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Limit formulas of period integrals for a certain symmetric pair II

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Abstract. Let $(G,H)=(U(p,q),U(p-1,q)\times U(1))$ and $\{\Gamma_n\}$ a tower of congruence uniform lattices in G. By the period integrals of automorphic forms on $\Gamma\backslash G$ along $\Gamma_n\cap H\backslash H$, we introduce a certain discrete measure $\mathrm{d}\mu_{\Gamma_n}^H$ on the H-spherical unitary dual of G. It is shown that the sequence of measures $\mathrm{d}\mu_{\Gamma_n}^H$ with growing n converges in a weak sense to the Plancherel measure $\mathrm{d}\mu^H$ for the symmetric space $H\backslash G$.

1. Introduction.

Let G be a connected reductive Lie group and Γ a lattice in G. Let $\mathrm{d}g$ be a Haar measure on G. Then, the right regular action of G on the L^2-space $L^2(\Gamma\backslash G)$ yields a unitary representation R_Γ, which is a central object in the theory of automorphic representations. When Γ is cocompact, it is known that $L^2(\Gamma\backslash G)$ is a discrete direct sum of irreducible closed invariant subspaces with finite multiplicities. For each $\pi\in\hat{G}$, let $\mathscr{V}_{\Gamma,\pi}$ be the π-isotypic component of $L^2(\Gamma\backslash G)$ defined as the image of the natural G-inclusion

$$\mathscr{I}_G(\pi\mid R_\Gamma)\otimes\mathscr{H}_\pi\ni T\otimes v\longrightarrow T(v)\in L^2(\Gamma\backslash G).$$

Here, $\mathscr{I}_G(\pi\mid R_\Gamma)$ denotes the $\boldsymbol{C}$-vector space of all the bounded G-intertwining operators from $\mathscr{H}_\pi$ to $L^2(\Gamma\backslash G)$, whose dimension $m_\Gamma(\pi)\in\boldsymbol{N}$ is the multiplicity of π in R_Γ.

Let H be a unimodular closed subgroup of G such that the inclusion $\Gamma\cap H\backslash H\hookrightarrow\Gamma\backslash G$ has a closed image. Let $\mathrm{d}h$ be a Haar measure on H. Then, for a smooth function $\phi\in L^2(\Gamma\backslash G)^\infty$, the integral

$$\int_{\Gamma\cap H\backslash H}\phi(h)\,\mathrm{d}h,\qquad \phi\in L^2(\Gamma\backslash G)^\infty \tag{1.1}$$

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is often called the H-period of ϕ. When Γ is an arithmetic lattice, this kind of integrals plays an important role in the study of automorphic L-functions. Our point of view in this paper and also in [**12**] is to regard (1.1) as a linear functional on the space $\mathscr{V}_{\Gamma,\pi}^{\infty}$ with varying $\phi \in \mathscr{V}_{\Gamma,\pi}^{\infty}$, which actually defines an H-invariant distribution vector $\mathscr{P}_{\Gamma}^{H}(\pi)$ of $\mathscr{V}_{\Gamma,\pi}$. Recall that an irreducible unitary representation π is said to be H-spherical if it admits a non-zero H-invariant distribution vector. Suppose $\pi \in \hat{G}$ satisfies the stronger condition $\dim_{\boldsymbol{C}}(\mathscr{H}_{\pi}^{-\infty})^{H} = 1$. Then, for a non zero element $l_{\pi}^{0} \in (\mathscr{H}_{\pi}^{-\infty})^{H}$, there exists a unique $T \in \mathscr{I}_{G}(\pi \mid R_{\Gamma})^{*}$ such that

$$\mathscr{P}_{\Gamma}^{H}(\pi) = T \otimes l_{\pi}^{0}$$

under the identification $\mathscr{V}_{\Gamma,\pi}^{-\infty} \cong \mathscr{I}_{G}(\pi \mid R_{\Gamma})^{*} \otimes \mathscr{H}_{\pi}^{-\infty}$. The space $\mathscr{I}_{G}(\pi \mid R_{\Gamma})$ has a natural hermitian inner product (see Lemma 33). Let us define $\|\mathscr{P}_{\Gamma}^{H}(\pi)/l_{\pi}^{0}\|^{2}$ to be the norm square of T with respect to the dual inner product on $\mathscr{I}_{G}(\pi \mid R_{\Gamma})^{*}$. The number $\|\mathscr{P}_{\Gamma}^{H}(\pi)/l_{\pi}^{0}\|^{2}$ is closely related to the number $\boldsymbol{P}_{\tau}^{H}(\Gamma)_{\pi}$ introduced in [**12**] when (G, H) is a symmetric pair. Unlike the latter quantity, the former one does not involve an irreducible representation τ of a maximal compact subgroup of G. In this paper, continuing the case study [**12**] on the asymptotic properties of $\boldsymbol{P}_{\tau}^{H}(\Gamma)_{\pi}$ with shrinking Γ for the unitary symmetric pair $(G, H) = (U(p, q), U(p-1, q) \times U(1))$ $(p, q \geq 2)$, we investigate the limiting behavior of a certain discrete measure $\mathrm{d}\mu_{\Gamma}^{H}$ on the unitary dual $\hat{G}$ associated with the numbers $\|\mathscr{P}_{\Gamma}^{H}(\pi)/l_{\pi}^{0}\|^{2}$ (see 7.3). In this article, we consider a tower $\{\Gamma_{n}\}$ of uniform lattices defined by a principal congruence condition with respect to some $\boldsymbol{Q}$-structure on G. Among other things, we prove that a sequence of measures $\mathrm{d}\mu_{\Gamma_{n}}^{H}$ with growing n converges in a weak sense to the Plancherel measure $\mathrm{d}\mu^{H}$ for the symmetric space $H\backslash G$ (Theorem 43).

Let us briefly explain the organization of this paper. The next Section 2 is a preliminary, where, in the first place, we introduce the unitary group $G \cong U(p, q)$, a maximal compact subgroup K and a symmetric subgroup $H \cong U(p-1, q) \times U(1)$. Then, recalling definitions made in [**12**], we give a realization of $H \cap K$-spherical representations of K on the space of harmonic polynomials. In Section 3, we state our substantial results (Theorems 8, 9, 11 and Corollary 10), whose proofs are given in Section 5. Our main tool of investigation here is a form of relative trace formula, which was developed in [**12**] to prove a discrete series analogue [**12**, Theorem 5] of the limit formula given in Theorem 11. The key on which the remaining results rely is Theorem 9, whose proof is given in the paragraph 5.1 by examining individual terms of the relative trace formula in detail. Once Theorem 9 is obtained, the limit formula (Theorem 11) is deduced by a similar argument as in DeGeorge-Wallach [**5**] (see also [**6**] and [**7**]). In Section 6, we give an application of Corollary 10 to have an asymptotic formula of a certain counting function associated to Hodge-Laplace eigenforms on an arithmetic quotient of $U(p,q)$. We obtain a formula (6.1) which resembles to the usual Weyl's law for the Hodge-Laplacian on forms. In the final section Section 7, we prove the main theorem (Theorem 43). Before that, following [**8**], we recall the parametrization of H-spherical irreducible unitary representations π of G, fix a normalization of H-invariant distribution vector for each π, define the Fourier transform for functions in $C_c^\infty(H\backslash G)$ and then give the Plancherel measure describing the inversion formula of Fourier transform. Having these basic materials in hands, we deduce Theorem 43 from Theorem 11 and [**12**, Theorem 5] by making a link between the two quantities $\|\mathscr{P}_\Gamma^H(\pi)/l_\pi^0\|^2$ and $\boldsymbol{P}_\tau^H(\Gamma)_\pi$ with a suitably chosen K-type τ. We should remark that our limit formulas Theorem 11 and Theorem 43 are still conditional when $p+q-1$ is odd. In the proof, we need to assume the existence of a spectral gap on the eigenvalues of Laplacian on the 'H-distinguished' automorphic forms in $L_\tau^2(\Gamma\backslash G/K)$ with varying Γ and with a fixed K-type τ. The importance of this kind of spectral gap condition is observed by Bergeron-Clozel [**2**] in a closely related context. Actually, in a geometric situation, the averaged period of automorphic forms considered in this article and in [**12**] was already considered by Bergeron ([**1**]). He proved the limit formula for a relative discrete series representation for the symmetric pair $(O(n,1), O(k,1)\times O(n-k))$ under a spectral gap hypothesis.

