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The finite group action and the equivariant determinant of elliptic operators II

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Abstract. Let M be an almost complex manifold and g a periodic automorphism of M of order p. Then the rotation angles of g around fixed points of g are naturally defined by the almost complex structure of M. In this paper, under the assumption that the fixed points of g^k $(1 \leq k \leq p-1)$ are isolated, a calculation formula is provided for the homomorphism $I_D : \mathbb{Z}_p \to \mathbb{R}/\mathbb{Z}$ defined in [**8**]. The formula gives a new method to study the periodic automorphisms of almost complex manifolds. As examples of the application of the formula, we show the nonexistence of the $\mathbb{Z}_p$-action of specific isotropy orders and examine whether specific rotation angles exist or not.

1. Introduction.

The problem whether a manifold with some geometric structure admits an action of a finite group which preserves the geometric structure is a basic problem in geometry, and the problem is well studied for compact Riemann surfaces.

Let M be a $2m$-dimensional closed oriented manifold and G a finite group acting on M. We assume that the action of G is effective. Let g be an element of G of order $p \geq 2$ and $\mathbb{Z}_p$ the cyclic group generated by g. In this paper, we set the following assumption.

Assumption 1.1. Some g^k $(1 \leq k \leq p-1)$ has a fixed point, and any fixed point of g^k is isolated for $1 \leq k \leq p-1$ if g^k has a fixed point.

Under the assumption above, let Ω be the union of the fixed points of g^k for $1 \leq k \leq p-1$ and suppose that the image $\pi(\Omega)$ consists of b points $y_1, \ldots, y_b \in M/\mathbb{Z}_p$ where $\pi : M \longrightarrow M/\mathbb{Z}_p$ is the projection. In this paper, the $\mathbb{Z}_p$-action is called the $\mathbb{Z}_p$-action of isotropy orders $(p_1, \ldots, p_b)$ if the isotropy group at a point $q_i \in \pi^{-1}(y_i)$ $(1 \leq i \leq b)$ is the cyclic group of order p_i. Then for $1 \leq i \leq b$ the isotropy group at any points in $\pi^{-1}(y_i)$ is the cyclic group of order p_i generated by g^{r_i} where $r_i = p/p_i$ and $\pi^{-1}(y_i)$ consists of r_i points q_i, $g \cdot q_i, \ldots, g^{r_i-1} \cdot q_i$. Note

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that $\pi : M \longrightarrow M/\mathbb{Z}_p$ is called a branched covering with branch points $y_1, \dots, y_b$ of order $(p_1, \dots, p_b)$ if $m = 1$.

In [**5**] Harvey gives the necessary and sufficient condition for the existence of the branched covering of a specific order, and the problem of examining the existence of an action of a cyclic group has been completely settled (see also [**3**], [**4**], [**7**]). But there still has been no known general method to examine the existence of an action of a cyclic group when $m \geq 2$.

In [**8**] we introduce a group homomorphism I_D by using an elliptic operator D adapted to a geometric structure of a manifold, whose dimension is not restricted.

Let D be a G-equivariant elliptic operator. Then a homomorphism I_D from G to $\mathbb{R}/\mathbb{Z}$ is defined by

$$I_D(g) = \frac{1}{2\pi\sqrt{-1}} \log \det(D, g) \in \mathbb{R}/\mathbb{Z}$$

for $g \in G$, where $\det(D, g)$ is defined by

$$\det(D, g) = \det(g \mid \ker D)/\det(g \mid \operatorname{coker} D) \in S^1 \subset \mathbb{C}^*$$

(see [**8**, Definition 2.1]). Then as we see in [**8**] (3) the next equality holds

$$I_D(g) \equiv \frac{p-1}{2p} \operatorname{Ind}(D) - \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - \xi_p^{-k}} \operatorname{Ind}(D, g^k) \pmod{\mathbb{Z}}, \tag{1}$$

where Ind is the Atiyah-Singer index (see [**2**]) and ξ_p is the primitive p-th root of unity defined by $\xi_p = e^{2\pi\sqrt{-1}/p}$.

We can express the value $I_D(g)$ by the fixed point data of the g^k-action $(1 \leq k \leq p-1)$ by using the equality (1) and the fixed point formula of Atiyah-Segal-Singer [**1**], [**2**].

Since I_D is a homomorphism, the equalities $I_D(g^z) = zI_D(g)$, $I_D(gh) = I_D(g) + I_D(h)$ hold for any g, $h \in G$ and any integer z because $\mathbb{R}/\mathbb{Z}$ is an abelian group. These properties of I_D impose conditions on the fixed point data and I_D can be used to examine the existence of a finite group action.

When M is a compact Riemann surface and the g-action preserves the complex structure of M, we give a formula to calculate $I_D(g)$ precisely for the $\otimes^\ell TM$-valued Dolbeault operator D over M in [**8**, Proposition 3.2].

Though the formula is useful to examine the existence of a finite group action on the Riemann surfaces, we need a formula to calculate the precise value of $I_D(g)$ for arbitrary m to examine the existence of a finite group action on higher dimensional manifolds. In this paper, we give a formula to calculate the precise value of $I_D(g)$ for $2m$-dimensional almost complex manifolds.

2. Main result.

Let M be a $2m$-dimensional almost complex manifold. Assume that $p \geq 2$ and that the action of $\mathbb{Z}_p = \langle g \rangle$ preserves the almost complex structure of M.

The main theorem of this paper is stated by using integers $f_{m,p}$, $\Lambda_{m,p}$ defined below.

For a nonnegative integer s, an integer $f_{m,p}(s)$ is defined by

$$f_{m,p}(s) = \sum_{k=0}^{m} \sum_{\ell=0}^{m-k} (-1)^{\ell} \binom{m-k}{\ell} \binom{-\ell p + s + m - p}{m} \times \sum_{u=k}^{m+1} \binom{s}{m+1-u} \sum_{v=0}^{k} (-1)^{v} \binom{k}{v} \binom{pv}{u}. \tag{2}$$

Let E be a complex $\mathbb{Z}_p$-vector bundle over M and D_E the E-valued Dolbeault operator over the almost complex manifold M, which is a $\mathbb{Z}_p$-equivariant elliptic operator.

Suppose that g^{r_i} acts on the tangent space of M at $q_i \in \pi^{-1}(y_i)$ via multiplication by a diagonal matrix with diagonal entries $\xi_{p_i}^{\tau_{i1}}, \dots, \xi_{p_i}^{\tau_{im}}$ and acts on the fiber $E|q_i$ via diagonal matrix with diagonal entries $\xi_{p_i}^{\mu_{i1}}, \dots, \xi_{p_i}^{\mu_{id}}$ where d is the rank of E, $1 \leq \tau_{ij}$, $\mu_{ic} \leq p_i - 1$ and τ_{ij} is prime to p_i. Then since g acts transitively on $\pi^{-1}(y_i)$, g^{r_i} acts on the tangent space of M or the fiber of E at each point in $\pi^{-1}(y_i)$ via multiplication by the same diagonal matrices. In this paper the set $\{\tau_{ij}\}$ is called the rotation angle of g^{r_i} around the points in $\pi^{-1}(y_i)$.

Since the fixed point set of g^k $(1 \leq k \leq p-1)$ exists if and only if k equals $r_i \kappa$ for $1 \leq i \leq b$, $1 \leq \kappa \leq p_i - 1$, it follows from Theorem (4.3), Theorem (4.6) in [**2**] (see also [**8**, Proposition 2.7, p. 101]) that

$$\mathrm{Ind}(D_E) = \mathrm{Ch}(E)\,\mathrm{Td}(M)[M],$$
$$\sum_{k=1}^{p-1} \frac{1}{1-\xi_p^{-k}} \mathrm{Ind}(D_E, g^k) = \sum_{i=1}^{b} r_i \sum_{c=1}^{d} \sum_{\kappa=1}^{p_i-1} \frac{\xi_{p_i}^{\kappa\mu_{ic}}}{1-\xi_{p_i}^{-\kappa}} \prod_{j=1}^{m} \frac{1}{1-\xi_{p_i}^{-\kappa\tau_{ij}}} \tag{3}$$

where $\mathrm{Ch}(E)$ is the Chern character of E, $\mathrm{Td}(M)$ is the Todd class of M and $[M]$ is the fundamental cycle of M.

Definition 2.1. For an integer λ which is prime to p, there exists an integer

$\overline{\lambda}$ which satisfies the following conditions:

$$1 \leq \overline{\lambda} \leq p-1, \quad \lambda\overline{\lambda} \equiv 1 \pmod{p}.$$

$\overline{\lambda}$ is called the mod p inverse of λ.

For any natural number z and any integers μ, s, an integer $\Lambda_{m,p}(z,\mu,s)$ is defined by

$$\Lambda_{m,p}(z,\mu,s) = \sum_{\lambda_1=0}^{z\theta_{i1}-1} \sum_{\lambda_{21},\lambda_{22}=0}^{\theta_{i2}-1} \cdots \sum_{\lambda_{m1},\dots,\lambda_{mm}=0}^{\theta_{im}-1} \delta_p(\zeta(z,\mu,s,\tau,\lambda)), \tag{4}$$

where τ, λ denote the sets $\{\tau_{ij} \mid 1 \leq j \leq m\}$, $\{\lambda_1, \lambda_{jk} \mid 2 \leq j \leq m,\ 1 \leq k \leq j\}$ respectively and $\delta_p(\zeta(z,\mu,s,\tau,\lambda))$ is defined by

$$\zeta(z,\mu,s,\tau,\lambda) = 1 + \lambda_1 + z\mu + z\sum_{j=1}^{m} \tau_{ij} + z\sum_{j=2}^{m} \tau_{i\,j-1}(\lambda_{j1} + \cdots + \lambda_{jj}) + sz\tau_{im},$$

$$\delta_p(\zeta(z,\mu,s,\tau,\lambda)) = \begin{cases} 1 & (\zeta(z,\mu,s,\tau,\lambda) \equiv 0 \pmod{p}) \\ 0 & (\text{otherwise}) \end{cases}.$$

Set $\theta_{i1} = \tau_{i1}$ and for $2 \leq j \leq m$ let θ_{ij} be a natural number such that $1 \leq \theta_{ij} \leq p_i - 1$, $\theta_{ij} \equiv \overline{\tau_{i\,j-1}}\tau_{ij} \pmod{p_i}$, where $\overline{\tau_{i\,j-1}}$ is the mod p_i inverse of $\tau_{i\,j-1}$.

