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Picard numbers of Delsarte surfaces

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Abstract. We give a classification of all complex Delsarte surfaces with only isolated ADE singularities. This results in 83 types of surfaces. For each of these types, we give a closed formula for the Picard number depending only on the degree.

1. Introduction.

In [**11**] Shioda introduced Delsarte surfaces as a generalization of Delsarte curves. In the same paper he introduced an algorithm to compute the Picard number of these surfaces. This algorithm works for arbitrary characteristic. In this text we will only consider the case where the surfaces are defined over the complex numbers. The computation of the Picard number is in general a hard problem. In [**9**], the algorithm of [**11**] is one of the methods used to find Picard numbers of quintic surfaces with only isolated ADE singularities. In the current paper we extend these results to Delsarte surfaces of any degree with only isolated ADE singularities. For non-singular Delsarte surfaces this has already been done in [**7**].

The main result of this paper is:

Theorem 1.1. *For any degree $n \geq 6$, there are up to isomorphism at most* 83 *Delsarte surfaces of degree $n \geq 6$ with only isolated* ADE *singularities. The possible cases, and for every case a closed formula for the Picard number, are given in Appendix* A.

Of particular interest to us are surfaces where the Picard number is the highest possible. These are called maximal (see 2). Using the list in Appendix A, and considering the cases with $n = 5$ as well, a short search gives the following.

Corollary 1.2. *There are up to isomorphism precisely three maximal Delsarte surfaces with degree $n \geq 5$. These are given by*:

$$X^3YZ + Y^3ZU + XZ^3U + XYU^3 = 0,$$
$$X^5Y + XY^5 + Z^5U + ZU^5 = 0 \text{ and}$$
$$X^6 + Y^6 + Z^6 + U^6 = 0.$$

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Note that we do not address the question of describing generators of the Néron-Severi groups of the surfaces described here; we refer to [**4**] for work on this for generators over $\mathbb{Z}$. See also [**3**] for work on generators on Fermat surfaces.

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2. Some general theory about Delsarte surfaces.

In this section we explain some of the theory on Delsarte surfaces that we will use. We use Shioda's definition of Delsarte surface [**11**].

Definition 2.1. A Delsarte surface is an irreducible two dimensional subvariety in $\mathbb{P}^3$, which is defined as the zero set of a polynomial consisting of the sum of four monomials, such that the exponent matrix is invertible.

Note that there always is a scaling of coordinates so that the constants in the polynomial are all one. Hence we will assume we are in this situation.

In [**11**] an algorithm is presented to compute the Picard number of these surfaces. For later use we state an adapted version of this algorithm as given in [**6**].

Let S be a projective Delsarte surface defined by the homogeneous polynomial

$$F = \sum_{i=1}^{4} X^{a_{i1}} Y^{a_{i2}} Z^{a_{i3}} U^{a_{i4}}.$$

We construct the exponent matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

We define three row vectors $e_1 = (1,0,0,-1)$, $e_2 = (0,1,0,-1)$ and $e_3 = (0,0,1,-1)$. From these we construct three vectors $\tilde{v} = e_1 A^{-1}$, $\tilde{w} = e_2 A^{-1}$ and $\tilde{u} = e_3 A^{-1}$. Let V be the $\mathbb{Z}$-module given by $V = \{(v_1, v_2, v_3, v_4) \in (\mathbb{Q}/\mathbb{Z})^4 : v_1 + v_2 + v_3 + v_4 \equiv 0(1)\}$. We will view $\tilde{v}$, $\tilde{w}$ and $\tilde{u}$ as elements of V. These vectors generate a finite $\mathbb{Z}$-module

$$L := \{i\tilde{v} + j\tilde{w} + k\tilde{u} \in (\mathbb{Q}/\mathbb{Z})^4 : i, j, k \in \mathbb{Z}\}.$$

We construct the set L_0 as the subset of L where at least one of the coordinates is zero and L_1 as its complement.

$$L_0 := \{v = (v_1, v_2, v_3, v_4) \in L : \exists i : v_i = 0\}.$$

$$L_1 := \{v = (v_1, v_2, v_3, v_4) \in L : \forall i : v_i \neq 0\}.$$

We now define $\Lambda \subseteq L_1$ by the following property. An element $v = (v_1, v_2, v_3, v_4) \in L_1$

is an element of Λ precisely when there exists a $t \in \mathbb{Z}$ such that $\sum\{tv_j\} \neq 2$ and for all i we have $\mathrm{ord}_+(tv_i) = \mathrm{ord}_+(v_i)$. Here $\{\cdot\}$ is the natural bijection between the set $\mathbb{Q}/\mathbb{Z}$ and the interval $[0,1) \cap \mathbb{Q}$ and $\mathrm{ord}_+(\cdot)$ stands for the order in the additive group $\mathbb{Q}/\mathbb{Z}$.

The Lefschetz number $\lambda = \lambda(S)$ is defined as follows. Fix a desingularisation $\tilde{S}$ of S. Then λ is the difference between the second Betti number of $\tilde{S}$ and the Picard number $\rho(\tilde{S})$: $\rho(\tilde{S}) = b_2(\tilde{S}) - \lambda$. This is independent of the choice of $\tilde{S}$. The main result of [**11**], rephrased in terms of Λ, is that the Lefschetz number can be computed by

$$\lambda = \#\Lambda.$$

Let $\tilde{X}$ be the minimal resolution of the surface X in $\mathbb{P}^3$ over $\mathbb{C}$ of degree n with only ADE singularities. Then $b_2(\tilde{X}) = n^3 - 4n^2 + 6n - 2$. The Hodge number $h^{1,1}(\tilde{X}) = (2n^3 + 7)/3 - 2n^2$ is an upper bound for the Picard number over the complex numbers: $\rho(\tilde{X}) \leq h^{1,1}(\tilde{X})$. We call the surface X maximal when we have equality in the last equation. See [**9**] for more details.

Remark 2.2. It is important to note that if two surfaces are birationally equivalent, then they have the same Lefschetz number. In particular we can compute the Lefschetz number on a singular (Delsarte) surface and use it to compute the Picard number of the desingularization.

The set $L_1 \setminus \Lambda$ consists of elements of a specific form. This has been shown originally in [**10**] for Fermat surfaces, but the results extend trivially to Delsarte surfaces. There are three different types of elements $v \in L_1 \setminus \Lambda$.

- v is such that $v_1 + v_j \equiv 0(1)$ for some $j \in \{2,3,4\}$. In this case v is called decomposable. The set of decomposable elements will be denoted by D.
- v is a permutation of a vector of the form $(a, 1/2, 1/2+a, -2a)$ or $(a, 1/2+a, 1/2+2a, -4a)$ or $(a, 1/3+a, 2/3+a, -3a)$. Here a should be such that none of the coordinates is zero and the vector is not decomposable. These v are called regularly indecomposable. The set of indecomposable elements will be denoted by R.
- v is an exceptional element. It was proven in [**1**] and correctly formulated in [**2**] that there are only a finite number of exceptional cases for all Fermat surfaces. One can easily see that these give all the possible cases for a Delsarte surface as well. The set of exceptional elements is explicitly known and has cardinality 22080. We will refer to these cases as irregularly indecomposable. In any given example below, the set of all such v is denoted by I.

