

## A necessary condition for Chow semistability of polarized toric manifolds

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**Abstract.** Let  $\Delta \subset \mathbf{R}^n$  be an  $n$ -dimensional Delzant polytope. It is well-known that there exist the  $n$ -dimensional compact toric manifold  $X_\Delta$  and the very ample  $(\mathbf{C}^\times)^n$ -equivariant line bundle  $L_\Delta$  on  $X_\Delta$  associated with  $\Delta$ . In the present paper, we show that if  $(X_\Delta, L_\Delta^i)$  is Chow semistable then the sum of integer points in  $i\Delta$  is the constant multiple of the barycenter of  $\Delta$ . Using this result we get a necessary condition for the polarized toric manifold  $(X_\Delta, L_\Delta)$  being asymptotically Chow semistable. Moreover we can generalize the result in [4] to the case when  $X_\Delta$  is not necessarily Fano.

### 1. Introduction.

Let  $X$  be a compact complex variety and  $L$  an ample line bundle on  $X$ . We call the pair  $(X, L)$  a polarized variety. When we study the moduli space of polarized varieties it is important to consider stability of  $(X, L)$  in the sense of geometric invariant theory, see, for example, [9], [15]. In this paper, we deal with Chow stability of polarized varieties.

The concept of Chow stability is also significant for Kähler geometry: Let  $(X, L)$  be an  $n$ -dimensional polarized manifold. One of the main subjects in Kähler geometry is the existence problem of Kähler metrics with constant scalar curvature in the first Chern class  $c_1(L)$  of  $L$ . In [1] Donaldson proved that if a polarized manifold  $(X, L)$  admits a constant scalar curvature Kähler metric (cscK metric for short) in  $c_1(L)$  and if the automorphism group  $\text{Aut}(X, L)$  of  $(X, L)$  is discrete then  $(X, L)$  is asymptotically Chow stable. This result was extended by Mabuchi [8] when  $\text{Aut}(X, L)$  is not discrete. Namely, Mabuchi proved that if the obstruction introduced in [7] vanishes and  $(X, L)$  admits a cscK metric in  $c_1(L)$  then  $(M, L)$  is asymptotically Chow polystable. The obstruction introduced in [7] is an obstruction for  $(X, L)$  to be asymptotically Chow semistable. This obstruction was reformulated by Futaki in [3] to the vanishing of a collection of integral

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invariants  $\mathcal{F}_{\text{Td}^{(1)}}, \dots, \mathcal{F}_{\text{Td}^{(n)}}$ . Futaki, Sano and the author [4] also reformulated the Mabuchi’s obstruction as the vanishing of the derivation of the Hilbert series when  $(X, L)$  is a toric Fano manifold with the anticanonical polarization. We can compute Futaki’s integral invariants and the derivation of the Hilbert series for some toric Fano manifolds. Especially, Sano, Yotsutani and the author [14] proved the following.

**THEOREM 1.1** ([14]). *There exists a 7-dimensional toric Kähler-Einstein manifold  $X$  such that  $(X, -K_X)$  is asymptotically Chow unstable.*

Therefore, unlike Donaldson’s result in [1], the existence of cscK metric in  $c_1(L)$  does not necessarily imply asymptotic Chow semistability of a polarized manifold  $(X, L)$  when the automorphism group  $\text{Aut}(X, L)$  is not discrete.

In the present paper, we give an obstruction for Chow semistability of polarized toric manifolds from a different viewpoint. Let  $\Delta \subset \mathbf{R}^n$  be an  $n$ -dimensional integral Delzant polytope. Namely,  $\Delta$  satisfies the following conditions (in [13] a polytope satisfying these conditions are called absolutely simple):

1. The vertices  $\mathbf{w}_1, \dots, \mathbf{w}_d$  of  $\Delta$  are contained in  $\mathbf{Z}^n$ .
2. For each vertex  $\mathbf{w}_l$ , there are  $n$  edges  $e_{l,1}, \dots, e_{l,n}$  of  $\Delta$  emanating from  $\mathbf{w}_l$ .
3. The primitive vectors with respect to the edges  $e_{l,1}, \dots, e_{l,n}$  generate the lattice  $\mathbf{Z}^n$  over  $\mathbf{Z}$ .