THEOREM 2.2. *Let z be an integer such that $1 \leq z \leq p-1$ and that z is prime to p. Then the next equality holds as elements of $\mathbb{R}/\mathbb{Z}$.*

$$I_{D_E}(g^z) = \frac{p-1}{2p}\operatorname{Ch}(E)\operatorname{Td}(M)[M] + \sum_{i=1}^{b} \frac{1}{p_i^{m+2}} \Biggl\{ dz \Biggl(\prod_{j=1}^{m} \theta_{ij}^{j} \Biggr) \sum_{s=0}^{p_i-1} f_{m,p_i}(s) - p_i \sum_{c=1}^{d} \sum_{s=0}^{p_i-1} f_{m,p_i}(s)\Lambda_{m,p_i}(z,\mu_{ic},s) \Biggr\}.$$

PROOF. Since z is prime to p_i, the fixed point set of g^{zr_i} coincides with that of g^{r_i}, and g^{zr_i} acts on $T_{q_i}M$ via multiplication by the diagonal matrix with diagonal entries $\xi_{p_i}^{z\tau_{i1}}, \dots, \xi_{p_i}^{z\tau_{im}}$ and acts on the fiber E_{q_i} via multiplication by the diagonal matrix with diagonal entries $\xi_{p_i}^{z\mu_{i1}}, \dots, \xi_{p_i}^{z\mu_{id}}$. Hence it follows from (1), (3) that

$$I_{D_E}(g^z) = \frac{p-1}{2p}\operatorname{Ch}(E)\operatorname{Td}(M)[M] - \sum_{i=1}^{b}\frac{1}{p_i}\sum_{c=1}^{d}\sum_{\kappa=1}^{p_i-1}\frac{\xi_{p_i}^{\kappa z\mu_{ic}}}{1-\xi_{p_i}^{-\kappa}}\prod_{j=1}^{m}\frac{1}{1-\xi_{p_i}^{-\kappa z\tau_{ij}}}. \quad (5)$$

Therefore it suffices to show that the equality

$$\sum_{k=1}^{p-1}\frac{\xi_p^{kz\mu}}{1-\xi_p^{-k}}\prod_{j=1}^{m}\frac{1}{1-\xi_p^{-kz\tau_{ij}}}$$
$$= \frac{1}{p^{m+1}}\left\{p\sum_{s=0}^{p-1} f_{m,p}(s)\Lambda_{m,p}(z,\mu,s) - z\left(\prod_{j=1}^{m}\theta_{ij}^{j}\right)\sum_{s=0}^{p-1} f_{m,p}(s)\right\} \quad (6)$$

holds for any natural number p with $p \geq 2$ and any integer μ. To prove the equality (6) we need several lemmas.

For integers i, j define the number $\delta(i,j)$ by

$$\delta(i,j) = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

Lemma 2.3. *For $1 \leq k,\ \ell \leq m+1$ set*

$$a_{k\ell} = \binom{\ell-1-k}{\ell-1}.$$

Then we have

$$a_{k\ell} = (-1)^{\ell-1}\binom{k-1}{\ell-1}, \qquad \sum_{\ell=1}^{m+1} a_{k\ell}a_{\ell s} = \delta(k,s).$$

Proof. Note that $a_{k\ell} = 0$ if $k < \ell$. For $f(x) = (e^x - 1)^{k-1}$ we have

$$f(x) = \sum_{\ell=0}^{k-1}\binom{k-1}{\ell}(-1)^{k-1-\ell}e^{\ell x}$$

and hence $(-1)^{k-1}f^{(j)}(0)$ is equal to

$$\sum_{\ell=0}^{k-1}\binom{k-1}{\ell}(-1)^{\ell}\ell^{j} = \begin{cases} 0 & \text{if } 0 \leq j < k-1 \\ (-1)^{k-1}(k-1)! & \text{if } j = k-1 \end{cases}. \quad (7)$$

Since

$$a_{k\ell} = \frac{(\ell - 1 - k)\cdots(1-k)}{(\ell-1)!} = (-1)^{\ell-1}\binom{k-1}{\ell-1},$$

it follows from the equality (7) above that

$$\begin{aligned}\sum_{\ell=1}^{m+1} a_{k\ell}a_{\ell s} &= \sum_{\ell=1}^{k}(-1)^{\ell-1}\binom{k-1}{\ell-1}\binom{s-1-\ell}{s-1} \\ &= \sum_{\ell=0}^{k-1}(-1)^{\ell}\binom{k-1}{\ell}\frac{(-\ell)^{s-1} + \text{lower order terms}}{(s-1)!} \\ &= \begin{cases}(-1)^{k-1}(k-1)!\dfrac{(-1)^{k-1}}{(k-1)!} = 1 & (s=k) \\ 0 & (s<k)\end{cases}. \qquad \square\end{aligned}$$

Let p be a natural number with $p \geq 2$.

Lemma 2.4. *For any nonnegative integers j, s the next equality holds*:

$$\binom{pj+s+m}{m} = \sum_{k=1}^{m+1}\sum_{\ell=1}^{k}\binom{j+k-1}{k-1}\binom{\ell-1-k}{\ell-1}\binom{-\ell p+s+m}{m}.$$

Proof. Define a polynomial $P(x)$ of degree m by

$$P(x) = \frac{(px+s+m)\cdots(px+s+1)}{m!} - \gamma_1 - \sum_{k=2}^{m+1}\gamma_k\frac{(x+k-1)\cdots(x+1)}{(k-1)!}$$

where γ_k is an integer defined by

$$\gamma_k = \sum_{\ell=1}^{k}\binom{\ell-1-k}{\ell-1}\binom{-\ell p+s+m}{m}.$$

Then for any natural number j it follows from Lemma 2.3 that

$$P(-j) = \binom{-pj+s+m}{m} - \sum_{k=1}^{m+1}\binom{k-1-j}{k-1}\sum_{\ell=1}^{k}\binom{\ell-1-k}{\ell-1}\binom{-\ell p+s+m}{m}$$

$$= \binom{-pj+s+m}{m} - \sum_{\ell=1}^{k} \delta(j,\ell) \binom{-\ell p+s+m}{m} = 0,$$

which implies that $P(x) = 0$ for any x. Hence we have $P(j) = 0$ for any nonnegative integer j. □

For a nonnegative integer s set

$$h_s(t) = \sum_{k=1}^{m+1} \sum_{\ell=0}^{k-1} (-1)^\ell \binom{k-1}{\ell} \binom{-\ell p+s+m-p}{m} t^s (1-t^p)^{m+1-k}.$$

Lemma 2.5. *Let a be a complex number such that $a^p = 1$. Then for $|t| < 1$ we have*

$$\frac{1}{(1-at)^{m+1}} = \frac{1}{(1-t^p)^{m+1}} \sum_{s=0}^{p-1} a^s h_s(t).$$

Proof. Set

$$f(t) = (1-at)^{-1} = \sum_{i=0}^{\infty} a^i t^i.$$

Then we have

$$\begin{aligned} \frac{f^{(m)}(t)}{m!a^m} &= (1-at)^{-m-1} = \sum_{i=0}^{\infty} \binom{i+m}{m} a^i t^i \\ &= \sum_{j=0}^{\infty} \sum_{s=0}^{p-1} \binom{pj+s+m}{m} a^s t^{pj+s} = \sum_{s=0}^{p-1} a^s t^s \sum_{j=0}^{\infty} \binom{pj+s+m}{m} t^{pj}. \end{aligned}$$

The same argument shows that

$$(1-t^p)^{-k} = \sum_{j=0}^{\infty} \binom{j+k-1}{k-1} t^{pj}.$$

Hence it follows from Lemma 2.3 and Lemma 2.4 that

$$
\begin{aligned}
(1-at)^{-m-1} &= \sum_{s=0}^{p-1} a^s t^s \sum_{k=1}^{m+1} \sum_{\ell=1}^{k} \sum_{j=0}^{\infty} \binom{j+k-1}{k-1} t^{pj} \binom{\ell-1-k}{\ell-1} \binom{-\ell p+s+m}{m} \\
&= \sum_{s=0}^{p-1} a^s t^s \sum_{k=1}^{m+1} (1-t^p)^{-k} \sum_{\ell=1}^{k} (-1)^{\ell-1} \binom{k-1}{\ell-1} \binom{-\ell p+s+m}{m} \\
&= \frac{1}{(1-t^p)^{m+1}} \sum_{s=0}^{p-1} a^s h_s(t). \qquad \square
\end{aligned}
$$

Lemma 2.6. *Let a be a complex number such that $a^p = 1$, $a \neq 1$. Then we have*

$$
(1-a)^{-m-1} = \frac{(-1)^{m+1}}{p^{m+1}} \sum_{s=0}^{p-1} a^s f_{m,p}(s).
$$

Proof. Let q, r be nonnegative integers. Then we have

$$
\begin{aligned}
\frac{d^q}{dt^q}\{t^s(1-t^p)^r\} &= \sum_{u=0}^{q} \binom{q}{u} (t^s)^{(q-u)} \left\{ \sum_{v=0}^{r} (-1)^v \binom{r}{v} t^{pv} \right\}^{(u)} \\
&= \sum_{u=0}^{q} \binom{q}{u} \binom{s}{q-u} (q-u)!\, t^{s-q+u} \sum_{v=0}^{r} (-1)^v \binom{r}{v} \binom{pv}{u} u!\, t^{pv-u},
\end{aligned}
$$

$$
\lim_{t\to 1} \{(1-t^p)^r\}^{(u)} = 0 \text{ if } u < r,
$$

and hence it follows that

$$
\lim_{t\to 1} \frac{d^q}{dt^q}\{t^s(1-t^p)^r\} = q! \sum_{u=r}^{q} \binom{s}{q-u} \sum_{v=0}^{r} (-1)^v \binom{r}{v} \binom{pv}{u}.
$$

Therefore we have

$$
\begin{aligned}
h_s^{(m+1)}(1) &= \sum_{k=1}^{m+1} \sum_{\ell=0}^{k-1} (-1)^\ell \binom{k-1}{\ell} \binom{-\ell p+s+m-p}{m} \lim_{t\to 1} \{t^s(1-t^p)^{m+1-k}\}^{(m+1)} \\
&= \sum_{r=0}^{m} \sum_{\ell=0}^{m-r} (-1)^\ell \binom{m-r}{\ell} \binom{-\ell p+s+m-p}{m} \lim_{t\to 1} \{t^s(1-t^p)^r\}^{(m+1)}
\end{aligned}
$$

$$
\begin{aligned}
&= (m+1)! \sum_{r=0}^{m} \sum_{\ell=0}^{m-r} (-1)^{\ell} \binom{m-r}{\ell} \binom{-\ell p + s + m - p}{m} \\
&\quad \times \sum_{u=r}^{m+1} \binom{s}{m+1-u} \sum_{v=0}^{r} (-1)^{v} \binom{r}{v} \binom{pv}{u} \\
&= (m+1)! f_{m,p}(s).
\end{aligned}
$$

Moreover direct computation shows that

$$
\lim_{t \to 1} \{(1-t^p)^{m+1}\}^{(m+1)} = (-1)^{m+1}(m+1)! p^{m+1}.
$$

Hence it follows from Lemma 2.5 that

$$
\begin{aligned}
\sum_{s=0}^{p-1} a^s f_{m,p}(s) &= \frac{1}{(m+1)!} \sum_{s=0}^{p-1} a^s h_s^{(m+1)}(1) \\
&= \frac{1}{(m+1)!} \lim_{t \to 1} \{(1-at)^{-m-1}(1-t^p)^{m+1}\}^{(m+1)} \\
&= \frac{1}{(m+1)!} (1-a)^{-m-1} \lim_{t \to 1} \{(1-t^p)^{m+1}\}^{(m+1)} \\
&= (1-a)^{-m-1}(-1)^{m+1} p^{m+1}. \qquad \square
\end{aligned}
$$

