J. Math. Soc. Japan
Vol. 63, No. 4 (2011) pp. 1155–1186
doi: 10.2969/jmsj/06341155

Hypersurfaces with isotropic Blaschke tensor

By Zhen Guo, Jianbo Fang and Limiao Lin

(Received May 8, 2010)

Abstract. Let M^m be an m-dimensional submanifold without umbilical points in the $m+1$-dimensional unit sphere $\boldsymbol{S}^{m+1}$. Three basic invariants of M^m under the Möbius transformation group of $\boldsymbol{S}^{m+1}$ are a 1-form Φ called Möbius form, a symmetric (0,2) tensor $\boldsymbol{A}$ called Blaschke tensor and a positive definite (0,2) tensor $\boldsymbol{g}$ called Möbius metric. We call the Blaschke tensor is isotropic if there exists a function λ such that $\boldsymbol{A} = \lambda \boldsymbol{g}$. One of the basic questions in Möbius geometry is to classify the hypersurfaces with isotropic Blaschke tensor. When λ is constant, the classification was given by Changping Wang and others. When λ is not constant, all hypersurfaces with dimensional $m \geq 3$ and isotropic Blaschke tensor are explicitly expressed in this paper. Therefore, for the dimensional $m \geq 3$, the above basic question is completely answered.

1. Introduction.

Let $x : M^m \to \boldsymbol{S}^{m+1}$ be an m-dimensional hypersurface without umbilical pints in the $(m+1)$-dimensional unit sphere $\boldsymbol{S}^{m+1}$ and $\{e_i\}$ be a local orthonormal tangent frame field of x for the standard metric $I = dx \cdot dx$ with dual frame field $\{\theta_i\}$. Let $II = \sum_{i,j} h_{ij}\theta_i\theta_j$ be the second fundamental form and H the mean curvature of x. Define $\rho^2 = m/(m-1)|II - HI|^2$, then positive definite 2-form $\boldsymbol{g} = \rho^2 I$ is invariant under Möbius group of $\boldsymbol{S}^{m+1}$ and is called Möbius metric of x. Two basic Möbius invariants of x, Möbius form $\Phi = \sum_i \rho C_i\theta_i$, Blaschke tensor $\boldsymbol{A} = \sum_{ij} \rho^2 A_{ij}\theta_i\theta_j$, are defined by (cf. [**Wa**])

$$C_i = -\rho^{-2}\left(e_i(H) + \sum_j (h_{ij} - H\delta_{ij})e_j(\log\rho)\right), \tag{1.1}$$

$$\begin{aligned} A_{ij} = {} & -\rho^{-2}(\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - Hh_{ij}) \\ & - \frac{1}{2}\rho^{-2}(\|\nabla\log\rho\|^2 - 1 + H^2)\delta_{ij}, \end{aligned} \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary 53A30; Secondary 53B25.
Key Words and Phrases. Möbius geometry, Blaschke tensor.
The authors are supported by the project (No. 10561010) of NSFC and the project (No. 10861013) of NSFC.

where (Hess_{ij}) and ∇ are the Hessian-matrix and the gradient operator with respect to the induced metric $I = dx \cdot dx$. We call the *Blaschke tensor* $\boldsymbol{A}$ *is isotropic* if there exists a function λ on M^m, such that

$$\boldsymbol{A} = \lambda \boldsymbol{g}. \tag{1.3}$$

Huili Liu, Changping Wang and Guosong Zhao defined a Möbius isotropic submanifold as a submanifold satisfying two conditions: $\boldsymbol{A} = \lambda \boldsymbol{g}$ and $\Phi = 0$. They showed that the conformal Gauss map of a Möbius isotropic submanifolds is harmonic, and proved that a Möbius isotropic submanifold is conformally equivalent to a minimal submanifold with constant scalar in $\boldsymbol{S}^n$ (if $\lambda > 0$), or in Euclidean space $\boldsymbol{R}^n$ (if $\lambda = 0$), or in hyperbolic space $\boldsymbol{H}^n$ (if $\lambda < 0$) (cf. [**LWZ**]). This result give the unified Möbius characterization of the minimal submanifolds with constant scalar in the three space forms by two conditions: Blaschke tensor $\boldsymbol{A}$ is isotropic and Φ vanishes. In the fact, $\Phi = 0$ implies that λ is constant. From then, people want to know the hypersurface whose Blaschke tensor is isotropic and corresponding λ is non-constant. In this paper, we will give all hypersurfaces with isotropic Blaschke tensor and non-constant λ.

For the purpose to make our main result intuitional, we use the following notations: $\boldsymbol{R}_1^{m+3}$ denotes Lorentz space with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle Y, Z \rangle = -y_0 z_0 + y_1 z_1 + \cdots + y_{m+2} z_{m+2},$$

where $Y = (y_0, y_1, \ldots, y_{m+2})$, $Z = (z_0, z_1, \ldots, z_{m+2}) \in \boldsymbol{R}^{m+3}$. $\boldsymbol{C}_+^{m+2}$ and $\boldsymbol{Q}^{m+1}$ denote the positive light cone and the quadric in real projection space $\boldsymbol{R}P^{m+2}$, which are defined as follows:

$$\boldsymbol{C}_+^{m+2} = \big\{X \in \boldsymbol{R}_1^{m+3} : \langle X, X \rangle = 0, x_0 > 0\big\},$$
$$\boldsymbol{Q}^{m+1} = \big\{[Y] \in \boldsymbol{R}P^{m+2} : \langle Y, Y \rangle = 0\big\}.$$

We use map

$$\pi : \boldsymbol{C}_+^{m+2} \to \boldsymbol{Q}^{m+1} \tag{1.4}$$

to denote the nature projection. For a hypersurface $x : M^m \to \boldsymbol{S}^{m+1}$, we have map

$$X := \pi(1, x) : M^m \to \boldsymbol{Q}^{m+1}, \tag{1.5}$$

which is determined by the immersion x and is called the *natural map* of x. It is known from a classical theorem that two hypersurfaces $x, \tilde{x} : M^m \to \boldsymbol{S}^{m+1}$ are Möbius equivalent if and only if whose natural maps $X, \tilde{X} : M^m \to \boldsymbol{Q}^{m+1}$ are equivalent under the action of Lorentz group $O(m+2,1)$. We use σ to denote the inverse stereographic projection from $\boldsymbol{R}^{m+1} \cup \{\infty\}$ to $\boldsymbol{S}^{m+1}$. As σ is a conformal homeomorphism, two hypersurfaces $\tau,\ \tilde{\tau} : M^m \to \boldsymbol{R}^{m+1} \cup \{\infty\}$ are Möbius equivalent if and only if $x = \sigma \circ \tau,\ \tilde{x} = \sigma \circ \tilde{\tau} : M^m \to \boldsymbol{S}^{m+1}$ are Möbius equivalent. Now we state the main theorem as follows:

THEOREM 1.1. *Let $x : M^m \to \boldsymbol{S}^{m+1}$ be an $m(\geq 3)$-dimensional hypersurface without umbilical points in the $(m+1)$-dimensional unit sphere $\boldsymbol{S}^{m+1}$. If $\boldsymbol{A} = \lambda \boldsymbol{g}$ and function λ is non-constant, then for a connected open set U of M, with $\nabla\lambda \neq 0$, the function λ is a single variate function and is the implicit function determined by function equation*:

$$\int \frac{me^{m\lambda} d\lambda}{\sqrt{ae^{2m\lambda} - 2\lambda - \dfrac{1}{m^2}}} = s, \tag{1.6}$$

where $s \in l$, l is an interval in $\boldsymbol{R}^1$ and a is a constant. Moreover, x is one of the following cases:
(i) *for $a = 0$, up to the Möbius transformations of $\boldsymbol{S}^{m+1}$,*

$$x(U) = \sigma(\Gamma_1 \times \boldsymbol{R}^{m-1}), \quad \Gamma_1 \subset \boldsymbol{R}^2, \tag{1.7}$$

where Γ_1 is a curve with arc-length parameter s in $\boldsymbol{R}^2$ and its position vector ξ in $\boldsymbol{R}^2$ is given by

$$\begin{aligned}\xi(u) = \Bigg(&2\int \exp\left(-\frac{u^2}{2m} + bu + c\right)\cos u du,\\ &2\int \exp\left(-\frac{u^2}{2m} + bu + c\right)\sin u du\Bigg),\end{aligned} \tag{1.8}$$

where b is a constant, $c = -(m/2)(b^2 + (1/m^2))$, $u = \int e^{-m\lambda(s)} ds$;
(ii) *for $a < 0$, up to the transformations in $O(m+2,1)$,*

$$X(U) = \pi\left(\boldsymbol{H}^{m-1}\left(\frac{1}{\sqrt{-a}}\right) \times \Gamma_2\right), \quad \Gamma_2 \subset \boldsymbol{S}^2\left(\frac{1}{\sqrt{-a}}\right), \tag{1.9}$$

where Γ_2 is a curve with arc-length parameter s in $\boldsymbol{S}^2(1/\sqrt{-a})$ and its position

vector $\xi(s)$ in $\boldsymbol{R}^3$ is a solution of the following equation (1.11);
(iii) *for $a > 0$, up to the transformations in $O(m+2,1)$,*

$$X(U) = \pi\left(\Gamma_3 \times \boldsymbol{S}^{m-1}\left(\frac{1}{\sqrt{a}}\right)\right), \quad \Gamma_3 \subset \boldsymbol{H}^2\left(\frac{1}{\sqrt{a}}\right), \tag{1.10}$$

where Γ_3 is a curve with arc-length parameter s in $\boldsymbol{H}^2(1/\sqrt{a})$ and its position vector $\xi(s)$ in $\boldsymbol{R}^3_1$ is a solution of the following equation (1.11) :

$$\frac{d}{ds}\left(e^{m\lambda}\left(\frac{d^2\xi}{ds^2} - a\xi\right)\right) = -e^{-m\lambda}\frac{d\xi}{ds}. \tag{1.11}$$

For the purpose of making the procedure of the proof of Theorem 1.1 clear, we organize the main content of the paper as four parts. In Section 2 we give the structure equations, the Möbius invariants of general m-dimensional hypersurfaces in $\boldsymbol{S}^{m+1}$. In Section 3 we concentrate on getting the local expression of the basic Möbius invariants of the hypersurfaces with isotropic Blaschke tensor. The main result is Theorem 3.1 in this part. In Section 4 we obtain the differential equations characterizing the hypersurfaces with isotropic Blaschke tensor. The main result is Theorem 4.1 in this part. In Section 5 we treat the differential equations given in Section 4 and classify the hypersurfaces with isotropic Blaschke tensor.