Finally, we should remark that the parallel argument is possible for real rank one unitary groups, which are excluded from our consideration here only for simplicity. The relevant spectral gap hypothesis at the end point of the complementary series is already established by Bergeron-Clozel [**2**, Theorem 3]. In particular, Theorem 43, when extended to the real rank one case, is true unconditionally.

2. Preliminary.

2.0.

Let G be the unitary group of a non-degenerate hermitian form $\langle\,,\,\rangle$ on an N-dimensional $\boldsymbol{C}$-vector space W:

$$G = \{g \in GL_{\boldsymbol{C}}(W) \mid \langle g\boldsymbol{x}, g\boldsymbol{y}\rangle = \langle \boldsymbol{x}, \boldsymbol{y}\rangle \text{ for any } \boldsymbol{x}, \boldsymbol{y} \in W\}.$$

(Our convention is that a hermitian form $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ is anti-linear in $\boldsymbol{y}$, i.e., $\langle \boldsymbol{x}, a\boldsymbol{y}\rangle = \bar{a}\,\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ for any $\boldsymbol{x}, \boldsymbol{y} \in W$ and for any $a \in \boldsymbol{C}$.) We assume that the signature of $\langle\,,\,\rangle$ is $(p+, q-)$ with $\inf(p,q) \geq 2$, $N = p+q$. The integer $p+q-1$, which occurs frequently in this paper, is denoted by ρ_0, i.e.,

$$\rho_0 = p+q-1.$$

Fix a negative definite subspace $W^- \subset W$ of maximal dimension and denote by W^+ the orthogonal complement of W^- in W. Thus, $\dim_{\boldsymbol{C}} W^- = q$ and $\dim_{\boldsymbol{C}} W^+ = p$. Let I be the element of G defined by

$$I \mid W^+ = \mathrm{id}, \qquad I \mid W^- = -\mathrm{id}$$

and consider the positive definite hermitian form $(\boldsymbol{v} \mid \boldsymbol{u}) = \langle \boldsymbol{v}, I(\boldsymbol{u}) \rangle$ on W. Let K be the stabilizer of W^- in G; then K is a maximal compact subgroup of G preserving the inner-product $(\,|\,)$. The Cartan involution of G corresponding to K is the inner automorphism $\theta(g) = IgI^{-1}$.

Let $H = H_\ell$ be a closed subgroup of G obtained as the stabilizer of a one dimensional subspace $\ell \subset W^+$. Let J_ℓ be the element of G defined by

$$J_\ell \mid \ell = \mathrm{id}, \quad J_\ell \mid \ell^\perp = -\mathrm{id}.$$

Then H is the fixed point subgroup of the inner automorphism $\sigma_\ell(g) = J_\ell g J_\ell^{-1}$ of G. Note that σ_ℓ is an involution of G which commutes with θ. Thus, $K_H = H \cap K$ is a maximal compact subgroup of H. We have

$$G \cong U(p,q), \quad K \cong U(p) \times U(q), \quad H \cong U(p-1,q) \times U(1).$$

2.1. Automorphic forms and H-periods.

For a uniform lattice $\Gamma \subset G$ and an irreducible unitary representation (τ, V) of K, recall that $L^2_\tau(\Gamma\backslash G/K)$ is the Hilbert space of all the square integrable functions $\phi : \Gamma\backslash G \longrightarrow V$ possessing the K-equivariance $\phi(gk) = \tau(k)^{-1}\phi(g)$, $k \in K$ ([**12**, 3.2]). Let $\Omega_{\mathfrak{g}}$ be the Casimir element of G defined as an element of $\mathrm{U}(\mathfrak{g}_{\boldsymbol{C}})$ such that $\Omega_{\mathfrak{g}} = \sum_j X_j X^j$ for any $\boldsymbol{R}$-basis $\{X_j\}$ and $\{X^j\}$ of $\mathfrak{g}$ satisfying $2^{-1}\mathrm{tr}(X_j X^i) = \delta_{ij}$. For $\nu \in \boldsymbol{C}$, let $\mathscr{A}_\tau(\Gamma; \nu)$ denote the $(\rho_0^2 - \nu^2)$-eigenspace of the Laplacian $\Delta_\tau = -\Omega_{\mathfrak{g}}$ acting on $L^2_\tau(\Gamma\backslash G/K)$. Then, $\mathscr{A}_\tau(\Gamma;\nu)$ is a finite dimensional subspace of $C^\infty(G/K;\tau)^\Gamma$. Let $S_\tau(\Gamma)$ be the set of all $\nu \in \boldsymbol{C}$ such that $\mathrm{Re}(\nu) \geq 0$ and $\mathscr{A}_\tau(\Gamma;\nu) \neq \{0\}$.

Definition. A uniform lattice $\Gamma \subset G$ is said to be H-admissible if $\sigma_\ell(\Gamma) = \Gamma$ and Γ is torsion free. □

Lemma 1. *If Γ is an H-admissible lattice of G, then $\Gamma_H = \Gamma \cap H$ is a uniform lattice of H. In particular, the image of $\Gamma_H\backslash H \hookrightarrow \Gamma\backslash G$ is compact. Moreover, the natural map $\Gamma_H\backslash H/K_H \longrightarrow \Gamma\backslash G/K$ is injective.*

Proof. This is well-known as Jaffee's lemma. For convenience, we give a proof. Let us show that $H\Gamma$ is closed in G, which implies the first assertion. Suppose a sequence $g_n = h_n\gamma_n$ ($n \in \boldsymbol{N}$) of points in $H\Gamma$ converges to $g \in G$. Then, $g_n^{-1}\sigma_\ell(g_n) = \gamma_n^{-1}\sigma_\ell(\gamma_n)$ converges to $g^{-1}\sigma_\ell(g)$ in G on the one hand. On the other hand, the points $\gamma_n^{-1}\sigma_\ell(\gamma_n)$ belongs to the discrete set Γ by H-admissibility of Γ. Hence, there exists $n \in \boldsymbol{N}$ such that $\gamma_n^{-1}\sigma_\ell(\gamma_n) = g^{-1}\sigma_\ell(g)$, or equivalently $g\gamma_n^{-1} \in H$. Thus, $g \in H\Gamma$.