Now the equality (6) is proved as follows. Set $\nu = 1 + z\mu + z\sum_{j=1}^{m} \tau_{ij}$. Then it follows from Lemma 2.6 that

$$
\begin{aligned}
&\sum_{k=1}^{p-1} \frac{\xi_p^{kz\mu}}{1-\xi_p^{-k}} \prod_{j=1}^{m} \frac{1}{1-\xi_p^{-kz\tau_{ij}}} \\
&\quad = \sum_{k=1}^{p-1} \frac{(-1)^{m+1}\xi_p^{k\nu}}{(1-\xi_p^{k})(1-\xi_p^{kz\tau_{i1}})\cdots(1-\xi_p^{kz\tau_{im}})} \\
&\quad = (-1)^{m+1} \sum_{k=1}^{p-1} \xi_p^{k\nu} \frac{1-\xi_p^{kz\theta_{i1}}}{1-\xi_p^{k}} \left(\frac{1-\xi_p^{kz\tau_{i1}\theta_{i2}}}{1-\xi_p^{kz\tau_{i1}}}\right)^2 \\
&\qquad\qquad \cdots \left(\frac{1-\xi_p^{kz\tau_{i\,m-1}\theta_{im}}}{1-\xi_p^{kz\tau_{i\,m-1}}}\right)^m \frac{1}{(1-\xi_p^{kz\tau_{im}})^{m+1}}
\end{aligned}
$$

$$= (-1)^{m+1} \sum_{k=1}^{p-1} \xi_p^{k\nu} \sum_{\lambda_1=0}^{z\theta_{i1}-1} \xi_p^{k\lambda_1} \Bigg(\sum_{\lambda_2=0}^{\theta_{i2}-1} \xi_p^{kz\tau_{i1}\lambda_2} \Bigg)^2 \cdots \Bigg(\sum_{\lambda_m=0}^{\theta_{im}-1} \xi_p^{kz\tau_{i\,m-1}\lambda_m} \Bigg)^m \frac{1}{(1-\xi_p^{kz\tau_{im}})^{m+1}}$$

$$= (-1)^{m+1} \sum_{k=1}^{p-1} \xi_p^{k\nu} \sum_{\lambda_1=0}^{z\theta_{i1}-1} \xi_p^{k\lambda_1} \sum_{\lambda_{21},\lambda_{22}=0}^{\theta_{i2}-1} \xi_p^{kz\tau_{i1}(\lambda_{21}+\lambda_{22})} \cdots \sum_{\lambda_{m1},\dots,\lambda_{mm}=0}^{\theta_{im}-1} \xi_p^{kz\tau_{i\,m-1}(\lambda_{m1}+\cdots+\lambda_{mm})} \frac{1}{(1-\xi_p^{kz\tau_{im}})^{m+1}}$$

$$= (-1)^{m+1} \sum_{k=1}^{p-1} \xi_p^{k\nu} \sum_{\lambda_1=0}^{z\theta_{i1}-1} \xi_p^{k\lambda_1} \sum_{\lambda_{21},\lambda_{22}=0}^{\theta_{i2}-1} \xi_p^{kz\tau_{i1}(\lambda_{21}+\lambda_{22})} \cdots \sum_{\lambda_{m1},\dots,\lambda_{mm}=0}^{\theta_{im}-1} \xi_p^{kz\tau_{i\,m-1}(\lambda_{m1}+\cdots+\lambda_{mm})} \frac{(-1)^{m+1}}{p^{m+1}} \sum_{s=0}^{p-1} \xi_p^{ksz\tau_{im}} f_{m,p}(s)$$

$$= \frac{1}{p^{m+1}} \sum_{s=0}^{p-1} f_{m,p}(s) \sum_{\lambda_1=0}^{z\theta_{i1}-1} \sum_{\lambda_{21},\lambda_{22}=0}^{\theta_{i2}-1} \cdots \sum_{\lambda_{m1},\dots,\lambda_{mm}=0}^{\theta_{im}-1} \sum_{k=1}^{p-1} \xi_p^{k\zeta(z,\mu,s,\tau,\lambda)}$$

$$= \frac{1}{p^{m+1}} \sum_{s=0}^{p-1} f_{m,p}(s) \sum_{\lambda_1=0}^{z\theta_{i1}-1} \sum_{\lambda_{21},\lambda_{22}=0}^{\theta_{i2}-1} \cdots \sum_{\lambda_{m1},\dots,\lambda_{mm}=0}^{\theta_{im}-1} \sum_{k=1}^{p} \xi_p^{k\zeta(z,\mu,s,\tau,\lambda)}$$

$$- \frac{1}{p^{m+1}} \sum_{s=0}^{p-1} f_{m,p}(s) \sum_{\lambda_1=0}^{z\theta_{i1}-1} \sum_{\lambda_{21},\lambda_{22}=0}^{\theta_{i2}-1} \cdots \sum_{\lambda_{m1},\dots,\lambda_{mm}=0}^{\theta_{im}-1} \xi_p^{p\zeta(z,\mu,s,\tau,\lambda)}$$

$$= \frac{1}{p^{m+1}} \Bigg\{ p \sum_{s=0}^{p-1} f_{m,p}(s) \Lambda_{m,p}(z,\mu,s) - z\theta_{i1}\theta_{i2}^2 \cdots \theta_{im}^m \sum_{s=0}^{p-1} f_{m,p}(s) \Bigg\}.$$

This completes the proof of the equality (6) and hence completes the proof of Theorem 2.2. □

REMARK 2.7. Using Proposition 2.6 in [**8**] and the equality (6), we can obtain a calculation formula of $I_D(g)$ for the Dirac operator D and a periodic automorphism g of a Spin^c-manifold under Assumption 1.1.

PROPOSITION 2.8. *There exists a polynomial $g_{m,p}(s)$ with integer coefficients which satisfies the equality below*:

$$f_{m,p}(s) = \frac{(-p)^m}{m!(m+1)!} g_{m,p}(s).$$

PROOF. It follows from the equality (7) that the equalities

$$\sum_{\ell=0}^{m-k} (-1)^\ell \binom{m-k}{\ell} \binom{-\ell p + s + m - p}{m} = \frac{1}{m!} \sum_{\ell=0}^{m-k} (-1)^\ell \binom{m-k}{\ell} \sum_{i=m-k}^{m} (-p\ell)^i Q_i(s),$$

$$\sum_{v=0}^{k} (-1)^v \binom{k}{v} \binom{pv}{u} = \frac{1}{u!} \sum_{v=0}^{k} (-1)^v \binom{k}{v} \sum_{j=k}^{u} (pv)^j S_j(u)$$

hold where $Q_i(s)$, $S_j(u)$ are polynomials with integer coefficients. Hence we have

$$\begin{aligned} f_{m,p}(s) &= \sum_{k=0}^{m} \frac{1}{m!} \sum_{\ell=0}^{m-k} (-1)^\ell \binom{m-k}{\ell} \sum_{i=m-k}^{m} (-p\ell)^i Q_i(s) \\ &\quad \times \sum_{u=k}^{m+1} \binom{s}{m+1-u} \frac{1}{u!} \sum_{v=0}^{k} (-1)^v \binom{k}{v} \sum_{j=k}^{u} (pv)^j S_j(u) \\ &= \sum_{k=0}^{m} \frac{(-p)^{m-k}}{m!} R_k(s) \sum_{u=k}^{m+1} \frac{s \cdots (s-m+u)}{(m+1-u)!} \frac{(-p)^k}{u!} T_k(u) \\ &= \frac{(-p)^m}{m!(m+1)!} \sum_{k=0}^{m} R_k(s) \sum_{u=k}^{m+1} \binom{m+1}{u} T_k(u) \{s \cdots (s-m+u)\} \end{aligned}$$

where $R_k(s)$, $T_k(u)$ are polynomials with integer coefficients. □

EXAMPLE 2.9. Direct computation shows that

$$\begin{aligned} g_{1,p}(s) &= s^2 - (p-2)s - (p-1)^2, \\ g_{2,p}(s) &= 2s^3 - 3(p-3)s^2 + (p^2 - 9p + 12)s + 9(p-1)^2(p-2), \\ \frac{1}{2} g_{3,p}(s) &= 3s^4 - 6(p-4)s^3 + 3(p^2 - 12p + 22)s^2 + 6(p-4)(2p-3)s \\ &\quad - (p-1)^2(73p^2 - 274p + 265). \end{aligned}$$

Note that $g_{1,p}(s)$ coincides with $f_p(s)$ in [**8**, Proposition 3.2].

Corresponding to the irreducible representations of the unitary group, complex vector bundles are defined by using the almost complex structure of M.

Definition 2.10. Let L be the subset of $\mathbb{Z}^m$ defined by

$$L = \{(\ell_1, \dots, \ell_{m-1}, \ell_m) \in \mathbb{Z}^m \mid \ell_j \geq 0 \ (1 \leq j \leq m-1)\}.$$

For $(\ell_1, \dots, \ell_m) \in L$, let $E_{\ell_1,\dots,\ell_m}$ be a complex vector bundle defined by

$$E_{\ell_1,\dots,\ell_m} = \bigotimes_{j=1}^{m} \Big(\bigotimes^{\ell_j} \Big(\bigwedge\nolimits_{\mathbb{C}}^{j} TM \Big) \Big)$$

and $D_{\ell_1,\dots,\ell_m}$ the $E_{\ell_1,\dots,\ell_m}$-valued Dolbeault operator with respect to the almost complex structure of M.

Let b_j denote the binomial coefficient $\binom{m}{j}$ hereafter. Then we have

$$d = \operatorname{rank}_{\mathbb{C}} E_{\ell_1,\dots,\ell_m} = \prod_{j=1}^{m} (b_j)^{\ell_j},$$

$$\sum_{c=1}^{d} \xi_{p_i}^{kz\mu_{ic}} = \prod_{j=1}^{m} (\sigma_{ij})^{\ell_j} \quad (1 \leq i \leq b) \tag{8}$$

where σ_{ij} is the j-th elementary symmetric polynomial in $\xi_{p_i}^{kz\tau_{i1}}, \dots, \xi_{p_i}^{kz\tau_{im}}$.

Let $c_i(M)$ be the i-th Chern class of M. Then we have the next formula (see [**6**]).

Formula 2.11. *Up to higher order terms, the following equalities hold*:

$$\operatorname{Td}(M) = 1 + \frac{1}{2}c_1(M) + \frac{1}{12}(c_1(M)^2 + c_2(M)) + \frac{1}{24}c_1(M)c_2(M),$$

$$\begin{aligned}\operatorname{Ch}(TM) &= m + c_1(M) + \frac{1}{2}(c_1(M)^2 - 2c_2(M)) \\ &\quad + \frac{1}{6}(c_1(M)^3 - 3c_1(M)c_2(M) + 3c_3(M)),\end{aligned}$$

$$\operatorname{Ch}\Big(\bigwedge\nolimits_{\mathbb{C}}^{m} TM\Big) = 1 + c_1(M) + \frac{1}{2}c_1(M)^2 + \frac{1}{6}c_1(M)^3.$$

Let e, σ denote the Euler number and the signature of M respectively.