2. Möbius invariants for a hypersurface in $\boldsymbol{S}^{m+1}$.

In this section we define Möbius invariants and recall the structure equations for hypersurfaces in $\boldsymbol{S}^{m+1}$. For the detail we refer to [**Wa**].

Let $\boldsymbol{R}^{m+3}_1$ be the Lorentz space with inner product $\langle\cdot,\cdot\rangle$ defined in Section 1, $x : M^m \to \boldsymbol{S}^{m+1} \subset \boldsymbol{R}^{m+2}$ be a hypersurface without umbilical points in $\boldsymbol{S}^{m+1}$. We define the Möbius position vector $Y : M^m \to \boldsymbol{R}^{m+3}_1$ of x by

$$Y = \rho(1,x) : M^m \to \boldsymbol{R}^{m+3}_1, \;\; \rho^2 = \frac{m}{m-1}(\|II\|^2 - mH^2) > 0.$$

Theorem 2.1 ([**Wa**]). *Two hypersurfaces $x, \tilde{x} : M^m \to \boldsymbol{S}^{m+1}$ are Möbius equivalent if and only if there exists T in the Lorentz group $O(m+2,1)$ acting on $\boldsymbol{R}^{m+3}_1$, such that $Y = \tilde{Y}T$.*

Since the Möbius group in $\boldsymbol{S}^{m+1}$ is isomorphic to the subgroup $O^+(m+2,1)$ of $O(m+2,1)$, which preserves the positive part of the light cone in $\boldsymbol{R}^{m+3}_1$, from Theorem 2.1 we know that 2-form

$$\boldsymbol{g} = \langle dY, dY \rangle = \rho^2 dx \cdot dx \tag{2.1}$$

is a Möbius invariant (cf. [**Bl**], [**Ch**], [**Wi**], [**PW**], [**Wa**] and [**GLW2**]). We call $\boldsymbol{g}$ the Möbius metric or the Möbius first fundamental form induced by x. Let Δ denote the Laplacian of $\boldsymbol{g}$, then we have

$$\langle \Delta Y, \Delta Y \rangle = 1 + m^2 R,$$

where R is the normalized scalar curvature of $\boldsymbol{g}$ (cf. [**Wa**]). By defining

$$N = -\frac{1}{m}\Delta Y - \frac{1}{2m^2}(1 + m^2 R)Y, \tag{2.2}$$

we have

$$\langle Y, Y \rangle = \langle N, N \rangle = 0, \quad \langle Y, N \rangle = 1. \tag{2.3}$$

Moreover, if we take a local orthonormal basis $\{E_i\}$ with respect to $\boldsymbol{g}$ with dual basis $\{\omega_i\}$. Denoting $E_i(Y)$ by Y_i, we have

$$\langle Y_i, Y_j \rangle = \delta_{ij}, \quad \langle Y_i, Y \rangle = \langle Y_i, N \rangle = 0, \quad 1 \le i, j \le m. \tag{2.4}$$

Let V be the orthogonal complement of $\operatorname{span}\{Y, N, Y_i\}$ in R_1^{m+3}. Then we have the orthogonal decomposition

$$\boldsymbol{R}_1^{m+3} = \operatorname{span}\{Y, N\} \oplus \operatorname{span}\{Y_1, \dots Y_m\} \oplus V. \tag{2.5}$$

Let E be an unit vector field of V. Then

$$\{Y, N, Y_1, \dots, Y_m, E\} \tag{2.6}$$

forms a moving frame in $\boldsymbol{R}_1^{m+3}$ along M. We make use of the following convention on the ranges of indices: $1 \le i, j, k, \ldots \le m$; and we shall agree that repeated indices are summed over respective ranges. Then the structure equations are given by

$$dY = \omega_i Y_i, \tag{2.7}$$

$$dN = A_{ij}\omega_j Y_i + C_i \omega_i E, \tag{2.8}$$

$$dY_i = -A_{ij}\omega_j Y - \omega_i N + \omega_{ij} Y_j + B_{ij}\omega_j E, \tag{2.9}$$

$$dE = -C_i\omega_i Y - B_{ij}\omega_j Y_i, \tag{2.10}$$

where ω_{ij} is the connection form of the Möbius metric $\boldsymbol{g}$, and A_{ij} and B_{ij} are symmetric with respect to (ij). It is clear that

$$\boldsymbol{A} = A_{ij}\omega_i \otimes \omega_j, \quad \boldsymbol{B} = B_{ij}\omega_i \otimes \omega_j, \quad \Phi = C_i\omega_i \tag{2.11}$$

are Möbius invariants and called the Blaschke tensor, the Möbius second fundamental form and the Möbius form, respectively.

REMARK 2.1. The relations among $\boldsymbol{A}, \boldsymbol{B}, \Phi$ and Euclidean invariants of x are given by (1.1), (1.2) and

$$B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij}). \tag{2.12}$$

Define covariant derivative of A_{ij}, B_{ij} and C_i by

$$\sum_k A_{ij,k}\omega_k = dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki}, \tag{2.13}$$

$$\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}, \tag{2.14}$$

$$\sum_k C_{i,k}\omega_k = dC_i + \sum_k C_k\omega_{ki}. \tag{2.15}$$

By exterior differentiation of structure equations, we can get the integrability conditions for structure equations as follows (cf. [**Wa**]):

$$A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k; \tag{2.16}$$

$$C_{i,j} - C_{j,i} = \sum_k \big(B_{ik}A_{kj} - B_{jk}A_{ki}\big); \tag{2.17}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j; \tag{2.18}$$

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}; \tag{2.19}$$

$$R_{ij} = -\sum_k B_{ik}B_{kj} + \mathrm{tr}(A)\delta_{ij} + (m-2)A_{ij}; \tag{2.20}$$

$$\mathrm{tr}(\boldsymbol{A}) = \frac{1}{2m}(1 + m^2R); \quad \sum_i B_{ii} = 0; \quad \sum_{i,j}(B_{ij})^2 = \frac{m-1}{m}, \tag{2.21}$$

where R_{ijkl} and R_{ij} denote the curvature tensor and Ricci curvature tensor of $\boldsymbol{g}$, respectively. $R = 1/(m(m-1))\sum_{i,j} R_{ijij}$ is the normalized scalar curvature. From (2.18) and (2.21) we have

$$\sum_i B_{ij,i} = -(m-1)C_j. \tag{2.22}$$

We define Möbius shape operator B by using

$$\boldsymbol{g}(BX, Y) = \boldsymbol{B}(X, Y), \quad X, Y \in T_p(M), \tag{2.23}$$

at each point $p \in M$.

REMARK 2.2. Some recent results about the Möbius geometry of submanifolds can be found in [**AG1**], [**AG2**], [**GLW1**], [**HL1**], [**HL2**], [**LLWZ**], [**LW1**], [**LW2**] and [**LWW**].

3. The Möbius invariants of the hypersurfaces with isotropic Blaschke tensor in $\boldsymbol{S}^{m+1}$.

Let $x : M^m \to \boldsymbol{S}^{m+1}$ be an $m(\geq 3)$-dimensional hypersurface with isotropic Blaschke tensor and without umbilical point in unit sphere $\boldsymbol{S}^{m+1}$. On the following we assume that λ is not constant. Let U be a connected open set on which we have $\nabla\lambda \neq 0$. We prove first the following proposition.

PROPOSITION 3.1. *Möbius shape operator B has two distinct principal curvatures $(m-1)/m$ and $-1/m$ of multiplicities 1 and $m-1$ on U.*

For each point $p \in U$ we can choose a field of tangent orthonormal frames $E_1, \ldots, E_m$ of $(M, \boldsymbol{g})$ on a neighborhood of p such that

$$B_{ij} = B_{ii}\delta_{ij}, \quad 1 \leq i, j \leq m, \tag{3.1}$$

at the point p. Let $f_i = E_i(f)$ for a function f on M. Then, at the point p, we can write equation (2.16) as

$$\lambda_k\delta_{ij} - \lambda_j\delta_{ik} = B_{ii}\delta_{ik}C_j - B_{ii}\delta_{ij}C_k, \quad 1 \leq i, j, k \leq m. \tag{3.2}$$

By taking i, j, k such that $i = j \neq k$ in (3.2), we have

$$\lambda_k = -B_{ii}C_k, \quad i \neq k. \tag{3.3}$$

By taking j such that $j = i$, making summation for i from 1 to m and using (2.21) we have

$$(m-1)\lambda_k = B_{kk}C_k, \quad 1 \le k \le m. \tag{3.4}$$

(3.3) and (3.4) imply

$$[B_{kk} + (m-1)B_{ii}]\lambda_k = 0, \quad i \ne k. \tag{3.5}$$

The assumption $\nabla\lambda \ne 0$ implies that at least one of λ_i's is not zero at p. Without losing generality, we assume $\lambda_1 \ne 0$ at point p. Then from (3.5) and the last identity of (2.21) we see that

$$B_{11} = \frac{m-1}{m}, \quad B_{ii} = -\frac{1}{m}, \quad 2 \le i \le m. \tag{3.6}$$