To prove the second assertion, suppose $\Gamma hK = \Gamma h_1 K$ for $h, h_1 \in H$. Then there exist $\gamma \in \Gamma$ and $k \in K$ such that $h_1 = \gamma\, h\, k$. Since $\sigma_\ell(h_1) = h_1$, we obtain $\sigma_\ell(\gamma)\, h\, \sigma_\ell(k) = \gamma\, h\, k$, or equivalently $\gamma^{-1}\sigma_\ell(\gamma) = hk\sigma_\ell(k)^{-1}h^{-1}$, which, in turn, implies that $\gamma\sigma_\ell(\gamma)^{-1}$ belongs to $\Gamma \cap hKh^{-1}$. Since Γ is torsion free, $\Gamma \cap hKh^{-1} = \{e\}$. Thus, $\gamma\sigma_\ell(\gamma)^{-1} = e$, or equivalently $\gamma \in \Gamma_H$. In combination with $h_1 = \gamma hk$, the relation $\gamma \in \Gamma_H$ yields $k \in K_H$. $\square$

For an H-admissible lattice Γ in G, the H-period integral of $\phi \in \mathscr{A}_\tau(\Gamma; \tau)$ is defined to be the function $\phi^H : G \longrightarrow V$ given by

$$\phi^H(g) = \int_{\Gamma_H \backslash H} \phi(hg)\, \mathrm{d}\dot{h}, \quad g \in G.$$

We focus on the value $\phi^H(e)$ at the identity, which belongs to $V^{H\cap K}$ ([**12**, Lemma 2]), and study the norm square of $\phi^H(e)$ collectively by taking summation over ϕ's belonging to an orthonormal basis $\mathscr{B}(\nu)$ of $\mathscr{A}_\tau(\Gamma; \tau)$:

$$\boldsymbol{P}_\tau^H(\Gamma; \nu) = \sum_{\phi \in \mathscr{B}(\nu)} \|\phi^H(e)\|^2.$$

By [**12**, Lemma 3], this number is independent of the choice of $\mathscr{B}(\nu)$. Note that the set

$$S_\tau^H(\Gamma) = \{\nu \in S_\tau(\Gamma) \mid \boldsymbol{P}_\tau^H(\Gamma; \nu) \neq 0\}$$

is empty unless $V^{H\cap K} \neq \{0\}$.

2.2. $\boldsymbol{H \cap K}$-spherical representations of $\boldsymbol{K}$.

An irreducible unitary representation of K having $H \cap K$-fixed vectors is realized on a space of certain harmonic polynomials; we recall the construction briefly. Let us fix a basis $\{\boldsymbol{v}_j\}_{1\le j\le N}$ of W orthonormal with respect to $(\,|\,)$ such that $W^+ = \langle \boldsymbol{v}_j \mid 1 \le j \le p\rangle_{\boldsymbol{C}}$, $W^- = \langle \boldsymbol{v}_{i+p} \mid 1 \le i \le q\rangle_{\boldsymbol{C}}$ and $\ell = \boldsymbol{C}\,\boldsymbol{v}_p$. Let $\{x_j\}$ be the dual $\boldsymbol{C}$-basis of $\{\boldsymbol{v}_j\}$, i.e., $x_j(\boldsymbol{v}_i) = \delta_{ij}$. Let $X = \mathrm{Hom}_{\boldsymbol{R}}(W, \boldsymbol{C})$ be the $\boldsymbol{C}$-vector space of all the $\boldsymbol{R}$-linear maps from W to $\boldsymbol{C}$. The complex conjugate $\bar{x}$ of $x \in X$ is defined by $\bar{x}(\boldsymbol{v}) = \overline{x(\boldsymbol{v})}$, $\boldsymbol{v} \in W$. Then, x_j's, together with their complex conjugates $\bar{x}_j$ $(1 \le j \le N)$, form a $\boldsymbol{C}$-basis of X, by which

$$(\boldsymbol{v} \mid \boldsymbol{u}) = \sum_{j=1}^{N} x_j(\boldsymbol{v})\bar{x}_j(\boldsymbol{u}), \quad \boldsymbol{v},\, \boldsymbol{u} \in W.$$

Let $\mathscr{P}$ be the symmetric algebra of X; it is identified with the polynomial algebra over $\boldsymbol{C}$ of the variables x_j, $\bar{x}_j$ $(1 \le j \le N)$. For $d \in \boldsymbol{N}$, let V_d be the set of all $P \in \mathscr{P}$ with the following properties.

(1) P belongs to $\boldsymbol{C}[x_1, \dots, x_p; \bar{x}_1, \dots, \bar{x}_p]$.
(2) P is homogeneous in the sense that $P(t_1\boldsymbol{x}, t_2\bar{\boldsymbol{x}}) = t_1^d t_2^d P(\boldsymbol{x}, \bar{\boldsymbol{x}})$ holds for any $t_1, t_2 \in \boldsymbol{R}^\times$. Here, $\boldsymbol{x} = (x_j)_{1 \le j \le p}$ and $\bar{\boldsymbol{x}} = (\bar{x}_j)_{1 \le j \le p}$.
(3) P is harmonic, i.e., $\sum_{j=1}^{p} \partial_j \bar{\partial}_j P = 0$. Here, $\partial_j = \partial/\partial x_j$ and $\bar{\partial}_j = \partial/\partial \bar{x}_j$.

The set V_d is a K-stable irreducible subspace of $\mathscr{P}$. Let τ_d be the action of K on V_d. Endowed with a K-invariant inner product, (τ_d, V_d) is a unitary representation of K.

Lemma 2. *(τ_d, V_d) is an irreducible representation of K and $V_d^{H \cap K} \ne \{0\}$. Up to equivalence, the irreducible unitary representations of K having $H \cap K$-fixed vectors are exhausted by $\{(\tau_d, V_d) \mid d \in \boldsymbol{N}\}$.*

2.3. $\boldsymbol{H}$-hyperbolic elements.

Recall that H is the stabilizer of a line $\ell \subset W^+$. Choosing a vector $\boldsymbol{u}$ such that $\ell = \boldsymbol{C}\boldsymbol{u}$, we define a bi H-invariant function $\xi_\ell : G \to \boldsymbol{R}$ by

$$\xi_\ell(g) = \left| \frac{\langle g\boldsymbol{u}, \boldsymbol{u} \rangle}{\langle \boldsymbol{u}, \boldsymbol{u} \rangle} \right|, \qquad g \in G.$$

Note that ξ_ℓ is independent of the choice of $\boldsymbol{u}$.

Definition. An element $g \in G - H$ satisfying $\xi_\ell(g) > 1$ is called *H-hyperbolic*.

2.4. Construction of $\boldsymbol{H}$-admissible lattices.

Let E be a subfield of $\boldsymbol{C}$ such that $E/\boldsymbol{Q}$ is a CM extension of finite degree, and $\mathscr{O}_E$ the ring of algebraic integers in E.

Let F be the maximal real subfield $\boldsymbol{R} \cap E$ of E. Then, F is a totally real extension of $\boldsymbol{Q}$ and E is a quadratic extension of F. Let $\iota_\nu : F \hookrightarrow \boldsymbol{R}$ $(1 \le \nu \le d_F)$ be all the distinct embeddings of F to $\boldsymbol{R}$ such that ι_1 is the natural inclusion. Then, each ι_ν can be extended to embeddings of E to $\boldsymbol{C}$ in two ways; we choose one of the extension and denote it by ι_ν also. From now on, we assume that there exists a $\boldsymbol{C}$-basis $\{\boldsymbol{u}_j\}$ of W satisfying the following conditions.