Example 2.12. When $m = 2$, we have

$$c_1^2 = 2e + 3\sigma, \quad c_2 = e \tag{9}$$

where $c_1^2 = c_1(M)^2[M]$, $c_2 = c_2(M)[M]$ are Chern numbers (see [**6**]). Hence it follows from Formula 2.11 that

$$\begin{aligned}
\mathrm{Ch}(E_{\ell_1,\ell_2})\,\mathrm{Td}(M)[M] &= \mathrm{Ch}(TM)^{\ell_1}\,\mathrm{Ch}\left(\bigwedge\nolimits^2_{\mathbb{C}} TM\right)^{\ell_2}\,\mathrm{Td}(M)[M] \\
&= 2^{\ell_1-3}\big\{(2\ell_1^2 + 8\ell_1\ell_2 + 8\ell_2^2 + 2\ell_1 + 8\ell_2 + 2)e \\
&\qquad + (3\ell_1^2 + 12\ell_1\ell_2 + 12\ell_2^2 + 9\ell_1 + 12\ell_2 + 2)\sigma\big\}. \quad (10)
\end{aligned}$$

Moreover we have

$$\sigma_1^{\ell_1}\sigma_2^{\ell_2} = \big(\xi_{p_i}^{kz\tau_{i1}} + \xi_{p_i}^{kz\tau_{i2}}\big)^{\ell_1}\big(\xi_{p_i}^{kz\tau_{i1}}\xi_{p_i}^{kz\tau_{i2}}\big)^{\ell_2} = \sum_{\gamma=0}^{\ell_1}\binom{\ell_1}{\gamma}\xi_{p_i}^{kz\mu_{i\gamma}}$$

where $\mu_{i\gamma} = \tau_{i1}(\ell_2 + \gamma) + \tau_{i2}(\ell_1 + \ell_2 - \gamma)$ and hence it follows from Theorem 2.2, Proposition 2.8 and Example 2.9 that

$$\begin{aligned}
&I_{D_{\ell_1,\ell_2}}(g^z) \\
&\quad = \frac{p-1}{2p}2^{\ell_1-3}\big\{(2\ell_1^2 + 8\ell_1\ell_2 + 8\ell_2^2 + 2\ell_1 + 8\ell_2 + 2)e \\
&\qquad\qquad + (3\ell_1^2 + 12\ell_1\ell_2 + 12\ell_2^2 + 9\ell_1 + 12\ell_2 + 2)\sigma\big\} \\
&\qquad + \sum_{i=1}^{b}\frac{1}{12p_i^2}\left\{2^{\ell_1}z\theta_{i1}\theta_{i2}^2\sum_{s=0}^{p_i-1}g_{2,p_i}(s) - p_i\sum_{\gamma=0}^{\ell_1}\binom{\ell_1}{\gamma}\sum_{s=0}^{p_i-1}g_{2,p_i}(s)\Lambda_{2,p_i}(z,\mu_{i\gamma},s)\right\}
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
\Lambda_{2,p_i}(z,\mu_{i\gamma},s) &= \sum_{\lambda_1=0}^{z\tau_{i1}-1}\ \sum_{\lambda_{21},\lambda_{22}=0}^{\theta_{i2}-1}\delta_{p_i}(\zeta(z,\mu_{i\gamma},s,\tau,\lambda)), \\
\zeta(z,\mu_{i\gamma},s,\tau,\lambda) &= 1 + \lambda_1 + z\tau_{i1}(\ell_2 + \gamma + \lambda_{21} + \lambda_{22} + 1) \\
&\qquad + z\tau_{i2}(s + \ell_1 + \ell_2 - \gamma + 1).
\end{aligned}$$

3. Nonexistence of a cyclic group action.

In this section we use Theorem 2.2 to examine whether a $\mathbb{Z}_p$-action with specific isotropy orders exists or not. Assume that $\mathbb{Z}_p = \langle g \rangle$ acts on a $2m$-dimensional almost complex manifold M and suppose that the isotropy orders of the $\mathbb{Z}_p$-action are $(p_1, \dots, p_b)$.

Since the Todd genus of 4-dimensional almost complex manifolds M is equal to $(e+\sigma)/4$ (see Formula 2.11 and the equality (9)), $e+\sigma$ is a multiple of 4. Conversely it follows from [**9**, Theorem 1] that there exists a closed connected almost complex manifold with $e = u$, $\sigma = v$ if $u+v$ is a multiple of 4. [**9**, Theorem 1] also asserts that there exists a closed connected complex manifold with $e = u$, $\sigma = v$ if $u+v$ is a multiple of 4 and $v \leq 0$.

Remark 3.1. Since $\mathbb{Z}_p$ acts freely on the punctured manifold $M_0 = M \setminus \{\bigcup_{i=1}^b \pi^{-1}(y_i)\}$, the next equality holds:

$$e \equiv \sum_{i=1}^{b} r_i \pmod{p}. \tag{12}$$

Example 3.2. In this example we consider the case that M is a 4-dimensional almost complex manifold with $e+\sigma = 0$. Suppose that $p = 6$, $b = 3$. First we set $(p_1, p_2, p_3) = (2, 2, 6)$. Then direct computation below shows that $I_{D_{0,0}}(g^5) \neq 5 I_{D_{0,0}}(g)$, which implies that M does not admit any $\mathbb{Z}_6$-action of isotropy orders $(2, 2, 6)$.

Since $(r_1, r_2, r_3) = (3, 3, 1)$ and $\ell_1 = \ell_2 = \gamma = 0$, $\mu_{10} = \mu_{20} = \mu_{30} = 0$ for the trivial complex line bundle $E_{0,0}$, it follows from (11) that

$$12 \cdot 6^2 I_{D_{0,0}}(g^z) = 432 I_{D_{0,0}}(g^z) = \sum_{i=1}^{3} r_i^2 f_i(z, \tau_{i1}, \tau_{i2})$$

where

$$f_i(z, \tau_{i1}, \tau_{i2}) \equiv z\theta_{i1}\theta_{i2}^2 \sum_{s=0}^{p_i-1} g_{2,p_i}(s) - p_i \sum_{s=0}^{p_i-1} g_{2,p_i}(s)\Lambda_{2,p_i}(z, 0, s) \pmod{432}$$

(see Example 2.9). For $i = 1$, 2, we have

$$\tau_{i1} = \tau_{i2} = 1 \implies \theta_{i1} = \theta_{i2} = 1, \quad g_{2,2}(0) = 0,\ g_{2,2}(1) = 3,$$

$$\Lambda_{2,2}(5,0,1)=\sum_{\lambda_1=0}^{4}\delta_2(\lambda_1+16)=3,\quad \Lambda_{2,2}(1,0,1)=\sum_{\lambda_1=0}^{0}\delta_2(\lambda_1+4)=1$$

and hence it follows that

$$f_i(5,\tau_{11},\tau_{12})=5\sum_{s=0}^{1}g_{2,2}(s)-2\sum_{s=0}^{1}g_{2,2}(s)\Lambda_{2,2}(5,0,s)=-3,$$

$$f_i(1,\tau_{11},\tau_{12})=\sum_{s=0}^{1}g_{2,2}(s)-2\sum_{s=0}^{1}g_{2,2}(s)\Lambda_{2,2}(1,0,s)=-3.$$

Therefore we have

$$\begin{aligned}&432(I_{D_{0,0}}(g^5)-5I_{D_{0,0}}(g))\\ &\quad\equiv 2\cdot 3^2(-3)+f_3(5,\tau_{31},\tau_{32})-5\{2\cdot 3^2(-3)+f_3(1,\tau_{31},\tau_{32})\}\pmod{432}.\end{aligned}$$

When $(\tau_{31},\tau_{32})=(1,1)$, we have $\theta_{31}=\theta_{32}=1$ and direct computation shows that $f_3(5,\tau_{31},\tau_{32})=-105$, $f_3(1,\tau_{31},\tau_{32})=135$. Hence we have

$$432(I_{D_{0,0}}(g^5)-5I_{D_{0,0}}(g))\equiv -564\not\equiv 0\pmod{432}.$$

When $(\tau_{31},\tau_{32})=(1,5)$, we have $\theta_{31}=1$, $\theta_{32}=5$ and direct computation shows that $f_3(5,\tau_{31},\tau_{32})=f_3(1,\tau_{31},\tau_{32})=-105$. Hence we have

$$432(I_{D_{0,0}}(g^5)-5I_{D_{0,0}}(g))\equiv 636\not\equiv 0\pmod{432}.$$

When $(\tau_{31},\tau_{32})=(5,5)$, we have $\theta_{31}=5$, $\theta_{32}=1$ and direct computation shows that $f_3(5,\tau_{31},\tau_{32})=135$, $f_3(1,\tau_{31},\tau_{32})=-105$. Hence we have

$$432(I_{D_{0,0}}(g^5)-5I_{D_{0,0}}(g))\equiv 876\not\equiv 0\pmod{432}.$$

These results imply that M does not admit the $\mathbb{Z}_6$-action of isotropy orders $(2,2,6)$.

Example 3.3. Let N be a 4-dimensional almost complex manifold with the Euler number $8n$ and the signature $-8n$ where n is a natural number. Then a 6-dimensional almost complex manifold M is defined by $M=N\times\mathbb{CP}^1$. We consider the case that $p=4$, $b=5$, $(p_1,p_2,p_3,p_4,p_5)=(2,2,2,4,4)$. Note that the condition (12) is satisfied in this case. Let a_i be the i-th Chern class of N, u the positive generator of $H^2(\mathbb{CP}^1;\mathbb{Z})=\mathbb{Z}$ and $c(M)$ the total Chern class of M.