Since p is arbitrary, we complete the proof of Proposition 3.1. $\square$

From Proposition 3.1 we know that multiplicity of each principal curvature is constant and so we can define two distributions V_1 and V_2 as follows:

$$V_1 = \bigcup_{p\in U} V_1(p), \quad V_2 = \bigcup_{p\in U} V_2(p), \tag{3.7}$$

where $V_1(p)$ and $V_2(p)$ are the eigenspaces corresponding to $(m-1)/m$ and $-1/m$, with $\dim(V_1(p)) = 1$ and $\dim(V_2(p)) = m-1$. Thus we have decomposition

$$T(U) = V_1 \oplus V_2. \tag{3.8}$$

We can choose a fields of orthonormal tangent frame $E_1, \ldots, E_m$ of $T(U)$ in a neighborhood of an point $p \in U$, such that E_1 is a basis of V_1 and $E_2, \ldots, E_m$ is a basis of V_2. For convenience, we make the following convention on the ranges of indices:

$$1 \le i, j, k, l \le m, \quad 2 \le \alpha, \beta, \gamma \le m. \tag{3.9}$$

Then under the basis $\{E_1, E_\alpha\}$ we have

$$B_{1j} = \frac{m-1}{m}\delta_{1j}, \quad B_{\alpha j} = -\frac{1}{m}\delta_{\alpha j}. \tag{3.10}$$

Therefore, (3.2), (3.3) and (3.4) hold in the neighborhood of p. From (3.3) and (3.4) we have

$$\lambda_1 = \frac{1}{m}C_1, \quad \lambda_\alpha = -\frac{m-1}{m}C_\alpha, \quad (m-1)\lambda_\alpha = -\frac{1}{m}C_\alpha, \tag{3.11}$$

which implies

$$C_\alpha = 0, \quad C_1 = m\lambda_1, \tag{3.12}$$

and

$$E_\alpha(\lambda) = 0, \quad E_1 = \frac{\nabla\lambda}{|\nabla\lambda|}, \quad 2 \le \alpha \le m. \tag{3.13}$$

Let $\{\omega_{ij}\}$ be the connection forms of $\boldsymbol{g}$ with respect to $\{\omega_i\}$. We have

$$\sum_{k=1}^m B_{1j,k}\omega_k = dB_{1j} + \sum_k B_{kj}\omega_{k1} + \sum_k B_{1k}\omega_{kj} = \left(B_{jj} - \frac{m-1}{m}\right)\omega_{j1}, \quad 1 \le j \le m, \tag{3.14}$$

which shows

$$B_{11,k} = 0, \quad \sum_k B_{1\alpha,k}\omega_k = \omega_{1\alpha}. \tag{3.15}$$

Thus, from (2.18) we have

$$B_{1\alpha,1} = B_{11,\alpha} + \delta_{1\alpha}C_1 - \delta_{11}C_\alpha = 0,$$

and from (3.15) we have

$$\omega_{1\alpha} = \sum_{\beta=2}^m B_{1\alpha\beta}\omega_\beta. \tag{3.16}$$

Since $B_{\alpha\beta} = -(1/m)\delta_{\alpha\beta}$, we have

$$\sum_{k=1}^m B_{\alpha\beta,k}\omega_k = dB_{\alpha\beta} + \sum_k B_{k\beta}\omega_{k\alpha} + \sum_k B_{\alpha k}\omega_{k\beta} = 0,$$

and so we have $B_{\alpha\beta,1} = 0$. Thus from (2.28) we have

$$B_{1,\alpha\beta} = B_{\alpha 1,\beta} = B_{\alpha\beta,1} + \delta_{\alpha 1}C_\beta - \delta_{\alpha\beta}C_1 = -\delta_{\alpha\beta}C_1. \tag{3.17}$$

From (3.16), (3.17) and (3.12) we get

$$\omega_{1\alpha} = -m\lambda_1\omega_\alpha, \quad \alpha \geq 2. \tag{3.18}$$

From (2.7) and (2.9), we see that the connection forms ω_{ij} is defined by the following equation:

$$d\omega_1 = \sum_{\alpha=2}^{m} \omega_\alpha \wedge \omega_{\alpha 1}; \quad d\omega_\alpha = \omega_1 \wedge \omega_{1\alpha} + \sum_{\beta=2}^{m} \omega_\beta \wedge \omega_{\beta\alpha}. \tag{3.19}$$

Thus the equation (3.18) and (3.19) imply

$$d\omega_1 = 0. \tag{3.20}$$

According to Frobenius theorem, (3.20) implies distribution $V_2 = \text{span}\{E_2, \ldots, E_m\}$ is locally integrable. Let V be the integral manifold of the distribution. Locally, M^m is diffeomorphic equivalent to product manifold $l \times V$, where interval $l \subset \boldsymbol{R}^1$. From (3.12) we see that, restricted on $\{t\} \times V$, λ is constant. From the Gauss equation (2.19), we have

$$R_{\alpha\beta\alpha\beta} = \frac{1}{m^2} + 2\lambda, \quad 2 \leq \alpha, \beta \leq m, \quad \alpha \neq \beta. \tag{3.21}$$

Since λ is constant on V, and (3.21) shows that the Möbius metric $\boldsymbol{g}$ has constant sectional curvature on $\{t\} \times V$ for each fixed $t \in l$, we can choose the local coordinates $(u, v_2, \ldots, v_m)$ of M such that

$$\boldsymbol{g} = du^2 + e^{2p(u,v_2,\ldots,v_m)}\left(dv_2^2 + \cdots + dv_m^2\right), \tag{3.22}$$

where $e^{(\cdot)}$ denotes exponential function exp, p is a smooth function on $l \times V$.

From (3.22), (3.18) and (3.12) we have

$$\omega_1 = du, \quad \omega_\alpha = e^p dv_\alpha, \quad \omega_{1\alpha} = -m\lambda_u e^p dv_\alpha, \quad \frac{\partial\lambda}{\partial v_\alpha} = 0. \tag{3.23}$$

LEMMA 3.1. *The function p in (3.22) satisfies*

$$p(u, v_2, \dots, v_m) = -m\lambda(u) + f(v_2, \dots, v_m), \tag{3.24}$$

and connection forms with respect to $\boldsymbol{g}$, except $\omega_{1\alpha}$, can be written as

$$\omega_{\alpha\beta} = \frac{\partial f}{\partial v_\alpha} dv_\beta - \frac{\partial f}{\partial v_\beta} dv_\alpha, \tag{3.25}$$

where f is a smooth function on V.

PROOF. One hand, we have

$$d\omega_\alpha = e^p dp \wedge dv_\alpha.$$

On the other hand, from (3.19) and (3.23) we have

$$d\omega_\alpha = -m\lambda_u e^p du \wedge dv_\alpha + e^p \sum_{\beta=2}^{m} dv_\beta \wedge \omega_{\beta\alpha}.$$

Thus we have equation

$$(dp + m\lambda_u du) \wedge dv_\alpha = \sum_{\gamma=2}^{m} \omega_{\alpha\gamma} \wedge dv_\gamma. \tag{3.26}$$

Noting that

$$(dp + m\lambda_u du) \wedge dv_\alpha \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v_\beta} \right) = (p_u + m\lambda_u)\delta_{\alpha\beta},$$

and

$$\sum_{\gamma=2}^{m} \omega_{\alpha\gamma} \wedge dv_\gamma \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v_\beta} \right) = \omega_{\alpha\beta} \left(\frac{\partial}{\partial u} \right),$$

we have

$$(p_u + m\lambda_u)\delta_{\alpha\beta} = \omega_{\alpha\beta} \left(\frac{\partial}{\partial u} \right),$$

which yields

$$p_u + m\lambda_u = 0, \quad \omega_{\alpha\beta}\left(\frac{\partial}{\partial u}\right) = 0. \tag{3.27}$$

From the first equation of (3.27) we have (3.24). From the second equation of (3.27) we see that $\omega_{\alpha\beta}$ can be expressed as

$$\omega_{\alpha\beta} = \sum_{\gamma=2}^{m} a_{\alpha\beta\gamma} dv_\gamma, \quad a_{\alpha\beta\gamma} = -a_{\beta\alpha\gamma}. \tag{3.28}$$

By putting (3.27) and (3.28) into (3.26) we have

$$\left(\frac{\partial p}{\partial v_\gamma}\delta_{\alpha\beta} + a_{\alpha\gamma\beta}\right) dv_\gamma \wedge dv_\beta = 0.$$

We get equation

$$\delta_{\alpha\beta}\frac{\partial p}{\partial v_\gamma} - \delta_{\alpha\gamma}\frac{\partial p}{\partial v_\beta} = a_{\alpha\beta\gamma} - a_{\alpha\gamma\beta}. \tag{3.29}$$

Since $a_{\alpha\beta\gamma} = -a_{\beta\alpha\gamma}$, from (3.29) we can get

$$a_{\alpha\beta\gamma} = \frac{\partial p}{\partial v_\alpha}\delta_{\beta\gamma} - \frac{\partial p}{\partial v_\beta}\delta_{\alpha\gamma}. \tag{3.30}$$

(3.30) and (3.28) imply (3.25). This proves Lemma 3.1. □

LEMMA 3.2. *The function $\lambda(u)$ and $f(v_2, \ldots, v_m)$ satisfy the following equations*:

$$m^2(\lambda')^2 + 2\lambda + \frac{1}{m^2} = ae^{2m\lambda}, \tag{3.31}$$

$$\frac{\partial^2 f}{\partial v_\alpha^2} + \frac{\partial^2 f}{\partial v_\beta^2} + \sum_{\gamma=2}^{m}\left(\frac{\partial f}{\partial v_\gamma}\right)^2 - \left(\frac{\partial f}{\partial v_\alpha}\right)^2 - \left(\frac{\partial f}{\partial v_\beta}\right)^2 = -ae^{2f}, \tag{3.32}$$

where a is a constant.

