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Strichartz estimates for Schrödinger equations with variable coefficients and potentials at most linear at spatial infinity

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Abstract. In the present paper we consider Schrödinger equations with variable coefficients and potentials, where the principal part is a long-range perturbation of the flat Laplacian and potentials have at most linear growth at spatial infinity. We then prove local-in-time Strichartz estimates, outside a large compact set centered at origin, without loss of derivatives. Moreover we also prove global-in-space Strichartz estimates under the non-trapping condition on the Hamilton flow generated by the kinetic energy.

1. Introduction.

In this paper we study the so called (local-in-time) *Strichartz estimates* for the solutions to d-dimensional time-dependent Schrödinger equations

$$i\partial_t u(t) = Hu(t),\ t \in \mathbb{R}; \quad u|_{t=0} = u_0 \in L^2(\mathbb{R}^d), \tag{1.1}$$

where $d \geq 1$ and H is a Schrödinger operator with variable coefficients:

$$H = -\frac{1}{2}\sum_{j,k=1}^{d} \partial_{x_j} a^{jk}(x) \partial_{x_k} + V(x).$$

Throughout the paper we assume that $a^{jk}(x)$ and $V(x)$ are real-valued and smooth on $\mathbb{R}^d$, and $(a^{jk}(x))$ is a symmetric matrix satisfying $(a^{jk}(x)) \geq C\,\mathrm{Id}$, $x \in \mathbb{R}^d$, with some $C > 0$. We also assume

Assumption 1. There exist constants $\mu, \nu \geq 0$ such that, for any $\alpha \in \mathbb{Z}_+^d$,

$$\left|\partial_x^\alpha (a^{jk}(x) - \delta_{jk})\right| \leq C_\alpha \langle x\rangle^{-\mu-|\alpha|}, \quad \left|\partial_x^\alpha V(x)\right| \leq C_\alpha \langle x\rangle^{2-\nu-|\alpha|}, \quad x \in \mathbb{R}^d,$$

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with some $C_\alpha > 0$.

We may assume $\mu < 1$ and $\nu < 2$ without loss of generality. It is well known that H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ under Assumption 1, and we denote the unique self-adjoint extension on $L^2(\mathbb{R}^d)$ by the same symbol H. By the Stone theorem, the solution to (1.1) is given by $u(t) = e^{-itH}u_0$, where e^{-itH} is a unique unitary group on $L^2(\mathbb{R}^d)$ generated by H and called the propagator.

Let us recall the (global-in-time) Strichartz estimates for the free Schrödinger equation which state that

$$\|e^{it\Delta/2}u_0\|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} \le C\|u_0\|_{L^2(\mathbb{R}^d)}, \tag{1.2}$$

where (p,q) satisfies the following *admissible* condition

$$2 \le p, q \le \infty, \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad (d,p,q) \ne (2,2,\infty). \tag{1.3}$$

For $d \ge 3$, $(p,q) = (2, 2d/(d-2))$ is called the endpoint. It is well known that these estimates are fundamental to study the local well-posedness of Cauchy problem of nonlinear Schrödinger equations (see, e.g., [**6**]). The estimates (1.2) were first proved by Strichartz [**23**] for a restricted pair of (p,q) with $p = q = 2(d+2)/d$, and have been extensively generalized for (p,q) satisfying (1.3) by [**12**], [**15**]. Moreover, in the flat case ($a^{jk} \equiv \delta_{jk}$), local-in-time Strichartz estimates

$$\|e^{itH}u_0\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \le C_T\|u_0\|_{L^2(\mathbb{R}^d)}, \tag{1.4}$$

have been extended to the case with potentials decaying at infinity [**25**] or increasing at most quadratically at infinity [**26**]. In particular, if $V(x)$ has at most quadratic growth at spatial infinity, i.e.,

$$V \in C^\infty(\mathbb{R}^d;\mathbb{R}), \quad |\partial_x^\alpha V(x)| \le C_\alpha \text{ for } |\alpha| \ge 2,$$

then it was shown by Fujiwara [**11**] that the fundamental solution $E(t,x,y)$ of the propagator e^{-itH} satisfies $|E(t,x,y)| \lesssim |t|^{-d/2}$ for all $x, y \in \mathbb{R}^d$ and $t \ne 0$ small enough. The estimates (1.4) are immediate consequences of this estimate and the TT^*-argument due to Ginibre-Velo [**12**] (see Keel-Tao [**15**] for the endpoint estimate). For the case with magnetic fields or singular potentials, we refer to Yajima [**26**], [**27**] and references therein.

On the other hand, local-in-time Strichartz estimates on manifolds have recently been proved by many authors under several conditions on the geometry.

Staffilani-Tataru [**22**], Robbiano-Zuily [**18**] and Bouclet-Tzvetkov [**2**] studied the case on the Euclidean space with the asymptotically flat metric under several settings. In particular, Bouclet-Tzvetkov [**2**] proved local-in-time Strichartz estimates without loss of derivatives under Assumption 1 with $\mu > 0$ and $\nu > 2$ and the non-trapping condition. Burq-Gérard-Tzvetkov [**4**] proved Strichartz estimates with a loss of derivative $1/p$ on any compact manifolds without boundaries. They also proved that the loss $1/p$ is optimal in the case of $M = \mathbb{S}^d$. Hassell-Tao-Wunsch [**13**] and the author [**17**] considered the case of non-trapping asymptotically conic manifolds which are non-compact Riemannian manifolds with an asymptotically conic structure at infinity. Bouclet [**1**] studied the case of an asymptotically hyperbolic manifold. Burq-Guillarmou-Hassell [**5**] recently studied the case of asymptotically conic manifolds with hyperbolic trapped trajectories of sufficiently small fractal dimension. For global-in-time Strichartz estimates, we refer to [**10**], [**8**] and the references therein in the case with electromagnetic potentials, and to [**3**], [**24**], [**16**] in the case of Euclidean space with an asymptotically flat metric.

The main purpose of the paper is to handle a mixed case of above two situations. More precisely, we show that local-in-time Strichartz estimates for long-range perturbations still hold (without loss of derivatives) if we add unbounded potentials which have at most linear growth at spatial infinity (i.e., $\nu \ge 1$), at least excluding the endpoint $(p, q) = (2, 2d/(d-2))$. To the best knowledge of the author, our result may be a first example on the case where both of variable coefficients and unbounded potentials in the spatial variable x are present.

To state the result, we recall the non-trapping condition. We denote by

$$H_0 = H - V = -\frac{1}{2}\sum_{j,k=1}^{d} \partial_{x_j} a^{jk}(x)\partial_{x_k}, \quad k(x,\xi) = \frac{1}{2}\sum_{j,k=1}^{d} a^{jk}(x)\xi_j\xi_k,$$

the principal part of H and the kinetic energy, respectively, and also denote by $(y_0(t,x,\xi), \eta_0(t,x,\xi))$ the Hamilton flow generated by $k(x,\xi)$:

$$\dot{y}_0(t) = \partial_\xi k(y_0(t), \eta_0(t)),\ \dot{\eta}_0(t) = -\partial_x k(y_0(t), \eta_0(t)); \quad (y_0(0), \eta_0(0)) = (x, \xi).$$

Note that the Hamiltonian vector field H_k, generated by k, is complete on $\mathbb{R}^{2d}$ since (a^{jk}) satisfies the uniform elliptic condition. Hence, $(y_0(t,x,\xi), \eta_0(t,x,\xi))$ exists for all $t \in \mathbb{R}$. We consider the following *non-trapping condition*:

$$\text{For any } (x,\xi) \in T^*\mathbb{R}^d \text{ with } \xi \neq 0,\ |y_0(t,x,\xi)| \to +\infty \text{ as } t \to \pm\infty. \tag{1.5}$$

We now state our main result.

Theorem 1.1. (i) *Suppose that H satisfies Assumption* 1 *with $\mu > 0$ and $\nu \ge 1$. Then, there exist $R_0 > 0$ large enough and $\chi_0 \in C_0^\infty(\mathbb{R}^d)$ with $\chi_0(x) = 1$ for $|x| < R_0$ such that, for any $T > 0$ and (p, q) satisfying* (1.3) *and $p \ne 2$, there exists $C_T > 0$ such that*

$$\|(1-\chi_0)e^{-itH}u_0\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \le C_T\|u_0\|_{L^2(\mathbb{R}^d)}. \tag{1.6}$$

(ii) *Suppose that H satisfies Assumption* 1 *with $\mu, \nu \ge 0$ and $k(x, \xi)$ satisfies the non-trapping condition* (1.5)*. Then, for any $\chi \in C_0^\infty(\mathbb{R}^d)$, $T > 0$ and (p, q) satisfying* (1.3) *and $p \ne 2$, we have*

$$\|\chi e^{-itH}u_0\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \le C_T\|u_0\|_{L^2(\mathbb{R}^d)}. \tag{1.7}$$

Moreover, combining with (1.6)*, we obtain global-in-space estimates*

$$\|e^{-itH}u_0\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \le C_T\|u_0\|_{L^2(\mathbb{R}^d)},$$

provided that $\mu > 0$ and $\nu \ge 1$.

We here display the outline of the paper and explain the idea of the proof of Theorem 1.1. By the virtue of the Littlewood-Paley theory in terms of H_0, the proof of (1.6) can be reduced to that of following *semi-classical Strichartz estimates*:

$$\|(1-\chi_0)\psi(h^2H_0)e^{-itH}u_0\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \le C_T\|u_0\|_{L^2(\mathbb{R}^d)}, \quad 0 < h \ll 1,$$

where $\psi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\psi \Subset (0, \infty)$ and $C_T > 0$ is independent of h. Moreover, there exists a smooth function $a \in C^\infty(\mathbb{R}^{2d})$ supported in a neighborhood of the support of $(1-\chi_0)\psi\circ k$ such that $(1-\chi_0)\varphi(h^2H_0)$ can be replaced with the semi-classical pseudodifferential operator $a(x, hD)$. In Section 2, we collect some known results on the semi-classical pseudo-differential calculus and prove such a reduction to semi-classical estimates. Rescaling $t \mapsto th$, we want to show dispersive estimates for e^{ithH} on a time scale of order h^{-1} to prove semi-classical Strichartz estimates. To prove dispersive estimates, we construct two kinds of parametrices, namely the Isozaki-Kitada and the WKB parametrices. Let $a^\pm \in S(1, dx^2/\langle x\rangle^2 + d\xi^2/\langle\xi\rangle^2)$ be symbols supported in the following outgoing and incoming regions:

$$\{(x, \xi); |x| > R_0,\ |\xi|^2 \in J,\ \pm x\cdot\xi > -(1/2)|x||\xi|\},$$

respectively, where $J \Subset (0,\infty)$ is an open interval so that $\pi_\xi(\operatorname{supp}\psi\circ k) \Subset J$ and π_ξ is the projection onto the ξ-space. If H is a long-range perturbation of $-(1/2)\Delta$, then the outgoing (resp. incoming) Isozaki-Kitada parametrix of $e^{-itH}a^+(x,hD)$ for $0 \le t \le h^{-1}$ (resp. $e^{-itH}a^-(x,hD)$ for $-h^{-1} \le t \le 0$) has been constructed by Robert [**20**] (see, also [**2**]). However, because of the unboundedness of V with respect to x, it is difficult to construct such parametrices of $e^{-ithH}a^\pm(x,hD)$. To overcome this difficulty, we use a method due to Yajima-Zhang [**29**] as follows. We approximate e^{-ithH} by e^{-ithH_h}, where $H_h = H - V + V_h$ and V_h vanishes in the region $\{x; |x| \gg h^{-1}\}$. Suppose that a^+ (resp. a^-) is supported in the intersection of the outgoing (resp. incoming) region and $\{x; |x| < h^{-1}\}$. In Section 3, we construct the Isozaki-Kitada parametrix of $e^{-ithH_h}a^\pm(x,hD)$ for $0 \le \pm t \le h^{-1}$ and prove the following justification of the approximation: for any $N > 0$,

$$\sup_{0\le \pm t\le h^{-1}} \big\|(e^{-ithH} - e^{-ithH_h})a^\pm(x,hD)f\big\|_{L^2} \le C_N h^N \big\|f\big\|_{L^2}, \quad 0 < h \ll 1.$$

In Section 4, we discuss the WKB parametrix construction of $e^{-ithH}a(x,hD)$ on a time scale of order h^{-1}, where a is supported in $\{(x,\xi); |x| > h^{-1},\ |\xi|^2 \in I\}$. Such a parametrix construction is basically known for the potential perturbation case (see, e.g., [**28**]) and has been proved by the author for the case on asymptotically conic manifolds [**17**]. Combining these results studied in Sections 2, 3 and 4 with the Keel-Tao theorem [**15**], we prove semi-classical Strichartz estimates in Section 5. Section 5 is also devoted to the proof of (1.7). The proof of (1.7) heavily depends on local smoothing effects due to Doi [**9**] and the Christ-Kiselev lemma [**7**] and the method of the proof is similar as that in Robbiano-Zuily [**18**]. Appendix A is devoted to prove some technical inequalities on the Hamilton flow needed for constructing the WKB parametrix.

