a_i, a_{i+1}$.*

In the case of the Euclidean space $\mathbb{R}^n = \{(u_1, \ldots, u_n)\}$ (the case where $A(\lambda) = 1$), the focal submanifold C_i is given by

$$C_i : \ u_i = 0, \quad \sum_{\substack{1 \le j \le n \\ j \ne i}} \frac{u_j^2}{a_j - a_i} = 1 \qquad (2 \le i \le n),$$

and N_i is the hyperplane $u_i = 0$ containing C_i. Also, the coordinate hypersurfaces $Q_i(p)$ are equal to the (connected component of) confocal quadrics

$$\mathcal{Q}(\lambda) : \ \sum_{i=1}^{n} \frac{u_i^2}{a_i - \lambda} = 1$$

for some $\lambda \in (a_{i+1}, a_i)$.

The submanifolds C_i and N_i have special meanings in the theory of Liouville manifold:

Proposition 3.3. (1) $C_i = \{p \in M \mid F_{i-1}|_{T_p^*M} = 0\}$.
(2) $N_i = \{p \in M \mid \operatorname{rank}(F_{i-1}|_{T_p^*M}) \le 1\}$.
(3) *$\bigcup_{j=2}^{n} C_j$ is identical with the branch locus of the branched covering $R \to M$.*

For the proof and the detailed explanation of these properties, see [**6**].

We put

$$L = \bigcap_{i=2}^{n} N_i,$$

which is a one-dimensional, connected, totally geodesic submanifold (i.e., a geodesic) diffeomorphic to $\mathbb{R}^1$. This submanifold is called *the core submanifold* of the Liouville manifold $(M, \mathcal{F})$. We identify L with a straight line $\mathbb{R} = \{s\}$ isometrically so that the origin $s = 0$ corresponds to the point

$$x = \left(\frac{\alpha_1}{4}, \frac{\alpha_2}{4}, \dots, \frac{\alpha_{n-1}}{4}, 0\right) \in L \subset M$$

and that the points $s = s_j \in \mathbb{R}$ $(2 \le j \le n)$ corresponding to

$$x = \left(0, \dots, 0, \frac{\alpha_j}{4}, \dots, \frac{\alpha_{n-1}}{4}, 0\right) \in L$$

satisfy

$$0 < s_2 < \cdots < s_n.$$

Then we have

Lemma 3.4. (1) *$C_j \cap L = \{\pm s_j\}$ $(2 \le j \le n)$.*
(2) *$C_2^+ \cap L = \{s_2\}$, $C_2^- \cap L = \{-s_2\}$.*
(3) *On the intervals (s_j, s_{j+1}) and $(-s_{j+1}, -s_j)$ for $j \ge 2$ $(s_{n+1} = \infty)$ and the interval $(-s_2, s_2)$ for $j = 1$, any coordinate function x_k except x_j is constant and the line element ds is given by*

$$ds^2 = (-1)^{n-j} \prod_{k=2}^{n} (a_k - f_j(x_j))\, dx_j^2 = \frac{A(f_j)^2\, df_j^2}{4(a_1 - f_j)}. \tag{3.5}$$

The proof is straightforward. For any point $p \in M$ which is not lying on $N_j \cup N_{j+1}$, the intersection $Q_j(p) \cap L$ consists of two points of the form $s = \pm r_j(p)$ for $j \ge 2$ and one point $s = r_1(p)$ for $j = 1$. We assume $r_j(p) > 0$ for $j \ge 2$. Clearly,

$$-s_2 \le r_1(p) \le s_2 \le \cdots \le s_j \le r_j(p) \le s_{j+1} \le \cdots \le s_n \le r_n(p) \tag{3.6}$$

and the functions $r_i(p)$ are continuously extended to the whole manifold M so that they satisfy the above inequalities.