PROOF. Since the connection forms $\{\omega_{1\alpha}, \omega_{\alpha\beta}\}$ of $\boldsymbol{g}$ are given by (3.23) and (3.25), we have

$$-\frac{1}{2}\sum_{i,j=1}^{m} R_{1\alpha ij}\omega_i\wedge\omega_j = d\omega_{1\alpha}-\sum_{\beta=2}^{m}\omega_{1\beta}\wedge\omega_{\beta\alpha}$$
$$= -me^{p}(\lambda''-m(\lambda')^2)du\wedge dv_\alpha, \tag{3.33}$$

which shows that

$$R_{1\alpha 1\alpha} = m(\lambda''-m(\lambda')^2). \tag{3.34}$$

From (3.25)we have

$$d\omega_{\alpha\beta} = \frac{\partial^2 f}{\partial v_\gamma \partial v_\alpha}dv_\gamma\wedge dv_\beta - \frac{\partial^2 f}{\partial v_\gamma \partial v_\beta}dv_\gamma\wedge dv_\alpha. \tag{3.35}$$

From (3.35) and (3.25) we have

$$d\omega_{\alpha\beta}-\sum_{\gamma=2}^{m}\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}$$
$$=\sum_{\gamma,\varepsilon=2}^{m}\Bigg\{\frac{\partial^2 f}{\partial v_\gamma\partial v_\alpha}\delta_{\beta\varepsilon}-\frac{\partial^2 f}{\partial v_\gamma\partial v_\beta}\delta_{\alpha\varepsilon}-\frac{\partial f}{\partial v_\alpha}\frac{\partial f}{\partial v_\gamma}\delta_{\beta\varepsilon}$$
$$+\frac{\partial f}{\partial v_\beta}\frac{\partial f}{\partial v_\gamma}\delta_{\alpha\varepsilon}+\left(\sum_{\alpha}\left(\frac{\partial f}{\partial v_\alpha}\right)^2\right)\delta_{\alpha\gamma}\delta_{\beta\varepsilon}\Bigg\}dv_\gamma\wedge dv_\varepsilon. \tag{3.36}$$

Since

$$-\frac{1}{2}\sum_{i,j=1}^{m}R_{\alpha\beta ij}\omega_i\wedge\omega_j = d\omega_{\alpha\beta}-\omega_{\alpha 1}\wedge\omega_{1\beta}-\sum_{\gamma=2}^{m}\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta},$$

from (3.36) and (3.23) we have

$$-e^{2p}R_{\alpha\beta\gamma\varepsilon} = \frac{\partial^2 f}{\partial v_\gamma\partial v_\alpha}\delta_{\beta\varepsilon}-\frac{\partial^2 f}{\partial v_\varepsilon\partial v_\alpha}\delta_{\beta\gamma}-\frac{\partial^2 f}{\partial v_\gamma\partial v_\beta}\delta_{\alpha\varepsilon}+\frac{\partial^2 f}{\partial v_\varepsilon\partial v_\beta}\delta_{\alpha\gamma}$$
$$-\frac{\partial f}{\partial v_\alpha}\frac{\partial f}{\partial v_\gamma}\delta_{\beta\varepsilon}+\frac{\partial f}{\partial v_\alpha}\frac{\partial f}{\partial v_\varepsilon}\delta_{\beta\gamma}+\frac{\partial f}{\partial v_\beta}\frac{\partial f}{\partial v_\gamma}\delta_{\alpha\varepsilon}-\frac{\partial f}{\partial v_\beta}\frac{\partial f}{\partial v_\varepsilon}\delta_{\alpha\gamma}$$
$$+\left(m^2(\lambda')^2e^{2p}+\sum_{t=2}^{m}\left(\frac{\partial f}{\partial v_t}\right)^2\right)(\delta_{\alpha\gamma}\delta_{\beta\varepsilon}-\delta_{\alpha\varepsilon}\delta_{\beta\gamma}), \tag{3.37}$$

which shows

$$e^{2p}R_{\alpha\beta\alpha\beta} = -\frac{\partial^2 f}{\partial v_\alpha^2} - \frac{\partial^2 f}{\partial v_\beta^2} - m^2(\lambda')^2 e^{2p} - \sum_{\gamma=2}^{m}\left(\frac{\partial f}{\partial v_\gamma}\right)^2 + \left(\frac{\partial f}{\partial v_\alpha}\right)^2 + \left(\frac{\partial f}{\partial v_\beta}\right)^2, \quad (3.38)$$

where $\alpha \neq \beta$.

On the other hand, from Gauss equation (2.19) we have

$$R_{1\alpha 1\alpha} = -\frac{m-1}{m^2} + 2\lambda. \quad (3.39)$$

Thus, (3.34) and (3.39) show that function λ satisfies

$$\lambda'' - m(\lambda')^2 = \frac{2}{m}\lambda - \frac{m-1}{m^3}. \quad (3.40)$$

Also from Gauss equation (2.19) we have (3.21). Thus, from (3.38) and (3.21) and (3.24) we have

$$\begin{aligned} & e^{-2m\lambda(u)}\left(m^2(\lambda')^2 + 2\lambda + \frac{1}{m^2}\right) \\ & \quad = -e^{-2f(v)}\left\{\left[\frac{\partial^2 f}{\partial v_\alpha^2} + \frac{\partial^2 f}{\partial v_\beta^2} + \sum_{\gamma=2}^{m}\left(\frac{\partial f}{\partial v_\gamma}\right)^2\right] - \left(\frac{\partial f}{\partial v_\alpha}\right)^2 - \left(\frac{\partial f}{\partial v_\beta}\right)^2\right\}, \quad (3.41) \end{aligned}$$

where v denotes the vector $(v_2, \ldots, v_m)$ in $\boldsymbol{R}^{m-1}$. We see that the left of the equation (3.41) is independent on v and the right of the equation is independent on u, which implies (3.31) and (3.32). This completes the proof of Lemma 3.2. □

REMARK 3.1. We note that the equation (3.31) is the first integral of (3.40). The function $\lambda(u)$ is decided by the following equation:

$$u = \pm \int \frac{m d\lambda}{\sqrt{a e^{2m\lambda} - 2\lambda - \frac{1}{m^2}}}. \quad (3.42)$$

We view quadratic form $\tilde{g} = e^{2f}(dv_2^2 + \cdots + dv_m^2)$ as a metric on manifold V. It is easy to see that the connection forms, denoted by $\tilde{\omega}_{\alpha\beta}$, with respect to this metric, are

$$\tilde{\omega}_{\alpha\beta} = \omega_{\alpha\beta} = \frac{\partial f}{\partial v_\alpha} dv_\beta - \frac{\partial f}{\partial v_\beta} dv_\alpha. \tag{3.43}$$

Let $\tilde{R}$ denote the curvature induced by $\tilde{\omega}_{\alpha\beta}$, which means

$$-\frac{1}{2}\sum_{\gamma\varepsilon} \tilde{R}_{\alpha\beta\gamma\varepsilon} e^{2f} dv_\gamma \wedge dv_\varepsilon = d\tilde{\omega}_{\alpha\beta} - \sum_\gamma \tilde{\omega}_{\alpha\gamma} \wedge \tilde{\omega}_{\gamma\beta}.$$

From the (3.36), we see

$$e^{2f} \tilde{R}_{\alpha\beta\alpha\beta} = -\frac{\partial^2 f}{\partial v_\alpha^2} - \frac{\partial^2 f}{\partial v_\beta^2} - \sum_{\gamma=2}^{m} \left(\frac{\partial f}{\partial v_\gamma}\right)^2 + \left(\frac{\partial f}{\partial v_\alpha}\right)^2 + \left(\frac{\partial f}{\partial v_\beta}\right)^2, \tag{3.44}$$

where $\alpha \neq \beta$. By making use of Lemma 3.2 we get

$$\tilde{R}_{\alpha\beta\alpha\beta} = a. \tag{3.45}$$

Now that metric $\tilde{g}$ on V has constant curvature a, there exist the local coordinates, for convenience, denoted by using same symbol $(v_2, \ldots, v_m)$ as above, such that

$$\tilde{g} = \frac{dv_2^2 + \cdots + dv_m^2}{\left(1 + \frac{a}{4}\|v\|^2\right)^2}, \tag{3.46}$$

where $\|v\|^2 = \sum_{\alpha=2}^m v_\alpha^2$.

By summing up above all, we come to the following conclusion:

THEOREM 3.1. *For an $m(\geq 3)$-dimensional hypersurface M^m with isotropic Blaschke in $\boldsymbol{S}^{m+1}$, there exist local coordinations (u, v_α), such that*

$$\boldsymbol{g} = du^2 + e^{-2m\lambda(u)}\left(e^{2f(v)}\sum_\alpha dv_\alpha^2\right), \tag{3.47}$$

$$\boldsymbol{B} = \frac{m-1}{m} du^2 - \frac{1}{m} e^{-2m\lambda(u)}\left(e^{2f(v)}\sum_\alpha dv_\alpha^2\right), \tag{3.48}$$

$$\Phi = m\lambda' du, \tag{3.49}$$

where function $\lambda(u)$ is given by (3.42) *and function $f(v)$ is given by*

$$f(v) = -\log\left(1 + \frac{a}{4}\|v\|^2\right). \tag{3.50}$$

Theorem 3.1 shows that the all Möbius invariants in structure equations are determined by function λ and f in (3.42) and (3.50), and so we can get the hypersurface M^m by integrating the structure equations. The procedure of integrating the structure equations is some complicated, however it is interesting, which shows the benefit of linearizing conformal group in studying the conformal geometry of submanifolds in space form. We will complete the procedure in the next sections.

4. The differential equations of the hypersurfaces with isotropic Blaschke tensor in S^{m+1}.

In this section we derive the differential equation of the Möbius position vector function Y.