Throughout the paper we use the following notations. For $A, B \ge 0$, $A \lesssim B$ means that there exists some universal constant $C > 0$ such that $A \le CB$. We denote the set of multi-indices by $\mathbb{Z}_+^d$. For Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the Banach space of bounded operators from X to Y, and we write $\mathcal{L}(X) := \mathcal{L}(X,X)$.

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2. Reduction to semi-classical estimates.

In this section we show that the estimate (1.6) follows from semi-classical Strichartz estimates. We first record known results on the pseudo-differential calculus and the L^p-functional calculus. For $a \in C^\infty(\mathbb{R}^{2d})$ and $h \in (0,1]$, we denote the semi-classical pseudo-differential operator (h-PDO for short) by $a(x, hD_x)$:

$$a(x, hD_x)u(x) = (2\pi h)^{-d} \int e^{i(x-y)\cdot\xi/h} a(x,\xi)u(y)dyd\xi, \quad u \in \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class. For the metric $g = dx^2/\langle x\rangle^2 + d\xi^2/\langle \xi\rangle^2$ on $T^*\mathbb{R}^d$, we consider Hörmander's symbol class $S(m,g)$ with a weighted function m, namely we write $a \in S(m,g)$ if $a \in C^\infty(\mathbb{R}^{2d})$ and

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha\beta} m(x,\xi)\langle x\rangle^{-|\alpha|}\langle \xi\rangle^{-|\beta|}, \quad x,\xi \in \mathbb{R}^d.$$

Let $a \in S(m_1,g)$, $b \in S(m_2,g)$. For any $N = 0,1,2,\dots,$ the symbol of the composition $a(x,hD)b(x,hD)$, denoted by $a\sharp b$, has an asymptotic expansion

$$a\sharp b(x,\xi) = \sum_{|\alpha|\le N}^{N} \frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha a(x,\xi)\cdot \partial_x^\alpha b(x,\xi) + h^{N+1} r_N(x,\xi) \tag{2.1}$$

with some $r_N \in S(\langle x\rangle^{-N-1}\langle\xi\rangle^{-N-1} m_1 m_2, g)$. For $a \in S(1,g)$, $a(x,hD_x)$ is extended to a bounded operator on $L^2(\mathbb{R}^d)$. Moreover, if $a \in S(\langle\xi\rangle^{-N}, g)$ for some $N > d$, then $a(x,hD)$ satisfies

$$\left\|a(x,hD)\right\|_{\mathcal{L}(L^q(\mathbb{R}^d),L^r(\mathbb{R}^d))} \le C_{qr} h^{-d(1/q-1/r)}, \quad 1 \le q \le r \le \infty,\ h \in (0,1], \tag{2.2}$$

where $C_{qr} > 0$ is independent of h. We follow the argument in [**2**]. We denote by $A_h(x,y)$ the distribution kernel of $a(x,hD)$:

$$A_h(x,y) = (2\pi h)^{-d} \int e^{(x-y)\cdot\xi/h} a(x,\xi)d\xi.$$

Since $|a(x,\xi)| \le C\langle\xi\rangle^{-N}$ with $N > d$, this integral is absolutely convergent and we can write $A_h(x,y) = (2\pi)^{-d/2} h^{-d}\hat{a}(x,(y-x)/h)$, where $\hat{a}$ is the Fourier transform of a with respect to the second variable. In particular, we have

$$\sup_{x,y} |A_h(x,y)| \le Ch^{-d}$$

which implies (2.2) for $(q,r) = (1,\infty)$. Since $|\hat{a}(x,\eta)| \le C_d\langle\eta\rangle^{-d-1}$ with $C_d > 0$ independent of x, a direct calculation yields

$$\sup_x \int |A_h(x,y)|dy + \sup_y \int |A_h(x,y)|dx \le C$$

for some $C > 0$ independent of h. The Schur lemma then implies (2.2) for $q = r$. Finally, for arbitrarily fixed $1 \le q \le r \le \infty$, we have the $\mathcal{L}(L^1, L^{r/q})$ bound by an interpolation between the $\mathcal{L}(L^1)$ and $\mathcal{L}(L^1, L^\infty)$ bounds. Interpolating between the $\mathcal{L}(L^1, L^{r/q})$ and $\mathcal{L}(L^\infty)$ bounds, we obtain the $\mathcal{L}(L^q, L^r)$ bound.

We next consider the L^p-functional calculus. The following lemma, which was proved by [**2**, Proposition 2.5], tells us that, for any $\varphi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \Subset (0,\infty)$, $\varphi(h^2 H_0)$ can be approximated in terms of the h-PDO.

Lemma 2.1. *Let $\varphi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\varphi \Subset (0,\infty)$ and $N \ge 0$ a non-negative integer. Then there exist symbols $a_j \in S(1,g)$, $j = 0, 1, \dots, N$, such that*

(i) *$a_0(x,\xi) = \varphi(k(x,\xi))$ and $a_j(x,\xi)$ are supported in the support of $\varphi(k(x,\xi))$.*
(ii) *For every $1 \le q \le r \le \infty$ there exists $C_{qr} > 0$ such that*

$$\left\|a_j(x, hD_x)\right\|_{\mathcal{L}(L^q(\mathbb{R}^d), L^r(\mathbb{R}^d))} \le C_{qr} h^{-d(1/q-1/r)},$$

uniformly with respect to $h \in (0,1]$.
(iii) *There exists a constant $N_0 \ge 0$ such that, for all $1 \le q \le r \le \infty$,*

$$\left\|\varphi(h^2 H_0) - a(x, hD_x)\right\|_{\mathcal{L}(L^q(\mathbb{R}^d), L^r(\mathbb{R}^d))} \le C_{Nqr} h^{N-N_0-d(1/q-1/r)}$$

uniformly with respect to $h \in (0,1]$, where $a = \sum_{j=0}^N h^j a_j$.

Remark 2.2. We note that Assumption 1 implies a stronger bounds on a_j:

$$\left|\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)\right| \le C_{\alpha\beta} \langle x\rangle^{-j-|\alpha|} \langle \xi\rangle^{-|\beta|},$$

though we do not use this estimate in the following argument.

We next recall the Littlewood-Paley decomposition in terms of $\varphi(h^2 H_0)$. Consider a 4-adic partition of unity with respect to $[1,\infty)$:

$$\sum_{j=0}^\infty \varphi(2^{-2j}\lambda) = 1, \quad \lambda \in [1,\infty),$$

where $\varphi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subset [1/4, 4]$ and $0 \le \varphi \le 1$.

Lemma 2.3. *Let $\chi \in C_0^\infty(\mathbb{R}^d)$. Then, for $q \in [2,\infty)$ with $0 \le d(1/2-1/q) \le 1$,*

$$\big\|(1-\chi)f\big\|_{L^q(\mathbb{R}^d)} \lesssim \big\|f\big\|_{L^2(\mathbb{R}^d)} + \bigg(\sum_{j=0}^\infty \big\|(1-\chi)\varphi(2^{-2j}H_0)f\big\|_{L^q(\mathbb{R}^d)}^2\bigg)^{1/2}.$$

This lemma can be proved similarly to the case of the Laplace-Beltrami operator on compact manifolds without boundaries (see [**4**, Corollary 2.3]). By using this lemma, we have the following:

Proposition 2.4. *Let χ_0 be as that in Theorem* 1.1*. Suppose that there exist $h_0, \delta > 0$ small enough such that, for any $\psi \in C_0^\infty((0,\infty))$ and any admissible pair (p,q) with $p>2$,*

$$\big\|(1-\chi_0)\psi(h^2H_0)e^{-itH}u_0\big\|_{L^p([-\delta,\delta];L^q(\mathbb{R}^d))} \le C\big\|u_0\big\|_{L^2(\mathbb{R}^d)}, \tag{2.3}$$

uniformly with respect to $h \in (0,h_0]$. Then, the statement of Theorem 1.1 (i) *holds.*

Proof. By Lemma 2.3 with $f = e^{-itH}u_0$, the Minkowski inequality and the unitarity of e^{-itH} on $L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}&\big\|(1-\chi_0)e^{-itH}u_0\big\|_{L^p([-\delta,\delta];L^q(\mathbb{R}^d))}\\ &\quad\lesssim \big\|u_0\big\|_{L^2(\mathbb{R}^d)} + \bigg(\sum_{j=0}^\infty \big\|(1-\chi_0)\varphi(2^{-2j}H_0)e^{-itH}u_0\big\|_{L^p([-\delta,\delta];L^q(\mathbb{R}^d))}^2\bigg)^{1/2}.\end{aligned}$$

For $0 \le j \le [-\log h_0]+1$, we have the bound

$$\begin{aligned}&\sum_{j=0}^{[-\log h_0]+1} \big\|(1-\chi_0)\varphi(2^{-2j}H_0)e^{-itH}u_0\big\|_{L^p([-\delta,\delta];L^q(\mathbb{R}^d))}^2\\ &\quad\lesssim \sum_{j=0}^{[-\log h_0]+1} \big\|\varphi(2^{-2j}H_0)\big\|_{\mathcal{L}(L^2(\mathbb{R}^d),L^q(\mathbb{R}^d))}\big\|e^{-itH}u_0\big\|_{L^\infty([-\delta,\delta];L^2(\mathbb{R}^d))}\\ &\quad\lesssim ([-\log h_0]+1)2^{([-\log h_0]+1)d(1/2-1/q)}\big\|u_0\big\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Choosing $\psi \in C_0^\infty(\mathbb{R})$ with $\psi \equiv 1$ on $\operatorname{supp}\varphi$, the Duhamel formula implies

$$\begin{aligned}
&\varphi(h^2H_0)e^{-itH}\\
&\quad= \psi(h^2H_0)e^{-itH}\varphi(h^2H_0) + \psi(h^2H_0)i\int_0^t e^{-i(t-s)H}[V,\varphi(h^2H_0)]e^{-isH}ds\\
&\quad=: \psi(h^2H_0)e^{-itH}\varphi(h^2H_0) + R(t,h).
\end{aligned}$$

Since $[H,\varphi(h^2H_0)] = [V,\varphi(h^2H_0)] = O(h)$ on $L^2(\mathbb{R}^d)$, $R(t,h)$ satisfies

$$\begin{aligned}
&\sup_{0\le t\le 1}\big\|R(t,h)\big\|_{\mathcal{L}(L^2(\mathbb{R}^d),L^q(\mathbb{R}^d))}\\
&\quad\lesssim \big\|\psi(h^2H_0)\big\|_{\mathcal{L}(L^2(\mathbb{R}^d),L^q(\mathbb{R}^d))}\big\|[V,\varphi(h^2H_0)]\big\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\\
&\quad\lesssim h^{-d(1/2-1/q)+1}.
\end{aligned}\tag{2.4}$$

We here note that $\gamma := -d(1/2-1/q)+1 = -2/p+1 > 0$ since $p > 2$. By (2.3), (2.4) with $h = 2^{-j}$ and the almost orthogonality of $\operatorname{supp}\varphi(2^{-2j}\cdot)$, we obtain

$$\begin{aligned}
&\sum_{j=[-\log h_0]}^{\infty}\big\|(1-\chi_0)\varphi(2^{-2j}H_0)e^{-itH}u_0\big\|^2_{L^p([-\delta,\delta];L^q(\mathbb{R}^d))}\\
&\lesssim \sum_{j=[-\log h_0]}^{\infty}\Big(\big\|\varphi(2^{-2j}H_0)u_0\big\|^2_{L^2(\mathbb{R}^d)} + 2^{-2\gamma j}\big\|u_0\big\|^2_{L^2(\mathbb{R}^d)}\Big) \lesssim \big\|u_0\big\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Combining with the bound for $0 \le j \le [-\log h_0]+1$, we have

$$\big\|(1-\chi_0)e^{-itH}u_0\big\|_{L^p([-\delta,\delta];L^q(\mathbb{R}^d))} \lesssim \big\|u_0\big\|_{L^2(\mathbb{R}^d)}.$$