(a) The matrix $Q = (\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle)$ has all its entries in $\mathscr{O}_E$.
(b) The hermitian matrix $Q^{(\nu)} = (\iota_\nu \langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle) \in \mathrm{M}_N(\boldsymbol{C})$ is positive definite if $2 \leq \nu \leq d_F$, and is of signature $(p+, q-)$ if $\nu = 1$.

We fix such a basis $\{\boldsymbol{u}_j\}$ once and for all, and let $\mathscr{L}$ be the $\mathscr{O}_E$-submodule of W generated by $\{\boldsymbol{u}_j\}$. Thus, $G \cong U(Q^{(1)}) \cong U(p, q)$. The following lemma is standard (*cf.* [**4**]).

Lemma 3. *Suppose $d_F = [F : \boldsymbol{Q}] \geq 2$. Then, $\Gamma_{\mathscr{L}} = \{g \in G \mid g\mathscr{L} = \mathscr{L}\}$ is a uniform lattice of G. For any $\mathscr{O}_E$-ideal $\mathscr{I}$,*

$$\Gamma_{\mathscr{L}}(\mathscr{I}) = \{\gamma \in \Gamma_{\mathscr{L}} \mid \gamma \boldsymbol{v} - \boldsymbol{v} \in \mathscr{I}\mathscr{L} \ (\forall \boldsymbol{v} \in \mathscr{L})\}$$

is a normal subgroup of $\Gamma_{\mathscr{L}}$ of finite index.

Lemma 4. *Suppose the basis $\{\boldsymbol{u}_j\}$ is taken such that $\ell = \boldsymbol{C}\boldsymbol{u}_1$. Then, for any $\mathscr{O}_E$-ideal $\mathscr{I}_0$, there exists an $\mathscr{O}_E$-ideal $\mathscr{I} \subset \mathscr{I}_0$ such that the lattice $\Gamma_{\mathscr{L}}(\mathscr{I})$ is H_ℓ-admissible.*

Proof. This is well-know; for convenience we include a proof. By assumption, we have $\mathscr{L} = (\mathscr{L} \cap \ell) \oplus (\mathscr{L} \cap \ell^\perp)$, which implies that the automorphism J_ℓ (see 2.0) preserves the lattice $\mathscr{L}$. Hence $\sigma(\Gamma_{\mathscr{L}}(\mathscr{I})) \subset \Gamma_{\mathscr{L}}(\mathscr{I})$ for any ideal $\mathscr{I}$. The existence of $\mathscr{I} \subset \mathscr{I}_0$ follows from [**3**, Proposition 17.6]. □

Lemma 5. *Let $\gamma \in \Gamma_{\mathscr{L}}(\mathscr{I})$. Then $\gamma \in H$ if and only if $\xi_\ell(\gamma) = 1$.*

Proof. Suppose $\xi_\ell(\gamma) = 1$ with $\gamma \in \Gamma$. Then, there exists $\lambda \in \boldsymbol{C}^{(1)}$ such that $\langle \gamma \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle = \lambda \langle \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle$. Since $\gamma \boldsymbol{u}_1 \in \mathscr{L}$, we have $\lambda \in E^\times$. Let $\tilde{\gamma} \in \boldsymbol{G}(\boldsymbol{Q})$ be the element such that $\mathrm{pr}_1(\tilde{\gamma}) = \lambda^{-1}\gamma$. Thus, we have the relation $\langle \tilde{\gamma}\boldsymbol{u}_1 - \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle = 0$, which implies $\tilde{\gamma}\boldsymbol{u}_1 = \boldsymbol{u}_1$ (*cf.* [**12**, Lemma 32]). Hence, $\gamma \boldsymbol{u}_1 = \lambda\, \boldsymbol{u}_1$, and $\gamma \in H$. □

For an $\mathscr{O}_E$-ideal $\mathscr{I}$, let $\delta(\mathscr{I})$ denote the minimal norm of varying elements $\lambda \in \mathscr{I} - \{0\}$ regarded in the Euclidean space $E \otimes_{\boldsymbol{Q}} \boldsymbol{R} \cong \boldsymbol{C}^{d_F}$.

Lemma 6. *For any $R > 0$, there exists a constant $c_0 > 0$ such that, if $\delta(\mathscr{I}) > c_0$ then* $\inf\{\xi_\ell(\gamma) \mid \gamma \in \Gamma_{\mathscr{L}}(\mathscr{I}) - H\} \geq R$.

Proof. In the proof of [**12**, Lemma 47], we showed that there exists a constant $C_1 > 0$ (independent of $\mathscr{I}$) such that the inequality

$$|\langle \gamma \boldsymbol{u}_1 - \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle| + C_1 \geq \delta(\mathscr{I})$$

holds for any $\gamma \in \Gamma_{\mathscr{L}}(\mathscr{I}) - H$. From this, we obtain the estimation

$$\xi_\ell(\gamma) \geq C_2 + C_3\,\delta(\mathscr{I}), \quad (\gamma \in \Gamma_{\mathscr{L}}(\mathscr{I}) - H)$$

with constants $C_2 \in \boldsymbol{R}$ and $C_3 \in \boldsymbol{R}^+$ independent of $\mathscr{I}$. The assertion is now obvious. $\square$

Lemma 7. *There exists a constant $c_1 > 0$ such that $\Gamma_{\mathscr{L}}(\mathscr{I}) - H$ consists of H-hyperbolic elements if $\delta(\mathscr{I}) > c_1$.*

Proof. This follows from Lemmas 5 and 6. $\square$

3. Results.

In this section, we fix $d \in \boldsymbol{N}$ and set $\tau = \tau_d$. For any H-admissible lattice Γ in G, set

$$S_\tau^H(\Gamma)_{\mathrm{ct}} = S_\tau^H(\Gamma) \cap \left\{\sqrt{-1}\boldsymbol{R}^+ \cup (0, \nu_0)\right\}.$$

Here, $\boldsymbol{R}^+ = [0, \infty)$ and $\nu_0 \in \{0, 1\}$ is the parity of the integer $\rho_0 = p + q - 1$. We remark that the subscript ct abbreviates "continuous", which means that this comes from the continuous part of the unitary dual of G. Then, $S_\tau^H(\Gamma)_{\mathrm{ct}}$ is a countable discrete subset of $\sqrt{-1}\boldsymbol{R}^+ \cup (0, \nu_0)$.

3.1. Counting functions.

Let us define the counting function

$$\mathrm{N}_\tau^H(\Gamma; x) = \sum_{\nu \in S_\tau^H(\Gamma)_{\mathrm{ct}};\, |\nu|^2 \leq x} \boldsymbol{P}_\tau^H(\Gamma; \nu), \quad x > 0. \tag{3.1}$$

We also need

$$\hat{\boldsymbol{P}}_\tau^H(\Gamma; T) = \sum_{\nu \in S_\tau^H(\Gamma)_{\mathrm{ct}}} \boldsymbol{P}_\tau^H(\Gamma; \nu)\, e^{\nu^2 T}, \quad T > 0. \tag{3.2}$$

Our first theorem gives an estimation of the counting function $\mathrm{N}_\tau^H(\Gamma; x)$ for large x, which is uniform for Γ in an 'H-admissible tower'.