From the structure equations (2.7)–(2.10), by using (3.23)–(3.25), (3.48) and (3.49) we have

$$dY = Y_u du + \sum_\alpha \frac{\partial Y}{\partial v_\alpha} dv_\alpha, \tag{4.1}$$

$$dN = (\lambda Y_u + m\lambda' E)du + \lambda \sum_\alpha \frac{\partial Y}{\partial v_\alpha} dv_\alpha, \tag{4.2}$$

$$dE = -\left(m\lambda' Y + \frac{m-1}{m} Y_u\right) du + \frac{1}{m} \sum_\alpha \frac{\partial Y}{\partial v_\alpha} dv_\alpha, \tag{4.3}$$

$$dY_u = -\left(\lambda Y + N + \frac{m-1}{m} E\right) du - m\lambda' \sum_\alpha \frac{\partial Y}{\partial v_\alpha} dv_\alpha, \tag{4.4}$$

$$\begin{aligned} dE_\alpha(Y) = {} & -e^p\left(\lambda Y + N + \frac{1}{m}E - m\lambda' Y_u + e^{-2p} \sum_\beta \frac{\partial f}{\partial v_\beta}\frac{\partial Y}{\partial v_\beta}\right) dv_\alpha \\ & + e^{-p} \sum_\beta \frac{\partial f}{\partial v_\alpha}\frac{\partial Y}{\partial v_\beta} dv_\beta. \end{aligned} \tag{4.5}$$

Noting that $E_\alpha = e^{-p}(\partial/\partial v_\alpha)$, from (4.1)–(4.5) we get

$$Y_{uu} = -\left(\lambda Y + N - \frac{m-1}{m} E\right); \tag{4.6}$$

$$N_u = \lambda Y_u + m\lambda' E, \quad \frac{\partial N}{\partial v_\alpha} = \lambda \frac{\partial Y}{\partial v_\alpha}; \tag{4.7}$$

$$E_u = -\Big(m\lambda' Y + \frac{m-1}{m} Y_u\Big), \quad \frac{\partial E}{\partial v_\alpha} = \frac{1}{m}\frac{\partial Y}{\partial v_\alpha}; \tag{4.8}$$

$$\frac{\partial^2 Y}{\partial u \partial v_\alpha} + m\lambda' \frac{\partial Y}{\partial v_\alpha} = 0; \tag{4.9}$$

$$\Big(\lambda Y + N + \frac{1}{m} E - m\lambda' Y_u\Big)\delta_{\alpha\beta} = -e^{m\lambda} F_{\alpha\beta}, \tag{4.10}$$

where we set

$$F_{\alpha\beta} = e^{m\lambda - 2f}\bigg(\frac{\partial^2 Y}{\partial v_\beta \partial v_\alpha} - \Big(\frac{\partial f}{\partial v_\beta}\frac{\partial Y}{\partial v_\alpha} + \frac{\partial f}{\partial v_\alpha}\frac{\partial Y}{\partial v_\beta}\Big) + \sum_\gamma \frac{\partial f}{\partial v_\gamma}\frac{\partial Y}{\partial v_\gamma}\delta_{\alpha\beta}\bigg). \tag{4.11}$$

On the following we concentrate on the solutions of the partial differential equations (4.6)–(4.10). From (4.6) and (4.10) we have

$$E = Y_{uu} + m\lambda' Y_u - e^{m\lambda} F_{\alpha\alpha}, \tag{4.12}$$

and

$$N = -\lambda Y + (m-1)\lambda' Y_u - \frac{1}{m} Y_{uu} - \frac{m-1}{m} e^{m\lambda} F_{\alpha\alpha}. \tag{4.13}$$

From (4.8), (4.9) and (4.12) we know that Y satisfies equations

$$Y_{uuu} + m\lambda' Y_{uu} + \Big(m\lambda'' + \frac{m-1}{m}\Big) Y_u + m\lambda' Y = \big(F_{\alpha\alpha} e^{m\lambda}\big)_u, \tag{4.14}$$

$$\Big(m\lambda'' + \frac{1}{m}\Big)\frac{\partial Y}{\partial v_\alpha} = -e^{m\lambda}\frac{\partial F_{\alpha\alpha}}{\partial v_\alpha}. \tag{4.15}$$

The equation (4.9) implies $(\partial^2/\partial v_\alpha \partial u)(e^{m\lambda} Y) = 0$. Thus we have

$$Y = e^{-m\lambda}(\xi(u) + \eta(v)). \tag{4.16}$$

Putting this into (4.15) and (4.11) we have

$$e^{-2m\lambda}\left(m\lambda'' + \frac{1}{m}\right)\frac{\partial \eta}{\partial v_\alpha} = -\frac{\partial F_{\alpha\alpha}}{\partial v_\alpha}, \tag{4.17}$$

$$F_{\alpha\beta} = e^{-2f}\left(\frac{\partial^2 \eta}{\partial v_\alpha \partial v_\beta} - \frac{\partial f}{\partial v_\alpha}\frac{\partial \eta}{\partial v_\beta} - \frac{\partial f}{\partial v_\beta}\frac{\partial \eta}{\partial v_\alpha} + \sum_\gamma \frac{\partial f}{\partial v_\gamma}\frac{\partial \eta}{\partial v_\gamma}\right). \tag{4.18}$$

Noting that (3.31) implies $e^{-2m\lambda}(m\lambda'' + (1/m)) = a$, we have

$$a\frac{\partial \eta}{\partial v_\alpha} = -\frac{\partial F_{\alpha\alpha}}{\partial v_\alpha}. \tag{4.19}$$

Let $\eta = e^f\zeta$, then we have

$$F_{\alpha\beta} = e^{-f}\left(\left(\frac{\partial^2 f}{\partial v_\alpha \partial v_\beta} - \frac{\partial f}{\partial v_\alpha}\frac{\partial f}{\partial v_\beta} + \sum_\gamma \left(\frac{\partial f}{\partial v_\gamma}\right)^2 \delta_{\alpha\beta}\right)\zeta + \frac{\partial^2 \zeta}{\partial v_\alpha \partial v_\beta} + \sum_\gamma \frac{\partial f}{\partial v_\gamma}\frac{\partial \zeta}{\partial v_\gamma}\delta_{\alpha\beta}\right). \tag{4.20}$$

From (4.10) we see that $F_{\alpha\beta}$ satisfies

$$F_{\alpha\alpha} = F_{\beta\beta}, \quad F_{\alpha\beta} = 0, \quad \alpha \neq \beta. \tag{4.21}$$

Thus, from (4.20) we have

$$\frac{\partial^2 \zeta}{\partial v_\alpha \partial v_\beta} = -\left(\frac{\partial^2 f}{\partial v_\alpha \partial v_\beta} - \frac{\partial f}{\partial v_\alpha}\frac{\partial f}{\partial v_\beta}\right)\zeta, \quad \alpha \neq \beta, \tag{4.22}$$

and

$$\frac{\partial^2 \zeta}{\partial v_\alpha^2} + \left(\frac{\partial^2 f}{\partial v_\alpha^2} - \frac{\partial f}{\partial v_\alpha}\frac{\partial f}{\partial v_\alpha}\right)\zeta = \frac{\partial^2 \zeta}{\partial v_\beta^2} + \left(\frac{\partial^2 f}{\partial v_\beta^2} - \frac{\partial f}{\partial v_\beta}\frac{\partial f}{\partial v_\beta}\right)\zeta. \tag{4.23}$$

Since the function f is given by (3.50), we have

$$\frac{\partial^2 f}{\partial v_\alpha \partial v_\beta} - \frac{\partial f}{\partial v_\alpha}\frac{\partial f}{\partial v_\beta} = 0, \quad \alpha \neq \beta, \tag{4.24}$$

and

$$\frac{\partial^2 f}{\partial v_\alpha^2} - \frac{\partial f}{\partial v_\alpha}\frac{\partial f}{\partial v_\alpha} = \frac{\partial^2 f}{\partial v_\beta^2} - \frac{\partial f}{\partial v_\beta}\frac{\partial f}{\partial v_\beta}. \tag{4.25}$$

From (4.22)–(4.25) we get

$$\frac{\partial^2 \zeta}{\partial v_\alpha \partial v_\beta} = 0, \quad \frac{\partial^2 \zeta}{\partial v_\alpha^2} = \frac{\partial^2 \zeta}{\partial v_\beta^2}, \quad \alpha \neq \beta, \tag{4.26}$$

which shows

$$\zeta = \|v\|^2 \vec{a} + \sum_{\alpha=2}^{m} v_\alpha \vec{b}_\alpha + \vec{c}, \tag{4.27}$$

where $\vec{a}, \vec{b}_\alpha$ and $\vec{c}$ are constant vectors. By putting (4.27) into (4.20) and using (3.50) we have

$$F_{\alpha\alpha} + a\eta = 2\vec{a} + \frac{a}{2}\vec{c}. \tag{4.28}$$

Now we consider the unknown function ξ. By substituting (4.16) into (4.14), and using (3.31) we have

$$\begin{aligned} &\xi''' - 2m\lambda'\xi'' - \left(ae^{2m\lambda} + 2\lambda - \frac{m^2+m-1}{m^2} \right)\xi' - ma\lambda' e^{2m\lambda}\xi \\ &\quad = m\lambda' e^{2m\lambda}(F_{\alpha\alpha} + a\eta). \end{aligned} \tag{4.29}$$

From (4.27), (4.28) and (4.29) we come to the following conclusion:

THEOREM 4.1. *Let Y be the Möbius position vector of the immersion $x : M^m \to \boldsymbol{S}^{m+1}$ with $\boldsymbol{A} = \lambda \boldsymbol{g}$. If λ is not constant, then*

$$Y = e^{-m\lambda(u)}(\xi(u) + \eta(v)),$$

and the function η and ξ satisfy

$$\eta(v) = e^{f(v)}\left(\|v\|^2 \vec{a} + \sum_{\alpha=2}^{m} v_\alpha \vec{b}_\alpha + \vec{c} \right), \tag{4.30}$$

$$\begin{aligned} &\xi''' - 2m\lambda'\xi'' - \left(ae^{2m\lambda} + 2\lambda - \frac{m^2+m-1}{m^2} \right)\xi' - ma\lambda' e^{2m\lambda}\xi \\ &\quad = m\lambda' e^{2m\lambda}\left(2\vec{a} + \frac{a}{2}\vec{c}\right), \end{aligned} \tag{4.31}$$

where $f(v)$ *is given by* (3.50) *and* $\lambda(u)$ *is given by* (3.42).

In next section we give the conditions satisfied by $\vec{a}, \vec{b}_\alpha, \vec{c}$.