3. 83 surfaces.

In this paper we consider Delsarte surfaces with only isolated ADE singularities over $\mathbb{C}$. There are several equivalent definitions of ADE singularities, see [**5**].

Definition 3.1. A singular point P on a complex surface S is ADE if it is locally isomorphic to one of the following types of singularities:

- $X^{n+1} + Y^2 + Z^2$; this is a singularity of type A_n, with $n \geq 1$,

- $X^{n-1} + XY^2 + Z^2$; this is a singularity of type D_n, with $n \geq 4$,
- $X^3 + Y^4 + Z^2$; this is a singularity of type E_6,
- $X^3 + XY^3 + Z^2$; this is a singularity of type E_7,
- $X^3 + Y^5 + Z^2$; this is a singularity of type E_8.

It is not hard to determine whether a given isolated singular point is ADE: compare, for example, [**8**].

THEOREM 3.2. *There are up to isomorphism at most* 83 *Delsarte surfaces of degree* $n \geq 6$ *with only isolated* ADE *singularities.*

REMARK 3.3. For most degrees the number of different Delsarte surfaces will in fact be 83. However, we did not check whether any of the surfaces we found might still be isomorphic for certain degrees.

PROOF. We will first bound the number of Delsarte surfaces of a fixed degree $n \geq 6$ with only isolated ADE singularities.

Let S be a Delsarte surface of degree n. Consider the point $P_x = (1 : 0 : 0 : 0)$. Unless one of the monomials equals X^n, the point P_x lies on S. A necessary (but not sufficient) condition for P_x to be either smooth or of ADE type is that one of the monomials defining S is divisible by X^{n-2}.

A similar argument applies for the points $(0 : 1 : 0 : 0)$, $(0 : 0 : 1 : 0)$, $(0 : 0 : 0 : 1)$. From this we see that if S has only isolated ADE singularities, then possibly after scaling it is defined by a polynomial of the form:

$$F = X^{n-2}M_X + Y^{n-2}M_Y + Z^{n-2}M_Z + U^{n-2}M_U. \tag{1}$$

Here M_X, M_Y, M_Z and M_U are degree 2 homogeneous monomials. This already implies that there are at most 10^4 possibilities for the equation F. We will now in various steps reduce this number to 83.

Now since $n \geq 6$ we see that the point $(1 : 0 : 0 : 0)$ is not of ADE type if M_X is one of Y^2, Z^2 or U^2. A similar condition applies to M_Y, M_Z and M_U.

For fixed n there are $7^4 = 2401$ degree n polynomials for which this does not occur. Hence we have at most 2401 degree n Delsarte surfaces with only ADE singularities.

Using a computer algebra package we remove all F's that are divisible by a coordinate, as well as duplicates after permutations of coordinates. This leaves us with 90 surfaces.

For these 90 surfaces we still have to check whether all the singular points are of ADE type. The following results will be helpful:

LEMMA 3.4. *Let S be a Delsarte surface given by*:

$$F = M_1 + M_2 + M_3 + M_4.$$

Here M_i are monomials in X, Y, Z and U. Let P be a singular point on S. Then

$$M_i(P) = 0, \text{ for } i = 1, 2, 3, 4.$$

Proof. Let A be the exponent matrix of S. By considering the partial derivatives of F we find for a singular point P that:

$$A \cdot \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} (P) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since A is nonsingular this means that we have $M_i = 0$ for all i. □

Corollary 3.5. *Let S be the Delsarte surface given by* (1)*. Let P be a singular point, then it has at least two zero coordinates.*

Proof. Assume that P has only one zero coordinate. We can assume this is the first coordinate. Then X divides all monomials M_i. This would imply that suface is reducible. □

We first consider the singular points with three coordinates zero. Without loss of generality we can assume that the singular point is the point $(0:0:0:1)$.

If $M_U = U^2$, then the point $(0:0:0:1)$ does not lie on the surface. If $M_U = XU, YU, ZU$ then this point is non-singular. Consider the case that M_U is one of XY, XZ or YZ. Without loss of generality we can then assume that $M_U = XY$. The local equation of the point will then be:

$$XY + Z^{n-2}M_Z + \text{ terms of higher order in } X \text{ and } Y.$$

The point $(0:0:0:1)$ is of ADE type if and only if $M_Z \neq XY$. For 7 of the 90 surfaces we find in this way a singular point not of ADE type.

The remaining possible singular points are permutations of the point $P = (\eta : 0 : 0 : 1)$, with η nonzero. We will assume that the singular point is the point P and consider what happens depending on M_X.

- Assume that $M_X \in \{X^2, XU\}$. For P to be a point on the surface this means that $M_U \in \{U^2, XU\}$. (Recall that $M_U = X^2, Y^2, Z^2$ is excluded since we assume $(0:0:0:1)$ is non-singular or of ADE type.) In this case η has to be a specific root of unity. It turns out that the partial derivatives with respect to X and U are then nonzero. So in this case, the point P is non-singular.
- Assume that $M_X \in \{XY, UY\}$. Then by considering the partial derivative with respect to Y we find that $M_U \in \{XY, UY\}$. Consider the affine chart with $U = 1$ and transform the singular point to the origin. Then we find a local equation of the form:

$$(\eta^a + \eta^b)Y + (a\eta^{a-1} + b\eta^{b-1})XY + \text{ higher order terms},$$

  with $a \neq b$. In case $\eta^a + \eta^b \neq 0$, this point is non-singular, otherwise it is of ADE type.

- The case $M_X \in \{XZ, UZ\}$ is symmetric to the previous case.
- The final case is $M_X = YZ$. If $M_U \neq YZ$, then by interchanging the roles of X and U we see from the previous cases that the point is not singular. In the case that $M_X = M_U = YZ$ we find that the singularity in the point P is not isolated. This case however has already been excluded by looking at the singularities with only one non-zero coordinate. □

The explicit equations of the 83 surfaces are given in the appendix.

REMARK 3.6. The table in Appendix A was computed using the condition $n \geq 6$. Also for $n = 5$ the entries in the table are Delsarte surfaces with at most ADE singularities. In case $n = 5$ there is (up to permutation) precisely one more Delsarte surface with only ADE singularities, namely the surface with $\rho = 25$ given by the equation:

$$Y^2X^3 + Z^2Y^3 + X^2Z^3 + U^5.$$

This surface has three E_8 singularities.

4. Computation of the Picard number: An example.

Here we illustrate how we computed the Picard numbers in the table. We do this for one example only. The same ideas work for the other cases. We compute the Picard number of the surface S given by the equation

$$X^n + Y^n + Z^{n-1}U + XYU^{n-2} = 0, \tag{2}$$

for all $n \geq 3$.