It is well-known that  $n$ -dimensional integral Delzant polytopes correspond to  $n$ -dimensional compact toric manifolds with  $(\mathbf{C}^\times)^n$ -equivariant very ample line bundles. The reader is referred to [13] for example. Hence we have the  $n$ -dimensional polarized toric manifold  $(X_\Delta, L_\Delta)$  associated with  $\Delta$ . The main result in this paper is the following.

**THEOREM 1.2.** *Let  $\Delta \subset \mathbf{R}^n$  be an  $n$ -dimensional integral Delzant polytope. If  $(X_\Delta, L_\Delta^i)$  is Chow semistable for a positive integer  $i$  then*

$$\sum_{\mathbf{a} \in i\Delta \cap \mathbf{Z}^n} \mathbf{a} = \frac{i\{\#\!(i\Delta \cap \mathbf{Z}^n)\}}{\text{Vol}(\Delta)} \int_\Delta \mathbf{x} \, dv, \tag{1}$$

where  $dv$  is the Euclidean volume form on  $\mathbf{R}^n$ .

**REMARK 1.3.** It is easy to see that the following two conditions are equivalent.

- The equality (1) holds for an integral Delzant polytope  $\Delta$ .
- The equality (1) holds for any  $\mathbf{Z}^n$ -translation of  $\Delta$ .

Note here that for any  $n$ -dimensional integral polytope  $P \subset \mathbf{R}^n$ , the number and the sum of the integer points in  $iP$  are well-behaved: There exists the polynomial  $E_P(t)$  of degree  $n$ , the so called Ehrhart polynomial of  $P$ , such that

$$E_P(t) = \text{Vol}(P)t^n + \sum_{j=0}^{n-1} E_{P,j}t^j, \quad E_P(i) = \#(iP \cap \mathbf{Z}^n). \tag{2}$$

Similarly it is known that there exists the  $\mathbf{R}^n$ -valued polynomial  $\mathbf{s}_P(t)$  such that

$$\mathbf{s}_P(t) = t^{n+1} \int_P \mathbf{x}dv + \sum_{j=1}^n t^j \mathbf{s}_{P,j}, \quad \mathbf{s}_P(i) = \sum_{\mathbf{a} \in iP \cap \mathbf{Z}^n} \mathbf{a}, \tag{3}$$

see [11]. Therefore Theorem 1.2 implies the following.

**THEOREM 1.4.** *Let  $\Delta \subset \mathbf{R}^n$  be an  $n$ -dimensional integral Delzant polytope. If  $(X_\Delta, L_\Delta)$  is asymptotically Chow semistable, then (1) holds for any positive integer  $i$ . If (1) does not hold for a positive integer  $i_0$ , then there exists a positive integer  $i_1$  such that  $(X_\Delta, L_\Delta^i)$  is Chow unstable for any  $i \geq i_1$ .*

We can rewrite the equality (1) as

$$\begin{aligned} & \text{Vol}(\Delta)\mathbf{s}_\Delta(i) - iE_\Delta(i) \int_\Delta \mathbf{x}dv \\ &= \sum_{j=1}^n i^j \left\{ \text{Vol}(\Delta)\mathbf{s}_{\Delta,j} - E_{\Delta,j-1} \int_\Delta \mathbf{x}dv \right\} = \mathbf{0}. \end{aligned} \tag{4}$$

Hence we have the following obstructions for asymptotic Chow semistability of  $(X_\Delta, L_\Delta)$ .

**COROLLARY 1.5.** *Let  $\Delta \subset \mathbf{R}^n$  be an  $n$ -dimensional integral Delzant polytope. If  $(X_\Delta, L_\Delta)$  is asymptotically Chow semistable then*

$$\mathcal{F}_{\Delta,j} := \text{Vol}(\Delta)\mathbf{s}_{\Delta,j} - E_{\Delta,j-1} \int_\Delta \mathbf{x}dv \in \mathbf{R}^n \tag{5}$$

vanishes for each  $j = 1, \dots, n$ .

These vectors  $\mathcal{F}_{\Delta,j}$  ( $j = 1, \dots, n$ ) are regarded as characters on the Lie algebra of the  $n$ -dimensional torus. On the other hand, Futaki’s integral invariants  $\mathcal{F}_{\text{Td}^{(p)}}$  ( $p = 1, \dots, n$ ) are characters on the Lie algebra of holomorphic vector fields

on  $X$ , [3].