Splitting the time interval $[-T,T]$ into $([T/\delta]+1)$ intervals with size 2δ, we obtain

$$\begin{aligned}
&\big\|(1-\chi_0)\psi(h^2H_0)e^{-itH}u_0\big\|_{L^p([-T,T];L^q(\mathbb{R}^d))}\\
&\quad\le \sum_{k=-[T/\delta]}^{[T/\delta]+1}\big\|(1-\chi_0)\psi(h^2H_0)e^{-itH}e^{-i(k+1)H}u_0\big\|_{L^p([-\delta,\delta];L^q(\mathbb{R}^d))}\\
&\quad\le C_T\big\|u_0\big\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

In the last inequality, we used the unitarity of $e^{-i(k+1)H}$ on $L^2(\mathbb{R}^d)$. $\square$

3. Isozaki-Kitada parametrix.

In this section we assume Assumption 1 with $0 < \mu = \nu < 1/2$ without loss of generality, and construct the Isozaki-Kitada parametrix. Since the potential V can grow at infinity, it is difficult to construct directly the Isozaki-Kitada parametrix for e^{-itH} even though we restrict it in an outgoing or incoming region. To overcome this difficulty, we approximate e^{-itH} as follows. Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be a cut-off function such that $\rho(x) = 1$ if $|x| \le 1$ and $\rho(x) = 0$ if $|x| \ge 2$. For a small constant $\varepsilon > 0$ and $h \in (0, 1]$, we define H_h by

$$H_h = H_0 + V_h, \quad V_h = V(x)\rho(\varepsilon h x).$$

We note that, for any fixed $\varepsilon > 0$,

$$h^2 \big|\partial_x^\alpha V_h(x)\big| \le C_\alpha h^2 \langle x\rangle^{2-\mu-|\alpha|} \le C_{\varepsilon,\alpha} \langle x\rangle^{-\mu-|\alpha|}, \quad x \in \mathbb{R}^d,$$

where $C_{\varepsilon,\alpha}$ may be taken uniformly with respect to $h \in (0, 1]$. Such a type modification has been used to prove Strichartz estimates and local smoothing effects for Schrödinger equations with super-quadratic potentials (see, Yajima-Zhang [**29**, Section 4]).

For $R > 0$, an open interval $J \Subset (0, \infty)$ and $-1 < \sigma < 1$, we define the outgoing and incoming regions by

$$\Gamma^\pm(R, J, \sigma) := \left\{(x, \xi) \in \mathbb{R}^{2d}; |x| > R,\ |\xi| \in J,\ \pm\frac{x \cdot \xi}{|x||\xi|} > -\sigma\right\},$$

respectively. Since $H_0 + h^2 V_h$ is a long-range perturbation of $-\Delta/2$, we have the following theorem due to Robert [**20**] and Bouclet-Tzvetkov [**2**].

Theorem 3.1. *Let J, J_0, J_1 and J_2 be relatively compact open intervals, $\sigma, \sigma_0, \sigma_1$ and σ_2 real numbers so that $J \Subset J_0 \Subset J_1 \Subset J_2 \Subset (0, \infty)$ and $-1 < \sigma < \sigma_0 < \sigma_1 < \sigma_2 < 1$. Fix arbitrarily $\varepsilon > 0$. Then there exist $R_0 > 0$ large enough and $h_0 > 0$ small enough such that the followings hold.*

(i) *There exist two families of smooth functions*

$$\big\{S_h^+; h \in (0, h_0], R \ge R_0\big\}, \quad \big\{S_h^-; h \in (0, h_0], R \ge R_0\big\} \subset C^\infty(\mathbb{R}^{2d}; \mathbb{R})$$

satisfying the Eikonal equation associated to $k + h^2 V_h$:

$$k\big(x, \partial_x S_h^{\pm}(x,\xi)\big) + h^2 V_h(x) = \frac{1}{2}|\xi|^2, \quad (x,\xi) \in \Gamma^{\pm}(R^{1/4}, J_2, \sigma_2), \quad h \in (0, h_0],$$

respectively, such that

$$\big|\partial_x^\alpha \partial_\xi^\beta (S_h^{\pm}(x,\xi) - x\cdot\xi)\big| \le C_{\alpha\beta}\langle x\rangle^{1-\mu-|\alpha|}, \quad \alpha,\beta \in \mathbb{Z}_+^d,\ x,\xi \in \mathbb{R}^d, \tag{3.1}$$

where $C_{\alpha\beta} > 0$ may be taken uniformly with respect to R and h.

(ii) *For every $R \ge R_0$, $h \in (0, h_0]$ and $N = 0, 1, \dots,$ we can find*

$$b_h^{\pm} = \sum_{j=0}^{N} h^j b_{h,j}^{\pm} \quad \textit{with} \quad b_{h,j}^{\pm} \in S(1,g),\ \operatorname{supp} b_{h,j}^{\pm} \subset \Gamma^{\pm}(R^{1/3}, J_1, \sigma_1),$$

such that, for every $a^{\pm} \in S(1,g)$ with $\operatorname{supp} a^{\pm} \subset \Gamma^{\pm}(R, J, \sigma)$, there exist

$$c_h^{\pm} = \sum_{j=0}^{N} h^j c_{h,j}^{\pm} \quad \textit{with} \quad c_{h,j}^{\pm} \in S(1,g),\ \operatorname{supp} c_{h,j}^{\pm} \subset \Gamma^{\pm}(R^{1/2}, J_0, \sigma_0),$$

such that, for all $\pm t \ge 0$,

$$e^{-ithH_h} a^{\pm}(x, hD) = U(S_h^{\pm}, b_h^{\pm}) e^{ith\Delta/2} U(S_h^{\pm}, c_h^{\pm})^* + Q_{\mathrm{IK}}^{\pm}(t, h, N),$$

respectively, where $U(S_h^{\pm}, w)$ are Fourier integral operators, with the phases $S_h^{\pm}$ and the amplitude w, defined by

$$U(S_h^{\pm}, w) f(x) = \frac{1}{(2\pi h)^d} \int e^{i(S_h^{\pm}(x,\xi) - y\cdot\xi)/h} w(x,\xi) f(y) dy d\xi,$$

respectively. Moreover, for any $s = 0, 1, 2, \dots,$ there exists $C_{N,s} > 0$ such that

$$\big\|(h^2 H_h + L)^s Q_{\mathrm{IK}}^{\pm}(t, h, N)\big\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C_{N,s} h^{N-1} \tag{3.2}$$

uniformly with respect to $h \in (0, h_0]$ and $0 \le \pm t \le h^{-1}$, where $L > 1$, independent of h, t and x, is a large constant so that $h^2 V_h + L \ge 1$.

(iii) *The distribution kernels $K_{\mathrm{IK}}^{\pm}(t, h, x, y)$ of $U(S_h^{\pm}, b_h^{\pm}) e^{-ith\Delta/2} U(S_h^{\pm}, c_h^{\pm})^*$ satisfy dispersive estimates*:

$$\big|K_{\mathrm{IK}}^{\pm}(t, h, x, y)\big| \le C|th|^{-d/2}, \quad 0 \le \pm t \le h^{-1}, \tag{3.3}$$

respectively, where $C>0$ *is independent of* $h\in(0,h_0]$, $0\le \pm t\le h^{-1}$ *and* $x,\xi\in\mathbb{R}^d$.

Proof. This theorem is basically known, and we only check (3.2) for the outgoing case. For the detail of the proof, we refer to [**20**, Section 4] and [**2**, Section 3]. We also refer to the original paper by Isozaki-Kitada [**14**].

The remainder $Q^+_{\mathrm{IK}}(t,h,N)$ consists of the following three parts:

$$
\begin{aligned}
&- h^{N+1}e^{-ithH_h}q_1(h,x,hD),\\
&- ih^N\int_0^t e^{-i(t-\tau)hH_h}U^+(S_h^+,q_2(h))e^{i\tau h\Delta/2}U^+(S_h^+,c_h^+)^*d\tau,\\
&- (i/h)\int_0^t e^{-i(t-\tau)hH_h}\widetilde{Q}(\tau,h)d\tau,
\end{aligned}
$$

where $\{q_1(h,\cdot,\cdot),q_2(h,\cdot,\cdot);h\in(0,h_0]\}\subset\bigcap_{M=1}^\infty S(\langle x\rangle^{-N}\langle\xi\rangle^{-M},g)$ is a bounded set, and $\widetilde{Q}(s,h)$ is an integral operator with a kernel $\tilde{q}(s,h,x,y)$ satisfying

$$
\left|\partial_x^\alpha\partial_\xi^\beta\tilde{q}(\tau,h,x,y)\right|\le C_{\alpha\beta}h^{M-|\alpha+\beta|}(1+|\tau|+|x|+|y|)^{-M+|\alpha+\beta|},\quad \tau\ge 0,
$$

for any $M\ge 0$. A standard L^2-boundedness of h-PDO and FIO then imply

$$
\left\|(h^2H_0+1)^s(q_1(h,x,hD)+U^+(S_h^+,q_2(h)))\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\le C_s,
$$

and a direct computation yields

$$
\left\|(h^2H_0+1)^s\widetilde{Q}(\tau,h)\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\le C_Mh^M.
$$

On the other hand, we choose a constant $L>0$ so large that $h^2V_h+L\ge 1$. Since $h^2V_h+L\lesssim 1$ by the definition of V_h, we have

$$
\left\|(h^2H_h+L)^s(h^2H_0+1)^{-s}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\le C_s,\quad s=1,2,\dots.
$$

Then (3.2) follows from the above three estimates since $(h^2H_h+L)^s$ commutes with e^{-ithH_h}. □

The following key lemma tells us that one can still construct the Isozaki-Kitada parametrix of the original propagator e^{-ithH} if we restrict the support of initial data in the region $\{x;|x|<h^{-1}\}$.

LEMMA 3.2. *Suppose that $\{a_h^\pm\}_{h\in(0,1]}$ are bounded sets in $S(1,g)$ and satisfy*

$$\operatorname{supp} a_h^\pm \subset \Gamma^\pm(R,J,\sigma)\cap\{x;|x|<h^{-1}\},$$

respectively. Then for any $M\geq 0$, $h\in(0,h_0]$ and $0\leq \pm t\leq h^{-1}$, we have

$$\left\|(e^{-ithH}-e^{-ithH_h})a_h^\pm(x,hD)\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\leq C_M h^M,$$

where $C_M>0$ is independent of h and t.

PROOF. We prove the lemma for the outgoing case only, and the proof of incoming case is completely analogous. We set $A=a_h^+(x,hD)$ and $W_h=V-V_h$. The Duhamel formula yields

$$\begin{aligned}
&\big(e^{-ithH}-e^{-ithH_h}\big)A\\
&\quad=-ih\int_0^t e^{-i(t-s)hH}W_h e^{-ishH_h}Ads\\
&\quad=-ih\int_0^t e^{-i(t-s)hH}e^{-ishH_h}W_h Ads\\
&\qquad-h^2\int_0^t e^{-i(t-s)hH}\int_0^s e^{-i(s-\tau)hH_h}[H_0,W_h]e^{-i\tau hH_h}Ad\tau ds.
\end{aligned}$$

Since $\operatorname{supp} a_h^+(\cdot,\xi)\subset\{x;|x|<h^{-1}\}$, we learn $\operatorname{supp} W_h\cap a_h^+(\cdot,\xi)=\emptyset$ if $\varepsilon<1$. Combining with the asymptotic formula (2.1), we see that this support property implies

$$\left\|W_hA\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\leq C_M h^M$$

for any $M\geq 0$. A direct computation yields that $[H_0,W_h]$ is of the form

$$\sum_{|\alpha|=0,1} a_\alpha(x)\partial_x^\alpha,\quad \operatorname{supp} a_\alpha\subset\operatorname{supp} W_h,\quad \left|\partial_x^\beta a_\alpha(x)\right|\leq C_{\alpha\beta}\langle x\rangle^{-\mu+|\alpha|-|\beta|}.$$

The support properties of W_h and a_h^+ again imply

$$\left\|[H_0,W_h]A\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\leq C_M h^M\quad\text{for any } M\geq 0.$$

We next consider $[H_h,[K,W_h]]$ which has the form

$$\sum_{|\alpha|=1,2} b_\alpha(x)\partial_x^\alpha + W_1(x),$$

where b_α and W_1 are supported in $\operatorname{supp} W_h$ and satisfy

$$\left|\partial_x^\beta b_\alpha(x)\right| \le C_{\alpha\beta}\langle x\rangle^{-2-\mu+|\alpha|-|\beta|}, \quad \left|\partial_x^\beta W_1(x)\right| \le C_{\alpha\beta}\langle x\rangle^{2-2\mu}.$$