The totally geodesic submanifold N_j $(2 \le j \le n)$ can be identified with the Liouville manifold constructed with the constants $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n$ and the function $A(\lambda)$ on $(-\infty, a_1]$. In fact, putting

$$\bar{a}_i = \begin{cases} a_i & (1 \le i \le j-1) \\ a_{i+1} & (j \le i \le n-1) \end{cases}$$

and denoting the symbols of the corresponding objects with bar; $\bar{\alpha}_i$, $\bar{x}_i$, $\bar{f}_i : \mathbb{R}/\bar{\alpha}_i \to [\bar{a}_{i+1}, \bar{a}_i]$ $(1 \le i \le n-2)$, $\bar{f}_{n-1} : \mathbb{R}/\bar{\alpha}_{n-1} \to (-\infty, \bar{a}_{n-1}]$, etc., one obtains the identification of the Liouville manifold with N_j by putting

$$\bar{f}_i(\bar{x}_i) = \begin{cases} f_i(x_i) & (1 \le i \le j-2) \\ f_{j-1}(x_{j-1}) + f_j(x_j) - a_j & (i = j-1) \\ f_{i+1}(x_{i+1}) & (j \le i \le n-1). \end{cases} \tag{3.7}$$

Note that $\bar{f}_{j-1}$ is actually equal to f_{j-1} or f_j, since $f_j = a_j$ or $f_{j-1} = a_j$ on N_j. It is easy to see that the focal submanifolds of N_j are $C_i \cap N_j$ $(2 \le i \le n,\ i \ne j)$ and that the core submanifold is L. The submanifold C_j is merely a coordinate hypersurface in the Liouville manifold N_j and it does not have special meaning any more.

4. Behavior of geodesics.

Geodesic equations on Liouville manifolds admit separation of variables. More precisely, we have the following theorem.

Theorem 4.1. *Let $\gamma(t) = (x_1(t), \dots, x_n(t))$ be a geodesic parametrized by arc length. Then, around points where $\gamma(t) \notin C_j$ for any j and $x_i'(t) \ne 0$ for any i, it satisfies the equations*

$$\sum_{i=1}^{n} \frac{(-1)^i G(f_i(x_i)) A(f_i(x_i))}{\sqrt{-\prod_{k=2}^{n}(f_i(x_i) - b_k) \cdot \prod_{k=1}^{n}(f_i(x_i) - a_k)}} \left| \frac{df_i(x_i(t))}{dt} \right| = 0, \tag{4.1}$$

$$\sum_{i=1}^{n} \frac{(-1)^{i-1} \tilde{G}(f_i(x_i)) A(f_i(x_i))}{2\sqrt{-\prod_{k=2}^{n}(f_i(x_i) - b_k) \cdot \prod_{k=1}^{n}(f_i(x_i) - a_k)}} \left| \frac{df_i(x_i(t))}{dt} \right| = 1, \tag{4.2}$$

where $G(\lambda)$ is any polynomial in λ of degree $\le n-2$, and $\tilde{G}(\lambda)$ is any monic polynomial of degree $n-1$. Here, $b_2 > \cdots > b_n$ are constants (depending on the geodesic) satisfying

$$f_1(x_1(t)) > b_2 > f_2(x_2(t)) > b_3 > \cdots > b_n > f_n(x_n(t)). \tag{4.3}$$

In particular,

$$a_{i-1} > b_i > a_{i+1}, \quad b_i > b_{i+1} \quad \textit{for any } i. \tag{4.4}$$

Proof. Since the first integrals F_i's are constant along each solution curves $(x_1(t), \dots, x_n(t), \xi_1(t), \dots, \xi_n(t))$ of the geodesic flow, we have

$$\sum_{j=1}^{n} b_{ij}(x_i(t)) c_j = \xi_i(t)^2 \tag{4.5}$$

for some constants $c_1, \dots, c_n$, in view of (3.3). Considered on the unit cotangent bundle,

c_n must be equal to 1. Since $(1/2)F_n$ is the Hamiltonian of the geodesic flow, we have

$$\frac{dx_i}{dt} = \frac{1}{2}\frac{\partial F_n}{\partial \xi_i} = \frac{(-1)^{n-i}\xi_i}{\prod_{j\neq i}(f_j(x_j) - f_i(x_i))}. \tag{4.6}$$