This corresponds to case 26 in the table of Appendix A.

We determine the exponent matrix:

$$A = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n-1 & 1 \\ 1 & 1 & 0 & n-2 \end{pmatrix}.$$

From this we find the three vectors in V that generate L:

$$\tilde{v} = \left(\frac{n-1}{n(n-2)}, \frac{1}{n(n-2)}, 0, \frac{-1}{n-2}\right),$$

$$\tilde{w} = \left(\frac{1}{n(n-2)}, \frac{n-1}{n(n-2)}, 0, \frac{-1}{n-2}\right),$$

$$\tilde{u} = \left(\frac{1}{(n-1)(n-2)}, \frac{1}{(n-1)(n-2)}, \frac{1}{n-1}, \frac{-n}{(n-1)(n-2)}\right).$$

We use the following formula to compute $\#\Lambda$:

$$\#\Lambda = \#L - \#(L_0 \cup D) - \#I - \#R.$$

We first determine the sets $N_i = \{v = (v_1, v_2, v_3, v_4) \in L : v_i = 0\}$, for $i = 1, 2, 3, 4$.

$$N_1 = \{i(n(n-2))\tilde{v} + j((1-n)\tilde{v} + \tilde{w}) + k((n-n^2)\tilde{v} + (n-1)\tilde{u}) : i, j, k \in \mathbb{Z}\},$$
$$N_2 = \{i(n(n-2))\tilde{v} + j((1-n)\tilde{v} + \tilde{w}) + k(-n\tilde{v} + (n-1)\tilde{u}) : i, j, k \in \mathbb{Z}\},$$
$$N_3 = \{i\tilde{v} + j\tilde{w} + k(n-1)\tilde{u} : i, j, k \in \mathbb{Z}\},$$
$$N_4 = \{i(n-2)\tilde{v} + j(-\tilde{v} + \tilde{w}) + k(-n\tilde{v} + (n-1)\tilde{u}) : i, j, k \in \mathbb{Z}\}.$$

We compute the intersection of these sets:

$$\bigcap_{i=1}^{4} N_i = \{i(n(n-2))\tilde{v} + j((1-n)\tilde{v} + \tilde{w}) + k((n-n^2)\tilde{v} + (n-1)\tilde{u}) : i, j, k \in \mathbb{Z}\}.$$

From this we see that L can be described by

$$L = \{i\tilde{v} + j\tilde{w} + k\tilde{u} : 0 \le i < n(n-2),\ j = 0,\ 0 \le k < n-1\}.$$

Note that we can assume that $j = 0$ since $(n-1)\tilde{v} = \tilde{w}$. This implies $\#L = n(n-1)(n-2)$ since this description gives a bijection between the set L and the set

$$\{(i, j, k) \in \mathbb{Z}^3 : 0 \le i < n(n-2),\ j = 0,\ 0 \le k < n-1\}.$$

We compute the number of elements of L_0 using

$$\#L_0 = \sum_{i=1}^{4} (-1)^{i+1} \sum_{1 \le d_1 < \cdots < d_i \le 4} \#L \cap \bigcap N_{d_i}.$$

This can be computed by a computer. $\#L_0 = n(n-2)$.

Let N_5, N_6 and N_7 consist of all elements of the form $(a, -a, b, -b)$, $(a, b, -a, -b)$, $(a, b, -b, -a)$. These are the decomposable elements comprising D. These sets are given by:

$$N_5 = \begin{cases} \left\{ i(n-2)\tilde{v} + j(\tilde{v} + \tilde{w}) + k\left(-\tilde{v} + \dfrac{n-1}{2}\tilde{u} \right) : i, j, k \in \mathbb{Z} \right\} & \text{if } n \equiv 1 \bmod 2 \\ \{i(n-2)\tilde{v} + j(\tilde{v} + \tilde{w}) + k(-2\tilde{v} + n - 1\tilde{u}) : i, j, k \in \mathbb{Z}\} & \text{if } n \equiv 0 \bmod 2 \end{cases}$$
$$N_6 = \{i(n(n-2))\tilde{v} + j((1-n)\tilde{v} + \tilde{w}) + k(-n(n-1)\tilde{v} + \tilde{u}) : i, j, k \in \mathbb{Z}\}.$$
$$N_7 = \{i(n(n-2))\tilde{v} + j((1-n)\tilde{v} + \tilde{w}) + k(-n\tilde{v} + \tilde{u}) : i, j, k \in \mathbb{Z}\}.$$

Note that for N_5 this result will depend on $n \bmod 2$. This behavior is quite standard and happens for more cases.

We use this to compute

$$\#(L_0 \cup D) = \sum_{i=1}^{7} (-1)^{i+1} \sum_{1 \le d_1 < \cdots < d_i \le 7} \#L \cap \bigcap N_{d_i} = \begin{cases} n^2 - 3 & \text{if } n \equiv 1(2) \\ n^2 - n - 2 & \text{if } n \equiv 0(2). \end{cases}$$

We now compute the number of irregularly indecomposable elements. To do this we consider vectors of the form $r = (a, 1/2, 1/2 + a, -2a)$. We determine for which a we have $r \in L$. If r belongs to L then there exists a vector $(k, l, m) \in \mathbb{Z}^3$ such that

$$(k, l, m) \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} A^{-1} = r.$$

Note that r and A are constructed in such a way that a rational solution for (k, l, m) always exists. We only need to find out whether this solution is integral.

Multiplying both sides of this equality on the right by A shows

$$(k, l, m) = (a, 1/2, 1/2 + a) \begin{pmatrix} n-1 & -1 & 0 \\ -1 & n-1 & 0 \\ -1 & -1 & n-1 \end{pmatrix}.$$

If (k, l, m) is computed in this way the last coordinate of r will always be $-2a$. No check is needed.

The vector (k, l, m) is only integral when $n \equiv 0(2)$ and $a = 1/2$. This corresponds to the solution $r = (1/2, 1/2, 0, 0)$. Since the third and fourth coordinates of r are zero we find that $r \in L_0$ and hence $r \notin R$.

We now consider the cases where r is a permutation of $(a, 1/2, 1/2 + a, -2a)$. It turns out that we only find the result $r = (1/2, 1/2, 0, 0)$, three more times, and no other integer solutions. So none of the permutations of $(a, 1/2, 1/2 + a, -2a)$ makes a contribution to the set R.

We move on to elements of R that are permutations of $(a, 1/2 + a, 1/2 + 2a, -4a)$. We will give three examples in detail.

Consider the case $r = (a, 1/2+a, 1/2+2a, -4a)$. With the same argument as before, we need to find a solution for

$$(k, l, m) = (a, 1/2 + a, 1/2 + 2a) \begin{pmatrix} n-1 & -1 & 0 \\ -1 & n-1 & 0 \\ -1 & -1 & n-1 \end{pmatrix},$$

with k, l and m integers. We find solutions with $a = 1/4, 3/4$ and $n \equiv 0(4)$, furthermore we find solutions with $a = 1/12, 5/12, 7/12, 11/12$ and $n \equiv 4(12)$.