5. The classification theorem of the hypersurfaces with isotropic Blaschke tensor in S^{m+1}.

In this section we classify the hypersurfaces with isotropic Blaschke tensor. We first determine the constant vectors $\vec{a}, \vec{b}_\alpha, \vec{c}$ in (4.30). It is from (2.3), Theorem 3.1 and Theorem 4.1 that the Möbius position vector Y of the immersion x satisfies the following conditions:

$$Y = e^{-m\lambda}(\xi(u) + \eta(v)), \quad \langle Y, Y\rangle = 0, \tag{5.1}$$

$$\left\langle \frac{\partial Y}{\partial u}, \frac{\partial Y}{\partial u} \right\rangle = 1, \quad \left\langle \frac{\partial Y}{\partial u}, \frac{\partial Y}{\partial v_\alpha} \right\rangle = 0, \tag{5.2}$$

$$\left\langle \frac{\partial Y}{\partial v_\alpha}, \frac{\partial Y}{\partial v_\beta} \right\rangle = e^{-2m\lambda+2f}\delta_{\alpha\beta}. \tag{5.3}$$

LEMMA 5.1. *Let* $\vec{a}, \vec{b}_\alpha, \vec{c}$ *and* ξ *are the quantities given in Theorem* 4.1. *Then we have following identities*:

$$\langle \xi + \vec{c}, \xi + \vec{c} \rangle = 0, \quad \langle \xi', \xi' \rangle = e^{2m\lambda}; \tag{5.4}$$

$$\langle \xi + \vec{c}, \vec{b}_\alpha \rangle = 0, \quad \left\langle \xi + \vec{c}, 2\vec{a} - \frac{a}{2}\vec{c} \right\rangle + 1 = 0; \tag{5.5}$$

$$\langle \vec{b}_\alpha, \vec{b}_\beta \rangle = \delta_{\alpha\beta}, \quad \left\langle 2\vec{a} - \frac{a}{2}\vec{c}, \vec{b}_\alpha \right\rangle = 0, \quad \left\langle 2\vec{a} - \frac{a}{2}\vec{c}, 2\vec{a} - \frac{a}{2}\vec{c} \right\rangle = a. \tag{5.6}$$

PROOF. (5.1) implies

$$\langle \xi + \eta, \xi + \eta \rangle = 0,$$

for all (u, v). In particular, if we take $v = 0$ then we have $f(0) = 0$, $\eta(0) = \vec{c}$ and so have

$$\langle \xi + \vec{c}, \xi + \vec{c} \rangle = 0,$$

which is the first equation of (5.4). Noting that the first equation of (5.2) implies

$$e^{-2m\lambda}\big(-2m\lambda' \langle \xi + \eta, \xi' \rangle + \langle \xi', \xi' \rangle \big) = 1.$$

As $\eta(0)=\vec{c}$ we have

$$e^{-2m\lambda}\big(-2m\lambda'\langle\xi+\vec{c},\xi'\rangle+\langle\xi',\xi'\rangle\big)=1.$$

Since the first equation of (5.4) implies $\langle\xi+\vec{c},\xi'\rangle=0$, we have

$$\langle\xi',\xi'\rangle=e^{2m\lambda},$$

which is the second equation of (5.4).

From the definitions of the f and η we have

$$\begin{aligned}\frac{\partial f}{\partial v_\alpha}&=-\frac{a}{2}v_\alpha e^f,\quad \frac{\partial\eta}{\partial v_\alpha}=-\frac{a}{2}v_\alpha e^f\eta+e^f\big(2v_\alpha\vec{a}+\vec{b}_\alpha\big),\\ \frac{\partial^2\eta}{\partial v_\alpha^2}&=e^f\bigg(-\frac{a}{2}\eta-\frac{a}{2}v_\alpha\frac{\partial f}{\partial v_\alpha}\eta-\frac{a}{2}v_\alpha\frac{\partial\eta}{\partial v_\alpha}-\frac{\partial f}{\partial v_\alpha}\big(2v_\alpha\vec{a}+\vec{b}_\alpha\big)+2\vec{a}\bigg).\end{aligned}\tag{5.7}$$

Hence we have

$$\frac{\partial\eta}{\partial v_\alpha}\bigg|_{v=0}=\vec{b}_\alpha,\quad \frac{\partial^2\eta}{\partial v_\alpha^2}\bigg|_{v=0}=2\vec{a}-\frac{a}{2}\vec{c}.\tag{5.8}$$

We see that (5.1) and (5.7) implies

$$0=\left\langle Y,\frac{\partial Y}{\partial v_\alpha}\right\rangle=e^{-2m\lambda}\left\langle\xi+\eta,\frac{\partial\eta}{\partial v_\alpha}\right\rangle,$$

which yields

$$\begin{aligned}\left\langle\xi+\eta,\frac{\partial\eta}{\partial v_\alpha}\right\rangle&=0,\\ \left\langle\frac{\partial\eta}{\partial v_\alpha},\frac{\partial\eta}{\partial v_\alpha}\right\rangle+\left\langle\xi+\eta,\frac{\partial^2\eta}{\partial v_\alpha^2}\right\rangle&=0.\end{aligned}\tag{5.9}$$

By putting (5.8) into (5.9) we have

$$\langle\xi+\vec{c},\vec{b}_\alpha\rangle=0,\quad \left\langle\xi+\vec{c},2\vec{a}-\frac{a}{2}\vec{c}\right\rangle+\langle\vec{b}_\alpha,\vec{b}_\alpha\rangle=0.\tag{5.10}$$

Noting that (5.3) implies

$$\begin{aligned}\delta_{\alpha\beta}e^{-2m\lambda+2f} &= \left\langle \frac{\partial Y}{\partial v_\alpha}, \frac{\partial Y}{\partial v_\beta} \right\rangle \\ &= e^{-2m\lambda+2f}\left\langle -\frac{a}{2}v_\alpha\eta + 2v_\alpha\vec{a} + \vec{b}_\alpha, -\frac{a}{2}v_\beta\eta + 2v_\beta\vec{a} + \vec{b}_\beta \right\rangle.\end{aligned}$$

We have

$$\left\langle -\frac{a}{2}v_\alpha\eta + 2v_\alpha\vec{a} + \vec{b}_\alpha, -\frac{a}{2}v_\beta\eta + 2v_\beta\vec{a} + \vec{b}_\beta \right\rangle = \delta_{\alpha\beta}. \tag{5.11}$$

By taking $v = 0$ in (5.11) we have

$$\langle \vec{b}_\alpha, \vec{b}_\beta \rangle = \delta_{\alpha\beta},$$

which is the first equation (5.6). By taking $\alpha = \beta$ and differentiating the two sides of the equation (5.11) by ∂/v_α, we have

$$\left\langle -\frac{a}{2}v_\alpha\eta + 2v_\alpha\vec{a} + \vec{b}_\alpha,\ -\frac{a}{2}\eta - \frac{a}{2}v_\alpha\frac{\partial\eta}{\partial v_\alpha} + 2\vec{a} \right\rangle = 0. \tag{5.12}$$

By differentiating the two sides of the equation (5.12) by ∂/v_α, we have

$$\begin{aligned}&\left\langle -\frac{a}{2}\eta - \frac{a}{2}v_\alpha\frac{\partial\eta}{\partial v_\alpha} + 2\vec{a},\ -\frac{a}{2}\eta - \frac{a}{2}v_\alpha\frac{\partial\eta}{\partial v_\alpha} + 2\vec{a} \right\rangle \\ &\quad + \left\langle -\frac{a}{2}v_\alpha\eta + 2v_\alpha\vec{a} + \vec{b}_\alpha,\ -2a\frac{\partial\eta}{\partial v_\alpha} - \frac{a}{2}v_\alpha\frac{\partial^2\eta}{\partial v_\alpha^2} \right\rangle = 0.\end{aligned} \tag{5.13}$$

By taking $v = 0$ and putting (5.8) into (5.12) and (5.13) we get the second equation and the third equation of (5.6). This completes the proof of Lemma 5.1. □

We need to discuss the cases $a = 0$, $a < 0$ and $a > 0$.

Case I: $a = 0$.
In this case, the equation (4.31) is reduced to the following equation:

$$\xi''' - 2m\lambda'\xi'' - \left(2\lambda - \frac{m^2+m-1}{m^2}\right)\xi' = m\lambda' e^{2m\lambda}. \tag{5.14}$$

We use $\bar{\xi}$ to denote the solution of the homogeneous equation corresponding to equation (5.14):

$$\bar{\xi}''' - 2m\lambda'\bar{\xi}'' - \left(2\lambda - \frac{m^2+m-1}{m^2}\right)\bar{\xi}' = 0. \tag{5.15}$$

Let y denote a component of the unknown vector value function ξ in the equation (5.14). We consider ordinary equation

$$y''' - 2m\lambda' y'' - \left(2\lambda - \frac{m^2+m-1}{m^2}\right)y' = m\lambda' e^{2m\lambda} \tag{5.16}$$

and the corresponding homogeneous equation:

$$y''' - 2m\lambda' y'' - \left(2\lambda - \frac{m^2+m-1}{m^2}\right)y' = 0. \tag{5.17}$$

In this case, if we set $z = y'$ then the equations (5.16) and (5.17) are reduced to

$$z'' - 2m\lambda' z' - \left(2\lambda - \frac{m^2+m-1}{m^2}\right)z = m\lambda' e^{2m\lambda}, \tag{5.18}$$

$$z'' - 2m\lambda' z' - \left(2\lambda - \frac{m^2+m-1}{m^2}\right)z = 0. \tag{5.19}$$

Noting λ satisfies the equation (3.31), we get two linear independent solutions of (5.19) as follows

$$z_1 = e^{m\lambda}\cos u, \quad z_2 = e^{m\lambda}\sin u, \tag{5.20}$$

where c is a nonzero constant. z_1 and z_2 have properties:

$$z_1' = m\lambda' z_1 - z_2, \quad z_2' = m\lambda' z_2 + z_1, \tag{5.21}$$

$$z_1 z_1' + z_2 z_2' = m\lambda'\big(z_1^2 + z_2^2\big), \quad z_1^2 + z_2^2 = e^{2m\lambda}. \tag{5.22}$$