Then we have $a_1^2u[M] = -8n$, $a_2u[M] = 8n$ (see (9)) and

$$c(M) = (1 + a_1 + a_2)(1 + 2u) = 1 + (a_1 + 2u) + (a_2 + 2a_1u) + 2a_2u,$$

$$\mathrm{Td}(M) = \mathrm{Td}(N)\,\mathrm{Td}(\mathbb{CP}^1) = 1 + \frac{1}{2}(a_1 + 2u) + \frac{1}{12}(a_1^2 + a_2 + 6a_1u) + \frac{1}{12}(a_1^2 + a_2)u,$$

$$\mathrm{Ch}(E_{0,0,\ell}) = \exp(\ell(a_1 + 2u)) = 1 + \ell(a_1 + 2u) + \frac{1}{2}\ell^2(a_1^2 + 4a_1u) + \ell^3a_1^2u$$

for any integer ℓ. Hence for $p = 4$ we have

$$\frac{p-1}{2p}\,\mathrm{Ch}(E_{0,0,\ell})\,\mathrm{Td}(M)[M] = -\frac{3}{2}n\ell(\ell+1)(2\ell+1),$$

which is an integer. Set $\mu_i = \ell(\tau_{i1} + \tau_{i2} + \tau_{i3})$. Then we have

$$\sigma_1^0\sigma_2^0\sigma_3^\ell = \left(\xi_{p_i}^{kz\tau_{i1}}\xi_{p_i}^{kz\tau_{i2}}\xi_{p_i}^{kz\tau_{i3}}\right)^\ell = \xi_{p_i}^{kz\mu_i}$$

and therefore it follows from Theorem 2.2 and Proposition 2.8 that

$$I_{D_{0,0,\ell}}(g^z) = -\sum_{i=1}^{5}\frac{1}{72p_i^2}\left\{z\theta_{i1}\theta_{i2}^2\theta_{i3}^3\sum_{s=0}^{p_i-1}h_{p_i}(s) - p_i\sum_{s=0}^{p_i-1}h_{p_i}(s)\Lambda_{3,p_i}(z,\mu_i,s))\right\}$$

where $h_p(s) = g_{3,p}(s)/2$ (see Example 2.9) and

$$\Lambda_{3,p_i}(z,\mu_i,s) = \sum_{\lambda_1=0}^{z\tau_{i1}-1}\ \sum_{\lambda_{21},\lambda_{22}=0}^{\theta_{i2}-1}\ \sum_{\lambda_{31},\lambda_{32},\lambda_{33}=0}^{\theta_{i3}-1}\delta_{p_i}(\zeta(z,\mu_i,s,\tau,\lambda)),$$

$$\begin{aligned}\zeta(z,\mu_i,s,\tau,\lambda) = {} & 1 + \lambda_1 + z\mu_i + z(\lambda_{21} + \lambda_{22} + 1)\tau_{i1} \\ & + z(\lambda_{31} + \lambda_{32} + \lambda_{33} + 1)\tau_{i2} + z(s+1)\tau_{i3}.\end{aligned}$$

Then for $i = 1, 2, 3$ we have $(\tau_{i1}, \tau_{i2}, \tau_{i3}) = (1, 1, 1)$ and it follows that

$$h_2(0) = -9, \quad h_2(1) = 0, \quad \Lambda_{3,2}(z,\mu_i,0) = \sum_{\lambda_1=0}^{z-1}\delta_2(1 + \lambda_1 + 3z(\ell+1)) = \frac{z + (-1)^\ell}{2}$$

$$\Longrightarrow\ -\frac{1}{72\cdot 2^2}\left(z\sum_{s=0}^{1}h_2(s) - 2\sum_{s=0}^{1}h_2(s)\Lambda_{3,2}(z,3\ell,s)\right) = \frac{(-1)^{\ell+1}}{32}$$

for $z = 1, 3$. Moreover since

$$-\frac{1}{72\cdot 4^2}\sum_{s=0}^{3} h_4(s) \equiv \frac{41}{64} \pmod{\mathbb{Z}},$$

$$\frac{1}{72\cdot 4}(h_4(0), h_4(1), h_4(2), h_4(3)) \equiv \left(-\frac{17}{32}, -\frac{20}{32}, -\frac{25}{32}, -\frac{20}{32}\right) \pmod{\mathbb{Z}},$$

we have

$$\begin{aligned}
I_{D_{0,0,\ell}}(g^z) &= -\frac{1}{72\cdot 2^2}\sum_{i=1}^{3}\left\{z\sum_{s=0}^{1} h_2(s) - 2\sum_{s=0}^{1} h_2(s)\Lambda_{3,2}(z, 3\ell, s)\right\} \\
&\quad -\frac{1}{72\cdot 4^2}\sum_{i=4}^{5}\left\{z\theta_{i1}\theta_{i2}^2\theta_{i3}^3\sum_{s=0}^{3} h_4(s) - 4\sum_{s=0}^{3} h_4(s)\Lambda_{3,4}(z, \mu_i, s)\right\} \\
&= (-1)^{\ell+1}\frac{3}{32} + \frac{41}{64}z\{\theta_{41}\theta_{42}^2\theta_{43}^3 + \theta_{51}\theta_{52}^2\theta_{53}^3\} \\
&\quad -\frac{1}{32}\sum_{i=4}^{5}\left\{\begin{array}{l} 17\Lambda_{3,4}(z,\mu_i,0) + 20\Lambda_{3,4}(z,\mu_i,1) \\ +25\Lambda_{3,4}(z,\mu_i,2) + 20\Lambda_{3,4}(z,\mu_i,3)\end{array}\right\}.
\end{aligned}$$

Set

$$\varphi_\ell(\tau_{41}, \tau_{42}, \tau_{43}, \tau_{51}, \tau_{52}, \tau_{53}) = 32I_{D_{0,0,\ell}}(g^3) - 3\cdot 32I_{D_{0,0,\ell}}(g).$$

Then direct computation shows that

$$\begin{aligned}
&\varphi_0(1,1,1,3,3,3) \equiv 0 - 3\cdot 0 = 0 \equiv 0 &&\pmod{32},\\
&\varphi_0(1,1,3,1,1,3) \equiv -12 - 3\cdot(-4) = 0 \equiv 0 &&\pmod{32},\\
&\varphi_0(1,3,3,1,3,3) \equiv -4 - 3\cdot(-12) = 32 \equiv 0 &&\pmod{32}
\end{aligned}$$

and $\varphi_0(\tau_{41}, \tau_{42}, \tau_{43}, \tau_{51}, \tau_{52}, \tau_{53}) \not\equiv 0 \pmod{32}$ for

$$\begin{aligned}
&(\tau_{41}, \tau_{42}, \tau_{43}, \tau_{51}, \tau_{52}, \tau_{53}) \\
&\quad = (1,1,1,1,1,1),\ (1,1,1,1,1,3),\ (1,1,1,1,3,3), \\
&\qquad (1,1,3,1,3,3),\ (1,1,3,3,3,3),\ (1,3,3,3,3,3),\ (3,3,3,3,3,3).
\end{aligned}$$

Direct computation also shows that

$$\begin{aligned}
\varphi_1(1,1,1,1,1,1) &\equiv 12 - 3\cdot 4 = 0 \equiv 0 \pmod{32},\\
\varphi_1(1,1,3,1,3,3) &\equiv 0 - 3\cdot 0 = 0 \equiv 0 \pmod{32},\\
\varphi_1(3,3,3,3,3,3) &\equiv 4 - 3\cdot 12 = -32 \equiv 0 \pmod{32}
\end{aligned}$$

and $\varphi_1(\tau_{41},\tau_{42},\tau_{43},\tau_{51},\tau_{52},\tau_{53}) \not\equiv 0 \pmod{32}$ for

$$\begin{aligned}
&(\tau_{41},\tau_{42},\tau_{43},\tau_{51},\tau_{52},\tau_{53})\\
&\quad = (1,1,1,1,1,3),\ (1,1,1,1,3,3),\ (1,1,1,3,3,3),\\
&\qquad (1,1,3,1,1,3),\ (1,1,3,3,3,3),\ (1,3,3,1,3,3),\ (1,3,3,3,3,3).
\end{aligned}$$

As we see above there does not exist $(\tau_{41},\tau_{42},\tau_{43},\tau_{51},\tau_{52},\tau_{53})$ such that

$$\varphi_\ell(\tau_{41},\tau_{42},\tau_{43},\tau_{51},\tau_{52},\tau_{53}) \equiv 0 \pmod{32}$$

for $\ell = 0$, 1, which implies that M does not admit the $\mathbb{Z}_4$-action of isotropy orders $(2,2,2,4,4)$.

4. Angle vectors.

In this section we assume that p is an odd prime number. In Example 3.2 we argued about the existence of a rotation angle of a $\mathbb{Z}_6$-action. In this section using the assumption above, we give a detailed examination of the existence of a rotation angle.

Let $\mathbb{Z}_p$ be the cyclic group of order p generated by g. Assume that $\mathbb{Z}_p$ acts on a $2m$-dimensional almost complex manifold M and that the action preserves the almost complex structure of M. Let $q_1,\dots,q_n$ be the fixed points of g. Then the fixed points of g^k coincides with those of g for $1 \le k \le p-1$.

In this section, a set of natural numbers $\{t_{ij}\}$ $(1 \le j \le m,\ 1 \le i \le n)$ is called an angle vector of type (m,n) and denoted by $\boldsymbol{t}(p)$ or $((t_{11},\dots,t_{1m}),\dots,(t_{n1},\dots,t_{nm}))$ when $0 < t_{ij} < p$ for any i, j. An angle vector of type (m,n) is regarded as an element of the vector space $\mathbb{Z}_p^{mn}$ over the field $\mathbb{Z}_p$. Note that a rotation angle $\{\tau_{ij}\}$ is an angle vector but an angle vector $\boldsymbol{t}(p)$ is not necessarily a rotation angle.

If $\boldsymbol{t}(p)$ is the rotation angle of the periodic automorphism g, it follows from the equalities (1), (3), (8) that the equality

$$F(z,\ell_1,\dots,\ell_m;\boldsymbol{t}(p)) \equiv I_{D_E}(g^z) \pmod{\mathbb{Z}} \tag{13}$$

holds where $F(z,\ell_1,\dots,\ell_m;\boldsymbol{t}(p))$ is a complex number defined below.

Definition 4.1. Let z be an integer such that $0 < z < p$, $(\ell_1, \ldots, \ell_m)$ an element of L, $\boldsymbol{t}(p) = \{t_{ij}\}$ an angle vector of type (m, n) and σ_{ij} the j-th elementary symmetric polynomial in $\xi_p^{kzt_{i1}}, \ldots, \xi_p^{kzt_{im}}$. Then $F(z, \ell_1, \ldots, \ell_m; \boldsymbol{t}(p)) \in \mathbb{C}$ is defined by

$$F(z, \ell_1, \ldots, \ell_m; \boldsymbol{t}(p)) = \frac{p-1}{2p} \operatorname{Ch}(E_{\ell_1, \ldots, \ell_m}) \operatorname{Td}(M)[M] - \frac{1}{p} \sum_{i=1}^{n} \sum_{k=1}^{p-1} \left(\prod_{j=1}^{m} (\sigma_{ij})^{\ell_j} \right) \frac{1}{1 - \xi_p^{-k}} \prod_{j=1}^{m} \frac{1}{1 - \xi_p^{-kzt_{ij}}}. \tag{14}$$