CONJECTURE 1.6.

$$\text{Lin}_{\mathbf{C}}\{\mathcal{F}_{\Delta,j}, j = 1, \dots, n\} = \text{Lin}_{\mathbf{C}}\{\mathcal{F}_{\text{Td}(P)} \mid \mathbf{C}^n, p = 1, \dots, n\} \subset \mathbf{C}^n, \quad (6)$$

where  $\text{Lin}_{\mathbf{C}}$  stands for the linear hull in  $\mathbf{C}^n$ .

As a special case, we consider reflexive polytopes. An  $n$ -dimensional integral polytope  $P \subset \mathbf{R}^n$  is called reflexive if  $P$  satisfies the following conditions:

1. For each codimension 1 face  $F \subset P$ , there is an  $\mathbf{n}_F \in \mathbf{Z}^n$  with  $F = \{\mathbf{x} \in P \mid \langle \mathbf{x}, \mathbf{n}_F \rangle = 1\}$ ,
2. The origin  $\mathbf{0} \in \mathbf{R}^n$  is contained in the interior of  $P$ .

It is well-known that reflexive Delzant polytopes correspond to toric Fano manifolds with the anticanonical polarization.

COROLLARY 1.7. *Let  $\Delta \subset \mathbf{R}^n$  be an  $n$ -dimensional reflexive Delzant polytope. Then the following conditions are equivalent.*

- (1)  $(X_{\Delta}, L_{\Delta})$  is asymptotically Chow semistable.
- (2) For all positive integer  $i$ , the equality

$$\mathbf{s}_{\Delta}(i) = \frac{iE_{\Delta}(i)}{\text{Vol}(\Delta)} \int_{\Delta} \mathbf{x} dv = \mathbf{0} \quad (7)$$

holds.

We next observe the relation between asymptotic Chow semistability of  $(X_{\Delta}, L_{\Delta})$  and the derivative of the Hilbert series. Let  $\Delta$  be an  $n$ -dimensional integral Delzant polytope and  $\mathbf{w}_1, \dots, \mathbf{w}_d \in \mathbf{Z}^n$  the vertices of  $\Delta$ . We put

$$\mathcal{C}(\Delta) := \{r_1(\mathbf{w}_1, 1) + \dots + r_d(\mathbf{w}_d, 1) \in \mathbf{R}^{n+1} \mid r_1, \dots, r_d \geq 0\}$$

and

$$\begin{aligned} C_{\Delta}(x_1, \dots, x_{n+1}) &:= \sum_{(a_1, \dots, a_{n+1}) \in \mathcal{C}(\Delta) \cap \mathbf{Z}^{n+1}} x_1^{a_1} \dots x_{n+1}^{a_{n+1}} \\ &=: \sum_{\mathbf{a} \in \mathcal{C}(\Delta) \cap \mathbf{Z}^{n+1}} \mathbf{x}^{\mathbf{a}}. \end{aligned} \quad (8)$$

We call  $C_\Delta$  the Hilbert series of  $\Delta$ . Since

$$\begin{pmatrix} \frac{\partial C_\Delta}{\partial x_1}(1, \dots, 1, t) \\ \vdots \\ \frac{\partial C_\Delta}{\partial x_n}(1, \dots, 1, t) \end{pmatrix} = \sum_{i=1}^{\infty} \mathbf{s}_\Delta(i) t^i \tag{9}$$

holds, the derivative of the Hilbert series at  $(1, \dots, 1, t)$  can be regarded as the generating function of  $\mathbf{s}_\Delta(i)$ . By (9) and Theorem 1.2, we see the following.

COROLLARY 1.8. *If  $(X_\Delta, L_\Delta)$  is asymptotically Chow semistable then*

$$\begin{pmatrix} \frac{\partial C_\Delta}{\partial x_1}(1, \dots, 1, t) \\ \vdots \\ \frac{\partial C_\Delta}{\partial x_n}(1, \dots, 1, t) \end{pmatrix} = \left( \sum_{i=1}^{\infty} i E_\Delta(i) t^i \right) \frac{\int_\Delta \mathbf{x} dv}{\text{Vol}(\Delta)}. \tag{10}$$

Moreover when  $\Delta$  is reflexive

$$\begin{pmatrix} \frac{\partial C_\Delta}{\partial x_1}(1, \dots, 1, t) \\ \vdots \\ \frac{\partial C_\Delta}{\partial x_n}(1, \dots, 1, t) \end{pmatrix} = \mathbf{0} \tag{11}$$

holds.