Therefore from the assumption it follows that $\xi_i(t) \neq 0$ for any i around the consideration point. Then, putting

$$\Theta(\lambda) = \sum_{j=1}^{n-1} c_j \prod_{\substack{1\leq k\leq n-1\\ k\neq j}} (\lambda - a_{k+1}) - \prod_{1\leq k\leq n-1} (\lambda - a_{k+1}), \tag{4.7}$$

we have

$$\xi_i(t)^2 = \sum_{j=1}^{n} b_{ij}(x_i(t))c_j = (-1)^i\Theta(f_i(x_i(t))) > 0. \tag{4.8}$$

Therefore the polynomial $\Theta(\lambda)$ has $n-1$ real roots $b_2 > \cdots > b_n$ satisfying

$$f_1(x_1(t)) > b_2 > f_2(x_2(t)) > b_3 > \cdots > b_n > f_n(x_n(t)).$$

This and the inequalities $a_i \geq f_i(x_i) \geq a_{i+1}$ imply (4.3) and (4.4).

Now, using (3.2), (4.6), and (4.8), we obtain the formula

$$\left|\frac{df_i(x_i(t))}{dt}\right| = \frac{2\sqrt{-\prod_{k=2}^{n}(f_i(x_i) - b_k)\cdot\prod_{k=1}^{n}(f_i(x_i) - a_k)}}{(-1)^{n-i}\prod_{j\neq i}(f_j(x_j) - f_i(x_i))A(f_i(x_i))}. \tag{4.9}$$

Then the formulas (4.1) and (4.2) follows from the identity:

$$\sum_{i=1}^{n} \frac{f_i^k}{\prod_{\substack{1\leq j\leq n\\ j\neq i}}(f_j - f_i)} = \begin{cases} 0 & (0 \leq k \leq n-2) \\ (-1)^{n-1} & (k = n-1). \end{cases}$$ □

In the above proof the functions $b_2, \ldots, b_n$ are defined for the unit covectors (x, ξ) such that x is not on the branch locus $\bigcup_i C_i$ and every $\xi_i \neq 0$. Clearly the set of such covectors are open and dense in the unit cotangent bundle and therefore the functions b_i are continuously extended to the whole unit cotangent bundle so that they are invariant under the geodesic flow; the range is given by

$$a_{i-1} \geq b_i \geq a_{i+1}, \quad b_i \geq b_{i+1} \quad (\text{any } i). \tag{4.10}$$

In case $b_{i+1} = a_i$ or $b_i = a_{i+1}$ or $b_i = b_{i+1}$, along the corresponding geodesic the coordinate function $x_i(t)$ remains constant. If $b_{i+1} = a_i$ or $b_i = a_{i+1}$, then the geodesic is totally contained in the submanifold N_i or N_{i+1} respectively.

Now, let us observe the behavior of each coordinate function $x_i(t)$ (and $f_i(x_i(t))$)

along a geodesic $\gamma(t)$. Put

$$a_i^+ = \max\{a_i, b_i\}, \quad a_i^- = \min\{a_i, b_i\} \quad (2 \le i \le n), \quad a_1^- = a_1,$$

and let Λ_i $(1 \le i \le n-1)$ (resp. Λ_n) be a connected component of the inverse image of $[a_{i+1}^+, a_i^-]$ (resp. $(-\infty, a_n^-]$) by the mapping

$$f_i : \mathbb{R}/\alpha_i\mathbb{Z} \to [a_{i+1}, a_i] \qquad (\text{resp. } f_n : (-\alpha_n, \alpha_n) \to (-\infty, a_n]).$$

Note that each Λ_i is an interval or the whole circle.