Note that if

$$\prod_{j=1}^{m} (\sigma_{ij})^{\ell_j} = \sum_{c=1}^{d} \xi_p^{kz\mu_{ic}} \quad (1 \leq i \leq n),$$

it follows from the equality (6) that

$$F(z, \ell_1, \ldots, \ell_m; \boldsymbol{t}(p)) = \frac{p-1}{2p} \operatorname{Ch}(E_{\ell_1, \ldots, \ell_m}) \operatorname{Td}(M)[M] + \frac{1}{p^{m+2}} \sum_{i=1}^{n} \left\{ dz \left(\prod_{j=1}^{m} \theta_{ij}^{j} \right) \sum_{s=0}^{p-1} f_{m,p}(s) - p \sum_{c=1}^{d} \sum_{s=0}^{p-1} f_{m,p}(s) \Lambda_{m,p}(z, \mu_{ic}, s) \right\} \tag{15}$$

where $1 \leq \theta_{ij} \leq p-1$, $\theta_{ij} \equiv \overline{t_{i\,j-1}} t_{ij} \pmod{p}$ and

$$\Lambda_{m,p}(z, \mu_{ic}, s) = \sum_{\lambda_1=0}^{z\theta_{i1}-1} \sum_{\lambda_{21}, \lambda_{22}=0}^{\theta_{i2}-1} \cdots \sum_{\lambda_{m1}, \ldots, \lambda_{mm}=0}^{\theta_{im}-1} \delta_p(\zeta(z, \mu_{ic}, s, \tau, \lambda)),$$

$$\zeta(z, \mu_{ic}, s, \tau, \lambda) = 1 + \lambda_1 + z\mu_{ic} + z \sum_{j=1}^{m} t_{ij} + z \sum_{j=2}^{m} t_{i\,j-1}(\lambda_{j1} + \cdots + \lambda_{jj}) + szt_{im}.$$

Proposition 4.2. *Assume that p is greater than $m+2$. Then the equalities*

$$F\big(z, \ell_1, \ldots, \ell_r + p(p-1), \ldots, \ell_m; \boldsymbol{t}(p)\big) \equiv F\big(z, \ell_1, \ldots, \ell_r, \ldots, \ell_m; \boldsymbol{t}(p)\big) \pmod{\mathbb{Z}} \qquad (1 \leq r \leq m),$$

$$F\big(z,\ell_1,\dots,\ell_{m-1},\ell_m+p;\boldsymbol{t}(p)\big)\equiv F\big(z,\ell_1,\dots,\ell_{m-1},\ell_m;\boldsymbol{t}(p)\big)\pmod{\mathbb{Z}}$$

hold for any integer z $(0<z<p)$ *and any* $(\ell_1,\dots,\ell_m)\in L$.

PROOF. Set $CT(\ell_1,\dots,\ell_m)=\mathrm{Ch}(E_{\ell_1,\dots,\ell_m})\,\mathrm{Td}(M)[M]$ and

$$C(\ell_1,\dots,\ell_m)=\sum_{i=1}^{n}\sum_{k=1}^{p-1}\left(\prod_{j=1}^{m}(\sigma_{ij})^{\ell_j}\right)\frac{1}{1-\xi_p^{-k}}\prod_{j=1}^{m}\frac{1}{1-\xi_p^{-kzt_{ij}}}.$$

Then we have

$$pF(z,\ell_1,\dots,\ell_m;\boldsymbol{t}(p))=\frac{p-1}{2}CT(\ell_1,\dots,\ell_m)-C(\ell_1,\dots,\ell_m). \tag{16}$$

Note that $CT(\ell_1,\dots,\ell_m)$ is an index and hence an integer for any $(\ell_1,\dots,\ell_m)\in L$.

Let f, g_j be polynomials defined by

$$\mathrm{Td}(M)=1+f(c_1(M),\dots,c_m(M)),$$
$$\mathrm{Ch}\left(\bigwedge\nolimits_{\mathbb{C}}^{j}TM\right)=b_j+g_j(c_1(M),\dots,c_m(M)).$$

Here it follows from the definition of the Chern character that the coefficients of $m!g_j$ are integers for $1\le j\le m$. Moreover since

$$\frac{x}{1-e^{-x}}=\left(x^{-1}-\sum_{i=0}^{m+1}\frac{(-1)^i}{i!}x^{i-1}\right)^{-1}=1+\sum_{j=1}^{m}\left(\sum_{k=1}^{m}\frac{(-1)^{k+1}}{(k+1)!}x^k\right)^j$$

up to higher order terms, the coefficients of $\{(m+1)!\}^{m^2}f$ are integers. Therefore we have

$$\begin{aligned}&CT(\ell_1,\dots,\ell_m)\\&=\frac{1}{m!}\lim_{t\to 0}\left(\frac{d}{dt}\right)^m\left[\{1+f(tc_1,\dots,t^mc_m)\}\prod_{j=1}^{m}\{b_j+g_j(tc_1,\dots,t^mc_m)\}^{\ell_j}\right]\\&=\frac{1}{\nu}\left(\prod_{j=1}^{m}b_j^{\ell_j}\right)P(\ell_1,\dots,\ell_m)\end{aligned}$$

where $c_1^{i_1}\cdots c_m^{i_m}$ $(i_1+\cdots+mi_m=m)$ are Chern numbers, $P(\ell_1,\dots,\ell_m)$ is a polynomial with integer coefficients and ν is an integer defined by

$$\nu = \{(m+1)!\}^{m^2}\{m!\}^m \prod_{j=1}^{m} b_j^m.$$

Since the assumption that $p > m+2$ implies that ν is not a multiple of p, there exists the mod p inverse $\overline{\nu}$ of ν. Then for $1 \leq r \leq m$ we have

$$\begin{aligned} CT(\ell_1, \ldots, \ell_r + p, \ldots, \ell_m) &\equiv \nu\overline{\nu} CT(\ell_1, \ldots, \ell_r + p, \ldots, \ell_m) \pmod p \\ &= b_r^p \overline{\nu} \left(\prod_{j=1}^{m} b_j^{\ell_j} \right) P(\ell_1, \ldots, \ell_r + p, \ldots, \ell_m) \\ &\equiv b_r^p CT(\ell_1, \ldots, \ell_r, \ldots, \ell_m) \pmod p \end{aligned}$$

which implies the equality

$$CT(\ell_1, \ldots, \ell_r + p(p-1), \ldots, \ell_m) \equiv CT(\ell_1, \ldots, \ell_r, \ldots, \ell_m) \pmod p \tag{17}$$

because the assumption implies that b_r is not a multiple of p and hence that $b_r^{p-1} \equiv 1 \pmod p$. When $r = m$, since $b_m = 1$ we have

$$CT(\ell_1, \ldots, \ell_m + p) \equiv CT(\ell_1, \ldots, \ell_m) \pmod p. \tag{18}$$

Let $Q_i(s)$, $R_k(s)$ be the integral polynomials in the proof of Proposition 2.8. Then since the degree of $Q_j(s)$ with respect to s is less than or equal to $m - j$, the degree of $R_k(s)$ is less than or equal to k, and hence the degree of $g_{m,p}(s)$ is less than or equal to $m+1$. Here for any nonnegative integer j since

$$\begin{aligned} (j+1)! p^{j+2} &= (j+1)! \left(\sum_{s=1}^{p} s^{j+2} - \sum_{s=0}^{p-1} s^{j+2} \right) = (j+1)! \sum_{s=0}^{p-1} \left((s+1)^{j+2} - s^{j+2} \right) \\ &= (j+2)! \sum_{s=0}^{p-1} s^{j+1} + \sum_{i=0}^{j} \frac{(j+1)!}{(i+1)!} \binom{j+2}{i} (i+1)! \sum_{s=0}^{p-1} s^i, \end{aligned}$$

the induction on j shows that

$$(j+1)! \sum_{s=0}^{p-1} s^j \equiv 0 \pmod p.$$

Hence there exists an integer λ_1 such that

$$(m+2)!\sum_{s=0}^{p-1} g_{m,p}(s) = p\lambda_1,$$

and therefore it follows from the assumption that there exists an integer λ_2 such that

$$\sum_{s=0}^{p-1} g_{m,p}(s) = p\lambda_2.$$

Hence it follows from Proposition 2.8 that

$$(-1)^m m!(m+1)!\sum_{s=0}^{p-1} f_{m,p}(s) = p^{m+1}\lambda_2,$$

and therefore it follows from the assumption that

$$h_m(p) := \frac{1}{p^{m+1}}\sum_{s=0}^{p-1} f_{m,p}(s)$$

is an integer. Moreover it also follows from Proposition 2.8 that

$$h_{m,p}(s) := \frac{f_{m,p}(s)}{p^m}$$

is an integer. Hence it follows from the equality (6) that

$$\begin{aligned}
&\sum_{k=1}^{p-1} \frac{\xi_p^{kz\mu}}{1-\xi_p^{-k}} \prod_{j=1}^{m} \frac{1}{1-\xi_p^{-kzt_{ij}}} \\
&\quad = \frac{1}{p^{m+1}}\left\{p\sum_{s=0}^{p-1} f_{m,p}(s)\Lambda_{m,p}(z,\mu,s) - z\left(\prod_{j=1}^{m}\theta_{ij}^{j}\right)\sum_{s=0}^{p-1} f_{m,p}(s)\right\} \\
&\quad = \sum_{s=0}^{p-1} h_{m,p}(s)\Lambda_{m,p}(z,\mu,s) - z\left(\prod_{j=1}^{m}\theta_{ij}^{j}\right)h_m(p)
\end{aligned}$$

is an integer for any integers z $(0 < z < p)$ and μ. Therefore $C(\ell_1, \dots, \ell_m)$ is an integer for any $\ell_1, \dots, \ell_m$ and

$$\sum_{k=1}^{p-1} f\big(\xi_p^{kzt_{i1}}, \dots, \xi_p^{kzt_{im}}\big) \frac{1}{1-\xi_p^{-k}} \prod_{j=1}^{m} \frac{1}{1-\xi_p^{-kt_{ij}}}$$

is an integer for any polynomial $f(x_1, \dots, x_m)$ with integer coefficients.

Here there exist polynomials $g(x_1, \dots, x_m)$, $h(x_1, \dots, x_m)$ with integer coefficients such that

$$\begin{aligned}\{(\sigma_{ir})^p - b_r\} \prod_{j=1}^{m} (\sigma_{ij})^{\ell_j} &= \Bigg\{\bigg(\sum_{1 \le j_1 < \cdots < j_r \le m} \xi_p^{kz(t_{ij_1}+\cdots+t_{ij_r})}\bigg)^p - b_r\Bigg\} \prod_{j=1}^{m} (\sigma_{ij})^{\ell_j} \\ &= p \sum \frac{(p-1)!}{i_1! \cdots i_{b_r}!} g\big(\xi_p^{kzt_{i1}}, \dots, \xi_p^{kzt_{im}}\big) \prod_{j=1}^{m} (\sigma_{ij})^{\ell_j} \\ &= ph\big(\xi_p^{kzt_{i1}}, \dots, \xi_p^{kzt_{im}}\big)\end{aligned}$$

where $\sum$ denotes the summation over $0 \le i_1, \dots, i_{b_r} < p$ such that $i_1 + \cdots + i_{b_r} = p$ because $(p-1)!$ is a multiple of $i_1! \cdots i_{b_r}!$ for $0 \le i_1, \dots, i_{b_r} < p$. Hence it follows that

$$C(\ell_1, \dots, \ell_r + p, \dots, \ell_m) \equiv b_r C(\ell_1, \dots, \ell_m) \pmod{p},$$

and therefore we have

$$\begin{gathered} C(\ell_1, \dots, \ell_r + p(p-1), \dots, \ell_m) \equiv C(\ell_1, \dots, \ell_m)\ (1 \le r \le m) \pmod{p}, \\ C(\ell_1, \dots, \ell_m + p) \equiv C(\ell_1, \dots, \ell_m) \pmod{p}. \end{gathered} \tag{19}$$

Now the equality in the proposition follows from the equalities (16), (17), (18), (19). □

Definition 4.3. An equivalence relation between angle vectors is defined as follows. Two angle vectors $\{t_{ij}\}$, $\{t'_{ij}\}$ are defined to be equivalent if there exists an integer w $(0 < w < p)$, a permutation ρ of $\{1, \dots, n\}$ and permutations η_i $(1 \le i \le n)$ of $\{1, \dots, m\}$ such that $t'_{ij} \equiv wt_{\rho(i)\eta_i(j)} \pmod{p}$.