The equality (11) is equivalent to the necessary condition for asymptotic Chow semistability of toric Fano manifolds proved in [4]. Hence Corollary 1.8 is a generalization of the result in [4] to the case when  $X_\Delta$  is not necessarily Fano.

This paper is organized as follows. In Section 2, we first review the definition and some results of semistability in the sense of geometric invariant theory [10]. We next give the definition of Chow form of projective varieties. In Section 3, we give a proof of our main theorem, Theorem 1.2, based on the results in [5]. We also provide some examples of Chow unstable polarized toric manifolds.

**2. Preliminaries.**

Let  $G$  be a reductive Lie group. Suppose that  $G$  acts a complex vector space  $V$  linearly. We call a nonzero vector  $v \in V$   $G$ -semistable if the closure of the orbit  $Gv$  does not contain the origin. Similarly we call  $p \in \mathbf{P}(V)$   $G$ -semistable if any representative of  $p$  in  $V \setminus \{0\}$  is  $G$ -semistable. It is well-known that there is the following good criterion for  $v$  being  $G$ -semistable, see [10].

PROPOSITION 2.1 (Hilbert-Mumford criterion, [10]).  *$p \in \mathbf{P}(V)$  is  $G$ -semistable if and only if  $p$  is  $H$ -semistable for each maximal torus  $H \subset G$ .*

Hence it is important to study  $G$ -semistability when  $G$  is isomorphic to an algebraic torus  $(\mathbf{C}^\times)^n$ . Let  $G$  be isomorphic to  $(\mathbf{C}^\times)^n$ . Then a  $G$ -module  $V$  is decomposed as

$$V = \sum_{\chi \in \chi(G)} V_\chi, \quad V_\chi := \{v \in V \mid t \cdot v = \chi(t)v, \forall t \in G\}, \tag{12}$$

where  $\chi(G) \simeq \mathbf{Z}^n$  is the character group of the torus  $G$ .

DEFINITION 2.2. Let  $v = \sum_{\chi \in \chi(G)} v_\chi$  be a nonzero vector in  $V$ . The weight polytope  $\text{Wt}_G(v) \subset \chi(G) \otimes_{\mathbf{Z}} \mathbf{R}$  of  $v$  is the convex hull of  $\{\chi \in \chi(G) \mid v_\chi \neq 0\}$  in  $\chi(G) \otimes_{\mathbf{Z}} \mathbf{R}$ .

The following fact about  $G$ -semistability is standard.

PROPOSITION 2.3. *Let  $G$  be isomorphic to  $(\mathbf{C}^\times)^n$ . Suppose that  $G$  acts a complex vector space  $V$  linearly. Then a nonzero vector  $v \in V$  is  $G$ -semistable if and only if the weight polytope  $\text{Wt}_G(v)$  contains the origin.*

Let  $G = (\mathbf{C}^\times)^{n+1}$  and  $H$  be the subtorus

$$H = \{(t_1, \dots, t_n, (t_1 \cdots t_n)^{-1}) \mid (t_1, \dots, t_n) \in (\mathbf{C}^\times)^n\} \simeq (\mathbf{C}^\times)^n. \tag{13}$$

Then the weight polytope  $\text{Wt}_H(v) \subset \chi(H) \otimes_{\mathbf{Z}} \mathbf{R} \simeq \mathbf{R}^n$  equals to  $\pi(\text{Wt}_G(v))$ , where the linear map  $\pi : \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$  is given as  $(x_1, \dots, x_n, x_{n+1}) \mapsto (x_1 - x_{n+1}, \dots, x_n - x_{n+1})$ .

Therefore we see the following.

PROPOSITION 2.4. *If  $v$  is  $H$ -semistable then there exists  $t \in \mathbf{R}$  such that  $(t, \dots, t) \in \text{Wt}_G(v)$ .*

We next define the Chow form of irreducible projective varieties. See [5] for

more detail.

DEFINITION 2.5. Let  $X \subset \mathbf{C}P^N$  be an  $n$ -dimensional irreducible subvariety of degree  $d$ . It is easy to see that the subset  $Z_X$  of the Grassmannian  $\text{Gr}(N - n - 1, \mathbf{C}P^N)$  defined by

$$Z_X = \{L \in \text{Gr}(N - n - 1, \mathbf{C}P^N) \mid L \cap X \neq \emptyset\}$$

is an irreducible hypersurface of degree  $d$ . Hence  $Z_X$  is given by the vanishing of a degree  $d$  element  $R_X \in \mathbf{P}(\mathcal{B}_d(N - n - 1, \mathbf{C}P^N))$ , where  $\mathcal{B}(N - n - 1, \mathbf{C}P^N) = \bigoplus_d \mathcal{B}_d(N - n - 1, \mathbf{C}P^N)$  is the graded coordinate ring of the Grassmannian. We call  $R_X$  the Chow form of  $X$ .