Setting $I_1 = \sum_{|\alpha|=1,2} b_\alpha(x)\partial_x^\alpha$ and $N_\mu := [1/\mu]+1$, we iterate this procedure N_μ times with W_h replaced by W_1. $(e^{-ithH} - e^{-ithH_h})A$ then can be brought to a linear combination of the following forms (modulo $O(h^M)$ on $L^2(\mathbb{R}^d)$):

$$\int_{t\ge s_1\ge\cdots\ge s_j\ge 0} e^{-i(t-s_1)hH} e^{-i(s_1-s_j)hH_h} I_{j/2} e^{-is_jhH_h} A ds_j \cdots ds_1$$

for $j=2m,\ m=1,2,\dots,N_\mu$, and

$$\int_{t\ge s_1\ge\cdots\ge s_{N_\mu}\ge 0} e^{-i(t-s_1)hH} e^{-i(s_1-s_{N_\mu})hH_h} W_{N_\mu} e^{-is_{N_\mu}hH_h} A ds_{2N_\mu} \cdots ds_1,$$

where I_k are second order differential operators with smooth and bounded coefficients, and W_{N_μ} is a bounded function since $2-2\mu N_\mu < 0$. Moreover, they are supported in $\{x; |x| > (\varepsilon h)^{-1}\}$. Therefore, it is sufficient to show that, for any $h\in(0,h_0]$, $0\le\tau\le h^{-1}$, $\alpha\in\mathbb{Z}_+^d$ and $M\ge 0$,

$$\left\|(1-\rho(\varepsilon hx))\partial_x^\alpha e^{-i\tau hH_h} A\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C_{M,\alpha} h^{M-|\alpha|}. \tag{3.4}$$

We now apply Theorem 3.1 to $e^{-i\tau hH_h}A$ and obtain

$$e^{-i\tau hH_h}A = U(S_h^+, b_h^+)e^{i\tau h\Delta/2}U(S_h^+, c_h^+)^* + Q_{\mathrm{IK}}^+(t,h,N).$$

Recall that the elliptic nature of H_0 implies, for every $s\ge 0$,

$$\left\|\langle D\rangle^s (h^2H_0+1)^{-s/2} f\right\|_{L^2(\mathbb{R}^d)} \le Ch^{-s}\|f\|_{L^2(\mathbb{R}^d)},$$
$$\left\|(h^2H_0+1)^{s/2}(h^2H_h+L)^{-s/2} f\right\|_{L^2(\mathbb{R}^d)} \le C\|f\|_{L^2(\mathbb{R}^d)},$$

if $L>0$ so large that $h^2H_h+L\ge 1$. Combining these estimates with (3.2), the remainder satisfies

$$\big\|\langle D\rangle^s Q^+_{\mathrm{IK}}(t,h,N)f\big\|_{L^2(\mathbb{R}^d)} \le C_{N,s}h^{N-1-s}\big\|f\big\|_{L^2(\mathbb{R}^d)}, \quad s \ge 0.$$

The main term can be handled in terms of the non-stationary phase method as follows. The distribution kernel of the main term is given by

$$(2\pi h)^{-d}(1-\rho(\varepsilon hx))\partial_x^\alpha \int e^{i\Phi_h^+(\tau,x,y,\xi)/h} b_h^+(x,\xi)\overline{c_h^+(y,\xi)}d\xi, \tag{3.5}$$

where $\Phi_h^+(\tau,x,y,\xi) = S_h^+(x,\xi) - (1/2)\tau|\xi|^2 - S_h^+(y,\xi)$. We here claim that

$$\operatorname{supp} c_h^+ \subset \big\{(x,\xi)\in\mathbb{R}^{2d}; a_h^+(x,\partial_\xi S_h^+(x,\xi)) \neq 0\big\}. \tag{3.6}$$

This property follows from the construction of $c_h^+ = \sum_{j=0}^N h^j c_{h,j}^+$. We set

$$\widetilde{S}_h^+(x,y,\xi) = \int_0^1 \partial_x S_h^+(y+\theta(x-y),\xi)d\theta.$$

Let $\xi \mapsto [\widetilde{S}_h^+]^{-1}(x,y,\xi)$ be the inverse map of $\xi \mapsto \widetilde{S}_h^+(x,y,\xi)$, and we denote their Jacobians by $A_1 = |\det \partial_\xi \widetilde{S}_h^+(x,y,\xi)|$ and $A_2 = |\det \partial_\xi [\widetilde{S}_h^+]^{-1}(x,y,\xi)|$. $c_{h,j}^+$ then satisfy the following triangular system:

$$\overline{c_{h,j}^+(x,\xi)} = b_{h,0}^+(x,\xi)^{-1}\big(r_{h,j}^+\big(x,\widetilde{S}_h^+(x,y,\xi)\big)A_1\big)\Big|_{y=x}, \quad j=0,1,\dots,N,$$

where $r_{h,0}^+ = a_h^+(x,\widetilde{S}_h^+(x,y,\xi))$ and, for each $j\ge 1$, $r_{h,j}^+$ is a linear combination of

$$\frac{1}{i^{|\alpha|}\alpha!}\big(\partial_\xi^\alpha\partial_y^\alpha b_{h,k_0}^+\big(x,[\widetilde{S}_h^+]^{-1}(x,y,\xi)\big)c_{h,k_1}^+\big(y,[\widetilde{S}_h^+]^{-1}(x,y,\xi)\big)A_2\big)\Big|_{y=x},$$

where $\alpha\in\mathbb{Z}_+^d$ and $k_0,k_1 = 0,1,\dots,j$ so that $0\le|\alpha|\le j$, $k_0+k_1 = j-|\alpha|$ and $k_1 \le j-1$. Therefore, we inductively obtain

$$\operatorname{supp} c_{h,0}^+ \subset \operatorname{supp} r_0^+|_{y=x}, \quad \operatorname{supp} c_{h,j}^+ \subset \operatorname{supp} c_{h,j-1}^+(h),\ j=1,2,\dots,N,$$

and (3.6) follows. In particular, c_h^+ vanishes in the region $\{x; |x|\ge h^{-1}\}$. By using (3.1), we have

$$\partial_\xi \Phi_h^+(\tau,x,y,\xi) = (x-y)(\mathrm{Id} + O(R^{-\mu/3})) - \tau\xi,$$

which implies

$$\left|\partial_\xi \Phi_h^+(\tau,x,y,\xi)\right| \ge \frac{|x|}{2} - |y| - |\tau\xi|$$

as long as $R \ge 1$ large enough. We now set $\varepsilon = (2\sqrt{\sup J_2} + 2)^{-1}$. Since $|x| > (\varepsilon h)^{-1}$, $|y| < h^{-1}$ and $|\xi|^2 \in J_2$ on the support of the amplitude, we have

$$\left|\partial_\xi \Phi_h^+(\tau,x,y,\xi)\right| \gtrsim (|x| + h^{-1}) > c(1+|x|+|y|+|\tau|), \quad 0 \le \tau \le h^{-1},$$

for some $c > 0$ independent of h. Therefore, integrating by parts (3.5) with respect to $-ih\left|\partial_\xi \Phi_h^+\right|^{-2}(\partial_\xi \Phi_h^+) \cdot \partial_\xi$, we obtain

$$\begin{aligned}&\left|(2\pi h)^{-d}(1-\rho(\varepsilon h x))\partial_x^\alpha \partial_y^\beta \int e^{i\Phi_h^+(\tau,x,y,\xi)/h} b_h^+(x,\xi)\overline{c_h^+(y,\xi)} d\xi\right| \\ &\quad \le C_{\alpha\beta M} h^{M-d-|\alpha+\beta|}(1+|x|+|y|+\tau)^{-M},\end{aligned}$$

for all $M \ge 0$, $0 \le \tau \le h^{-1}$ and $\alpha, \beta \in \mathbb{Z}_+^d$. (3.4) follows from this inequality and the L^2-boundedness of FIOs. □

4. WKB parametrix.

In the previous section we proved that e^{-ithH} is well approximated in terms of an Isozaki-Kitada parametrix on a time scale of order h^{-1} if we localize the initial data in regions $\Gamma^\pm(R,J,\sigma) \cap \{x; R < |x| < h^{-1}\}$. Therefore, it remains to control e^{-ithH} on a region $\{x; |x| \gtrsim h^{-1}\}$. In this section we construct the WKB parametrix for $e^{-ithH}a(x,hD)$, where $a \in S(1,g)$ with $\operatorname{supp} a \subset \{(x,\xi) \in \mathbb{R}^{2d}; |x| \gtrsim h^{-1},\ |\xi|^2 \in J\}$. In what follows we assume that H satisfies Assumption 1 with $\mu \ge 0$ and $\nu = 1$.

We first consider the phase function of the WKB parametrix, that is a solution to the time-dependent Hamilton-Jacobi equation generated by $p_h(x,\xi) = k(x,\xi) + h^2 V(x)$. For $R > 0$ and an open interval $J \Subset (0,\infty)$, we set

$$\Omega(R,J) := \{(x,\xi) \in \mathbb{R}^{2d}; |x| > R/2,\ |\xi|^2 \in J\}.$$

We note that $\Omega(R_1,J_1) \subset \Omega(R_2,J_2)$ if $R_1 > R_2$ and $J_1 \subset J_2$.

PROPOSITION 4.1. *Choose arbitrarily an open interval $J \Subset (0,\infty)$. Then, there exist $\delta_0 > 0$ and $h_0 > 0$ small enough such that, for all $h \in (0, h_0]$, $0 < R \le h^{-1}$ and $0 < \delta \le \delta_0$, we can construct a family of smooth functions*

$$\{\Psi_h(t,x,\xi)\}_{h\in(0,h_0]} \subset C^\infty((-\delta R, \delta R) \times \mathbb{R}^{2d})$$

such that $\Psi_h(t,x,\xi)$ satisfies the Hamilton-Jacobi equation associated to p_h:

$$\begin{cases} \partial_t \Psi_h(t,x,\xi) = -p_h(x, \partial_x \Psi_h(t,x,\xi)), & 0 < |t| < \delta R,\ (x,\xi) \in \Omega(R,J), \\ \Psi_h(0,x,\xi) = x\cdot\xi, & (x,\xi) \in \Omega(R,J). \end{cases} \tag{4.1}$$

Moreover, for all $|t| \le \delta R$ and $\alpha, \beta \in \mathbb{Z}_+^d$, $\Psi_h(t,x,\xi)$ satisfies

$$\left|\partial_x^\alpha \partial_\xi^\beta (\Psi_h(t,x,\xi) - x\cdot\xi)\right| \le C\delta R^{1-|\alpha|}, \qquad x,\xi \in \mathbb{R}^d,\ |\alpha+\beta| \ge 2, \tag{4.2}$$

$$\left|\partial_x^\alpha \partial_\xi^\beta (\Psi_h(t,x,\xi) - x\cdot\xi + tp_h(x,\xi))\right| \le C_{\alpha\beta}\delta R^{-|\alpha|}|t|, \quad x,\xi \in \mathbb{R}^d. \tag{4.3}$$

PROOF. We give the proof in Appendix A. □

We next define the corresponding FIO. Let $0 < R \le h^{-1}$, $J \Subset J_1 \Subset (0,\infty)$ open intervals and Ψ_h defined by the previous proposition with R, J replaced by $R/4, J_1$, respectively. We suppose that $\{a_h(t,\cdot,\cdot)\}_{h\in(0,h_0],0\le t\le\delta R}$ is bounded in $S(1,g)$ and supported in $\Omega(R,J)$, and consider the time-dependent FIO with the phase $\Psi_h(t)$ and amplitude $a_h(t)$, namely

$$U(\Psi_h(t), a_h(t))u(x) = \frac{1}{(2\pi h)^d}\int e^{i(\Psi_h(t,x,\xi)-y\cdot\xi)/h} a_h(t,x,\xi)u(y)dyd\xi.$$

LEMMA 4.2. *Let $\Psi_h(t)$ and $a_h(t)$ be as above. $U(\Psi_h(t),a(t))$ then is bounded on $L^2(\mathbb{R}^d)$ uniformly with respect to R, h and t:*

$$\sup_{h\in(0,h_0],0\le t\le\delta R} \left\|U(\Psi_h(t),a(t))\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C.$$

PROOF. For $|t| \le \delta R$, we define the map $\widetilde{\Xi}(t,x,y,\xi)$ on $\mathbb{R}^{3d}$ by

$$\widetilde{\Xi}(t,x,y,\xi) = \int_0^1 (\partial_x \Psi_h)(t, y+\lambda(x-y), \xi)d\lambda.$$