For example, when $p = 3$, $m = n = 2$,

$$((t_{11}, t_{12}), (t_{21}, t_{22})) \sim ((t'_{11}, t'_{12}), (t'_{21}, t'_{22})) = ((2t_{22}, 2t_{21}), (2t_{11}, 2t_{12})).$$

Definition 4.4. Let L_p be the finite subset of L defined by

$$L_p = \{(\ell_1, \dots, \ell_{m-1}, \ell_m) \in \mathbb{Z}^m \mid 0 \leq \ell_j < p(p-1) \ (1 \leq j < m), \ 0 \leq \ell_m < p\}.$$

In this paper, an angle vector $\boldsymbol{t}(p)$ is called a necessary angle vector if

$$F(z, \ell_1, \dots, \ell_m; \boldsymbol{t}(p)) \equiv z\, F(1, \ell_1, \dots, \ell_m; \boldsymbol{t}(p)) \pmod{\mathbb{Z}}$$

for any integer z such that $0 < z < p$ and any element $(\ell_1, \dots, \ell_m)$ of L_p and is called a proper angle vector if $F(z, \ell_1, \dots, \ell_m; \boldsymbol{t}(p))$ is an integer for any integer z such that $0 < z < p$ and any element $(\ell_1, \dots, \ell_m)$ of L_p.

Note that an angle vector $\boldsymbol{t}(p)$ is a necessary angle vector if $\boldsymbol{t}(p)$ is the rotation angle of a periodic automorphim of order p (see (13)).

Proposition 4.5. *An angle vector $\boldsymbol{t}(p)$ is necessary or proper if $\boldsymbol{t}(p)$ is equivalent to a necessary or proper angle vector, respectively.*

Proof. It is clear that

$$F(z, \ell_1, \dots, \ell_m; \{wt_{\rho(i)\eta_i(j)}\}) = F(wz, \ell_1, \dots, \ell_m; \{t_{ij}\})$$

for any integer w $(0 < w < p)$ and permutations ρ, η_i $(1 \leq i \leq n)$. Hence if $\{t_{ij}\}$ is a proper angle vector, $\{wt_{\rho(i)\eta_i(j)}\}$ is also a proper angle vector because

$$F(z, \ell_1, \dots, \ell_m; \{wt_{\rho(i)\eta_i(j)}\}) = F(wz, \ell_1, \dots, \ell_m; \{t_{ij}\}) \equiv 0 \pmod{\mathbb{Z}}.$$

If $\{t_{ij}\}$ is a necessary angle vector, $\{wt_{\rho(i)\eta_i(j)}\}$ is also a necessary angle vector because

$$\begin{aligned} &F(z, \ell_1, \dots, \ell_m; \{wt_{\rho(i)\eta_i(j)}\}) = wzF(1, \ell_1, \dots, \ell_m; \{t_{ij}\}) \\ &\quad = zF(w, \ell_1, \dots, \ell_m; \{t_{ij}\}) = zF(1, \ell_1, \dots, \ell_m; \{wt_{\rho(i)\eta_i(j)}\}). \end{aligned}$$ □

First we consider the case that $m = 1$.

Proposition 4.6. *When $m = 1$, an angle vector $\{t_i\}$ is a rotation angle of a periodic automorphim of order p if and only if $\{t_i\}$ is a necessary angle vector.*

Proof. Let Σ^γ be the compact Riemann surface of genus $\gamma \geq 2$ and U the universal covering of Σ^γ. Then there exists a Fuchsian group Γ with compact orbit space generated by $a_1, \dots, a_\gamma$, $b_1, \dots, b_\gamma$, $x_1, \dots, x_n$ with the relation

$$x_1^p = \cdots = x_n^p = 1, \qquad \prod_{i=1}^{\gamma}[a_i, b_i]x_1 \cdots x_n = 1$$

such that $\Sigma^\gamma = U/\Gamma$. If the equality

$$\sum_{i=1}^{n} \bar{t}_i \equiv 0 \pmod{p}, \tag{20}$$

holds, $\phi(x_i) = \overline{t_i}$ defines a homomorphism $\phi : \Gamma \to \mathbb{Z}_p$ such that the order of $\phi(x_i)$ is p for $1 \leq i \leq n$. Then $\mathbb{Z}_p = \Gamma / \ker\phi$ acts on $U/\ker\phi = \Sigma^\rho$ with rotation angle $\{t_1, \ldots, t_n\}$, where the genus ρ is determined by the Riemann-Hurwitz equation

$$\rho = p(\gamma - 1) + \frac{n(p-1)}{2} + 1. \tag{21}$$

(For details see [**5**].) So it suffices to show that the equality (20) holds under the assumption that $\{t_i\}$ is a necessary angle vector.

We have

$$\mathrm{Td}(M)[M] = \frac{1}{2}c_1(M)[M] = 1 - \rho \equiv \frac{n(1-p)}{2} \pmod{p}$$

(see (21)) and hence it follows that

$$\begin{aligned}
&pzF(1, 0; \{t_i\}) - pF_p(z, 0; \{t_i\}) \pmod{p} \\
&\quad \equiv \frac{1}{4}(1-z)n(p-1)^2 + \sum_{i=1}^{n}\sum_{k=1}^{p-1} \frac{1}{1-\xi_p^{-k}}\left(\frac{1}{1-\xi_p^{-kzt_i}} - z\frac{1}{1-\xi_p^{-kt_i}}\right) \pmod{p}.
\end{aligned}$$

Here as we show in Appendix, the equality

$$\sum_{k=1}^{p-1} \frac{1}{1-\xi_p^{-k}}\left(\frac{1}{1-\xi_p^{-kzt_i}} - z\frac{1}{1-\xi_p^{-kt_i}}\right) \equiv \varphi_p(z)\bar{t}_i + \frac{1}{4}(1-z)(p^2-1) \pmod{p} \tag{22}$$

holds where $\varphi_p(z)$ is an integer defined by

$$\varphi_p(z) = \sum_{k=1}^{p-1} k\left[\frac{kz}{p}\right]$$

where $[x]$ is the largest integer which satisfies $[x] \leq x$.

Therefore if $\{t_i\}$ is a necessary angle vector, the equalities

$$\varphi_p(z)\sum_{i=1}^{n}\bar{t}_i + p(1-z)n\frac{p-1}{2} \equiv \varphi_p(z)\sum_{i=1}^{n}\bar{t}_i \equiv 0 \pmod{p}$$

hold for $2 \leq z \leq p-1$. Here we have

$$\varphi_p(2) = \sum_{k=1}^{p-1} k\left[\frac{2k}{p}\right] = \sum_{k=(p+1)/2}^{p-1} k = \frac{(p-1)(3p-1)}{8},$$

which is not a multiple of p. Hence the equality (20) holds. □

Next we consider the case that $m = 2$. Then it follows from (15) that

$$\begin{aligned} &F(z,\ell_1,\ell_2;\boldsymbol{t}(p)) \\ &\quad= \frac{p-1}{2p}2^{\ell_1-3}\big\{(2\ell_1^2+8\ell_1\ell_2+8\ell_2^2+2\ell_1+8\ell_2+2)e \\ &\qquad\qquad + (3\ell_1^2+12\ell_1\ell_2+12\ell_2^2+9\ell_1+12\ell_2+2)\sigma\big\} \\ &\qquad + \frac{1}{12p^2}\sum_{i=1}^{n}\left\{2^{\ell_1}z\theta_{i1}\theta_{i2}^2\sum_{s=0}^{p-1}g_{2,p}(s) - p\sum_{\gamma=0}^{\ell_1}\binom{\ell_1}{\gamma}\sum_{s=0}^{p-1}g_{2,p}(s)\Lambda_{2,p}(z,\mu_{i\gamma},s)\right\} \end{aligned} \tag{23}$$

(see (11)), where

$$\Lambda_{2,p}(z,\mu_{i\gamma},s) = \sum_{\lambda_1=0}^{zt_{i1}-1}\sum_{\lambda_{21},\lambda_{22}=0}^{\theta_{i2}-1}\delta_p(\zeta(z,\mu_{i\gamma},s,\tau,\lambda)),$$

$$\zeta(z,\mu_{i\gamma},s,\tau,\lambda) = 1+\lambda_1+zt_{i1}(\ell_2+\gamma+\lambda_{21}+\lambda_{22}+1)+zt_{i2}(s+\ell_1+\ell_2-\gamma+1).$$

Let M be the 2-dimensional complex projective space $\mathbb{CP}^2$. Then it follows from the Lefschetz fixed point formula that $n = 3$. Moreover since $e = 3$, $\sigma = 1$, we have

$$\begin{aligned} &F(z,\ell_1,\ell_2;\boldsymbol{t}(p)) \\ &\quad= \frac{p-1}{2p}2^{\ell_1-3}\big\{9\ell_1^2+36\ell_1\ell_2+36\ell_2^2+15\ell_1+36\ell_2+8\big\} \end{aligned}$$

$$+\frac{1}{12p^2}\sum_{i=1}^{3}\left\{2^{\ell_1}z\theta_{i1}\theta_{i2}^2\sum_{s=0}^{p-1}g_{2,p}(s)-p\sum_{\gamma=0}^{\ell_1}\binom{\ell_1}{\gamma}\sum_{s=0}^{p-1}g_{2,p}(s)\Lambda_{2,p}(z,\mu_{i\gamma},s)\right\}. \tag{24}$$

Proposition 4.7. *Assume that g preserves the standard integrable complex structure of $\mathbb{CP}^2$. Then the rotation angle $\{\tau_{ij}\}$ of g is proper.*

Proof. The set of automorphims of $\mathbb{CP}^2$ which preserve the standard complex structure is known to be the factor group $PGL(3;\mathbb{C}) = GL(3;\mathbb{C})/\mathbb{C}^*$. Any element of $PGL(3;\mathbb{C})$ is expressed as $[S]$ by $S \in GL(3;\mathbb{C})$. Since the cyclic group $\mathbb{Z}_p = \langle g\rangle$ is a compact subgroup of $PGL(3;\mathbb{C})$, there exists elements $h \in PGL(3;\mathbb{C})$ such that $h^{-1}gh$ is represented by an element of the special unitary group $SU(3)$, and there exists $u \in PGL(3;\mathbb{C})$ such that $g' = u^{-1}h^{-1}ghu$ is represented by a periodic diagonal matrix

$$S = \begin{pmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & e^{i\theta_3}\end{pmatrix} \quad (\theta_1+\theta_2+\theta_3=0).$$

Note that the rotation angle of g is the same as that of g' because the eigenvalues of the action of g on the tangent space at q_i are the same as those of the action of g' on the tangent space at $(hu)^{-1}\cdot q_i$.