Since the special linear group  $SL(N + 1, \mathbf{C})$  acts naturally on  $\mathcal{B}_d(N - n - 1, \mathbf{C}P^N)$ , we can consider the  $SL(N + 1, \mathbf{C})$ -stability of the Chow form  $R_X$ .

DEFINITION 2.6. Let  $X \subset \mathbf{C}P^N$  be an  $n$ -dimensional irreducible subvariety of degree  $d$ . We call  $X$  Chow semistable if the Chow form  $R_X$  is  $SL(N + 1, \mathbf{C})$ -semistable. When  $X$  is not Chow semistable  $X$  is called Chow unstable.

DEFINITION 2.7. Let  $(X, L)$  be a polarized manifold.  $(X, L)$  is called asymptotically Chow semistable when  $\Psi_i(X) \subset \mathbf{P}(H^0(X; L^i)^*)$  is Chow semistable for each  $i \gg 1$ . Here  $\Psi_i : X \rightarrow \mathbf{P}(H^0(X; L^i)^*)$  is the Kodaira embedding.

### 3. Chow semistability of polarized toric manifolds.

In this section, we first introduce a necessary condition for Chow semistability of  $n$ -dimensional irreducible projective subvariety  $X \subset \mathbf{C}P^N$ . This condition is very simple and easy. However this leads us to our main theorem when  $X$  is toric.

Let  $(\mathbf{C}^\times)^{N+1} \subset GL(N + 1, \mathbf{C})$  be the  $(N + 1)$ -dimensional torus consists of invertible diagonal matrices. Then the subtorus  $H$  defined by (13) is a maximal torus of  $SL(N + 1, \mathbf{C})$ . Proposition 2.4 implies the following.

PROPOSITION 3.1. Let  $X \subset \mathbf{C}P^N$  be an irreducible subvariety. If  $X$  is Chow semistable, then there exists  $t \in \mathbf{R}$  such that

$$(t, \dots, t) \in \text{Wt}_{(\mathbf{C}^\times)^{N+1}}(R_X) \subset \text{Aff}_{\mathbf{R}}(\text{Wt}_{(\mathbf{C}^\times)^{N+1}}(R_X)), \tag{14}$$

where  $\text{Aff}_{\mathbf{R}}(\text{Wt}_{(\mathbf{C}^\times)^{N+1}}(R_X))$  is the affine hull of the weight polytope  $\text{Wt}_{(\mathbf{C}^\times)^{N+1}}(R_X)$  of  $R_X$  in  $\mathbf{R}^{N+1}$ .

Hence, if the dimension of  $\text{Aff}_{\mathbf{R}}(\text{Wt}_{(\mathbf{C}^\times)^{N+1}}(R_X))$  becomes smaller, then there is less chance for  $X$  to be Chow semistable. In [5] Chapter 6, Gelfand, Kapranov

and Zelevinsky gave the following dimension formula for  $\text{Aff}_{\mathbf{R}}(\text{Wt}_{(\mathbf{C}^\times)^{N+1}}(R_X))$ .

PROPOSITION 3.2 ([5, Chapter 6, Proposition 3.8]). *Let  $X \subset \mathbf{C}P^N$  be an  $n$ -dimensional irreducible subvariety. Suppose that  $X$  does not contained in any projective hyperplane. Then we have*

$$\dim \text{Aff}_{\mathbf{R}}(\text{Wt}_{(\mathbf{C}^\times)^{N+1}}(R_X)) = N + 1 - \dim \{t \in (\mathbf{C}^\times)^{N+1} \mid tX = X\}. \tag{15}$$

Therefore, if  $\dim\{t \in (\mathbf{C}^\times)^{N+1} \mid tX = X\}$  becomes larger, then there is less chance for  $X$  to be Chow semistable.