The solutions $a = 1/4, 3/4$ give respectively $r = (1/4, 3/4, 0, 0)$ and $r = (3/4, 1/4, 0, 0)$. Since both solutions have a zero coordinate they are elements of L_0, and hence not of R. The solutions $a = 1/12, 5/12, 7/12, 11/12$ give respectively $r = (1/12, 7/12, 2/3, 2/3)$,

$r = (5/12, 11/12, 1/3, 1/3)$, $r = (7/12, 1/12, 2/3, 2/3)$ and $r = (11/12, 5/12, 1/3, 1/3)$. These vectors are all elements of R.

Now we consider $r = (a, 1/2 + 2a, -4a, 1/2 + a)$. We find

$$(k, l, m) = (a, 1/2 + a, 1/2 + 2a) \begin{pmatrix} n & 0 & 0 \\ 0 & n-1 & 1 \\ 1 & 0 & n-2 \end{pmatrix}.$$

This equation has no integer solutions at all.

Let us now consider the vector $r = (a, 1/2 + 2a, 1/2 + a, -4a)$. We need integral solutions for

$$(k, l, m) = (a, 1/2 + a, 1/2 + 2a) \begin{pmatrix} n-1 & 0 & -1 \\ -1 & n-1 & -1 \\ -1 & 0 & n-1 \end{pmatrix}.$$

We find that the only solution is given by $a = \frac{1}{2}$ with $n \equiv 0(2)$. This solutions corresponds to the vector $(1/2, 0, 1/2, 0)$, and this is an element of L_0 and not of R.

Considering further permutations of $(a, 1/2 + a, 1/2 + 2a, -4a)$, yields no new elements of R.

We will now look at the solutions of the form $r = (a, 1/3 + a, 2/3 + a, -3a)$. As before r is an element of L precisely when

$$(k, l, m) = (a, 1/3 + a, 2/3 + a) \begin{pmatrix} n-1 & -1 & 0 \\ -1 & n-1 & 0 \\ -1 & -1 & n-1 \end{pmatrix}$$

has an integral solution for k, l and m. The only solutions occur when $a = 1/3$ and $n \equiv 0(3)$ or when $a = 5/6$ and $n \equiv 3(6)$. In the case $a = 1/3$ we find that $r = (1/3, 2/3, 0, 0)$, hence we find $r \in L_0$ and $r \notin R$. In the case $a = 5/6$ we find $r = (5/6, 1/6, 1/2, 1/2)$. This vector is decomposable, so we have $r \in D$. Both cases yield no elements of R.

Finally considering permutations of $r = (a, 1/3 + a, 2/3 + a, -3a)$ yields no elements of R.

In total there are only four potential elements of R namely:

$$\left(\frac{1}{12}, \frac{7}{12}, \frac{2}{3}, \frac{2}{3}\right), \left(\frac{5}{12}, \frac{11}{12}, \frac{1}{3}, \frac{1}{3}\right),$$
$$\left(\frac{7}{12}, \frac{1}{12}, \frac{2}{3}, \frac{2}{3}\right), \left(\frac{11}{12}, \frac{5}{12}, \frac{1}{3}, \frac{1}{3}\right).$$

These elements exist in Λ precisely when $n \equiv 4(12)$.

Finally we will consider the finite set of exceptional solutions. We have 22080 potential elements of the set I. For each of these 22080 elements r we can easily check whether indeed $r \in I$. We simply compute rA and if the result consists of only integers then r belongs to I, otherwise not. For example take $r = (1/24, 19/24, 1/3, 5/6)$, one of the 22080 elements. We find:

$$rA = \left(\frac{5}{6} + \frac{n}{24}, \frac{5}{6} + \frac{19n}{24}, \frac{n-1}{3}, \frac{5n}{6} + \frac{2}{3}\right).$$

These coefficients are all integers if and only if $n \equiv 4(24)$.

Since we have to check 22080 elements we resort to a computer search here. The number of exceptional solutions is given by the formula:

$$\begin{aligned}\#I = {} & 8\delta_{4,24} + 8\delta_{\{5,6,10\},30} + 16\delta_{6,40} + 12\delta_{9,42} \\ & + 16\delta_{10,48} + 16\delta_{\{5,6,10\},60} + 24\delta_{\{8,9\},84} + 32\delta_{22,120}.\end{aligned}$$

Here

$$\delta_{i,j} = \begin{cases} 0 & \text{if } n \not\equiv i(j) \\ 1 & \text{if } n \equiv i(j), \end{cases} \qquad \delta_{S,j} = \sum_{i \in S} \delta_{i,j}.$$

We now have enough information to compute the Lefschetz number using:

$$\#\Lambda = \#L - \#(L_0 \cup D) - \#I - \#R.$$

This gives:

$$\begin{aligned}\lambda = {} & n^3 - 4n^2 + 2n + 3 + (n-1)\delta_{0,2} - 4\delta_{4,12} - 8\delta_{4,24} - 8\delta_{\{5,6,10\},30} - 16\delta_{6,40} \\ & - 12\delta_{9,42} - 16\delta_{10,48} - 16\delta_{\{5,6,10\},60} - 24\delta_{\{8,9\},84} - 32\delta_{22,120}.\end{aligned}$$

For a surface of degree n with only isolated ADE singularities recall that the minimal resolution has $b_2 = n^3 - 4n^2 + 6n - 2$. So the Picard number of the resolution of the surface given by (2) is :

$$\begin{aligned}\rho = {} & 4n - 5 - (n-1)\delta_{0,2} + 4\delta_{4,12} + 8\delta_{4,24} + 8\delta_{\{5,6,10\},30} + 16\delta_{6,40} \\ & + 12\delta_{9,42} + 16\delta_{10,48} + 16\delta_{\{5,6,10\},60} + 24\delta_{\{8,9\},84} + 32\delta_{22,120}.\end{aligned}$$

A. Appendix.

In this table we will give the formula's of all Delsarte surfaces with only ADE singularities. The number on the left is simply an index. This is followed by the equation of the Delsarte surface and in the third column the singular points. Below the second and third column the equation for the Picard number is given. We assume that $n \geq 6$.

The case of the Fermat surface was already computed for many degrees by Shioda [**10**] and in general by Aoki [**1**]. Several examples of smooth Delsarte surfaces where given by [**11**], and a systematic treatment of all smooth Delsarte surfaces was given in [**7**]. These cases have been marked in the table.