Let P_2, P_3, V_k $(1 \le k \le 3)$ be the periodic elements of $GL(3;\mathbb{C})$ defined by

$$P_2 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1\end{pmatrix},\quad P_3 = \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0\end{pmatrix},\quad V_k = \begin{pmatrix} e^{i\theta_k} & & \\ & 1 & \\ & & 1\end{pmatrix}$$

and G the finite group generated by $[S]$, $[P_2]$, $[P_3]$, $[V_1]$, $[V_2]$, $[V_3]$. Then $I_{D_E}(g)$ is defined for $g \in G$. Since

$$S = V_1P_2V_2P_2^{-1}P_3V_3P_3^{-1},$$

it follows that

$$\begin{aligned} I_{D_E}(g') &= I_{D_E}([V_1P_2V_2P_2^{-1}P_3V_3P_3^{-1}]) \\ &= I_{D_E}([V_1]) + I_{D_E}([P_2]) + I_{D_E}([V_2]) \\ &\quad - I_{D_E}([P_2]) + I_{D_E}([P_3]) + I_{D_E}([V_3]) - I_{D_E}([P_3]) \\ &= I_{D_E}([V_1]) + I_{D_E}([V_2]) + I_{D_E}([V_3]) = I_{D_E}([V_1V_2V_3]) = I_{D_E}([E_3]) = 0 \end{aligned}$$

where E_3 is the unit matrix. Therefore it follows from (13) that

$$F(z,\ell_1,\ell_2;\{\tau_{ij}\}) \equiv I_{D_E}({g'}^z) = zI_{D_E}(g') = 0 \pmod{\mathbb{Z}}$$

for any integer z $(0 < z < p)$ and any element (ℓ_1,ℓ_2) of L. □

Remark 4.8. Using the argument above, we can show that the rotation angle of a periodic automorphism of $\mathbb{CP}^m$ is proper if the automorphism preserves the standard complex structure of $\mathbb{CP}^m$.

Let A be the set of angle vectors which satisfy the inequalities

$$1 = t_{11} \le t_{21} \le \cdots \le t_{n1}, \quad 1 \le t_{i1} \le t_{i2} \le \cdots \le t_{im} \le p-1 \ (1 \le i \le n).$$

Note that any angle vector is equivalent to an element of A because any t_{ij} has its $\operatorname{mod} p$ inverse. The number of angle vectors $\{t_{ij}\}$ which satisfies the second inequality is equal to $({}_{p-1}H_m)^n$ where ${}_{p-1}H_m$ is the repeated combination. And the number of mutually distinct angle vectors of the form $wt_{\rho(i)j}$ for $0 < w < p$ and permutations ρ is less than or equal to $(p-1)n!$ for any $\{t_{ij}\} \in A$ and less than $(p-1)n!$ for some $\{t_{ij}\} \in A$. Hence the number of the equivalence classes of angle vectors is greater than $L(p,m,n)$ where

$$L(p,m,n) = \min\left\{\lambda \in \mathbb{Z} \mid \lambda \ge \frac{({}_{p-1}H_m)^n}{(p-1)n!}\right\}.$$

For example, when $p = 3$, $m = 2$, $n = 3$, six angle vectors

$$\begin{gathered}((1,1),(1,1),(1,1)),\ ((1,1),(1,1),(1,2)),\ ((1,1),(1,1),(2,2)),\\ ((1,1),(1,2),(1,2)),\ ((1,1),(1,2),(2,2)),\ ((1,2),(1,2),(1,2))\end{gathered} \tag{25}$$

represent all angle vectors and we have $L(3,2,3) = 3 < 6$.

Example 4.9. Let M be a 4-dimensional almost complex manifold with $(e,\sigma) = (3,1)$, which is the same as (e,σ) of $\mathbb{CP}^2$. In this example, we examine the difference between the set of the rotation angles of $\mathbb{CP}^2$ and the set of the proper angle vectors of M and the set of angle vectors of M.

We assume that the action of $\mathbb{Z}_p = \langle g\rangle$ on $\mathbb{CP}^2$ preserves the standard complex structure of $\mathbb{CP}^2$. Then as we see in the proof of Proposition 4.7, the action of g is expressed by integers $1 \le \rho_0 < \rho_1 < \rho_2 \le p-1$ as

$$g \cdot [z_0 : z_1 : z_2] = \left[\xi_p^{\rho_0} z_0 : \xi_p^{\rho_1} z_1 : \xi_p^{\rho_2} z_2\right],$$

where $[z_0 : z_1 : z_2]$ is the homogeneous coordinate of $\mathbb{CP}^2$, whose rotation angle is

$$((\rho_1 - \rho_0, \rho_2 - \rho_0), (p + \rho_0 - \rho_1, \rho_2 - \rho_1), (p + \rho_0 - \rho_2, p + \rho_1 - \rho_2)).$$

Direct computation shows that the angle vectors of the form above are represented by the angle vectors listed below.

p	rotation angles for $\mathbb{CP}^2$
3	$((1,2),(1,2),(1,2))$
5	$((1,2),(1,4),(3,4))$
7	$((1,2),(1,6),(5,6))$, $((1,3),(2,6),(4,5))$

(26)

Moreover direct computation using the equality (24) shows that the proper angle vectors are represented by the angle vectors listed below.

p	proper angle vectors when $(e, \sigma, n) = (3, 1, 3)$	$L(p, 2, 3)$
3	$((1,2),(1,2),(1,2))$	3
5	$((1,2),(1,4),(3,4))$, $((1,2),(2,3),(3,4))$	42
7	$((1,2),(1,6),(5,6))$, $((1,2),(2,5),(5,6))$, $((1,2),(3,4),(5,6))$, $((1,3),(2,6),(4,5))$	258

(27)

EXAMPLE 4.10. Suppose that $p = n = 3$. Then it follows from (12) that e must be a multiple of 3. Here we consider the case that $e + \sigma$ is $0, 4, 8$ and e is $0, 3, 6$. When $(e, \sigma) = (0, 0)$, $(3, -3)$ or $(6, -6)$, direct computation shows that

$$F(2, 0, 1, \boldsymbol{t}(3)) - 2F(1, 0, 1, \boldsymbol{t}(3)) \not\equiv 0 \pmod{\mathbb{Z}}$$

for any angle vectors listed in (25). Hence M with $(e, \sigma) = (0, 0)$, $(3, -3)$, $(6, -6)$ does not admit any action of $\mathbb{Z}_3$ which satisfies Assumption 1.1 with three fixed points. When $(e, \sigma) = (0, 4)$, $(3, 1)$ or $(6, -2)$, the only one necessary angle vector in the list (25) is $((1, 2), (1, 2), (1, 2))$, and when $(e, \sigma) = (0, 8)$, $(3, 5)$ or $(6, 2)$, the only one necessary angle vector in the list (25) is $((1, 1), (1, 1), (1, 1))$.

5. Appendix.

Here we prove the equality (22). Let p be an odd prime number and a, b integers such that $0 < a$, $b < p$. Then we have the next formula of Zagier (see [**10**,

p. 100, p. 101]).

$$\sum_{k=1}^{p-1} \cot\frac{\pi ka}{p}\cot\frac{\pi kb}{p} = 4p\sum_{k=1}^{p-1}\left(\left(\frac{ka}{p}\right)\right)\left(\left(\frac{kb}{p}\right)\right), \quad \sum_{k=1}^{p-1}\left[\frac{ka}{p}\right] = \frac{(p-1)(a-1)}{2}$$

where

$$((x)) = \begin{cases} x-[x]-(1/2) & \text{if } x \text{ is not an integer} \\ 0 & \text{if } x \text{ is an integer} \end{cases}.$$

Since

$$\frac{1}{1-\xi_p^{-k}} = \frac{1}{2} - \frac{\sqrt{-1}}{2}\cot\frac{\pi k}{p},$$

it follows from the formula above that

$$\begin{aligned}
&\sum_{k=1}^{p-1}\frac{1}{1-\xi_p^{-k}}\left(\frac{1}{1-\xi_p^{-kzt_i}} - z\frac{1}{1-\xi_p^{-kt_i}}\right) \\
&\quad= \sum_{k=1}^{p-1}\frac{1}{1-\xi_p^{-k\bar{t}_i}}\left(\frac{1}{1-\xi_p^{-kz}} - z\frac{1}{1-\xi_p^{-k}}\right) \\
&\quad= \sum_{k=1}^{p-1}\left\{\text{Real part of } \frac{1}{1-\xi_p^{-k\bar{t}_i}}\left(\frac{1}{1-\xi_p^{-kz}} - z\frac{1}{1-\xi_p^{-k}}\right)\right\} \\
&\quad= \frac{1}{4}(p-1)(1-z) - \frac{1}{4}\sum_{k=1}^{p-1}\cot\frac{\pi kz}{p}\cot\frac{\pi k\bar{t}_i}{p} + \frac{1}{4}z\sum_{k=1}^{p-1}\cot\frac{\pi k}{p}\cot\frac{\pi k\bar{t}_i}{p} \\
&\quad= \frac{1}{4}(p-1)(1-z) + p\sum_{k=1}^{p-1}\left(-\left(\left(\frac{kz}{p}\right)\right) + z\left(\left(\frac{k}{p}\right)\right)\right)\left(\left(\frac{k\bar{t}_i}{p}\right)\right) \\
&\quad= \frac{1}{4}(p-1)(1-z) + p\sum_{k=1}^{p-1}\left(\left[\frac{kz}{p}\right] - \frac{1}{2}(z-1)\right)\left(\frac{k\bar{t}_i}{p} - \left[\frac{k\bar{t}_i}{p}\right] - \frac{1}{2}\right) \\
&\quad= \frac{1}{4}(p-1)(1-z) + \bar{t}_i\sum_{k=1}^{p-1}k\left[\frac{kz}{p}\right] - p\sum_{k=1}^{p-1}\left[\frac{kz}{p}\right]\left[\frac{k\bar{t}_i}{p}\right] - \frac{p}{2}\frac{(p-1)(z-1)}{2} \\
&\qquad - \frac{1}{2}(z-1)\bar{t}_i\sum_{k=1}^{p-1}k + \frac{p}{2}(z-1)\frac{(p-1)(\bar{t}_i-1)}{2} + \frac{p}{4}(z-1)\sum_{k=1}^{p-1}1
\end{aligned}$$

$$\equiv \varphi_p(z)\bar{t}_i + \frac{1}{4}(1-z)(p^2-1) \pmod{p}.$$

This completes the proof of the equality (22).

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