By (4.2), $\widetilde{\Xi}(t,x,y,\xi)$ satisfies

$$\big|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma(\widetilde{\Xi}(t,x,y,\xi)-\xi)\big| \le C_{\alpha\beta\gamma}\delta R^{-|\alpha+\beta|},\quad |t|\le\delta R,\ x,y\in\mathbb{R}^d,$$

and the map $\xi\mapsto\widetilde{\Xi}(t,x,\xi,y)$ hence is a diffeomorphism from $\mathbb{R}^d$ onto itself for all $|t|\le\delta R$ and $x,y\in\mathbb{R}^d$, provided that $\delta>0$ is small enough. Let $\xi\mapsto[\widetilde{\Xi}]^{-1}(t,x,y,\xi)$ be the corresponding inverse. $[\widetilde{\Xi}]^{-1}$ satisfies the same estimate as that for $\widetilde{\Xi}$:

$$\big|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma([\widetilde{\Xi}]^{-1}(t,x,y,\xi)-\xi)\big| \le C_{\alpha\beta\gamma}\delta R^{-|\alpha+\beta|}\quad\text{on}\quad[-\delta R,\delta R]\times\mathbb{R}^{3d}.$$

Using the change of variables $\xi\mapsto[\widetilde{\Xi}]^{-1}$, $U(\Psi_h(t),a(t))U(\Psi_h(t),a(t))^*$ can be regarded as a semi-classical PDO with a smooth and bounded amplitude

$$a_h\big(t,x,[\widetilde{\Xi}]^{-1}(t,x,y,\xi)\big)\overline{a_h(t,y,[\widetilde{\Xi}]^{-1}(t,x,y,\xi))}\big|\det\partial_\xi[\widetilde{\Xi}]^{-1}(t,x,y,\xi)\big|.$$

Therefore, the L^2-boundedness follows from the Calderón-Vaillancourt theorem. $\square$

We now state the main result in this section.

Theorem 4.3. *Let $J\Subset J_0\Subset J_1\Subset(0,\infty)$ be open intervals. Then there exist $\delta_0,h_0>0$ small enough such that, for all $h\in(0,h_0]$, $0<R\le h^{-1}$, $0<\delta\le\delta_0$, $N\ge0$ and all symbol $a\in S(1,g)$ with $\operatorname{supp}a\in\Omega(R,J)$, we can find a semi-classical symbol $b_h(t,x,\xi)=\sum_{j=0}^N h^jb_{h,j}(t,x,\xi)$ with*

$$\big\{b_{h,j}(t,\cdot,\cdot);h\in(0,h_0],0<R\le h^{-1},|t|\le\delta R\big\}\subset S(1,g)$$

and $\operatorname{supp}b_{h,j}(t,\cdot,\cdot)\subset\Omega(R/2,J_0)$ uniformly with respect to $h\in(0,h_0]$ and $|t|\le\delta R$, such that $e^{-ithH}a(x,hD_x)$ can be brought to the form

$$e^{-ithH}a(x,hD_x)=U(\Psi_h(t),b_h(t))+Q_{\mathrm{WKB}}(t,h,N),$$

where $U(\Psi_h(t),b_h(t))$ is the Fourier integral operator with the phase function $\Psi_h(t,x,\xi)$, defined in Proposition 4.1 with R,J replaced by $R/4,J_1$, respectively, and its distribution kernel satisfies the following bounds:

$$|K_{\mathrm{WKB}}(t,h,x,y)|\le C|th|^{-d/2},\quad h\in(0,h_0],\ 0<|t|\le\delta R,\ x,\xi\in\mathbb{R}^d.\tag{4.4}$$

Moreover the remainder $Q_{\mathrm{WKB}}(t,h,N)$ satisfies

$$\big\| Q_{\mathrm{WKB}}(t,h,N)\big\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C_N h^N |t|, \quad h\in(0,h_0],\ |t|\le \delta R.$$

Here the constants $C, C_N > 0$ can be taken uniformly with respect to h, t and R.

Remark 4.4. The essential point of Theorem 4.3 is to construct the parametrix on the time interval $|t|\le\delta R$. When $|t|>0$ is small and independent of R, such a parametrix construction is basically well known (see, e.g., [**19**]).

Proof of Theorem 4.3. We consider the case when $t\ge 0$ and the proof for $t<0$ is similar.

Construction of the amplitude. The Duhamel formula yields

$$\begin{aligned} &e^{-ithH}U(\Psi_h(0),b_h(0))\\ &\quad= U(\Psi_h(t),b_h(t)) + \frac{i}{h}\int_0^t e^{-i(t-s)hH}(hD_s + h^2H)U(\Psi_h(s),b_h(s))ds. \end{aligned}$$

Therefore, it suffices to show that there exist $b_{h,j}$ with $b_{h,0}|_{t=0}=a$ and $b_{h,j}|_{t=0}=0$ for $j\ge1$ such that

$$\big\|(hD_s+h^2H)U(\Psi_h(s),b_h(s))\big\|_{\mathcal{L}(L^2)} \le C_N h^{N+1}, \quad 0\le s\le \delta R. \tag{4.5}$$

Let $k+k_1$ be the full symbol of H_0: $H_0 = k(x,D)+k_1(x,D)$, and define a smooth vector field $\mathcal{X}_h(t)$ and a function $\mathcal{Y}_h(t)$ by

$$\mathcal{X}_h(t,x,\xi) := (\partial_\xi k)(x,\partial_x\Psi_h(t,x,\xi)),\quad \mathcal{Y}_h(t,x,\xi) := -(H_0\Psi_h)(t,x,\xi).$$

Symbols $\{b_{h,j}\}$ can be constructed in terms of the method of characteristics as follows. For all $0\le s,t\le\delta R$, we consider the flow $z_h(t,s,x,\xi)$ generated by $\mathcal{X}_h(t)$, that is the solution to the following ODE:

$$\partial_t z_h(t,s,x,\xi) = \mathcal{X}_h(z_h(t,s,x,\xi),\xi);\quad z_h(s,s)=x.$$

Choose R', R'' and two intervals J_0', J_0'' so that

$$R/2 > R' > R'' > R/4,\quad J_0 \Subset J_0' \Subset J_0'' \Subset (0,\infty).$$

(4.3) and the same argument as that in the proof of Lemmas A.1 and A.2 imply that there exists $\delta_0, h_0>0$ small enough such that, for all $0<\delta\le\delta_0$, $h\in(0,h_0]$ $0<R\le h^{-1}$ and $0\le s,t\le\delta R$, $z_h(t,s)$ is well defined on $\Omega(R'',J_0'')$ and satisfies

$$\left|\partial_x^\alpha \partial_\xi^\beta (z_h(t,s,x,\xi)-x)\right| \le C_{\alpha\beta}\delta R^{1-|\alpha|}. \tag{4.6}$$

In particular, $(z_h(t,s,x,\xi),\xi) \in \Omega(R',J')$ for $0 \le s,t \le \delta R$ if $\delta > 0$, depending only on J'', is small enough. We now define $\{b_{h,j}(t,x,\xi)\}_{0\le j\le N}$ inductively by

$$b_{h,0}(t,x,\xi) = a(z_h(0,t),\xi)\exp\biggl(\int_0^t \mathcal{Y}_h(s,z_h(s,t,x,\xi),\xi)ds\biggr),$$

$$b_{h,j}(t,x,\xi) = -\int_0^t (iH_0 b_{h,j-1})(s,z_h(s,t),\xi)\exp\biggl(\int_u^t \mathcal{Y}_h(u,z_h(u,t,x,\xi),\xi)du\biggr)ds.$$

Since $\operatorname{supp} a \in \Omega(R,J)$ and $z_h(t,s,\Omega(R,J)) \subset \{x; |x| > R/2\}$ for all $0 \le s,t \le \delta R$, $b_{h,j}(t)$ are supported in $\Omega(R/2,J_0)$. Thus, if we extend $b_{h,j}$ on $\mathbb{R}^{2d}$ so that

$$b_{h,j}(t,x,\xi) = 0, \quad (x,\xi) \notin \Omega(R/2,J_0),$$

then $b_{h,j}$ is still smooth in (x,ξ). By (4.3) and (4.6), we learn

$$\left|\partial_x^\alpha \partial_\xi^\beta \mathcal{Y}_h(s,z_h(s,t,x,\xi),\xi)\right| \le C\delta R^{-1-|\alpha|}, \quad 0 \le s,t \le \delta R.$$

$\{b_{h,j}(t,\cdot,\cdot); h \in (0,h_0],\ 0 < R \le h^{-1},\ t \in [0,\delta R],\ 0 \le j \le N\}$ thus is a bounded set in $S(1,g)$ and $\operatorname{supp} b_{h,j}(t,\cdot,\cdot) \subset \Omega(R/2,J_0)$ uniformly with respect to $h \in (0,h_0]$ and $0 \le t \le \delta R$. A standard Hamilton-Jacobi theory shows that $b_{h,j}(t)$ satisfy the following transport equations:

$$\begin{cases} \partial_t b_{h,0}(t) + \mathcal{X}_h(t)\cdot\partial_x b_{h,0}(t) + \mathcal{Y}_h(t) b_{h,0}(t) = 0, \\ \partial_t b_{h,j}(t) + \mathcal{X}_h(t)\cdot\partial_x b_{h,j}(t) + \mathcal{Y}_h(t) b_{h,j}(t) = -iH_0 b_{h,j-1}(t), \quad j \ge 1, \end{cases} \tag{4.7}$$

with the initial condition $b_{h,0}(0) = a$, $b_{h,j}(0) = 0$, $j = 1,2,\dots,N$. A direct computation then yields

$$e^{-i\Psi_h(s,x,\xi)/h}(hD_s + h^2 H)\biggl(e^{i\Psi_h(s,x,\xi)/h}\sum_{j=0}^N h^j b_{h,j}\biggr) = O(h^{N+1}) \text{ in } S(1,g)$$

which, combined with Lemma 4.2, implies (4.5).

Dispersive estimates. The distribution kernel of $U(\Psi_h(t),b_h(t))$ is given by

$$K_{\mathrm{WKB}}(t,h,x,y)=\frac{1}{(2\pi h)^d}\int e^{(i/h)(\Psi_h(t,x,\xi)-y\cdot\xi)}b_h(t,x,\xi)d\xi.$$

Since $b_h(t,x,\xi)$ has a compact support with respect to ξ,

$$|K_{\mathrm{WKB}}(t,h,x,y)|\le Ch^{-d}\le C|th|^{-d/2}\quad\text{for}\quad 0<t\le h.$$

We hence assume $h<t$ without loss of generality. Choose $\chi\in S(1,g)$ so that $0\le\chi\le 1$, $\chi\equiv 1$ on $\Omega(R/2,J_0)$ and $\operatorname{supp}\chi\subset\Omega(R/4,J_1)$, and set

$$\psi_h(t,x,y,\xi)=\frac{(x-y)}{t}\cdot\xi-p_h(x,\xi)+\chi(x,\xi)\biggl(\frac{\Psi_h(t,x,\xi)-x\cdot\xi}{t}+p_h(x,\xi)\biggr).$$

By the definition, we obtain

$$\psi_h(t,x,y,\xi)=\frac{\Psi_h(t,x,\xi)-y\cdot\xi}{t},\quad t\in[h,\delta R],\ (x,\xi)\in\Omega(R/2,J_1),\ y\in\mathbb{R}^d,$$

and (4.3) implies

$$\bigl|\partial_x^\alpha\partial_\xi^\beta\psi_h(t,x,y,\xi)\bigr|\le C_{\alpha\beta}\quad\text{on}\quad[0,\delta R]\times\mathbb{R}^{3d},\quad|\alpha+\beta|\ge 2.$$

Moreover, $\partial_\xi^2\psi_h(t,x,y,\xi)$ can be brought to the form

$$\partial_\xi^2\psi_h(t,x,y,\xi)=-(a^{jk}(x))_{j,k}+Q_h(t,x,\xi),$$

where the error term $Q_h(t,x,\xi)$ is a $d\times d$-matrix satisfying

$$\bigl|\partial_x^\alpha\partial_\xi^\beta Q_h(t,x,\xi)\bigr|\le C_{\alpha\beta}\delta h^{|\alpha|}\quad\text{on}\quad[0,\delta R]\times\mathbb{R}^{2d}.$$

Since $(a^{jk}(x))$ is uniformly elliptic, the stationary phase theorem implies

$$|K_{\mathrm{WKB}}(t,h,x,y)|\le Ch^{-d}|t/h|^{-d/2}=C|th|^{-d/2},\quad 0<t\le\delta R,$$

provided that $\delta>0$ is small enough. We complete the proof. □

5. Proof of Theorem 1.1.

In this section we complete the proof of Theorem 1.1.