Suppose that, putting $a_{n+1} = -\infty$,

$$a_{i+1} < b_i < a_{i-1} \quad \text{and} \quad b_i \ne a_i \quad \text{for any } i = 2, \dots, n. \tag{4.11}$$

Then a connected component N of the locus defined by $F_i = c_i$ $(1 \le i \le n)$ in the cotangent bundle T^*M is a Lagrangean submanifold diffeomorphic to the cylinder $(S^1)^{n-1} \times \mathbb{R}$, and its image $\pi(N) \subset M$ by the bundle projection $\pi : T^*M \to M$ is identical to the image of

$$\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n$$

by the quotient mapping $R \to M$. (Observe that $\Lambda_1 \times \cdots \times \Lambda_n$ is injectively mapped into M by the quotient mapping in this case.) It then turns out from the formula (4.9) that the derivative of each function $x_i(t)$ does not vanish on the interior of Λ_i and that if $x_i(t_0)$ is at the endpoint of Λ_i, then $x_i''(t_0) \ne 0$ and $x_i(t)$ reverses the direction when t passes over t_0.

Note, however, that when t goes to the infinity, the total variation of $x_i(t)$ $(i \le n-1)$ is finite in some cases and infinite in other cases; it depends on the growth rate of $A(\lambda)$ as λ tends to $-\infty$. In the four cases of Section 3.2, the case of elliptic paraboloid is the only one where the above-mentioned total variation is infinite. For an explanation to this phenomena, see [**5**].

In the case of the Euclidean space $\mathbb{R}^n$, i.e., the case where $A(\lambda) = 1$, the behavior of geodesics explained above is known as the result due to Chasles: Each geodesic (a straight line) $\gamma(t) = (\lambda_1(t), \dots, \lambda_n(t))$ is tangent to $n-1$ quadrics $\mathcal{Q}(b_i)$ $(2 \le i \le n)$ for some constants $b_2 \ge \cdots \ge b_n$, where the quadric $\mathcal{Q}(\lambda)$ is given by

$$\mathcal{Q}(\lambda) : \ \sum_{i=1}^{n} \frac{u_i^2}{a_i - \lambda} = 1,$$

and each $\lambda_i(t)$ (resp. $\lambda_n(t)$) moves on the interval $[a_{i+1}^+, a_i^-]$ (resp. $(-\infty, a_n^-]$). Namely, the boundary of $\Lambda_1 \times \cdots \times \Lambda_n \subset M$ is given by the quadrics $\mathcal{Q}(b_i)$'s in this case.

We now state our fundamental theorem:

Theorem 4.2. *Let $\gamma(t) = (x_1(t), \dots, x_n(t))$ be a general geodesic on M, i.e., it is not totally contained in any submanifold N_j. Then the length of any segment of it is* *equal to or greater than the sum of the distances that the corresponding n points $r_j(\gamma(t))$ $(1 \le i \le n)$ on L moved out. Moreover, the equality holds if and only if $b_i = a_i$ for every $i = 2, \dots, n$.*

Proof. The length t_0 of the geodesic $\gamma(t) = (x_1(t), \dots, x_n(t))$ $(0 \le t \le t_0)$ is given by

$$\sum_{i=1}^{n} \frac{(-1)^{i-1}\tilde{G}(f_i(x_i))A(f_i(x_i))}{2\sqrt{-\prod_{k=2}^{n}(f_i(x_i)-b_k)\cdot\prod_{k=1}^{n}(f_i(x_i)-a_k)}} \left|\frac{df_i(x_i(t))}{dt}\right| = 1,$$

where $\tilde{G}(\lambda)$ is any polynomial of the form

$$\tilde{G}(\lambda) = \lambda^{n-1} + (\text{lower order terms}).$$

Putting $\tilde{G}(\lambda) = \prod_{k=2}^{n}(\lambda - b_k)$, we have

$$\frac{1}{2}\sum_{i=1}^{n}\int_0^{t_0} \frac{A(f_i(x_i))}{\sqrt{a_1 - f_i(x_i)}} \sqrt{\frac{|\prod_{k=2}^{n}(f_i(x_i)-b_k)|}{|\prod_{k=2}^{n}(f_i(x_i)-a_k)|}} \left|\frac{df_i(x_i(t))}{dt}\right| dt = t_0. \tag{4.12}$$