1	$X^{n-2}YZ + Y^{n-2}ZU + Z^n + XU^{n-1}$	$(0:1:0:0)$ $(1:0:0:0)$
	$n^2 - 2n + 2 + 2\delta_{0,3}$	
2	$X^{n-2}YZ + Y^{n-2}ZU + Z^n + YU^{n-1}$	$(0:1:0:0)$ $(1:0:0:0)$
	$2n^2 - 3n - 2$	
3	$X^{n-2}YZ + XY^{n-2}Z + Z^n + XU^{n-1}$	$(0:1:0:0)$ $(1:0:0:0)$ $(1:\sqrt[n-3]{-1}:0:0)$
	$2n^2 - 5n + 4 + 6\delta_{7,9} + 6\delta_{13,18}$	
4	$X^{n-2}YZ + XY^{n-2}U + Z^n + XU^{n-1}$	$(0:1:0:0)$ $(1:0:0:0)$ $(0:1:0:\sqrt[n-2]{-1})$
	$n^2 - 2 + 4\delta_{4,10}$	
5	$X^{n-2}YZ + XY^{n-2}U + Z^n + YU^{n-1}$	$(0:1:0:0)$ $(1:0:0:0)$
	n^2	
6	$X^{n-2}YZ + XY^{n-2}U + Z^n + ZU^{n-1}$	$(0:1:0:0)$ $(1:0:0:0)$
	$2n^2 - 5n + 4$	
7	$X^{n-2}YU + Y^{n-2}ZU + Z^n + XU^{n-1}$	$(0:1:0:0)$ $(1:0:0:0)$
	$n^2 - n + \delta_{1,2}$	
8	$X^{n-2}YU + Y^{n-2}ZU + Z^n + YU^{n-1}$	$(0:1:0:0)$ $(1:0:0:0)$ $(\sqrt[n-2]{-1}:0:0:1)$
	$2n^2 - 3n - 2$	
9	$X^{n-2}YU + Y^{n-2}ZU + Z^n + ZU^{n-1}$	$(0:1:0:0)$ $(1:0:0:0)$ $(0:\sqrt[n-2]{-1}:0:1)$
	$2n^2 - 5n + 4 + (n-2)\delta_{0,2} + 6\delta_{17,18}$	
10	$X^{n-2}YU + XY^{n-2}U + Z^n + XU^{n-1}$	$(0:1:0:0)$ $(1:0:0:0)$ $(\sqrt[n-3]{-1}:1:0:0)$ $(0:\sqrt[n-2]{-1}:0:1)$
	$2n^2 - 5n + 4 + (n-2)\delta_{0,2} + 4\delta_{0,5} + 6\delta_{7,14} + 6\delta_{6,18} + 8\delta_{\{15,20\},30}$	
11	$X^{n-2}YU + XY^{n-2}U + Z^n + ZU^{n-1}$	$(0:1:0:0)$ $(1:0:0:0)$ $(\sqrt[n-3]{-1}:1:0:0)$
	$2n^2 - 5n + 4$	

12	$X^{n-1}Y+Y^{n-1}Z+Z^n+XYU^{n-2}$	$(0:0:0:1)$ $(1:0:0:\sqrt[n-2]{-1})$
	$2n^2-5n+4$	
13	$X^{n-1}Y+Y^{n-1}Z+Z^n+XZU^{n-2}$	$(0:0:0:1)$
	$2n^2-5n+4$	
14	$X^{n-1}Y+Y^{n-1}Z+Z^n+YZU^{n-2}$	$(0:0:0:1)$ $(0:1:0:\sqrt[n-2]{-1})$
	$2n^2-5n+4+4\delta_{11,12}+6\delta_{16,18}$	
15	$X^{n-1}Y+Y^{n-1}U+Z^n+XYU^{n-2}$	$(0:0:0:1)$ $(1:0:0:\sqrt[n-2]{-1})$
	$n^2-2n+2+n\delta_{0,2}+2\delta_{0,3}-2\delta_{2,4}+8\delta_{5,15}+8\delta_{\{6,8\},24}+8\delta_{12,30}$ $+12\delta_{14,42}+16\delta_{\{12,20\},60}$	
16	$X^{n-1}Y+Y^{n-1}U+Z^n+XZU^{n-2}$	$(0:0:0:1)$
	$n+4\delta_{3,5}$	
17	$X^{n-1}Y+Y^{n-1}U+Z^n+YZU^{n-2}$	$(0:0:0:1)$
	n^2-2	
18	$X^{n-1}Y+XY^{n-1}+Z^n+XYU^{n-2}$	$(0:0:0:1)$ $(1:0:0:\sqrt[n-2]{-1})$ $(0:1:0:\sqrt[n-2]{-1})$
	$2n^2-5n+4+(3n-10)\delta_{0,2}+4\delta_{2,4}+8\delta_{\{6,8\},12}+16\delta_{12,15}$ $+16\delta_{12,20}+16\delta_{\{14,18,20\},24}+16\delta_{\{12,20\},30}+32\delta_{\{32,42,50\},60}$	
19	$X^{n-1}Y+XY^{n-1}+Z^n+XZU^{n-2}$	$(0:0:0:1)$
	n^2-2	
20	$X^{n-1}Z+Y^{n-1}Z+Z^n+XYU^{n-2}$	$(0:0:0:1)$ $(1:\sqrt[n-1]{-1}:0:0)$
	$3n^2-10n+8+2\delta_{1,2}+n\delta_{0,2}+4\delta_{5,12}+12\delta_{8,14}$ $+24\delta_{17,24}+8\delta_{\{7,11\},30}+72\delta_{16,30}+16\delta_{17,40}+60\delta_{22,42}+16\delta_{17,48}$ $+48\delta_{\{37,41\},60}+16\delta_{46,60}+40\delta_{23,66}+48\delta_{40,78}+24\delta_{\{22,37\},84}+32\delta_{41,120}$	
21	$X^{n-1}Z+Y^{n-1}U+Z^n+XYU^{n-2}$	$(0:0:0:1)$
	$3n-2+8\delta_{4,15}+8\delta_{9,20}+10\delta_{5,22}$	
22	$X^{n-1}Z+Y^{n-1}U+Z^n+XZU^{n-2}$	$(0:0:0:1)$ $(1:0:0:\sqrt[n-2]{-1})$
	$n^2-2n+2+(n-1)\delta_{1,2}+6\delta_{4,18}$	
23	$X^{n-1}Z+Y^{n-1}U+Z^n+YZU^{n-2}$	$(0:0:0:1)$
	$n^2-2+4\delta_{9,10}$	
24	$X^{n-1}U+Y^{n-1}U+Z^n+XYU^{n-2}$	$(0:0:0:1)$ $(1:\sqrt[n-1]{-1}:0:0)$
	$n^2-n+\delta_{1,2}+n\delta_{0,2}+(2n-2)\delta_{1,3}$ $-2\delta_{4,6}+8\delta_{5,20}+8\delta_{6,30}+12\delta_{9,36}$ $+16\delta_{15,40}+12\delta_{10,42}+16\delta_{36,60}+24\delta_{21,84}$	
25	$X^{n-1}U+Y^{n-1}U+Z^n+XZU^{n-2}$	$(0:0:0:1)$ $(1:\sqrt[n-1]{-1}:0:0)$
	n^2-2	