Proof of Theorem 1.1 (i). Let $\chi_0 \in C_0^\infty(\mathbb{R}^d)$ with $\chi_0 \equiv 1$ on $\{|x| < R_0\}$ and $\psi \in C_0^\infty((0,\infty))$. A partition of unity argument and Lemma 2.1 show that there exist $a^\pm \in S(1,g)$ with $\operatorname{supp} a^\pm \subset \Gamma^\pm(R_0, J, 1/2)$ such that $(1-\chi_0)\psi(h^2 H_0)$ is approximated in terms of $a^\pm(x, hD)$:

$$(1-\chi_0)\psi(h^2 H_0) = a^+(x,hD)^* + a^-(x,hD)^* + Q_0(h),$$

where $J \Subset (0,\infty)$ is an open interval with $\pi_\xi(\operatorname{supp}\varphi \circ k) \Subset J$, and $Q_0(h)$ satisfies

$$\sup_{h\in(0,1]} \left\|Q_0(h)\right\|_{\mathcal{L}(L^2(\mathbb{R}^d), L^q(\mathbb{R}^d))} \le C_q,$$

for any $q \ge 2$. Let $b \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ be a cut-off function such that $b \equiv 1$ on a neighborhood of J. By the asymptotic formula (2.1), we can write

$$a^\pm(x,hD)^* = b(hD)a^\pm(x,hD)^* + Q_1(h)$$

where $Q_1(h)$ satisfies the same $\mathcal{L}(L^2, L^q)$-estimate as that of $Q_0(h)$. Therefore,

$$\left\|(Q_0(h)+Q_1(h))e^{-itH}u_0\right\|_{L^p([-\delta,\delta];L^q(\mathbb{R}^d))} \le C\left\|u_0\right\|_{L^2(\mathbb{R}^d)}, \quad h\in(0,1], \tag{5.1}$$

for any $p, q \ge 2$. Next, we shall prove the following estimate for the main terms:

$$\begin{aligned} &\left\|b(hD)a^\pm(x,hD)^* e^{-i(t-s)H} a^\pm(x,hD)b(hD)\right\|_{\mathcal{L}(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))} \\ &\quad \le C|t-s|^{-d/2} \end{aligned} \tag{5.2}$$

for $0 < |t-s| \le \delta$. We first consider the outgoing case. Let us fix $N > 1$ so large that $N \ge 2d+1$. After rescaling $t-s \mapsto (t-s)h$ and choosing $R_0 > 1$ large enough, we apply Theorem 3.1 with $R = R_0$, Lemma 3.2 and Theorem 4.3 with $R = h^{-1}$ to $e^{-i(t-s)hH}a^+(x,hD)$. Then, we can write

$$\begin{aligned} &e^{-i(t-s)hH}a^+(x,hD) \\ &\quad = U(S_h^+, b_h^+)e^{i(t-s)h\Delta/2}U(S_h^+, c_h^+)^* + U(\Psi_h(t-s), b_h(t-s)) + Q_2^+(t-s,h), \end{aligned}$$

where the distribution kernels of main terms satisfy dispersive estimates

$$\left|K_{\mathrm{IK}}^+(t-s,h,x,y)\right| + \left|K_{\mathrm{WKB}}(t-s,h,x,y)\right| \le C|(t-s)h|^{-d/2}, \tag{5.3}$$

uniformly with respect to $h \in (0, h_0]$, $0 < t - s \le \delta h^{-1}$ and $x, y \in \mathbb{R}^d$. Let $A(h, x, y)$ and $B(h, x, y)$ be the distribution kernels of $a(x, hD)^*$ and $b(hD)$, respectively. They clearly satisfy

$$\sup_x \int (|A(h,x,y)| + |B(h,x,y)|)dy + \sup_y \int (|A(h,x,y)| + |B(h,x,y)|)dx \le C$$

uniformly in $h \in (0, 1]$. By using this estimate and (5.3), we see that the distribution kernel of

$$b(hD)a^+(x, hD)^*\big(e^{-i(t-s)hH}a^+(x, hD) - Q_2^+(t - s, h)\big)b(hD)$$

satisfies the same dispersive estimates as (5.3) for $0 < t - s \le \delta h^{-1}$. On the other hand, $Q_2^+(t - s, h)$ satisfy

$$\big\|Q_2^+(t - s, h)\big\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \le C_N h^N, \quad h \in (0, h_0],\ 0 \le t - s \le \delta h^{-1}.$$

We here recall that $a^+(x, hD)^*$ is uniformly bounded on $L^2(\mathbb{R}^d)$ in $h \in (0, 1]$ and $b(hD)$ satisfies

$$\begin{aligned}
&\big\|b(hD)\big\|_{\mathcal{L}(H^{-s}(R^d), H^s(\mathbb{R}^d))}\\
&\quad\le \big\|\langle D\rangle^s\langle hD\rangle^{-s}\big\|_{\mathcal{L}(L^2(R^d))}\big\|\langle hD\rangle^s b(hD)\langle hD\rangle^s\big\|_{\mathcal{L}(L^2(R^d))}\big\|\langle hD\rangle^{-s}\langle D\rangle^s\big\|_{\mathcal{L}(L^2(R^d))}\\
&\quad\le C_s h^{-2s}.
\end{aligned}$$

$b(hD)a^+(x, hD)^*Q_2^+(t-s, h)b(hD)$ hence is a bounded operator in $\mathcal{L}(H^{-s}, H^s)$ for some $s > d/2$ and has the uniformly bounded distribution kernel $\widetilde{Q}_2^+(t - s, h, x, y)$ with respect to $h \in (0, h_0]$ and $0 \le t - s \le \delta h^{-1}$. Therefore,

$$\big|\widetilde{Q}_2^+(t - s, h, x, y)\big| \lesssim 1 \lesssim |(t - s)h|^{-d/2}, \quad h \in (0, h_0],\ 0 < t - s \le \delta h^{-1}.$$

The corresponding estimates for the incoming case also hold for $0 \le -(t - s) \le \delta h^{-1}$. Therefore, $b(hD)a^\pm(x, hD)^*e^{-i(t-s)hH}a^\pm(x, hD)b(hD)$ have distribution kernels $K^\pm(t - s, h, x, y)$ satisfying

$$\big|K^\pm(t - s, h, x, y)\big| \le C|(t - s)h|^{-d/2} \tag{5.4}$$

uniformly with respect to $h \in (0, h_0]$, $0 \le \pm(t - s) \le \delta h^{-1}$ and $x, y \in \mathbb{R}^d$, respectively.

We here use a simple trick due to Bouclet-Tzvetkov [**2**, Lemma 4.3]. If we set $U^{\pm}(t,h) = b(hD)a^{\pm}(x,hD)^* e^{-ithH} a^{\pm}(x,hD)b(hD)$, then

$$U^{\pm}(s-t,h) = U^{\pm}(t-s,h)^*,$$

and hence $K^{\pm}(s-t,h,x,y) = \overline{K^{\pm}(t-s,h,y,x)}$. Therefore, the estimates (5.4) also hold for $0 < \mp(t-s) \le \delta h^{-1}$ and $x,y \in \mathbb{R}^d$. Rescaling $(t-s)h \mapsto t-s$, we obtain the estimate (5.2).

Finally, since the $\mathcal{L}(L^2)$-boundedness of $a^{\pm}(x,hD)^* e^{-itH}$ is obvious, (5.1), (5.2) and the Keel-Tao theorem [**15**] imply the desired semi-classical Strichartz estimates:

$$\sup_{h\in(0,h_0]} \|(1-\chi_0)\psi_0(h^2H_0)e^{-itH}u_0\|_{L^p([-\delta,\delta];L^q(\mathbb{R}^d))} \le C\|u_0\|_{L^2(\mathbb{R}^d)}.$$

By the virtue of Proposition 2.4, we complete the proof. $\square$

We next give the proof of (ii). Suppose that H satisfies Assumption 1 with $\mu,\nu \ge 0$. We first recall the local smoothing effects for Schrödinger operators with at most quadratic potentials proved by Doi [**9**]. For any $s \in \mathbb{R}$, we set $\mathcal{B}^s := \{f \in L^2(\mathbb{R}^d); \langle x\rangle^s f \in L^2(\mathbb{R}^d), \langle D\rangle^s f \in L^2(\mathbb{R}^d)\}$, and define a symbol e_s by

$$e_s(x,\xi) := (k(x,\xi) + |x|^2 + L(s))^{s/2} \in S((1+|x|+|\xi|)^s, g).$$

We denote by E_s its Weyl quantization:

$$E_s f(x) = \frac{1}{2\pi}\int e^{i(x-y)\cdot\xi} e_s\left(\frac{x+y}{2},\xi\right) f(y)dyd\xi.$$

Here $L(s) > 1$ is a large constant depending on s. Then, for any $s \in \mathbb{R}$, there exists $L(s) > 0$ such that E_s is a homeomorphism from $\mathcal{B}^{r+s}$ to $\mathcal{B}^r$ for all $r \in \mathbb{R}$, and $(E_s)^{-1}$ is still a Weyl quantization of a symbol in $S((1+|x|+|\xi|)^{-s}, g)$.

Lemma 5.1 (The local smoothing effects [**9**]). *Suppose that the kinetic energy $k(x,\xi)$ satisfies the non-trapping condition* (1.5). *Then, for any $T > 0$ and $\sigma > 0$, there exists $C_{T,\sigma} > 0$ such that*

$$\|\langle x\rangle^{-1/2-\sigma} E_{1/2} u\|_{L^2([-T,T];L^2(\mathbb{R}^d))} \le C_{T,\sigma}\|u_0\|_{L^2}, \tag{5.5}$$

where $u = e^{-itH}u_0$.

REMARK 5.2. Let $\chi \in C_0^\infty(\mathbb{R}^d)$. (5.5) implies a usual local smoothing effect:

$$\left\| \langle D \rangle^{1/2} \chi u \right\|_{L^2([-T,T];L^2(\mathbb{R}^d))} \le C_T \left\| u_0 \right\|_{L^2(\mathbb{R}^d)}. \tag{5.6}$$

Indeed, let $\chi_1 \in C_0^\infty(\mathbb{R}^d)$ be such that $\chi_1 \equiv 1$ on $\operatorname{supp}\chi$. We split $\langle D \rangle^{1/2}\chi$ as follows:

$$\begin{aligned}
\langle D \rangle^{1/2}\chi &= \chi_1 \langle D \rangle^{1/2}\chi + \left[\langle D \rangle^{1/2}, \chi_1\right]\chi, \\
\chi_1 \langle D \rangle^{1/2}\chi &= \chi_1 \langle D \rangle^{1/2} (E_{1/2})^{-1} E_{1/2}\chi \\
&= \chi_1 \langle D \rangle^{1/2} (E_{1/2})^{-1} \chi_1 E_{1/2}\chi + \chi_1 \langle D \rangle^{1/2} (E_{1/2})^{-1} [E_{1/2}, \chi_1]\chi.
\end{aligned}$$

By a standard symbolic calculus, $[\langle D \rangle^{1/2}, \chi_1]\chi$, $\chi_1 \langle D \rangle^{1/2} (E_{1/2})^{-1}$ and $[E_{1/2}, \chi_1]\chi$ are bounded on $L^2(\mathbb{R}^d)$ since χ_1 has a compact support. Therefore, Lemma 5.1 implies

$$\begin{aligned}
\left\| \langle D \rangle^{1/2} \chi u \right\|_{L^2([-T,T];L^2(\mathbb{R}^d))} &\le C \left\| \chi_1 E_{1/2} \chi u \right\|_{L^2([-T,T];L^2(\mathbb{R}^d))} + C_T \left\| u \right\|_{L^2(\mathbb{R}^d)} \\
&\le C_T \left\| u_0 \right\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

PROOF OF THEOREM 1.1 (ii). We consider the case when $0 \le t \le T$ only, and the proof for the negative time is similar. We mimic the argument in [**18**, Section II. 2]. A direct computation yields

$$\begin{aligned}
(i\partial_t + \Delta)\chi u &= \Delta \chi u + \chi H u \\
&= \chi_1 (H + \Delta) \chi_1 \chi u + (\chi_1 [\chi, H] + [\Delta, \chi_1]\chi) u.
\end{aligned}$$