Theorem 8. *Let $\mathscr{L} = \bigoplus_{j=1}^N \mathscr{O}_E\, \boldsymbol{u}_j$ be an $\mathscr{O}_E$-lattice generated by a $\boldsymbol{C}$-basis $\{\boldsymbol{u}_j\}$ of W such that $\ell = \boldsymbol{C}\boldsymbol{u}_1$. Let $\{\mathscr{I}_n\}_{n \in \boldsymbol{N}}$ be a decreasing sequence of $\mathscr{O}_E$-ideals such that $\lim_{n\to\infty} \delta(\mathscr{I}_n) = +\infty$. Suppose $\Gamma_{\mathscr{L}}(\mathscr{I}_0)$ is torsion free, and set $\Gamma_n = \Gamma_{\mathscr{L}}(\mathscr{I}_n)$ for $n \in \boldsymbol{N}$. Then, there exist constants $C > 0$ and $n_0 \in \boldsymbol{N}$ such* *that*

$$\mathrm{vol}(\Gamma_n\cap H\backslash H)^{-1}\hat{\boldsymbol{P}}^H_\tau(\Gamma_n;T)\le C\,T^{-q},\qquad \forall T\in(0,1),\ \forall n\ge n_0, \tag{3.3}$$

$$\mathrm{vol}(\Gamma_n\cap H\backslash H)^{-1}\mathrm{N}^H_\tau(\Gamma_n;x)\le eC\,(1+x)^q,\quad \forall x>0,\ \forall n\ge n_0. \tag{3.4}$$

We give a proof of Theorem 8 in 5.2 relying on the next Theorem 9 which yields a more precise asymptotic behavior for individual counting function $\mathrm{N}^H_\tau(\Gamma;x)$ and the associated $\hat{\boldsymbol{P}}^H_\tau(\Gamma;T)$ with a fixed Γ.

Theorem 9. *Let Γ be an H-admissible lattice in G such that*

$$\heartsuit:\quad \inf\{\xi_\ell(\gamma)\mid \gamma\in\Gamma-\Gamma_H\}>1.$$

Then, there exists a smooth function $R(T)$ on $(0,1)$ satisfying

$$\lim_{T\to+0}\frac{\mathrm{d}^m}{\mathrm{d}T^m}R(T)=0,\quad \textit{for any}\quad m\in\boldsymbol{N}$$

such that

$$\hat{\boldsymbol{P}}^H_\tau(\Gamma;T)=\frac{\mathrm{vol}(\Gamma_H\backslash H)}{2\pi^q\Gamma(q)}\sum_{j=0}^{q-1}j!\,b_j\,T^{-j}+R(T),\quad T\in(0,1) \tag{3.5}$$

with b_j defined by

$$\prod_{j=1}^{q-1}\left\{\left(\frac{s}{2}\right)^2+\left(\frac{\rho_0}{2}+d-j\right)^2\right\}=\sum_{j=0}^{q-1}b_j\,s^{2j}.$$

We have a large time estimate

$$\hat{\boldsymbol{P}}^H_\tau(\Gamma;T)-\boldsymbol{P}^H_\tau(\Gamma;0)\prec e^{-aT},\quad T\ge 1$$

for some constant $a>0$.

A proof of Theorem 9 is given in 5.1.

Corollary 10. *Let Γ be as in Theorem 9. Then,*

$$\mathrm{N}^H_\tau(\Gamma;x)\sim\frac{\mathrm{vol}(\Gamma_H\backslash H)}{(4\pi)^q\Gamma(q+1)}\,x^q,\quad x\to+\infty.$$

Proof. This is deduced from Theorem 9 by a Tauberian theorem. □

Remark 1. By Lemma 7 (2), any lattice $\Gamma_{\mathscr{L}}(\mathscr{I})$ constructed in Lemma 4 with sufficiently large $\delta(\mathscr{I})$ satisfies the condition ♡ in the theorem.

Remark 2. For wave functions on real hyperbolic spaces, a result similar to Corollary 10 is proved in [**13**] by the same method.

3.2. Spectral measures.

The discrete set $S_{\tau}^{H}(\Gamma)_{\mathrm{ct}}$ yields a measure $\mu_{\tau}^{H}(\Gamma)$ on $\boldsymbol{R}^{+}$ given by

$$\mu_{\tau}^{H}(\Gamma)=\sum_{\nu\in S_{\tau}^{H}(\Gamma)_{\mathrm{ct}}}\frac{\boldsymbol{P}_{\tau}^{H}(\Gamma;\nu)}{\mathrm{vol}(\Gamma_{H}\backslash H)}\,\delta_{\nu_0^2-\nu^2},$$

where $\delta_{\nu_0^2-\nu^2}$ is the Dirac measure supported at the point $\nu_0^2-\nu^2$. Let $\{\Gamma_n\}$ be as in Theorem 8. By (3.7), the measure $\mu_{\tau}^{H}(\Gamma)$ with $n\geq n_0$ is actually a tempered distribution on $\boldsymbol{R}^{+}$. Then, the following theorem asserts that the sequence of measures $\{\mu_{\tau}^{H}(\Gamma_n)\}$ approximates the measure μ_{τ}^{H} (tempered distribution) on $\boldsymbol{R}^{+}$ defined by

$$\langle\mu_{\tau}^{H},f\rangle=\frac{\Gamma(q-1)}{8(q-1)\pi^{q+1}}\int_{\boldsymbol{R}}f(\nu_0^2+y^2)\frac{\mathrm{d}y}{|\boldsymbol{c}_d(\sqrt{-1}y)|^2},\quad f\in\mathscr{S}(\boldsymbol{R}^{+}),\tag{3.6}$$

where $\boldsymbol{c}_d(s)=\Gamma(q-1)\,\Gamma(s)\,\Gamma((s+\sigma_d)/2+q)^{-1}\,\Gamma((s-\sigma_d)/2)^{-1}$ and $\mathrm{d}y$ is the Lebesgue measure on $\boldsymbol{R}$.

Theorem 11. *Let $\mathscr{L}$, $\{\mathscr{I}_n\}_{n\in\boldsymbol{N}}$ and $\Gamma_n=\Gamma_{\mathscr{L}}(\mathscr{I}_n)$ be as in Theorem* 8. *If ρ_0 is even, then*

$$\lim_{n\to\infty}\langle\mu_{\tau}^{H}(\Gamma_n),f\rangle=\langle\mu_{\tau}^{H},f\rangle\quad\text{for any } f\in\mathscr{S}(\boldsymbol{R}^{+}).\tag{3.7}$$

If ρ_0 is odd and the following condition $\spadesuit(\tau)$ is satisfied, then (3.7) *is true.*

$$\spadesuit(\tau):\quad(\exists\epsilon\in(0,1))(\forall n\in\boldsymbol{N})(\forall\nu\in(1-\epsilon,1))(\boldsymbol{P}_{\tau}^{H}(\Gamma_n;\nu)=0).$$

We prove Theorem 11 in 5.3.

Remark. Several equivalent forms of the condition $\spadesuit(\tau)$ is given in Lemma 41.

4. Classification of double cosets.

In this section, we classify H-double cosets in G using the function ξ_ℓ defined in the paragraph 2.3.