We next investigate Chow semistability of projective varieties defined as follows. Let  $A := \{\mathbf{a}_1, \dots, \mathbf{a}_{N+1}\}$  be a finite subset in  $\mathbf{Z}^n$ . Suppose that  $A$  affinely generates the lattice  $\mathbf{Z}^n \subset \mathbf{R}^n$  over  $\mathbf{Z}$ . Then the closure of

$$X_A^0 := \{[\mathbf{x}^{\mathbf{a}_1} : \dots : \mathbf{x}^{\mathbf{a}_{N+1}}] \mid \mathbf{x} \in (\mathbf{C}^\times)^n\} \subset \mathbf{C}P^N$$

is an  $n$ -dimensional subvariety of  $\mathbf{C}P^N$ . We denote this by  $X_A = \overline{X_A^0}$ .

PROPOSITION 3.3 ([5, Chapter 7, Proposition 1.11] and [6]).

$$\begin{aligned} & \text{Aff}_{\mathbf{R}}(\text{Wt}_{(\mathbf{C}^\times)^{N+1}}(R_{X_A})) \\ &= \left\{ (\varphi_1, \dots, \varphi_{N+1}) \in \mathbf{R}^{N+1} \mid \begin{aligned} & \sum_{j=1}^{N+1} \varphi_j = (n+1)! \text{Vol}(Q), \\ & \sum_{j=1}^{N+1} \varphi_j \mathbf{a}_j = (n+1)! \int_Q \mathbf{x} dv \end{aligned} \right\}. \tag{16} \end{aligned}$$

Hence, by Propositions 3.1 and 3.3, we get a necessary condition for Chow semistability of  $X_A$ .

THEOREM 3.4. *If  $X_A$  is Chow semistable then*

$$\sum_{j=1}^{N+1} \mathbf{a}_j = \frac{N+1}{\text{Vol}(Q)} \int_Q \mathbf{x} dv. \tag{17}$$

PROOF. If  $X_A$  is Chow semistable then there exists  $t \in \mathbf{R}$  satisfying

$$(N+1)t = (n+1)! \text{Vol}(Q), \quad t \sum_{j=1}^{N+1} \mathbf{a}_j = (n+1)! \int_Q \mathbf{x} dv \tag{18}$$

by Propositions 3.1 and 3.3. Hence (17) holds. □

PROOF OF THEOREM 1.2. Let  $\Delta \subset \mathbf{R}^n$  be an  $n$ -dimensional integral Delzant polytope and  $A := i\Delta \cap \mathbf{Z}^n$ . Then  $X_A$  is the image of the Kodaira embedding  $X_\Delta \rightarrow \mathbf{P}(H^0(X_\Delta, L_\Delta^i)^*)$ . Therefore if  $(X_\Delta, L_\Delta^i)$  is Chow semistable then we have

$$s_\Delta(i) = \frac{E_\Delta(i)}{\text{Vol}(i\Delta)} \int_{i\Delta} \mathbf{x}dv = \frac{iE_\Delta(i)}{\text{Vol}(\Delta)} \int_\Delta \mathbf{x}dv$$

by Theorem 3.4. □

PROOF OF COROLLARY 1.7. By [3], when  $X_\Delta$  is asymptotically Chow semistable, the Futaki invariant of  $X_\Delta$ , in this case  $\int_\Delta \mathbf{x}dv$ , vanishes. Therefore  $s_\Delta(i) = \mathbf{0}$  for any positive integer  $i$ . Conversely, suppose that  $s_\Delta(i) = \int_\Delta \mathbf{x}dv = \mathbf{0}$  for any positive integer  $i$ . From the result of Wang and Zhu [16], there exists an Kähler-Einstein metrics on  $X_\Delta$ . Moreover we see that the derivative of Hilbert series  $C_\Delta$  vanishes by (9). Hence the integral invariant  $\mathcal{F}_{\text{Td}(p)}$  vanish for each  $p = 1, \dots, n$  [4]. The vanishing of these integral invariants is equivalent to the vanishing of the obstruction for Chow semistability in the sense of Mabuchi, [7]. This equivalence is proved by Futaki in [3]. Therefore by the result of Mabuchi [8],  $X_\Delta$  is asymptotically Chow semistable. □

We give some examples of polarized toric manifolds and investigate Chow semistability. We first investigate polarized toric surfaces. The equalities (2) and (3) imply the following.