We define a self-adjoint operator by $\widetilde{H} := -\Delta + \chi_1 (H + \Delta)\chi_1$, and set

$$\widetilde{U}(t) := e^{-it\widetilde{H}}, \quad F := (\chi_1[\chi, H] + [\Delta, \chi_1]\chi) u.$$

We here note that if H_0 satisfies the non-trapping condition then so does the principal part of $\widetilde{H}$. By the Duhamel formula, we can write

$$\chi u = \widetilde{U}(t)\chi u_0 + \int_0^t \widetilde{U}(t-s) F(s) ds.$$

Since $\chi_1 (H + \Delta)\chi_1$ is a compactly supported smooth perturbation, it was proved by Staffilani-Tataru [**22**] that $\widetilde{U}(t)$ is bounded from $L^2(\mathbb{R}^d)$ to $L^2([0,T]; H^{1/2}_{loc}(\mathbb{R}^d))$, and that its adjoint

$$\widetilde{U}^* f = \int_0^T U(-s) f(s,\cdot) ds$$

is bounded from $L^2([0,T]; H^{-1/2}_{loc}(\mathbb{R}^d))$ to $L^2(\mathbb{R}^d)$. Moreover, $\widetilde{U}(t)$ satisfies Strichartz estimates (for any admissible pair (p,q)):

$$\big\|\widetilde{U}(t)v\big\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \le C_T \big\|v\big\|_{L^2}.$$

Therefore, we have

$$\begin{aligned}\Big\| \int_0^T \widetilde{U}(t-s)F(s)ds \Big\|_{L^p([-T,T];L^q(\mathbb{R}^d))} &\le C_T \big\|\widetilde{U}^* F\big\|_{L^2(\mathbb{R}^d)} \\ &\le C_T \big\|\langle D\rangle^{-1/2} F\big\|_{L^2([-T,T];L^2(\mathbb{R}^d))}\end{aligned}$$

since F has a compact support with respect to x. The Christ-Kiselev lemma (see [**7**], [**21**]) then implies

$$\Big\| \int_0^t \widetilde{U}(t-s)F(s)ds \Big\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \le C_T \big\|\langle D\rangle^{-1/2} F\big\|_{L^2([-T,T];L^2(\mathbb{R}^d))},$$

provided that $p > 2$. We split F as

$$F = ([\chi, H]\chi_1 + [\Delta, \chi_1]\chi)u + [\chi_1, [\chi, H]]u =: F_1 + F_2.$$

Since $[\chi, H]$ is a first order differential operator with bounded coefficients, we see that $[\chi_1, [\chi, H]]$ is bounded on $L^2(\mathbb{R}^d)$, and $\big\|\langle D\rangle^{-1/2} F_2\big\|_{L^2([-T,T];L^2(\mathbb{R}^d))}$ is dominated by $C_T \big\|u_0\big\|_{L^2(\mathbb{R}^d)}$. We now use (5.6) and obtain

$$\begin{aligned}\big\|\langle D\rangle^{-1/2} F_1\big\|_{L^2([-T,T];L^2(\mathbb{R}^d))} &\le C \big\|\chi_1 u\big\|_{L^2([-T,T];H^{-1/2}(\mathbb{R}^d))} \\ &\le C \big\|\langle D\rangle^{1/2} \chi_1 u\big\|_{L^2([-T,T];L^2(\mathbb{R}^d))} \\ &\le C_T \big\|u_0\big\|_{L^2},\end{aligned}$$

which completes the proof. □

A. Proof of Proposition 4.1.

Assume Assumption 1 with $\mu, \nu \ge 0$. We here give the detail of the proof of Proposition 4.1. We first study the corresponding classical mechanics. Let $h \in (0,1]$ and consider the Hamilton flow $(X_h(t), \Xi_h(t)) = (X_h(t,x,\xi), \Xi_h(t,x,\xi))$ generated by the semi-classical total energy

$$p_h(x,\xi) = k(x,\xi) + h^2 V(x),$$

i.e., $(X_h(t), \Xi_h(t))$ is the solution to the Hamilton equations

$$\begin{cases} \dot{X}_{h,j}(t) = \displaystyle\sum_k a^{jk}(X_h(t))\Xi_{h,k}(t), \\ \dot{\Xi}_{h,j}(t) = -\dfrac{1}{2}\displaystyle\sum_{k,l} \dfrac{\partial a^{kl}}{\partial x_j}(X_h(t))\Xi_{h,k}(t)\Xi_{h,l}(t) - h^2 \dfrac{\partial V}{\partial x_j}(X_h(t)), \end{cases}$$

with the initial condition $(X_h(0), \Xi_h(0)) = (x,\xi)$, where $\dot{f} = \partial_t f$. We first prepare an a priori bound of the flow.

Lemma A.1. *For all $h \in (0,1]$, $|t| \lesssim h^{-1}$ and $(x,\xi) \in \mathbb{R}^{2d}$,*

$$|X_h(t) - x| \lesssim \big(|\xi| + h\langle x\rangle^{1-\nu/2}\big)|t|, \quad |\Xi_h(t)| \lesssim |\xi| + h\langle x\rangle^{1-\nu/2}.$$

Proof. We consider the case $t \ge 0$. The proof for the case $t < 0$ is analogous. Since the Hamilton flow conserves the total energy, namely

$$p_h(x,\xi) = p_h(X_h(t), \Xi_h(t)) \quad \text{for all} \quad t \in \mathbb{R},$$

we have

$$\begin{aligned} |\Xi_h(t)| &\lesssim \sqrt{p_0(X_h(t), \Xi_h(t))} \\ &\lesssim \sqrt{p_h(x,\xi) - h^2 V(X_h(t))} \\ &\lesssim |\xi| + h\langle x\rangle^{1-\nu/2} + h\langle X_h(t)\rangle^{1-\nu/2}. \end{aligned}$$

Applying the above inequality to the Hamilton equation, we have

$$|\dot{X}^h(t)| \lesssim |\Xi_h(t)| \lesssim |\xi| + h\langle x\rangle^{1-\nu/2} + h|X_h(t) - x|.$$

Integrating with respect to t and using Gronwall's inequality, we obtain the assertion since $e^{th} \lesssim |t|$ for $|t| \lesssim h^{-1}$. $\square$

Let $J \Subset (0,\infty)$ be an open interval. For sufficiently small $\delta > 0$ and for all $0 < R \le h^{-1}$, the above lemma implies

$$|x|/2 \le |X_h(t,x,\xi)| \le 2|x| \tag{A.1}$$

uniformly with respect to $h \in (0,1]$, $|t| \le \delta R$ and $(x,\xi) \in \Omega(R,J)$. By using this inequality, we have the following:

Lemma A.2. *Let J, δ be as above. Then, for $h \in (0,1]$, $0 < R \le h^{-1}$, $|t| \le \delta R$ and $(x,\xi) \in \Omega(R,J)$, $X_h(t,x,\xi)$ and $\Xi_h(t,x,\xi)$ satisfy*

$$\begin{cases} |X_h(t)-x| \le C(1+\delta h\langle x\rangle^{1-\nu})|t|, \\ |\Xi_h(t)-\xi| \le C(\langle x\rangle^{-1} + h^2\langle x\rangle^{1-\nu})|t|, \end{cases} \tag{A.2}$$

and, for $|\alpha+\beta| = 1$,

$$\begin{cases} \left|\partial_x^\alpha \partial_\xi^\beta (X_h(t)-x)\right| \le C_{\alpha\beta}\big(\langle x\rangle^{-|\alpha|} + h^{|\alpha|}\langle x\rangle^{-|\alpha|\nu/2}\big)|t|, \\ \left|\partial_x^\alpha \partial_\xi^\beta (\Xi_h(t)-\xi)\right| \le C_{\alpha\beta}\big(\langle x\rangle^{-1-|\alpha|} + h^{1+|\alpha|}\langle x\rangle^{-(1+|\alpha|)\nu/2}\big)|t|, \end{cases} \tag{A.3}$$

and, for $|\alpha+\beta| \ge 2$,

$$\begin{cases} \left|\partial_x^\alpha \partial_\xi^\beta (X_h(t)-x)\right| \le C_{\alpha\beta}\delta h^{|\alpha|}\langle x\rangle^{-1}R|t|, \\ \left|\partial_x^\alpha \partial_\xi^\beta (\Xi_h(t)-\xi)\right| \le C_{\alpha\beta} h^{|\alpha|}\langle x\rangle^{-1}|t|. \end{cases} \tag{A.4}$$

Moreover $C, C_{\alpha\beta} > 0$ may be taken uniformly with respect to R, h and t.

Proof. We only prove the case when $t \ge 0$, the proof for the case $t \le 0$ is similar. Applying Lemma A.1 and (A.1) to the Hamilton equation, we have

$$\begin{aligned} |\dot{\Xi}^h(t)| &\lesssim \langle X_h(t)\rangle^{-1}|\Xi_h(t)|^2 + h^2\langle X_h(t)\rangle^{1-\nu} \\ &\lesssim \langle x\rangle^{-1}(1+h^2\langle x\rangle^{2-\nu}) + h^2\langle x\rangle^{1-\nu} \\ &\lesssim \langle x\rangle^{-1} + h^2\langle x\rangle^{1-\nu}, \end{aligned}$$

$$|\dot{X}^h(t)| \lesssim |\Xi_h(t)| \lesssim 1 + \delta h\langle x\rangle^{1-\nu},$$

and (A.2) follows.

We next prove (A.3). By differentiating the Hamilton equation with respect to $\partial_x^\alpha \partial_\xi^\beta$, $|\alpha+\beta| = 1$, we have

$$\frac{d}{dt}\begin{pmatrix} \partial_x^\alpha \partial_\xi^\beta X_h \\ \partial_x^\alpha \partial_\xi^\beta \Xi_h \end{pmatrix} = \begin{pmatrix} \partial_x \partial_\xi p_h(X_h, \Xi_h) & \partial_\xi^2 p_h(X_h, \Xi_h) \\ -\partial_x^2 p_h(X_h, \Xi_h) & -\partial_\xi \partial_x p_h(X_h, \Xi_h) \end{pmatrix} \begin{pmatrix} \partial_x^\alpha \partial_\xi^\beta X_h \\ \partial_x^\alpha \partial_\xi^\beta \Xi_h \end{pmatrix}. \tag{A.5}$$

Define a weight function $w_h(x) = \langle x\rangle^{-1} + h\langle x\rangle^{-\nu/2}$. A direct computation and (A.2) then imply

$$\begin{aligned} &\big|(\partial_x^\alpha \partial_\xi^\beta p_h)(X_h(t), \Xi_h(t))\big| \le C_{\alpha\beta} w_h(x)^{|\alpha|}, && |\alpha+\beta| = 2, \\ &\big|(\partial_x^\alpha \partial_\xi^\beta p_h)(X_h(t), \Xi_h(t))\big| \le C_{\alpha\beta} \langle x\rangle^{2-|\alpha+\beta|} w_h(x)^{|\alpha|-1}, && |\alpha+\beta| \ge 3, \end{aligned}$$

for all $|t| \le \delta R$ and $(x,\xi) \in \Omega(R,J)$, and $\partial_\xi^\beta p_h \equiv 0$ on $\mathbb{R}^{2d}$ for $|\beta| \ge 3$. By integrating (A.5) with respect to t, we have

$$\begin{aligned} &w_h(x)\big|\partial_x^\alpha \partial_\xi^\beta (X_h(t) - x)\big| + \big|\partial_x^\alpha \partial_\xi^\beta (\Xi_h(t) - \xi)\big| \\ &\quad \lesssim \int_0^t \Big(w_h(x)\big(w_h(x)\big|\partial_x^\alpha \partial_\xi^\beta (X_h(t) - x)\big| + \big|\partial_x^\alpha \partial_\xi^\beta (\Xi_h(t) - \xi)\big|\big) + w_h(x)^{1+|\alpha|}\Big) d\tau. \end{aligned}$$

Using Gronwall's inequality, we have (A.3) since $|t| \le \delta R$.