LEMMA 12. *The map $\xi_\ell : G \to \boldsymbol{R}^+$ induces an injection $\tilde{\xi}_\ell$ from $H\backslash(G - H)/H$ into $\boldsymbol{R}^+$.*

PROOF. There exists a unitary character $\chi : H \to \boldsymbol{C}^{(1)}$ such that $h\,\boldsymbol{u} = \chi(h)\boldsymbol{u}$ for any $h \in H$. From this, it is obvious that ξ_ℓ is H-invariant both from left and from right, and thus $\tilde{\xi}_\ell : H\backslash(G - H)/H \to \boldsymbol{R}^+$ is well-defined by $\tilde{\xi}_\ell(HgH) = \xi_\ell(g)$. To show the injectivity of $\tilde{\xi}_\ell$, let $g_1, g_2 \in G - H$ satisfy $\xi_\ell(g_1) = \xi_\ell(g_2)$, and $\boldsymbol{v}_i$ $(i = 1, 2)$ the orthogonal projection of $g_i\boldsymbol{u}$ to $\ell^\perp$. Since $g_i \notin H$, we have $\boldsymbol{v}_i \neq 0$. Moreover, a computation yields $\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle = \langle \boldsymbol{u}, \boldsymbol{u} \rangle \{1 - \xi_\ell(g_i)^2\}$. Hence $\langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle = \langle \boldsymbol{v}_2, \boldsymbol{v}_2 \rangle$. By Witt's theorem, there exists an element $h \in U(\ell^\perp)$ such that $h\,\boldsymbol{v}_1 = \boldsymbol{v}_2$. Let $\tilde{h} \in G$ be the $\boldsymbol{C}$-linear extension of h to W fixing the vector $\boldsymbol{u}$ up to a scalar. Then, $\tilde{h}g_1\,\boldsymbol{u} = g_2\,\boldsymbol{u}$, which implies $Hg_1H = Hg_2H$. □

Recall the basis $\{\boldsymbol{v}_j\}_{1 \le j \le p+q}$ fixed in 2.2. In particular, the vector $\boldsymbol{v}_p$ spans the line ℓ defining H, i.e.,

$$\ell = \boldsymbol{C}\boldsymbol{v}_p.$$

For $1 \le j \le p$ and $1 \le i \le q$, define the one parameter subgroup $\{a_t^{(j,i)} \mid t \in \boldsymbol{R}\}$ of G by

$$\begin{aligned} a_t^{(j,i)}(\boldsymbol{v}_j) &= \cosh t\boldsymbol{v}_j + \sinh t\boldsymbol{v}_{p+i}, \\ a_t^{(j,i)}(\boldsymbol{v}_{p+i}) &= \cosh t\boldsymbol{v}_{p+i} + \sinh t\boldsymbol{v}_j, \\ a_t^{(j,i)}(\boldsymbol{u}) &= \boldsymbol{u}, \qquad \bigl(\forall \boldsymbol{u} \in \boldsymbol{v}_j^\perp \cap \boldsymbol{v}_{p+i}^\perp\bigr). \end{aligned}$$

Set $a_t = a_t^{(p,1)}$. Then, G is a disjoint union of double cosets Ha_tK with $t \ge 0$ ([**9**, Theorem 2.4]). It is easy to see that an element $g \in G$ belongs to the double coset Ha_tK if and only if $\cosh^2 t = 2^{-1}\{\|g^{-1}\boldsymbol{v}_p\|^2 + 1\}$.

LEMMA 13. *There exists $g_0 \in G$ such that $g_0\boldsymbol{u} - \boldsymbol{u}$ is a non-zero isotropic vector in $\ell^\perp$ for any $\boldsymbol{u} \in \ell - \{0\}$.*

PROOF. Since the signature of $\ell^\perp$ is $((p-1)+, q-)$ and since we assume $\inf(p, q) \ge 2$, the space $\ell^\perp$ contains a non-zero isotropic vector, say $\boldsymbol{e}$. A computation shows $\langle \boldsymbol{v}_p + \boldsymbol{e}, \boldsymbol{v}_p + \boldsymbol{e} \rangle = \langle \boldsymbol{v}_p, \boldsymbol{v}_p \rangle$. Thus, by Witt's theorem, there exists $g_0 \in G$ such that $g_0\,\boldsymbol{v}_p = \boldsymbol{v}_p + \boldsymbol{e}$. The element g_0 has the required property. □

Lemma 14. *Let $g \in G - H$.*

(1) *If $\xi_\ell(g) > 1$, then $g \in Ha_tH$ with the unique $t > 0$ such that $\cosh t = \xi_\ell(g)$.*
(2) *If $0 \le \xi_\ell(g) < 1$, then $g \in Hk_\theta H$ with the unique $\theta \in (0, \pi/2)$ such that $\cos\theta = \xi_\ell(g)$. Here, k_θ is the element of K defined by*

$$\begin{aligned} k_\theta \boldsymbol{v}_{p-1} &= \cos\theta\, \boldsymbol{v}_{p-1} - \sin\theta\, \boldsymbol{v}_p, \\ k_\theta \boldsymbol{v}_p &= \cos\theta\, \boldsymbol{v}_p + \sin\theta\, \boldsymbol{v}_{p-1}, \\ k_\theta \boldsymbol{w} &= \boldsymbol{w}, \quad \big(\boldsymbol{w} \in \boldsymbol{v}_p^\perp \cap \boldsymbol{v}_{p-1}^\perp\big). \end{aligned}$$

(3) *If $\xi_\ell(g) = 1$, then $g \in Hg_0H$.*

Proof. Let us show (1). If $\xi_\ell(g) > 1$, then $\cosh t = \xi_\ell(g)$ determines the unique $t > 0$. Since $\tilde{\xi}_\ell(Ha_tH) = \cosh t = \tilde{\xi}_\ell(HgH)$, we have $HgH = Ha_tH$ by Lemma 12. The remaining assertions (2) and (3) are proved in the same way by $\tilde{\xi}_\ell(Hk_\theta H) = |\cos\theta|$ and by $\tilde{\xi}_\ell(Hg_0H) = 1$. □

Corollary 15. *The map $\tilde{\xi}_\ell : H\backslash(G - H)/H \to \boldsymbol{R}^+$ is a bijection.*

Proof. This follows from Lemmas 12 and 14. □

On the subgroup H, the function ξ_ℓ is equal to the constant 1. The complement $G - H$ is divided to three classes according to the values of ξ_ℓ on them. An element $g \in G$, as well as the corresponding double coset HgH, is called *H-hyperbolic*, *H-elliptic* or *H-unipotent* according to $\xi_\ell(g) > 1$, $\xi_\ell(g) < 1$ or $\xi_\ell(g) = 1$, respectively. In this article, we consider only those uniform lattices Γ such that $\Gamma - \Gamma_H$ consists of H-hyperbolic elements.

5. Proofs.

Fix $d \in \boldsymbol{N}$, and set $\sigma_d = \rho_0 - 2(q-d)$. Let $\Gamma \subset G$ be an H-admissible uniform lattice satisfying the condition $\heartsuit$ in Theorem 9; thus, the whole $\Gamma - \Gamma_H$ consists of H-hyperbolic elements.