LEMMA 3.5. *Let  $P \subset \mathbf{R}^2$  be an integral polygon. Then*

$$E_P(t) = \text{Vol}(P)t^2 + (E_P(2) - E_P(1) - 3 \text{Vol}(P))t + 2E_P(1) - E_P(2) + 2 \text{Vol}(P) \tag{19}$$

and

$$s_P(t) = t^3 \int_P \mathbf{x}dv + \frac{t^2}{2} \left( s_P(2) - 2s_P(1) - 6 \int_P \mathbf{x}dv \right) + \frac{t}{2} \left( 4s_P(1) - s_P(2) + 4 \int_P \mathbf{x}dv \right) \tag{20}$$

hold.

For example, let  $\Delta_k$  be the convex hull of  $\{(0, 0), (0, 1), (1, 1), (k, 0)\} \subset \mathbf{R}^2$  for  $k \geq 2$ . The corresponding toric surface  $X_{\Delta_k}$  is the  $(k - 1)$ -th Hirzebruch surface.  $X_{\Delta_k}$  is not Fano for  $k \geq 3$ . By Theorem 1.2 we see Chow unstability of  $(X_{\Delta_k}, L_{\Delta_k}^i)$ .

PROPOSITION 3.6. *For each integers  $k \geq 2$  and  $i \geq 1$ ,  $(X_{\Delta_k}, L_{\Delta_k}^i)$  is Chow unstable.*

PROOF. It is easy to see that

$$\int_{\Delta_k} \mathbf{x}dv = \frac{1}{6} \begin{pmatrix} k^2 + k + 1 \\ k + 2 \end{pmatrix}, \tag{21}$$

$$\text{Vol}(\Delta_k) = \frac{1}{2}(k + 1), \tag{22}$$

$$E_{\Delta_k}(1) = k + 3, \quad E_{\Delta_k}(2) = 3k + 6 \tag{23}$$

and

$$\mathbf{s}_{\Delta_k}(1) = \frac{1}{2} \begin{pmatrix} k^2 + k + 2 \\ 4 \end{pmatrix}, \quad \mathbf{s}_{\Delta_k}(2) = \frac{1}{2} \begin{pmatrix} 5k^2 + 5k + 8 \\ 2k + 16 \end{pmatrix} \tag{24}$$

hold. Hence by Lemma 3.5,

$$E_{\Delta_k}(t) = \frac{1}{2}\{(k + 1)t^2 + (k + 3)t + 2\} \tag{25}$$

and

$$\mathbf{s}_{\Delta_k}(t) = \frac{t}{12} \begin{pmatrix} 2(k^2 + k + 1)t^2 + 3(k^2 + k + 2)t + k^2 + k + 4 \\ 2(k + 2)t^2 + 12t + 8 - 2k \end{pmatrix}. \tag{26}$$

Therefore

$$\text{Vol}(\Delta_k)\mathbf{s}_{\Delta_k}(i) - iE_{\Delta_k}(i) \int_{\Delta_k} \mathbf{x}dv = \frac{i(i + 1)k(k - 1)}{24} \begin{pmatrix} k - 1 \\ -2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

holds for any  $k \geq 2$  and  $i \geq 1$ . □

REMARK 3.7. In this case, note that  $\mathcal{F}_{\Delta_k,1}$  equals to  $\mathcal{F}_{\Delta_k,2}$ :

$$\mathcal{F}_{\Delta_k,1} = \mathcal{F}_{\Delta_k,2} = \frac{k(k-1)}{24} \begin{pmatrix} k-1 \\ -2 \end{pmatrix}.$$

When  $k = 2$ , that is,  $X_{\Delta_2}$  is the one point blow-up of the projective plane, we can easily calculate Futaki's integral invariants  $\mathcal{F}_{\text{Td}(1)}|_{\mathcal{C}^2}$  and  $\mathcal{F}_{\text{Td}(2)}|_{\mathcal{C}^2}$ :

$$\mathcal{F}_{\text{Td}(1)}|_{\mathcal{C}^2} = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathcal{F}_{\text{Td}(2)}|_{\mathcal{C}^2} = C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}, C_1, C_2 \neq 0.$$

Therefore, in this case (6) is right:

$$\text{Lin}_{\mathcal{C}}\{\mathcal{F}_{\Delta_2,1}, \mathcal{F}_{\Delta_2,2}\} = \text{Lin}_{\mathcal{C}}\{\mathcal{F}_{\text{Td}(1)}|_{\mathcal{C}^2}, \mathcal{F}_{\text{Td}(2)}|_{\mathcal{C}^2}\} = \mathcal{C} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \subset \mathcal{C}^2.$$