26	$X^n+Y^n+Z^{n-1}U+XYU^{n-2}$	$(0:0:0:1)$
	$3n-5+n\delta_{1,2}+\delta_{0,2}+4\delta_{4,12}$ $+8\delta_{4,24}+8\delta_{\{5,6,10\},30}+16\delta_{6,40}+12\delta_{9,42}$ $+16\delta_{10,48}+16\delta_{\{5,6,10\},60}+24\delta_{\{8,9\},84}+32\delta_{22,120}$	
27	$X^n+Y^n+Z^{n-1}U+XZU^{n-2}$	$(0:0:0:1)$
	$3n-2+4\delta_{4,10}$	
28	$X^n+Y^n+XZ^{n-1}+XYU^{n-2}$	$(0:0:0:1)$
	$n^2-3+(n-2)\delta_{1,2}+\delta_{0,2}+4\delta_{10,12}+8\delta_{\{22,26\},30}$	
29	$X^n+Y^n+XZ^{n-1}+XZU^{n-2}$	$(0:0:0:1)$ $(0:0:\sqrt[n-2]{-1}:1)$
	$2n^2-5n+4+(n-2)\delta_{0,2}$	
30	$X^n+Y^n+XZ^{n-1}+YZU^{n-2}$	$(0:0:0:1)$
	$n^2-2+4\delta_{5,12}+8\delta_{\{7,11\},30}$	
31	$X^{n-1}Y+Y^{n-1}Z+XZ^{n-2}U+XYU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$ $(1:0:0:\sqrt[n-2]{-1})$
	$3n^2-10n+7+n\delta_{1,2}+3\delta_{0,2}+8\delta_{\{14,20\},30}$	
32	$X^{n-1}Y+Y^{n-1}Z+XZ^{n-2}U+XZU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$ $(0:0:1:\sqrt[n-3]{-1})$
	$2n^2-5n+4$	
33	$X^{n-1}Y+Y^{n-1}Z+XZ^{n-2}U+YZU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$ $(0:1:0:\sqrt[n-2]{-1})$
	$3n^2-10n+10$	
34	$X^{n-1}Y+Y^{n-1}Z+YZ^{n-2}U+XZU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$
	$2n^2-5n+6$	
35	$X^{n-1}Y+Y^{n-1}Z+XYZ^{n-2}+XZU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$ $(1:0:\sqrt[n-2]{-1}:0)$
	$3n^2-10n+10+4\delta_{0,8}+8\delta_{20,24}$	
36	$X^{n-1}Y+XY^{n-1}+XZ^{n-2}U+YZU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$
	$4n^2-15n+16+4\delta_{10,12}+8\delta_{\{22,26\},30}$	
37	$X^{n-1}Z+Y^{n-1}Z+XZ^{n-2}U+XYU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$ $(1:\sqrt[n-1]{-1}:0:0)$
	$2n^2-5n+4$	
38	$X^{n-1}Z+Y^{n-1}U+XZ^{n-2}U+XYU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$
	$2n^2-5n+4$	

39	$X^{n-1}Z + Y^{n-1}U + XZ^{n-2}U + XZU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$ $(1:0:0:\sqrt[n-2]{-1})$ $(0:0:1:\sqrt[n-3]{-1})$
	$4n^2 - 15n + 16 + 4\delta_{10,12} + 8\delta_{\{22,26\},30}$	
40	$X^{n-1}Z + Y^{n-1}U + XZ^{n-2}U + YZU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$
	$3n^2 - 7n + \delta_{1,2}$	
41	$X^{n-1}Z + Y^{n-1}U + YZ^{n-2}U + XYU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$ $(0:\sqrt[n-2]{-1}:1:0)$
	$3n^2 - 10n + 10$	
42	$X^{n-1}Z + Y^{n-1}U + YZ^{n-2}U + XZU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$ $(1:0:0:\sqrt[n-2]{-1})$ $(0:1:\sqrt[n-2]{-1}:0)$
	$4n^2 - 15n + 16 + (n-4)\delta_{0,2} + 2\delta_{2,4}$ $+8\delta_{8,12} + 12\delta_{\{8,14\},18} + 16\delta_{12,20} + 16\delta_{14,24} + 32\delta_{32,60}$	
43	$X^n + Y^n + XZ^{n-2}U + XYU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$
	$n^2 - 2 + 6\delta_{5,9} + 6\delta_{8,18}$	
44	$X^n + Y^n + XZ^{n-2}U + XZU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$ $(0:0:1:\sqrt[n-3]{-1})$
	$2n^2 - 5n + 4 + 2\delta_{4,6} + 16\delta_{9,24} + 8\delta_{21,24}$ $+8\delta_{6,30} + 32\delta_{21,60} + 32\delta_{55,120}$	
45	$X^n + Y^n + XZ^{n-2}U + YZU^{n-2}$	$(0:0:0:1)$ $(0:0:1:0)$
	$n^2 + n - 8 + n\delta_{1,2} + 4\delta_{0,2} + 2\delta_{1,4} + 4\delta_{4,12}$ $+8\delta_{7,24} + 8\delta_{\{6,10,13\},30} + 12\delta_{10,42} + 16\delta_{13,60}$	
46	$X^n + Y^n + Z^{n-1}U + XU^{n-1}$	First calculated in [**11**] Case VII in [**7**]
	$n + 4\delta_{3,12} + 6\delta_{4,18}$	
47	$X^n + Y^n + Z^{n-1}U + ZU^{n-1}$	Case III in [**7**]
	$n^2 - n + 1 + (4n-9)\delta_{0,2} + (4n-4)\delta_{2,3} + 8\delta_{0,4}$ $-4\delta_{2,6} - 8\delta_{4,8} + 8\delta_{0,10} - 4\delta_{8,12} + 12\delta_{0,12} + 32\delta_{\{6,12\},24} + 16\delta_{8,24}$ $+24\delta_{14,28} + 24\delta_{\{12,20\},30} + 16\delta_{0,30} + 24\delta_{18,36} + 64\delta_{20,40}$ $+48\delta_{14,42} + 64\delta_{24,48} + 144\delta_{\{12,20\},60} + 192\delta_{30,60} + 40\delta_{44,66} + 48\delta_{54,72}$ $+216\delta_{42,84} + 72\delta_{56,84} + 64\delta_{\{30,72\},120} + 128\delta_{60,120} + 96\delta_{78,156} + 96\delta_{72,180}$	
48	$X^n + Y^n + XZ^{n-1} + XU^{n-1}$	$(0:0:\sqrt[n-1]{-1}:1)$
	$2n^2 - 5n + 4\delta_{1,2} + 2n\delta_{0,2} - 2\delta_{4,6} + 8\delta_{9,12} + 12\delta_{10,18} + 8\delta_{9,24} + 24\delta_{16,30}$ $+48\delta_{21,30} + 32\delta_{25,30} + 48\delta_{22,42} + 32\delta_{\{21,25\},60} + 48\delta_{55,90} + 32\delta_{25,120}$	