For $|\alpha+\beta| \ge 2$, we shall prove the estimate for $\partial_{\xi_1}^2 X_h(t)$ only. Proofs for other cases are similar, and for higher derivatives follow from an induction on $|\alpha+\beta|$. By the Hamilton equation and (A.3), we learn

$$\partial_{\xi_1}^2 X_h = \partial_x \partial_\xi p_h(X_h, \Xi_h) \partial_{\xi_1}^2 X_h + \partial_\xi^2 p_h(X_h, \Xi_h) \partial_{\xi_1}^2 \Xi_h + Q(h, x, \xi)$$

where $Q(h,x,\xi)$ satisfies

$$\begin{aligned} Q(h,x,\xi) &\le C \sum_{|\alpha+\beta|=3, |\beta|=1,2} \big(\partial_x^\alpha \partial_\xi^\beta p\big)(X_h, \Xi_h)(\partial_{\xi_1} X_h)^{|\alpha|}(\partial_{\xi_1} \Xi_h)^{|\beta|} \\ &\le C\langle x\rangle^{-1} \sum_{|\alpha|=1,2,3} w_h(x)^{|\alpha|-1} |t|^{|\alpha|} \\ &\le C\delta\langle x\rangle^{-1} R. \end{aligned}$$

We similarly obtain

$$\partial_{\xi_1}^2 \Xi_h = -\partial_x^2 p_h(X_h, \Xi_h)\partial_{\xi_1}^2 X_h - \partial_\xi \partial_x p_h(X_h, \Xi_h)\partial_{\xi_1}^2 \Xi_h + O(\langle x\rangle^{-1}),$$

and these estimates and Gronwall's inequality imply

$$\begin{aligned}
&(\delta R)^{-1}\big|\partial_{\xi_1}^2 X_h(t)\big| + \big|\partial_{\xi_1}^2 \Xi_h(t)\big| \\
&\lesssim \int_0^t w_h(x)\big((\delta R)^{-1}\big|\partial_{\xi_1}^2 X_h(t)\big| + \big|\partial_{\xi_1}^2 \Xi_h(t)\big|\big) + \langle x\rangle^{-1} d\tau \\
&\lesssim \langle x\rangle^{-1}|t|
\end{aligned}$$

for $0 \le t \le \delta R$. We hence have the assertion. $\square$

Remark A.3. If $\nu \ge 1$, then Lemma A.2 implies that for any $\alpha, \beta \in \mathbb{Z}_+^d$, there exists $C_{\alpha\beta}$ such that

$$\big|\partial_x^\alpha \partial_\xi^\beta (X_h(t) - x)\big| \le C_{\alpha\beta}\delta R^{1-|\alpha|}, \quad \big|\partial_x^\alpha \partial_\xi^\beta (\Xi_h(t) - \xi)\big| \le C_{\alpha\beta}\delta R^{-|\alpha|}, \tag{A.6}$$

uniformly with respect to $h \in (0,1]$, $0 < R \le h^{-1}$, $|t| \le \delta R$ and $(x,\xi) \in \Omega(R,J)$.

Lemma A.4. *Suppose that $\nu = 1$ and let $J_1 \Subset J_1' \Subset (0,\infty)$ be open intervals. Then there exists $\delta > 0$ small enough such that, for any fixed $|t| \le \delta R$, the map*

$$g_h(t) : (x,\xi) \mapsto (X_h(t,x,\xi), \xi)$$

is a diffeomorphism from $\Omega(R/2, J_1')$ onto its range. Moreover, we have

$$\Omega(R, J_1) \subset g^h(t, \Omega(R/2, J_1')), \quad |t| \le \delta R. \tag{A.7}$$

Proof. We choose J_1'' so that $J_1' \Subset J_1'' \Subset (0,\infty)$. Choosing $\chi \in S(1,g)$ such that

$$0 \le \chi \le 1, \quad \operatorname{supp}\chi \subset \Omega(R/3, J_1''), \quad \chi \equiv 1 \text{ on } \Omega(R/2, J_1'),$$

we define $X_h^\chi(t,x,\xi) := (1-\chi(x,\xi))x + \chi(x,\xi)X_h(t,x,\xi)$ and set

$$g_h^\chi(t,x,\xi) = (X_h^\chi(t,x,\xi), \xi).$$

We also define $(z,\xi) \mapsto \tilde{g}_h^\chi(t,z,\xi)$ by

$$\tilde{g}_h^\chi(t,z,\xi) = \big(\tilde{X}_h^\chi(t,z,\xi),\xi\big) := \big(X_h^\chi(t,Rz,\xi)/R,\xi\big).$$

By (A.6), there exists $\delta > 0$ so small that, for $|t| \le \delta R$, $(z,\xi) \in \mathbb{R}^{2d}$,

$$\big|\partial_z^\alpha \partial_\xi^\beta (\tilde{X}_h^\chi(t,z,\xi) - z)\big| \lesssim \delta R^{-|\alpha|}, \quad \big|\partial_z^\alpha \partial_\xi^\beta (J(\tilde{g}_h^\chi)(t,z,\xi) - \mathrm{Id})\big| \le C_{\alpha\beta}\delta < 1/2,$$

where $J(\tilde{g}_h^\chi)$ is the Jacobi matrix with respect to (z,ξ). The Hadamard global inverse mapping theorem then shows that $\tilde{g}_h^\chi(t)$ is a diffeomorphism from $\mathbb{R}^{2d}$ onto itself if $|t| \le \delta R$. By definition, $g_h(t)$ is a diffeomorphism from $\Omega(R/2, J_1')$ onto its range.

We next prove (A.7). Since $g_h(t) = g_h^\chi(t)$ and $g_h^\chi(t)$ is bijective on $\Omega(R/2, J_1')$, it suffices to check that

$$\Omega(R, J_1)^c \supset g_h^\chi(t, \Omega(R/2, J_1')^c).$$

Suppose that $(x,\xi) \in \Omega(R/2, J_1')^c$. If $(x,\xi) \in \Omega(R/3, J_1'')^c$, then

$$g_h^\chi(t,x,\xi) = (x,\xi) \in \Omega(R/3, J_1'')^c \subset \Omega(R, J_1)^c.$$

Suppose that $(x,\xi) \in \Omega(R/3, J_1'') \setminus \Omega(R/2, J_1')$. By (A.2) and the support property of χ, we have

$$|X_h^\chi(t)| \le |x| + |\chi(X_h(t) - x)| \le R/2 + C\delta R$$

for some $C > 0$ independent of R and h. Choosing δ satisfying $1/2 + C\delta < 1$, we obtain $g_h^\chi(t,x,\xi) \in \Omega(R, J_1)^c$. □

Let $\Omega(R, J_1) \ni (x,\xi) \mapsto (Y_h(t,x,\xi),\xi)$ be the inverse of $\Omega(R/2, J_1') \in (x,\xi) \mapsto (X_h(t,x,\xi),\xi)$.

Lemma A.5. *Let δ, J_1 as above and $\nu = 1$. Then, for all $h \in (0,1]$, $0 < R \le h^{-1}$, $0 < |t| \le \delta R$ and $(x,\xi) \in \Omega(R, J_1)$, we have*

$$\big|\partial_x^\alpha \partial_\xi^\beta (Y_h(t,x,\xi) - x)\big| \le C_{\alpha\beta}\delta R^{1-|\alpha|},$$
$$\big|\partial_x^\alpha \partial_\xi^\beta (\Xi_h(t, Y_h(t,x,\xi)) - \xi)\big| \le C_{\alpha\beta}\delta R^{-|\alpha|}.$$

Proof. We prove the inequalities for Y_h only. Proofs for $\Xi_h(t, Y_h(t,x,\xi),\xi)$ are similar. Since $(Y_h(t,x,\xi),\xi) \in \Omega(R/2, J_1')$,

$$
\begin{aligned}
|Y_h(t,x,\xi)-x| &= |X_h(0,Y_h(t,x,\xi),\xi)-X_h(t,Y_h(t,x,\xi),\xi)| \\
&\le \sup_{(x,\xi)\in\Omega(R/2,J_1')} |X_h(t,x,\xi)-x| \\
&\lesssim \delta R.
\end{aligned}
$$

Next, let $\alpha,\beta\in\mathbb{Z}_+^d$ with $|\alpha+\beta|=1$ and apply $\partial_x^\alpha\partial_\xi^\beta$ to the equality

$$
x = X_h(t,Y_h(t,x,\xi),\xi).
$$

We then have the following equality

$$
A(t,Z_h(t))\partial_x^\alpha\partial_\xi^\beta(Y_h(t,x,\xi)-x) = \partial_y^\alpha\partial_\eta^\beta(y-X_h(t,y,\eta))|_{(y,\eta)=Z_h(t)}, \tag{A.8}
$$

where $Z_h(t,x,\xi)=(Y_h(t,x,\xi),\xi)$ and $A(t,Z)=(\partial_x X_h)(t,Z)$. By (A.2) and a similar argument as that in the proof of Lemma A.4, we learn that $A(Z^h(t))$ is invertible, and that $A(Z^h(t))$ and $A(Z^h(t))^{-1}$ are uniformly bounded with respect to $h\in(0,1]$, $|t|\le\delta R$ and $(x,\xi)\in\Omega(R,J_1)$. Therefore,

$$
\begin{aligned}
\left|\partial_x^\alpha\partial_\xi^\beta(Y_h(t,x,\xi)-x)\right| &\le \sup_{(x,\xi)\in\Omega(R/2,J_1')} \left|\partial_y^\alpha\partial_\eta^\beta(y-X_h(t,y,\eta))\right| \\
&\le C_{\alpha\beta}\delta R^{1-|\alpha|}.
\end{aligned}
$$

The proof for higher derivatives is obtained by an induction on $|\alpha+\beta|$, and we omit the details. $\square$

Proof of Proposition 4.1. We consider the case when $t\ge0$, and the proof for $t\le0$ is similar. Choosing $J\Subset J_1\Subset(0,\infty)$, we define the action integral $\widetilde{\Psi}_h(t,x,\xi)$ on $[0,\delta R]\times\Omega(R/2,J_1)$ by

$$
\widetilde{\Psi}_h(t,x,\xi) := x\cdot\xi + \int_0^t L_h\big(X_h(s,Y_h(t,x,\xi),\xi),\Xi_h(s,Y_h(t,x,\xi),\xi)\big)ds,
$$

where $L_h(x,\xi)=\xi\cdot\partial_\xi p_h(x,\xi)-p_h(x,\xi)$ is the Lagrangian associated to p_h and Y_h is defined by the above argument with $R>0$ replaced by $R/2$. The smoothness property of $\widetilde{\Psi}_h$ follows from corresponding properties of X_h, Ξ_h and Y_h. By the standard Hamilton-Jacobi theory, $\widetilde{\Psi}_h(t,x,\xi)$ solves the Hamilton-Jacobi equation (4.1) on $\Omega(R/2,J_1)$ and satisfies

$$\partial_x \widetilde{\Psi}_h(t,x,\xi) = \Xi_h(t, Y_h(t,x,\xi),\xi), \quad \partial_\xi \widetilde{\Psi}_h(t,x,\xi) = Y_h(t,x,\xi).$$

In particular, we obtain the following energy conservation law:

$$p_h(x, \partial_x \widetilde{\Psi}_h(t,x,\xi)) = p_h(Y_h(t,x,\xi),\xi).$$

This energy conservation and Lemma A.5 imply

$$\begin{aligned}
&\left|p_h(\partial_x \widetilde{\Psi}_h(t,x,\xi) - p_h(x,\xi)\right| \\
&\quad\le |Y_h(t,x,\xi) - x| \int_0^1 |\partial_x p_h(\lambda x + (1-\lambda)Y_h(t,x,\xi),\xi)| d\lambda \\
&\quad\le C\delta R(\langle x\rangle^{-1} + h^2) \\
&\quad\le C\delta.
\end{aligned}$$

By using Lemma A.5, we also obtain

$$\left|\partial_x^\alpha \partial_\xi^\beta (p_h(x, \partial_x \widetilde{\Psi}_h(t,x,\xi)) - p_h(x,\xi))\right| \le C_{\alpha\beta}\delta R^{-|\alpha|}, \quad \alpha,\beta \in \mathbb{Z}_+^d.$$

Therefore,

$$\left|\partial_x^\alpha \partial_\xi^\beta \left(\widetilde{\Psi}_h(t,x,\xi) - x\cdot\xi + tp_h(x,\xi)\right)\right| \le C_{\alpha\beta}\delta R^{-|\alpha|}|t|.$$

Choosing a cut-off function $\chi \in S(1,g)$ so that $0 \le \chi \le 1$, $\chi \equiv 1$ on $\Omega(R,J)$ and $\operatorname{supp}\chi \subset \Omega(R/2, J_1)$, we define

$$\Psi_h(t,x,\xi) := x\cdot\xi - tp_h(x,\xi) + \chi(x,\xi)(\widetilde{\Psi}_h(t,x,\xi) - x\cdot\xi + tp_h(x,\xi)).$$

Clearly, $\Psi_h(t,x,\xi)$ satisfies the statement of Proposition 4.1. $\square$

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