In [**12**], for any $\alpha(s) \in \boldsymbol{C}[s^2]$, we introduced a Poincaré series

$$\hat{\Phi}^{(d)}(\alpha, T; g) = \sum_{\gamma \in \Gamma_H\backslash\Gamma} \hat{\varphi}^{(d)}(\alpha, T; \gamma g), \qquad g \in G,\ T > 0,$$

which is absolutely convergent as was shown in [**12**, Lemma 20]. Here $\hat{\varphi}^{(d)}(\alpha, T; g)$ is the 'relative heat kernel' defined by the integral

$$\hat{\varphi}^{(d)}(\alpha, T; g) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \varphi_s^{(d)}(g)\alpha(s)e^{s^2T}s\mathrm{d}s, \quad (g \in G - HK,\ T > 0) \quad (5.1)$$

of the secondary spherical function $\varphi_s^{(d)} : G - HK \to V_d$ constructed in [**12**, Proposition 6]. Recall that, for any $T > 0$, the function $g \mapsto \hat{\varphi}^{(d)}(\alpha, T; g)$ on $G - HK$ is extended to a smooth function on the whole group G ([**12**, Proposition 18]) whose value at the identity is evaluated in [**12**, Proposition 13]. Our main tool of investigation is a version of 'relative trace formula', which is the identity

$$\begin{aligned} &2 \sum_{\nu \in S_\tau^H(\Gamma)} \alpha(\nu)e^{\nu^2 T}\boldsymbol{P}_\tau^H(\Gamma; \nu) \\ &\quad = \frac{\Gamma(q-1)}{\|\theta_d\|^2\pi^q}\big\{\mathrm{vol}(\Gamma_H\backslash H)(\hat{\varphi}^{(d)}(\alpha, T; e) \mid \theta_d) + R_d(\Gamma; \alpha, T)\big\}, \quad T > 0 \end{aligned} \quad (5.2)$$

obtained by computing the H-period integral

$$\hat{\boldsymbol{P}}_\tau^H(\Gamma; \alpha, T) = \|\theta_d\|^{-2}\int_{\Gamma_H\backslash H} (\hat{\Phi}^{(d)}(\alpha, T; h) \mid \theta_d)\mathrm{d}\dot{h}$$

in two ways ([**12**, Propositions 29 and 30]). Here, $R_d(\Gamma; \alpha, T)$ is the 'H-hyperbolic term' defined as

$$R_d(\Gamma; \alpha, T) = \sum_{[\gamma] \in \Gamma_H\backslash(\Gamma - \Gamma_H)/\Gamma_H} \mathrm{vol}(\Gamma_H \cap \Gamma_H^\gamma\backslash H \cap H^\gamma)\, \hat{I}_\gamma(T), \quad (5.3)$$

$$\hat{I}_\gamma(T) = \int_{H\cap H^\gamma\backslash H} (\hat{\varphi}^{(d)}(\alpha, T; \gamma h) \mid \theta_d)\mathrm{d}_\gamma\dot{h} \quad (5.4)$$

with $\mathrm{d}^\gamma h$ a Haar measure of $H \cap H^\gamma$ (which will be specified later) and $\mathrm{d}_\gamma\dot{h} = \mathrm{d}h/\mathrm{d}^\gamma h$ is the quotient measure of $\mathrm{d}h$ by $\mathrm{d}^\gamma h$.

5.1. Proof of Theorem 9.

By Lemma 14, for any $\gamma \in \Gamma - \Gamma_H$, there exists a unique $t_\gamma > 0$ such that $\gamma \in Ha_{t_\gamma}H$, and

$$\inf\{\sinh t_\gamma \mid \gamma \in \Gamma - \Gamma_H\} > 1. \quad (5.5)$$

We analyze the right-hand-side of (5.2) in detail. Fix a $\boldsymbol{C}$-basis $\{\boldsymbol{v}_j\}$ as in 2.2 such that $\ell = \boldsymbol{C}\boldsymbol{v}_p$.

5.1.1. Computation of H-hyperbolic orbital integrals.

To study the H-orbital integrals (5.4), it is important to know about the group $H\cap H^{a_t}$ and the homogeneous space $(H\cap H^{a_t})\backslash H$ for $t>0$.

Lemma 16. *Let $\tilde{J}$ be the element of H which fixes $\boldsymbol{v}_j$ for $j\neq p+1$ and maps $\boldsymbol{v}_{p+1}$ to $-\boldsymbol{v}_{p+1}$. Let $t>0$. The group $H\cap H^{a_t}$ is the fixed point subgroup of the involution $h\mapsto \tilde{J}h\tilde{J}^{-1}$ of H. If we set $a_r^H=a_r^{(p-1,1)}$, then*

$$H=(H\cap H^{a_t})\{a_r^H\mid r\geq 0\}\ (H\cap K).$$

There exists an H-invariant measure $\mathrm{d}_{a_t}\dot{h}$ of $(H\cap H^{a_t})\backslash H$ such that

$$\int_{(H\cap H^{a_t})\backslash H} f(h)\mathrm{d}_{a_t}\dot{h}=\int_0^\infty\int_{H\cap K} f(a_r^H k_0)\varrho^H(r)\mathrm{d}r\mathrm{d}k_0,\qquad f\in C_{\mathrm{c}}((H\cap H^{a_t})\backslash H),\tag{5.6}$$

where $\varrho^H(r)=(\sinh r)^{2p-3}(\cosh r)^{2q-1}$, $\mathrm{d}r$ is the Lebesgue measure and $\mathrm{d}k_0$ is the Haar measure of $H\cap K$ such that $\mathrm{vol}(H\cap K)=1$.

Proof. The first assertion is proved by a direct matrix computation by the basis $\{\boldsymbol{v}_j\}$. The remaining assertions then result from [**9**, Theorems 2.4 and 2.5]. □

Let $\gamma\in\Gamma-\Gamma_H$. If we write

$$\gamma=h'_\gamma\,a_{t_\gamma}\,h_\gamma,\quad \big(h'_\gamma,\,h_\gamma\in H\big),$$

then

$$H\cap H^\gamma=H\cap h_\gamma^{-1}a_{t_\gamma}^{-1}Ha_{t_\gamma}h_\gamma=h_\gamma^{-1}\big(H\cap a_{t_\gamma}^{-1}Ha_{t_\gamma}\big)h_\gamma.$$

Thus, $h\mapsto h_\gamma^{-1}h$ induces an H-isomorphism from $(H\cap H^{a_{t_\gamma}})\backslash H$ onto $(H\cap H^\gamma)\backslash H$, by which we transfer the measure $\mathrm{d}_{a_{t_\gamma}}\dot{h}$ fixed by Lemma 16 on the former space to that on the latter space. By the transported measure $\mathrm{d}_\gamma\dot{h}$ on $(H\cap H^\gamma)\backslash H$, we normalize the Haar measure $\mathrm{d}^\gamma h$ on $H\cap H^\gamma$ so that $\mathrm{d}h/\mathrm{d}^\gamma h=\mathrm{d}_\gamma\dot{h}$. Having these normalization of measures, using (5.6), we compute

$$
\begin{aligned}
\hat{I}_\gamma(T) &= \int_{(H\cap H^{a_{t_\gamma}})\backslash H} \big(\hat{\varphi}^{(d)}(T; a_{t_\gamma} h) \mid \theta_d\big)\mathrm{d}_{a_{t_\gamma}} \dot{h} \\
&= \int_0^\infty \big(\hat{\varphi}^{(d)}(T; a_{t_\gamma} a_r^H k_0) \mid \theta_d\big)\varrho^H(r)\mathrm{d}r\, \mathrm{d}k_0 \\
&= \int_0^\infty \big(\hat{\varphi}^{(d)}(T; a_{t_\gamma} a_r^H) \mid \theta_d\big)\varrho^H(r)\mathrm{d}r \\
&= \int_0^\infty \left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \big(\varphi_s^{(d)}(a_{t_\gamma} a_r^H) \mid \theta_d\big)e^{s^2T}\, s\mathrm{d}s\right\}\varrho^H(r)\mathrm{d}r
\end{aligned} \tag{5.7}
$$

by the $H\cap K$-invariance of θ_d to have the third equality and by the formula (5.1) to have the last equality.