It is easy to check that

$$\psi(u) = z_1 \int z_1 du + z_2 \int z_2 du \tag{5.23}$$

is a special solution of the non-homogeneous equation (5.18). The general solution can be expressed by

$$z = \psi + Az_1 + Bz_2. \tag{5.24}$$

We use the notations y_0, y_1 and y_2, which are defined as

$$y_0 = \int \psi, \quad y_1 = \int z_1, \quad y_2 = \int z_2. \tag{5.25}$$

Then the general solution of (5.18) can be expressed by

$$y = y_0 + Ay_1 + By_2, \tag{5.26}$$

where A, B are constant. Thus the general solution of the vector equation (5.14) and (5.15) can be expressed as follows:

$$\xi = 2\vec{a}y_0 + y_1\vec{A} + y_2\vec{B}, \tag{5.27}$$

$$\bar{\xi} = y_1\vec{A} + y_2\vec{B}. \tag{5.28}$$

We can get the analytic representations of z_1 and z_2. In fact, since $a = 0$, from (3.31) or (3.42) we have

$$\lambda = -\frac{u^2}{2m^2} + \frac{b}{m}u - \frac{1}{2}\Big(b^2 + \frac{1}{m^2}\Big), \tag{5.29}$$

where b is constant. We have

$$\begin{aligned} z_1 &= \cos u \exp\Big(-\frac{u^2}{2m} + bu - \frac{m}{2}\Big(b^2 + \frac{1}{m^2}\Big)\Big), \\ z_2 &= \sin u \exp\Big(-\frac{u^2}{2m} + bu - \frac{m}{2}\Big(b^2 + \frac{1}{m^2}\Big)\Big). \end{aligned} \tag{5.30}$$

We are going to determine the constant vectors $\vec{a}, \vec{b}_\alpha, \vec{c}$ in (4.30) and $\vec{A}, \vec{B}$ in (5.27). Since $a = 0$, from Lemma 5.1 we have

$$\langle \vec{a}, \vec{a} \rangle = 0, \quad \langle \vec{a}, \vec{b}_\alpha \rangle = 0, \quad \langle \vec{b}_\alpha, \vec{b}_\beta \rangle = \delta_{\alpha\beta}, \tag{5.31}$$

$$\langle \vec{b}_\alpha, \vec{A} \rangle y_1 + \langle \vec{b}_\alpha, \vec{B} \rangle y_2 + \langle \vec{b}_\alpha, \vec{c} \rangle = 0, \tag{5.32}$$

$$\langle \vec{a}, \vec{A} \rangle y_1 + \langle \vec{a}, \vec{B} \rangle y_2 + \langle \vec{c}, \vec{a} \rangle = -\frac{1}{2}, \tag{5.33}$$

$$\langle \vec{A}, \vec{A} \rangle z_1^2 + \langle \vec{B}, \vec{B} \rangle z_2^2 + 4\langle \vec{a}, \vec{A} \rangle \psi z_1 + 4\langle \vec{a}, \vec{A} \rangle \psi z_2 + 2\langle \vec{A}, \vec{B} \rangle z_1 z_2 = e^{2m\lambda}, \tag{5.34}$$

$$\begin{aligned} y_1^2\langle \vec{A}, \vec{A} \rangle + y_2^2\langle \vec{B}, \vec{B} \rangle + 4\langle \vec{a}, \vec{A} \rangle y_0 y_1 + 4\langle \vec{a}, \vec{A} \rangle y_0 y_2 + 2\langle \vec{A}, \vec{B} \rangle y_1 y_2 \\ +4\langle \vec{a}, \vec{c} \rangle y_0 + 2\langle \vec{A}, \vec{c} \rangle y_1 + 2\langle \vec{B}, \vec{c} \rangle y_2 + \langle \vec{c}, \vec{c} \rangle = 0. \end{aligned} \tag{5.35}$$

Since z_1, z_2 are linear independent, we know that $y_1, y_2, 1$ are linear independent. From (5.32) and (5.33) we have

$$\begin{aligned} \langle \vec{b}_\alpha, \vec{A} \rangle = \langle \vec{b}_\alpha, \vec{B} \rangle = \langle \vec{b}_\alpha, \vec{c} \rangle = 0, \\ \langle \vec{a}, \vec{A} \rangle = \langle \vec{a}, \vec{B} \rangle = 0, \quad \langle \vec{c}, \vec{a} \rangle = -\frac{1}{2}. \end{aligned} \tag{5.36}$$

From (5.22), (5.34) and (5.36) we have

$$(\langle \vec{A}, \vec{A} \rangle - 1) z_1^2 + (\langle \vec{B}, \vec{B} \rangle - 1) z_2^2 + 2\langle \vec{A}, \vec{B} \rangle z_1 z_2 = 0.$$

Since (5.20) implies that z_1^2, z_2^2 and $z_1 z_2$ are linear independent, we have

$$\langle \vec{A}, \vec{A} \rangle = \langle \vec{B}, \vec{B} \rangle = 1, \quad \langle \vec{A}, \vec{B} \rangle = 0. \tag{5.37}$$

From (5.35), (5.36) and (5.37) we have

$$y_1^2 + y_2^2 + \langle \vec{c}, \vec{c} \rangle = 2y_0. \tag{5.38}$$

Since $\{\vec{a}, \vec{b}_\alpha, \vec{c}, \vec{A}, \vec{B}, \}$ satisfies (5.31), (5.36) and (5.37), we can take them, up to a Lorentz transformation in R_1^{m+3}, as following fixed vectors:

$$\begin{aligned} \vec{a} = (1, -1, 0, 0, \ldots, 0), \quad \vec{c} = (c_1, c_2, 0, 0, \ldots, 0), \\ \vec{A} = (0, 0, 1, 0, \ldots, 0), \quad \vec{B} = (0, 0, 0, 1, \ldots, 0), \\ \vec{b}_\alpha = (\underbrace{0, \ldots, 0}_{\alpha+2}, 1, 0, \ldots, 0), \quad 2 \le \alpha \le m. \end{aligned}$$

Noting $\langle \vec{a}, \vec{c} \rangle = -1/2$ (see (5.33)), we have $c_1 + c_2 = 1/2$ and so have

$$c_1 + \langle \vec{c}, \vec{c} \rangle = c_1 - c_1^2 + c_2^2 = \frac{1}{4}, \quad c_2 - \langle \vec{c}, \vec{c} \rangle = \frac{1}{4}. \tag{5.39}$$

From (5.38) and (5.39) we have

$$\begin{aligned}\xi+\eta &= \big(2y_0+c_1+\|v\|^2,\ \ -2y_0+c_2-\|v\|^2,\ y_1,y_2,v_2,\dots,v_m\big)\\ &= \left(\frac{1}{4}+y_1^2+y_2^2+\|v\|^2,\ \frac{1}{4}-\big(y_1^2+y_2^2+\|v\|^2\big),y_1,y_2,v\right).\end{aligned}$$

Noting

$$\rho(1,x)=e^{-m\lambda}(\xi+\eta),$$

we have

$$\rho=e^{-m\lambda}\left(y_1^2+y_2^2+\|v\|^2+\frac{1}{4}\right), \tag{5.40}$$

and

$$x=\left(\frac{1-4\big(y_1^2+y_2^2+\|v\|^2\big)}{4\big(y_1^2+y_2^2+\|v\|^2\big)+1},\ 2\frac{2(y_1,y_2,v)}{4\big(y_1^2+y_2^2+\|v\|^2\big)+1}\right), \tag{5.41}$$

where

$$\begin{aligned} y_1 &= \int \cos u \exp\left(-\frac{1}{2m}u^2+bu-\frac{m}{2}\left(b^2+\frac{1}{m^2}\right)\right),\\ y_2 &= \int \sin u \exp\left(-\frac{1}{2m}u^2+bu-\frac{m}{2}\left(b^2+\frac{1}{m^2}\right)\right).\end{aligned} \tag{5.42}$$

Since $\bar{\xi}\in\operatorname{span}\{\vec{A},\vec{B}\}\cong \boldsymbol{R}^2$, we can write

$$\bar{\xi}(u)=(2y_1(u),2y_2(u)),\quad u\in l\subset \boldsymbol{R}^1,$$

which is also the solution of equation (5.15). We can take $U=l\times\boldsymbol{R}^{m-1}$, let $\Gamma_1=\bar{\xi}(l)$, define an m-dimensional cylinder hypersurface in $\boldsymbol{R}^{m+1}$ as follows:

$$\Im(u,v)=(\bar{\xi}(u),2v),\quad (u,v)\in U, \tag{5.43}$$

and denote the inverse stereographic projection by σ which is defined in Section 1. Then, from (5.41) we have

$$x(U) = \sigma \circ \Im(U) = \sigma(\Gamma_1 \times \boldsymbol{R}^{m-1}). \tag{5.44}$$

CASE II: $a \neq 0$.
Let

$$\bar{\xi} = \xi + \vec{c} + \frac{1}{a}\left(2\vec{a} - \frac{a}{2}\vec{c}\right), \tag{5.45}$$

then we can write equation (4.43) as

$$\bar{\xi}''' - 2m\lambda'\bar{\xi}'' - \left(ae^{2m\lambda} + 2\lambda - \frac{m^2 + m - 1}{m^2}\right)\bar{\xi}' - ma\lambda' e^{2m\lambda}\bar{\xi} = 0. \tag{5.46}$$

From Lemma 5.1 we have

$$\langle \bar{\xi}, \bar{\xi} \rangle = -\frac{1}{a}, \quad \left\langle \bar{\xi}, 2\vec{a} - \frac{a}{2}\vec{c} \right\rangle = 0, \quad \langle \bar{\xi}, \vec{b}_\alpha \rangle = 0, \quad \langle \bar{\xi}', \bar{\xi}' \rangle = e^{2m\lambda}. \tag{5.47}$$

SUBCASE II-1: $a < 0$.
In this case, (5.6) shows that $2\vec{a} - (a/2)\vec{c}$ is a time-like vector. We can take, up to a transformation in $O(1, m+2)$, it and $\vec{b}_\alpha$ as follows:

$$\begin{aligned} 2\vec{a} - \frac{a}{2}\vec{c} &= \left(\sqrt{-a}, 0, \ldots, 0\right), \\ \vec{b}_\alpha &= (\underbrace{0, \ldots, 0}_{\alpha-1}, 1, \ldots, 0), \quad 2 \le \alpha \le m. \end{aligned} \tag{5.48}$$