Putting $\tilde{G}(\lambda) = \prod_{k=2}^{n}(\lambda - a_k)$, we also have

$$\frac{1}{2}\sum_{i=1}^{n}\int_0^{t_0} \frac{A(f_i(x_i))}{\sqrt{a_1 - f_i(x_i)}} \sqrt{\frac{|\prod_{k=2}^{n}(f_i(x_i)-a_k)|}{|\prod_{k=2}^{n}(f_i(x_i)-b_k)|}} \left|\frac{df_i(x_i(t))}{dt}\right| dt = t_0. \tag{4.13}$$

Now, let us take the arithmetic mean of the above two formulas. Since

$$\frac{1}{2}\left(\sqrt{\frac{|\prod_{k=2}^{n}(\lambda_i-b_k)|}{|\prod_{k=2}^{n}(\lambda_i-a_k)|}} + \sqrt{\frac{|\prod_{k=2}^{n}(\lambda_i-a_k)|}{|\prod_{k=2}^{n}(\lambda_i-b_k)|}}\right) \ge 1,$$

it therefore follows that

$$t_0 \ge \frac{1}{2}\sum_{i=1}^{n}\int_0^{t_0} \frac{A(f_i(x_i))}{\sqrt{a_1 - f_i(x_i)}} \left|\frac{df_i(x_i(t))}{dt}\right| dt. \tag{4.14}$$

By Lemma 3.4, the right-hand side is equal to the sum of the distances that the n points $r_j(\gamma(t))$ $(1 \le i \le n)$ moved out. Clearly, the equality holds in (4.14) if and only if $b_i = a_i$ for any i $(2 \le i \le n)$. □

Let us observe the detailed behavior of geodesics such that $b_i = a_i$ for every i, which will be necessary in the next section. Let p_0 be a point in M which is not contained in any hypersurfaces N_j $(2 \le j \le n)$. We shall first show the following lemmas.

Lemma 4.3. *There are just 2^n unit covectors at p_0 satisfying $b_i = a_i$ for any i $(2 \le i \le n)$.*

Proof. Let us consider a unit covector $\mu \in T^*_{p_0}M$ satisfying $b_i = a_i$ for any i. By the identities (4.7) and

$$\Theta(\lambda) = \prod_{i=2}^{n}(\lambda - b_i),$$

we see that $b_i = a_i$ for every i if and only if $c_j = 0$ (i.e., $F_j(\mu) = 0$) for every $j = 1, \dots, n-1$. Then, by the formula (3.3), the coordinates (x, ξ) of μ satisfy

$$(-1)^{i+1} \prod_{2 \le k \le n} (f_i(x_i) - a_k) = \xi_i^2 \quad (1 \le i \le n).$$

Since the left-hand side does not vanish, we have two choices of ξ_i for each i. □

Let $\gamma(t) = (x_1(t), \dots, x_n(t))$ be a geodesic such that $\gamma(0)$ is not contained in any N_j for any j $(2 \le j \le n)$.

Lemma 4.4. *If $\gamma(t)$ passes every focal submanifolds C_j $(2 \le j \le n)$, then $b_j = a_j$ for any j.*

Proof. In view of Proposition 3.3 (1), the assumption indicates $c_i = 0$ for $i = 1, \dots, n-1$. Hence the assertion follows. □

In spite of the above lemmas, it is not necessarily true that any geodesic satisfying $b_j = a_j$ $(2 \le j \le n)$ pass every focal submanifold C_j $(2 \le j \le n)$. It *is* true for elliptic paraboloids, but not true for the Euclidean space nor for the hyperbolic space. It may be said that in the latter cases some geodesics go away to the infinity before reaching some focal submanifolds. However, we have the following proposition, which is enough for our purpose.