49	$X^n + Y^n + XZ^{n-1} + YU^{n-1}$	Case IV in [**7**]
	$2n^2 - 5n + 4 + 4\delta_{4,12} + 24\delta_{6,30} + 8\delta_{10,30}$	
50	$X^{n-2}YZ + Y^{n-2}ZU + XZ^{n-2}U + XU^{n-1}$	$(1:0:0:0)$ $(0:1:0:0)$ $(0:0:1:0)$ $(0:0:\sqrt[n-2]{-1}:1)$
	$4n^2 - 15n + 16 + 4\delta_{10,12} + 8\delta_{22,30}$	
51	$X^{n-2}YZ + Y^{n-2}ZU + XZ^{n-2}U + YU^{n-1}$	$(1:0:0:0)$ $(0:1:0:0)$ $(0:0:1:0)$
	$3n^2 - 10n + 10$	
52	$X^{n-2}YZ + Y^{n-2}ZU + YZ^{n-2}U + XU^{n-1}$	$(1:0:0:0)$ $(0:1:0:0)$ $(0:0:1:0)$ $(0:\sqrt[n-3]{-1}:1:0)$
	$3n^2 - 10n + 10 + (n-3)\delta_{1,2} + 8\delta_{\{9,15\},15} + 8\delta_{\{15,19\},20}$	
53	$X^{n-2}YZ + Y^{n-2}ZU + XYZ^{n-2} + XU^{n-1}$	$(1:0:0:0)$ $(0:1:0:0)$ $(0:0:1:0)$ $(\sqrt[n-3]{-1}:0:1:0)$
	$4n^2 - 15n + 16 + 4\delta_{0,12} + 6\delta_{17,18}$	
54	$X^{n-2}YZ + Y^{n-2}ZU + XZ^{n-2}U + U^n$	$(1:0:0:0)$ $(0:1:0:0)$ $(0:0:1:0)$
	$2n^2 - 5n + 4 + 4\delta_{0,5}$	
55	$X^{n-2}YZ + Y^{n-2}ZU + YZ^{n-2}U + U^n$	$(1:0:0:0)$ $(0:1:0:0)$ $(0:0:1:0)$ $(0:\sqrt[n-3]{-1}:1:0)$
	$3n^2 - 8n + 4\delta_{1,2} + n\delta_{0,2} + 4\delta_{11,12} + 8\delta_{23,24}$ $+8\delta_{\{23,27,28\},30} + 16\delta_{37,40} + 12\delta_{36,42} + 16\delta_{41,48}$ $+16\delta_{\{53,57,58\},60} + 24\delta_{\{78,79\},84} + 32\delta_{101,120}$	
56	$X^{n-2}YZ + Y^{n-2}ZU + XYZ^{n-2} + U^n$	$(1:0:0:0)$ $(0:1:0:0)$ $(0:0:1:0)$ $(\sqrt[n-3]{-1}:0:1:0)$
	$2n^2 - 3n - 2 + 4\delta_{9,10}$	

57	$X^{n-2}YZ + XY^{n-2}Z + XYZ^{n-2} + U^n$	$(1:0:0:0)$ $(0:1:0:0)$ $(0:0:1:0)$ $(0:\sqrt[n-3]{-1}:1:0)$ $(\sqrt[n-3]{-1}:0:1:0)$ $(\sqrt[n-3]{-1}:1:0:0)$
	$3n^2 - 9n - 4 + 11\delta_{1,2} + 3n\delta_{0,2} + (2n-6)\delta_{0,3} - 6\delta_{0,6} + 12\delta_{10,14} + 36\delta_{15,18}$ $+24\delta_{15,20} + 24\delta_{24,28} + 72\delta_{18,30} + 36\delta_{27,36} + 48\delta_{35,40}$ $+60\delta_{24,42} + 48\delta_{48,60} + 48\delta_{42,78} + 72\delta_{63,84}$	
58	$X^{n-1}Y + Y^{n-1}Z + Z^{n-1}U + XYU^{n-2}$	$(0:0:0:1)$ $(1:0:0:\sqrt[n-2]{-1})$
	$2n^2 - 5n + 4 + 4\delta_{6,12} + 8\delta_{12,30}$	
59	$X^{n-1}Y + Y^{n-1}Z + Z^{n-1}U + XZU^{n-2}$	$(0:0:0:1)$
	$n^2 - 2n + 4$	
60	$X^{n-1}Y + Y^{n-1}Z + Z^{n-1}U + YZU^{n-2}$	$(0:0:0:1)$ $(0:1:0:\sqrt[n-2]{-1})$
	$3n^2 - 10n + 10$	
61	$X^{n-1}Y + Y^{n-1}Z + XZ^{n-1} + XYU^{n-2}$	$(0:0:0:1)$ $(1:0:0:\sqrt[n-2]{-1})$
	$2n^2 - 5n + 4 + 2\delta_{0,6}$	
62	$X^{n-1}Y + Y^{n-1}Z + YZ^{n-1} + XZU^{n-2}$	$(0:0:0:1)$ $(1:0:\sqrt[n-2]{-1}:0)$
	$3n^2 - 10n + 10$	
63	$X^{n-1}Y + Y^{n-1}U + Z^{n-1}U + XYU^{n-2}$	$(0:0:0:1)$ $(1:0:0:\sqrt[n-2]{-1})$ $(0:1:\sqrt[n-1]{-1}:0)$
	$3n^2 - 10n + 10$	
64	$X^{n-1}Y + Y^{n-1}U + Z^{n-1}U + XZU^{n-2}$	$(0:0:0:1)$ $(0:1:\sqrt[n-1]{-1}:0)$
	$2n^2 - 5n + 4 + 4\delta_{4,12} + 6\delta_{5,18}$	
65	$X^{n-1}Y + Y^{n-1}U + Z^{n-1}U + YZU^{n-2}$	$(0:0:0:1)$ $(0:1:\sqrt[n-1]{-1}:0)$
	$2n^2 - 5n + 4$	
66	$X^{n-1}Y + Y^{n-1}U + YZ^{n-1} + XZU^{n-2}$	$(0:0:0:1)$ $(1:0:\sqrt[n-1]{-1}:0)$
	$n^2 - 2n + 2 + (n-1)\delta_{1,2} + 8\delta_{\{4,10\},15} + 8\delta_{\{5,9\},20}$	
67	$X^{n-1}Y + XY^{n-1} + Z^{n-1}U + XYU^{n-2}$	$(0:0:0:1)$ $(1:0:0:\sqrt[n-2]{-1})$ $(0:1:0:\sqrt[n-2]{-1})$
	$4n^2 - 15n + 13 + n\delta_{1,2} + 3\delta_{0,2} + 4\delta_{10,12} + 12\delta_{9,14} + 24\delta_{10,24}$ $+72\delta_{17,30} + 8\delta_{\{22,26\},30} + 16\delta_{26,40} + 60\delta_{23,42} + 16\delta_{34,48} + 16\delta_{17,60}$ $+48\delta_{\{22,26\},60} + 40\delta_{46,66} + 48\delta_{41,78} + 24\delta_{\{50,65\},84} + 32\delta_{82,120}$	