Let

$$\vec{A} = (0, \ldots, 0, 1, 0, 0), \quad \vec{B} = (0, \ldots, 0, 0, 1, 0), \quad \vec{C} = (0, \ldots, 0, 0, 1).$$

Then we see that $\{2\vec{a} - (a/2)\vec{c}, \vec{A}, \vec{B}, \vec{C}, \vec{b}_\alpha\}$ is a Lorentz orthonormal basis in $\boldsymbol{R}_1^{m+3}$. From (5.47) and (5.48) we see that

$$\bar{\xi} \in \boldsymbol{S}^2\left(\frac{1}{\sqrt{-a}}\right) \subset \operatorname{span}\{\vec{A}, \vec{B}, \vec{C}\} \cong \boldsymbol{R}^3.$$

Noting that f is defined by (3.50), we can write

$$\begin{aligned}\eta-\vec{c}&=e^{f}\left(\|v\|^{2}\vec{a}+\sum_{\alpha}\vec{b}_{\alpha}v_{\alpha}+(1-e^{-f})\vec{c}\right)\\&=e^{f}\left(\frac{1}{2}\left(2\vec{a}-\frac{a}{2}\vec{c}\right)\|v\|^{2}+\sum_{\alpha}\vec{b}_{\alpha}v_{\alpha}\right).\end{aligned}\tag{5.49}$$

From (5.47) and (5.49) we have

$$\begin{aligned}\xi+\eta&=-\frac{1-\frac{a}{4}\|v\|^{2}}{a\left(1+\frac{a}{4}\|v\|^{2}\right)}\left(2\vec{a}-\frac{a}{2}\vec{c}\right)+\bar{\xi}+e^{f}\sum_{\alpha}\vec{b}_{\alpha}v_{\alpha}\\&=\left(\frac{1-\frac{a}{4}\|v\|^{2}}{\sqrt{-a}\left(1+\frac{a}{4}\|v\|^{2}\right)},\ \frac{v}{1+\frac{a}{4}\|v\|^{2}},\ \bar{\xi}\right).\end{aligned}\tag{5.50}$$

The Möbius position vector Y of the immersion x is

$$Y=\rho(1,x)=e^{-m\lambda}\left(\frac{1-\frac{a}{4}\|v\|^{2}}{\sqrt{-a}\left(1+\frac{a}{4}\|v\|^{2}\right)},\ \frac{v}{1+\frac{a}{4}\|v\|^{2}},\ \bar{\xi}\right),\tag{5.51}$$

where

$$\rho=e^{-m\lambda}\frac{1-\frac{a}{4}\|v\|^{2}}{\sqrt{-a}\left(1+\frac{a}{4}\|v\|^{2}\right)},\tag{5.52}$$

and

$$x=\left(\frac{\sqrt{-a}v}{1-\frac{a}{4}\|v\|^{2}},\ \frac{1+\frac{a}{4}\|v\|^{2}}{1-\frac{a}{4}\|v\|^{2}}\sqrt{-a}\bar{\xi}\right).\tag{5.53}$$

We can take $U=l\times V$, where $l\subset \boldsymbol{R}^{1}$ and $V=\{v:v\in\boldsymbol{R}^{m-1},\|v\|<2/\sqrt{-a}\}$. Let map

$$\tau:V\to\boldsymbol{H}^{m-1}\left(\frac{1}{\sqrt{-a}}\right),$$

denote the inverse stereographic projection to hyperbolic space which is defined by

$$\tau(v) = \left(\frac{1 - \frac{a}{4}\|v\|^2}{\sqrt{-a}\left(1 + \frac{a}{4}\|v\|^2\right)},\ \frac{v}{1 + \frac{a}{4}\|v\|^2} \right). \tag{5.54}$$

Then (5.50) implies

$$(\xi + \eta)(U) = \tau(V) \times \bar{\xi}(l) = \boldsymbol{H}^{m-1}\left(\frac{1}{\sqrt{-a}}\right) \times \Gamma_2. \tag{5.55}$$

Hence we have

$$X(U) = \pi((1, x)(U)) = \pi((\xi + \eta)(U)) = \pi\left(\boldsymbol{H}^{m-1}\left(\frac{1}{\sqrt{-a}}\right) \times \Gamma_2\right), \tag{5.56}$$

where X and π are the maps defined in Section 1.

Subcase II-2: $a > 0$.

In this case, (5.6) shows that $2\vec{a} - (a/2)\vec{c}$ is a space-like vector. We can take, up to a transformation in $O(1, m+2)$, it and $\vec{b}_\alpha$ as follows:

$$\begin{aligned} 2\vec{a} - \frac{a}{2}\vec{c} &= \big(0, 0, 0, -\sqrt{a}, 0, \ldots, 0\big), \\ \vec{b}_\alpha &= (\underbrace{0, \ldots, 0}_{\alpha+2}, 1, \ldots, 0), \quad 2 \le \alpha \le m. \end{aligned} \tag{5.57}$$

Let

$$\vec{A} = (1, 0, 0, 0, \ldots, 0), \quad \vec{B} = (0, 1, 0, 0, \ldots, 0), \quad \vec{C} = (0, 0, 1, 0, \ldots, 0).$$

Then we see that $\{2\vec{a} - (a/2)\vec{c}, \vec{A}, \vec{B}, \vec{C}, \vec{b}_\alpha\}$ is a Lorentz orthonormal basis in $\boldsymbol{R}_1^{m+3}$. From (5.47) we see that

$$\bar{\xi} \in \boldsymbol{H}^2\left(\frac{1}{\sqrt{a}}\right) \subset \operatorname{span}\{\vec{A}, \vec{B}, \vec{C}\} \cong \boldsymbol{R}_1^3.$$

From (5.47), (4.30), (3.51) and (5.57) we have

$$\xi+\eta=-\frac{1-\dfrac{a}{4}\|v\|^2}{a\left(1+\dfrac{a}{4}\|v\|^2\right)}\left(2\vec{a}-\frac{a}{2}\vec{c}\right)+\alpha\vec{A}+\beta\vec{B}+\gamma\vec{C}+e^f\sum_\alpha\vec{b}_\alpha v_\alpha$$

$$=\left(\bar{\xi},\ \frac{1-\dfrac{a}{4}\|v\|^2}{\sqrt{a}\left(1+\dfrac{a}{4}\|v\|^2\right)},\ \frac{v}{1+\dfrac{a}{4}\|v\|^2}\right). \tag{5.58}$$

Let map

$$\phi:\boldsymbol{R}^{m-1}\cup\{\infty\}\to\boldsymbol{S}^{m-1}\left(\frac{1}{\sqrt{a}}\right)$$

denote the inverse stereographic projection which is defined by

$$\phi(v)=\left(\frac{1-\dfrac{a}{4}\|v\|^2}{\sqrt{a}\left(1+\dfrac{a}{4}\|v\|^2\right)},\ \frac{v}{1+\dfrac{a}{4}\|v\|^2}\right).$$

Let $V=\boldsymbol{R}^{m-1}\cup\{\infty\}$. We can take $U=l\times V$, where $l\subset\boldsymbol{R}^1$. Then (5.58) implies

$$(\xi+\eta)(U)=\xi(l)\times\phi(V)=\Gamma_3\times\boldsymbol{S}^{m-1}\left(\frac{1}{\sqrt{a}}\right). \tag{5.59}$$

Hence we have

$$X(U)=\pi((1,x)(U))=\pi((\xi+\eta)(U))=\pi\left(\Gamma_3\times\boldsymbol{S}^{m-1}\left(\frac{1}{\sqrt{a}}\right)\right). \tag{5.60}$$

Finally, we will rewriting equations (5.46). From the last equation of (5.47) we see that the arc-length parameter s is given by $s=\int e^{m\lambda}du$. From (3.42) and (5.46) we know that λ and $\bar{\xi}$ satisfy the equations (1.6) and (1.11). We complete the proof of Theorem 1.1. □

We will show the relation between the curve $\bar{\xi}$ and a principal sphere on the following. As one of the Möbius principal curvature of x is $-1/m$, the curvature sphere corresponding to this principal curvature is

$$\mathscr{P} = E - \frac{1}{m}Y : M^m \to \boldsymbol{S}_1^{m+2}, \tag{5.61}$$

where $\boldsymbol{S}_1^{m+2} = \{Z \in \boldsymbol{R}_1^{m+3} : \langle Z, Z\rangle = 1\}$, called $m+2$-dimensional de Sitter space. The second equation of (4.8) shows

$$\frac{\partial \mathscr{P}}{\partial v_\alpha} = 0, \quad 2 \le \alpha \le m. \tag{5.62}$$

This shows that the curvature sphere $\mathscr{P}$ degenerates into a curve in $\boldsymbol{S}_1^{m+2}$. From the first equation of (4.8) we have

$$\left\langle \frac{d\mathscr{P}}{du}, \frac{d\mathscr{P}}{du} \right\rangle = 1. \tag{5.63}$$

Hence u is the arc-length parameter of the curvature sphere $\mathscr{P}$. For the case of $a \ne 0$, there is a correspondence between curvature sphere $\mathscr{P}$ and curve $\bar{\xi}$. We give this correspondence as follows:

$$\mathscr{P} = -ae^{m\lambda}\bar{\xi} + \left(e^{-m\lambda}\bar{\xi}'\right)', \tag{5.64}$$

$$\mathscr{P}' = -e^{-m\lambda}\bar{\xi}', \tag{5.65}$$

$$\bar{\xi} = -\frac{1}{a}e^{-m\lambda}(\mathscr{P} + \mathscr{P}''). \tag{5.66}$$

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Zhen Guo
Department of Mathematics
Yunnan Normal University
Kunming 650092, P. R. of China
E-mail: gzh2001y@yahoo.com

Jianbo Fang
Department of Mathematics
Yunnan Normal University
Kunming 650092, P. R. of China
E-mail: fjbwcj@126.com

Limiao Lin
Department of Mathematics
Yunnan Normal University
Kunming 650092, P. R. of China
E-mail: 83343055@163.com