Proposition 4.5. *Let $\gamma(t)$ be a geodesic such that $\gamma(0)$ is not contained in any N_j and $b_j = a_j$ for any j $(2 \le j \le n)$. Then*:

(1) *If $\gamma(t)$ passes some N_j, then the intersection point is on C_j.*
(2) *$\gamma(t)$ passes the focal submanifold C_n.*

Proof. (1) Suppose that $\gamma(t) \notin N_j$ for $0 \le t < t_1$ and $\gamma(t_1) \in N_j$. Then $f_j(x_j(t)) < a_j < f_{j-1}(x_{j-1}(t))$ for $0 \le t < t_1$ and either $f_j(x_j(t_1)) = a_j$ or $f_{j-1}(x_{j-1}(t_1)) = a_j$. We first assume that

$$f_j(x_j(t_1)) = a_j < f_{j-1}(x_{j-1}(t_1)).$$

We can choose $0 \le t_0 < t_1$ such that $df_j(x_j(t))/dt > 0$ on $[t_0, t_1)$. Then, from the formula (4.9) we have

$$\frac{df_j(x_j(t))}{dt} = \frac{2\sqrt{a_1 - f_j(x_j)}\prod_{k=2}^{n}(f_j(x_j) - a_k)}{\prod_{k \ne j}(f_j(x_j) - f_k(x_k))A(f_j(x_j))}$$

on the interval $[t_0, t_1)$. Now, let us observe the integrals:

$$\int_{f_j(x_j(t_0))}^{a_j} \frac{df_j}{a_j - f_j} = \int_{t_0}^{t_1} \frac{-2\sqrt{a_1 - f_j(x_j)}\prod_{k\neq j}(f_j(x_j) - a_k)}{\prod_{k\neq j}(f_j(x_j) - f_k(x_k))A(f_j(x_j))}\,dt.$$

Since $|f_j(x_j(t)) - f_{j-1}(x_{j-1}(t))|$ is bounded away from 0 on $[t_0, t_1]$, the integral of the right-hand side is finite, while the left-hand side is ∞; a contradiction. Hence this case does not occur. The other case is similar.

(2) Suppose that $\gamma(t)$ does not pass the focal submanifold C_n. We may assume that $df_n(x_n(t))/dt > 0$ at $t = 0$. Then $f_n(x_n(t))$ remains in the interval $[f_n(x_n(0)), a_n)$ when t goes to $+\infty$. Therefore, by Theorem 4.2, there is some j $(1 \leq j \leq n-1)$ such that $f_j(x_j(t))$ oscillates on the interval $[a_{j+1}, a_j]$ infinitely many times when t increases up to $+\infty$. This means $f_j(x_j(t))$ takes the value a_{j+1} at infinitely many times, and so is $f_{j+1}(x_{j+1}(t))$ in view of (1). Therefore $f_{j+1}(x_{j+1}(t))$ also oscillates infinitely many times on the interval $[a_{j+2}, a_{j+1}]$. Consequently, one knows that the function $f_{n-1}(x_{n-1}(t))$ take the value a_n at infinitely many times when $t \to +\infty$. However, since $f_n(x_n(t))$ does not reach a_n when $t \to +\infty$, this contradicts (1). The other case is similar. □

5. Thread construction in Liouville manifolds.

Let M be a Liouville manifold just explained in the previous sections. In this section we shall describe "thread construction" in M. The object to be constructed is a coordinate hypersurface defined by x_i =constant for some i which is not totally contained in any hypersurfaces N_j.

Let $p_0 \in M$ be a point which is not contained in any N_j $(2 \leq j \leq n)$. As in the case of $\mathbb{R}^3$, we shall consider $2(n-1)$ broken geodesic segments joining p_0 and $\pm s_j$ $(2 \leq j \leq n)$: Let $p_{j,\pm}(t)$ be a minimal broken geodesic segment which starts from p_0 at $t = 0$, passes every focal submanifold C_j $(2 \leq j \leq n)$, and reaches the point $\pm s_j \in L$ at, say, $t = t_{j,\pm}$. Note that the broken points only appear at focal submanifolds; otherwise, one can find shorter curve. Also, the points $\{r_k(p_{j,\pm}(t))\}$ on L $(1 \leq k \leq n)$ are well-defined and continuous in t even when $p_{j,\pm}(t)$ is moving in the intersection of some N_i's. Applying Theorem 4.2 to those intersections (Liouville manifolds), one knows that Theorem 4.2 is valid for any segment of $\{p_{j,\pm}(t)\}$.