We next consider the example given by Nill and Paffenholz in [12]. It is the toric Fano 7-fold corresponding to the reflexive Delzant polytope

$$\Delta_{NP} := \{\mathbf{x} \in \mathbf{R}^7 \mid \langle \mathbf{x}, \mathbf{v}_i \rangle \geq -1, i = 1, \dots, 12\}. \tag{27}$$

Here  $\mathbf{v}_1, \dots, \mathbf{v}_{12} \in \mathbf{R}^7$  are given by

$$\begin{aligned} & (\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4 \ \mathbf{v}_5 \ \mathbf{v}_6 \ \mathbf{v}_7 \ \mathbf{v}_8 \ \mathbf{v}_9 \ \mathbf{v}_{10} \ \mathbf{v}_{11} \ \mathbf{v}_{12}) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 2 & 1 & -1 \end{pmatrix}. \end{aligned}$$

Sano, Yotsutani and the author showed the following in [14].

**THEOREM 3.8 ([14]).** *Let  $\Delta_{NP}$  be the 7-dimensional reflexive Delzant polytope given above. Then*

$$\int_{\Delta_{NP}} \mathbf{x} dv = \mathbf{0} \tag{28}$$

and

$$\begin{pmatrix} \frac{\partial C_{\Delta_{NP}}}{\partial x_1}(1, \dots, 1, t) \\ \vdots \\ \frac{\partial C_{\Delta_{NP}}}{\partial x_n}(1, \dots, 1, t) \end{pmatrix} \neq \mathbf{0} \quad (29)$$

*hold.*

Therefore  $X_{\Delta_{NP}}$  admits Kähler-Einstein metrics by the theorem of Wang and Zhu [16], but  $(X_{\Delta_{NP}}, -K_{X_{\Delta_{NP}}})$  is asymptotically Chow unstable. Moreover, by Theorem 1.4, we see that there exists a positive integer  $i_1$  such that  $(X_{\Delta_{NP}}, (-K_{\Delta_{NP}})^i)$  is Chow unstable for any  $i \geq i_1$ .

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### References

- [1] S. K. Donaldson, Scalar curvature and projective embeddings, I, *J. Differential Geom.*, **59** (2001), 479–522.
- [2] W. Fulton, Introduction to Toric Varieties, Number 131, In: *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ, 1993.
- [3] A. Futaki, Asymptotic Chow semi-stability and integral invariants, *Internat. J. Math.*, **15** (2004), 967–979.
- [4] A. Futaki, H. Ono and Y. Sano, Hilbert series and obstructions to asymptotic semistability, *Adv. Math.*, **226** (2011), 254–284.
- [5] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1994.
- [6] M. M. Kapranov, B. Sturmfels and A. V. Zelevinsky, Chow polytopes and general resultants, *Duke Math. J.*, **67** (1992), 189–218.
- [7] T. Mabuchi, An obstruction to asymptotic semistability and approximate critical metrics, *Osaka J. Math.*, **41** (2004), 463–472.
- [8] T. Mabuchi, An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds, I, *Invent. Math.*, **159** (2005), 225–243.
- [9] D. Mumford, Stability of projective varieties, *Enseign. Math. (2)*, **23** (1977), 39–110.
- [10] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory*, Third edition, *Ergeb. Math. Grenzgeb. (2)*, **34**, Springer-Verlag, Berlin, 1994.
- [11] B. Nill, Gorenstein toric Fano varieties, Dissertation, Universität Tübingen, 2005.
- [12] B. Nill and A. Paffenholz, Examples of non-symmetric Kähler-Einstein toric Fano manifolds, arXiv:0905.2054.
- [13] T. Oda, Convex bodies and algebraic geometry, *Ergeb. Math. Grenzgeb. (3)*, **15**, Springer-Verlag, Berlin, 1988.
- [14] H. Ono, Y. Sano and N. Yotsutani, An example of asymptotically Chow unstable manifolds with constant scalar curvature, to appear in *Ann. Inst. Fourier*, arXiv:0906.3836

- [15] E. Viehweg, *Quasi-Projective Moduli for Polarized Manifolds*, *Ergeb. Math. Grenzgeb.* (3), **30**, Springer-Verlag, Berlin, 1995.
- [16] X.-J. Wang and X. Zhu, *Kähler-Ricci solitons on toric manifolds with positive first Chern class*, *Adv. Math.*, **188** (2004), 87–103.

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