68	$X^{n-1}Y + XY^{n-1} + Z^{n-1}U + XZU^{n-2}$	$(0:0:0:1)$
	$2n^2 - 3n - 2 + 8\delta_{14,30}$	
69	$X^{n-1}Y + Y^{n-1}Z + Z^{n-1}U + U^n$	Case IX in [**7**]
	$n + 4\delta_{3,8} + 8\delta_{7,24}$	
70	$X^{n-1}Y + Y^{n-1}Z + XZ^{n-1} + U^n$	First calculated in [**11**] Case VIII in [**7**]
	$1 + \delta_{0,2} + 2n\delta_{0,3} + 6\delta_{\{4,6\},14}$ $+12\delta_{\{4,20\},28} + 12\delta_{\{6,18\},42} + 24\delta_{\{18,24\},78}$	
71	$X^{n-1}Y + Y^{n-1}Z + YZ^{n-1} + U^n$	$(1:0:\sqrt[n-1]{-1}:0)$
	$n^2 - 2n + 2 + (n-1)\delta_{1,2} + 4\delta_{4,12} + 8\delta_{\{6,10\},30}$	
72	$X^{n-1}Y + Y^{n-1}U + Z^{n-1}U + U^n$	$(0:1:\sqrt[n-1]{-1}:0)$
	$2n^2 - 5n + 4 + (n-3)\delta_{1,2} + 2\delta_{1,4} + 8\delta_{7,12}$ $+12\delta_{\{7,13\},18} + 16\delta_{11,20} + 16\delta_{13,24} + 32\delta_{31,60}$	
73	$X^{n-1}Y + Y^{n-1}U + YZ^{n-1} + U^n$	$(1:0:\sqrt[n-1]{-1}:0)$
	$2n^2 - 5n + 4$	
74	$X^{n-1}Y + XY^{n-1} + Z^{n-1}U + U^n$	Case V in [**7**]
	$n^2 - 2n + 2\delta_{1,2} + 2\delta_{0,3} + 2\delta_{\{2,4\},6}$ $+8\delta_{4,24} + 8\delta_{5,30} + 32\delta_{6,120}$	
75	$X^n + Y^n + Z^n + XYU^{n-2}$	$(0:0:0:1)$
	$3n - 2 + (3n-8)\delta_{0,2} + 4\delta_{0,4} + 8\delta_{\{6,8\},12} + 16\delta_{5,15}$ $+16\delta_{10,20} + 16\delta_{\{6,8,12\},24} + 16\delta_{\{12,20\},30} + 32\delta_{\{12,20,32\},60}$	
76	$X^n + Y^n + Z^n + XU^{n-1}$	Case II in [**7**]
	$n^2 - 2n + 2n\delta_{1,2} + 2\delta_{0,2} - 2\delta_{3,6} + 8\delta_{4,12} + 12\delta_{9,18}$ $+8\delta_{16,24} + 32\delta_{6,30} + 48\delta_{10,30} + 24\delta_{15,30} + 48\delta_{21,42} + 32\delta_{\{36,40\},60}$ $+48\delta_{36,90} + 32\delta_{96,120}$	
77	$X^{n-2}YZ + Y^{n-2}ZU + XZ^{n-2}U + XYU^{n-2}$	$(1:0:0:0)$ $(0:1:0:0)$ $(0:0:1:0)$ $(0:0:0:1)$
	$5n^2 - 21n + 24 + \delta_{1,2} + 8\delta_{\{16,20\},20} + 8\delta_{\{20,26\},30} + 16\delta_{\{20,26,50,56\},60}$ $+32\delta_{\{86,110\},120}$	
78	$X^{n-1}Y + Y^{n-1}Z + Z^{n-1}U + XU^{n-1}$	First calculated in [**11**] Case X in [**7**]
	$n^2 - n + \delta_{1,2} + 8\delta_{\{4,8\},20} + 8\delta_{\{8,14\},30} + 16\delta_{\{8,14,38,44\},60} + 32\delta_{\{14,38\},120}$	
79	$X^{n-1}Y + Y^{n-1}Z + Z^{n-1}U + YU^{n-1}$	$(\sqrt[n-1]{-1}:0:0:1)$
	$n^2 - 2n + 2 + 2\delta_{3,6}$	
80	$X^{n-1}Y + Y^{n-1}Z + Z^{n-1}U + ZU^{n-1}$	$(0:\sqrt[n-1]{-1}:0:1)$
	$2n^2 - 5n + 4$	
81	$X^{n-1}Y + Y^{n-1}Z + YZ^{n-1} + ZU^{n-1}$	$(1:0:\sqrt[n-1]{-1}:0)$ $(0:\sqrt[n-1]{-1}:0:1)$
	$4n^2 - 15n + 16 + 4\delta_{10,12} + 8\delta_{22,30} + 24\delta_{26,30}$	

82	$X^{n-1}Y + XY^{n-1} + Z^{n-1}U + ZU^{n-1}$	Case VI in [**7**]
	$3n^2 - 9n + 7 + \delta_{0,2} + 8\delta_{\{6,8\},12} + 16\delta_{12,20} + 16\delta_{\{6,8,14\},24} + 32\delta_{12,30}$ $+16\delta_{20,30} + 48\delta_{30,42} + 96\delta_{12,60} + 32\delta_{\{20,32,42,50\},60} + 48\delta_{14,84} + 64\delta_{72,120}$	
83	$X^n + Y^n + Z^n + U^n$	The Fermat surface described in [**1**] and [**11**] Case I in [**7**]
	$3n^2 - 9n + 7 + (24n - 47)\delta_{0,2} + (8n - 24)\delta_{0,3} - 48\delta_{0,4} - 96\delta_{0,6}$ $-48\delta_{0,8} - 48\delta_{0,10} + 144\delta_{0,12} + 48\delta_{0,14} + 192\delta_{0,15}$ $+432\delta_{0,18} + 624\delta_{0,20} + 288\delta_{0,21} + 912\delta_{0,24} + 240\delta_{0,28}$ $+2256\delta_{0,30} + 432\delta_{0,36} + 384\delta_{0,40} + 3984\delta_{0,42} + 384\delta_{0,48}$ $+4896\delta_{0,60} + 720\delta_{0,66} + 288\delta_{0,72} + 768\delta_{0,78} + 1584\delta_{0,84}$ $+576\delta_{0,90} + 1728\delta_{0,120} + 576\delta_{0,156} + 576\delta_{0,180}$	

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