THEOREM 5.1. (1) *The minimal broken geodesic $p_{j,\pm}(t)$ uniquely exists.*
(2) *The length $t_{j,\pm}$ of $\{p_{j,\pm}(t)\}$ is equal to*

$$t_{j,\pm} = \mp r_1 + r_n + \sum_{k=2}^{j-1}(r_k - s_k) + \sum_{k=j}^{n-1}(s_{k+1} - r_k),$$

where $r_k = r_k(p_0)$ $(1 \leq k \leq n)$.

PROOF. First, let us consider the case of $p_{j,+}(t)$. Since the end point s_j lies on $I_j \cap L = I_j \cap \bigcap_{k=2}^n N_k$, we have

$$r_k(p_{j,+}(t_{j,+})) = \begin{cases} s_{k+1} & (1 \le k \le j-1) \\ s_k & (j \le k \le n). \end{cases}$$

Also, since $p_{j,+}(t)$ meets every focal submanifold C_i, it follows that each $r_k(p_{j,+}(t))$ $(2 \le k \le n-1)$ takes both values s_k and s_{k+1} when t moves from 0 to $t_{j,+}$. Therefore the distance that the point $r_k(p_{j,+}(t))$ moves out during $0 \le t \le t_{j,+}$ is at least $s_2 - r_1$ $(k=1)$, $r_n - s_n$ $(k=n)$, and

$$\begin{aligned} &r_k - s_k + s_{k+1} - s_k && (2 \le k \le j-1), \\ &s_{k+1} - r_k + s_{k+1} - s_k && (j \le k \le n-1). \end{aligned}$$

Thus the length of the broken geodesic $\{p_{j,+}(t)\}$ is, by Theorem 4.2, at least

$$\begin{aligned} &(s_2 - r_1) + \sum_{k=2}^{j-1}(r_k - s_k + s_{k+1} - s_k) + \sum_{k=j}^{n-1}(s_{k+1} - r_k + s_{k+1} - s_k) + (r_n - s_n) \\ &\quad = -r_1 + r_n + \sum_{k=2}^{j-1}(r_k - s_k) + \sum_{k=j}^{n-1}(s_{k+1} - r_k). \end{aligned}$$

Now we shall show that there is a unique broken geodesic segment from p_0 to s_j whose length is equal to the above value. We prove this by an induction on the dimension n of M. If $n = 2$, there is nothing to prove. Suppose $n \ge 3$ and the assertion is true for the manifolds of dimension less than n.

By (the proof of) Lemma 4.3 we see that there is a unique unit covector at p_0 such that $b_i = a_i$ for any $2 \le i \le n$ and that the corresponding geodesic $\gamma(t) = (x_1(t), \ldots, x_n(t))$ satisfies

$$\frac{d}{dt} f_k(x_k(t))\bigg|_{t=0} \begin{cases} > 0 & (2 \le k \le j-1,\ k = n) \\ < 0 & (k = 1,\ j \le k \le n-1). \end{cases}$$

Then we have

$$\frac{d}{dt} r_k(\gamma(t))\bigg|_{t=0} \begin{cases} < 0 & (2 \le k \le j-1,\ k = n) \\ > 0 & (k = 1,\ j \le k \le n-1). \end{cases} \tag{5.1}$$

From this one can see that the first focal submanifold which $\gamma(t)$ meets must be C_2 or C_n (or both at the same time). In fact, $k = 2, n$ are only k such that both $r_{k-1}(\gamma(t))$ and $r_k(\gamma(t))$ move toward s_k. Also, note that the geodesic $\gamma(t)$ must meet C_n in view of Proposition 4.5 and that it occurs at a positive time by (5.1).

Now, suppose that $\gamma(t)$ first meets C_n at $t = t_1 > 0$ and $\gamma(t) \notin C_2$ for any $t \in [0, t_1]$. Other cases will be similar. Putting $p_1 = \gamma(t_1) \in C_n \subset N_n$, we have

$$r_1(p_0) < r_1(p_1) < s_2, \quad s_k < r_k(p_1) < r_k(p_0) \quad (2 \le k \le j-1),$$
$$r_k(p_0) < r_k(p_1) < s_{k+1} \quad (j \le k \le n-2), \quad r_{n-1}(p_1) = s_n = r_n(p_1).$$

Then by the induction assumption there is a unique broken geodesic segment in N_n from p_1 to $s_j \in L$ which pass every focal submanifold $C_k \cap N_n$ $(k \neq n)$ and whose length is equal to

$$-r_1(p_1) + r_{n-1}(p_1) + \sum_{k=2}^{j-1}(r_k(p_1) - s_k) + \sum_{k=j}^{n-2}(s_{k+1} - r_k(p_1)).$$

Adding the length of the geodesic segment $\gamma(t)$ $(0 \le t \le t_1)$:

$$r_1(p_1) - r_1(p_0) + \sum_{k=2}^{j-1}(r_k(p_0) - r_k(p_1)) + \sum_{k=j}^{n-1}(r_k(p_1) - r_k(p_0)) + r_n(p_0) - s_n$$

to the length of the broken geodesic in N_n, we obtain the length in (2).

The uniqueness is proved as follows. The first geodesic segment $\gamma(t)$ from p_0 up to a point in C_n must satisfy $b_i = a_i$ for any i in view of Theorem 4.2. Then the possibility of initial direction (unit covector) of the geodesic is 2^n. Among those the only one direction corresponds to the initial behavior of $r_i(\gamma(t))$ $(1 \le i \le n)$ should be. □

REMARK 5.2. It is clear that the broken geodesic $\gamma_{n,\pm}$ passes the focal submanifolds in the order: $C_2, C_3, \ldots, C_n$. Also, the broken geodesic $\gamma_{2,\pm}$ passes the focal submanifolds in the order: $C_n, C_{n-1}, \ldots, C_2$. However, for $2 < j < n$, it is not clear in which order the focal submanifolds are passed by the broken geodesic $\gamma_{j,\pm}$. It is, at least, certain that $C_2, C_3, \ldots, C_j$ are passed in this order and so are $C_n, C_{n-1}, \ldots, C_j$, but it is not clear how the two groups are merged into one order; it depends on the initial point p_0.

We now state the thread construction in this setting. We first have

$$t_{n,-} + t_{2,+} = 2r_n + s_n - s_2,$$

which implies that the sum of the length of the two broken geodesic segments $\gamma_{n,-}$ and $\gamma_{2,+}$ depends only on the x_n-coordinate of the point p_0. Therefore, this value remains constant when p_0 moves on the coordinate hypersurface $x_n = x_n(p_0)$. Namely, we can *draw* that coordinate hypersurface in this way.

Next, we have

$$t_{j,-} - t_{j,+} = 2r_1,$$
$$t_{j+1,+} - t_{j,+} = 2r_j - s_j - s_{j+1}.$$

The former one indicates that the difference of the lengths of the two broken geodesic segments $\gamma_{j,-}$ and $\gamma_{j,+}$ depends only on the x_1-coordinate of the point p_0. Therefore it is constant when p_0 moves on the coordinate hypersurface $x_1 = x_1(p_0)$. Also, the latter one indicates that the difference of the lengths of the two broken geodesic segments $\gamma_{j+1,+}$ and $\gamma_{j,+}$ depends only on the x_j-coordinate of the point p_0. Therefore it is constant when p_0 moves on the coordinate hypersurface $x_j = x_j(p_0)$.

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Jin-ichi Itoh
Department of Mathematics
Faculty of Education
Kumamoto University
Kumamoto 860-8555, Japan
E-mail: j-itoh@gpo.kumamoto-u.ac.jp

Kazuyoshi Kiyohara
Department of Mathematics
Faculty of Science
Okayama University
Okayama 700-8530, Japan
E-mail: kiyohara@math.okayama-u.ac.jp