Lemma 17. *Let $t>0$ and $r\in \boldsymbol{R}$. If $a_t\, a_r^H = h a_v k$ with $h\in H$, $v\in \boldsymbol{R}$ and $k\in K$, then*

$$
\cosh^2 v = 1 + \sinh^2 t\, \cosh^2 r, \tag{5.8}
$$

Proof. Set $g = a_t\, a_r^H$. Then, by the decomposition $G = \bigcup_{v\geq 0} H a_v K$, we can write $g = h a_v k$ with $(h,v,k)\in H\times[0,\infty)\times K$. By $g = a_t\, a_r^H$, we have

$$
\begin{aligned}
\left\|g^{-1}\boldsymbol{v}_p\right\|^2 &= \left\|a_{-r}^H(\cosh t\, \boldsymbol{v}_p - \sinh t\, \boldsymbol{v}_{p+1})\right\|^2 \\
&= \left\|\cosh t\, \boldsymbol{v}_p - \sinh t(\cosh r\, \boldsymbol{v}_{p+1} - \sinh r\, \boldsymbol{v}_{p-1})\right\|^2 \\
&= \cosh^2 t + (\sinh t\, \cosh r)^2 + (\sinh t\, \sinh r)^2 \\
&= \cosh^2 t + \sinh^2 t(\cosh^2 r + \sinh^2 r) \\
&= \cosh^2 t + \sinh^2 t(2\cosh^2 r - 1) \\
&= 1 + 2\sinh^2 t\, \cosh^2 r
\end{aligned}
$$

on the one hand. On the other hand, by $g\in H a_v K$, we compute $\|g^{-1}\boldsymbol{v}_p\|^2 = 2\cosh^2 v - 1$. Thus, $2\cosh^2 v - 1 = 1 + 2\sinh^2 t \cosh^2 r$, which gives us (5.8). □

For each $r\in\boldsymbol{R}$, let $k_r\in K$ be an element of K such that $a_{t_\gamma} a_r^H \in H a_v k_r$, where v is the number determined by (5.8). Although k_r is not unique, it turns out that the vector $\tau_d(k_r)\theta_d$ is well defined as a continuous function in r. First by [**12**, Formula (5.3)] and then by applying Lemma 17 to the last formula of $\hat{I}_\gamma(T)$, we continue as follows.

$$\begin{aligned}\hat{I}_\gamma(T) &= \int_0^\infty \left\{ \frac{1}{2\pi i} \int_{L_c} \{s\boldsymbol{c}_d(s)\}^{-1} (\cosh v)^{-(s+\rho_0)} \right. \\ &\qquad\qquad \left. \cdot F\left(\frac{s+\sigma_d}{2} + q, \frac{s-\sigma_d}{2}; s+1; \frac{1}{\cosh^2 v} \right) e^{s^2 T} s \mathrm{d}s \right\} \\ &= \int_0^\infty Q_0(T; \cosh^2 v)(\theta_d \mid \tau_d(k_r)\theta_d)\rho^H(r)\,\mathrm{d}r,\end{aligned}$$

where, for $m \in \boldsymbol{N}$, $T > 0$ and $u > 1$, we set

$$\begin{aligned}Q_m(T; u) &= \frac{1}{2\pi i} \int_{L_c} B_m(s; u) u^{-s/2}\, e^{s^2 T} \mathrm{d}s, \\ B_m(s; u) &= \{s\boldsymbol{c}_d(s)\}^{-1} s^{2m}\, u^{-\rho_0/2}\, {}_2F_1\left(\frac{s+\sigma_d}{2} + q, \frac{s-\sigma_d}{2}; s+1; \frac{1}{u} \right).\end{aligned}$$

Lemma 18. *Let $\gamma \in \Gamma$ be such that $\gamma \in Ha_{t_\gamma}H$ with $t_\gamma > 0$. Then, the function $T \mapsto \hat{I}_\gamma(T)$ is of class C^∞ on $(0, \infty)$, and*

$$\frac{\mathrm{d}^m}{\mathrm{d}T^m} \hat{I}_\gamma(T) = \int_0^\infty Q_m\bigl(T; 1 + \sinh^2 t_\gamma \cosh^2 r\bigr)(\theta_d \mid \tau_d(k_r)\theta_d)\rho^H(r)\mathrm{d}r$$

for any $m \in \boldsymbol{N}$.

Proof. Fix $c > \sigma_d$. Set $f(s) = |\Gamma((s+\sigma_d)/2 + q)\Gamma((s+\sigma_d)/2 + 1)^{-1}|$. Then, by the estimation

$$\bigl\|\varphi_s^{(d)}(a_v)\bigr\| \prec f(s)(\cosh v)^{-(c+\rho_0)}, \quad s \in c + i\boldsymbol{R},\, v \in \boldsymbol{R},$$

established in the proof of [**12**, Lemma 11], we have

$$\begin{aligned}&\int_0^\infty \left\{ \int_{c-i\infty}^{c+i\infty} \bigl|(\varphi_s^{(d)}(a_t a_r^H) \mid \theta_d)\bigr| \bigl|e^{s^2 T} s^{2m+1}\bigr| \,|\mathrm{d}s| \right\} \varrho^H(r)\mathrm{d}r \\ &\quad \prec \left\{ \int_{\boldsymbol{R}} f(c+iy)(c^2+y^2)^{(2m+1)/2} e^{(c^2-y^2)T} \mathrm{d}y \right\} \\ &\qquad \cdot \left\{ \int_0^\infty (1 + \sinh^2 t_\gamma \cosh^2 r)^{-(c+\rho_0)/2} \varrho^H(r)\mathrm{d}r \right\}. \qquad (5.9)\end{aligned}$$

Here we use Lemma 17 and the obvious estimate $|(\tau_d(k_r)\theta_d \mid \theta_d)| \leq \|\theta_d\|^2$ for $k \in K$. By Stirling's formula, the function $y \mapsto f(c+iy)$ is of polynomial growth as $|y| \to \infty$. Hence the first integral on the right-hand-side of the estimate (5.9) is convergent. The second integral is also convergent provided $c > \rho_0$. Thus, by applying Fubini's theorem to the expression (5.7), we have that $T \mapsto \hat{I}_\gamma(T)$ is of class C^∞ on $T > 0$ and its m-th derivative is given by

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\int_0^\infty \left(\varphi_s^{(d)}(a_{t_\gamma}a_r^H) \mid \theta_d\right)\varrho^H(r)\mathrm{d}r\right\}s^{2m+1}e^{s^2T}\mathrm{d}s.$$

Then, by the computation after Lemma 17, we have the conclusion. □

LEMMA 19. *Given $u_0 > 1$, we have the estimation*

$$|B_m(s;u)| \prec (1+|s|^2)^{m+(q-1)/2}u^{-\rho_0/2}, \quad u > u_0,\ \mathrm{Re}(s) > \sigma_d + 1. \tag{5.10}$$

PROOF. Set $\alpha = (s+\sigma_d)/2 + q$ and $\beta = (s-\sigma_d)/2$. By the integral representation

$${}_2F_1\left(\alpha,\beta;\alpha+\beta-q+1;\frac{1}{u}\right) = \frac{\Gamma(\alpha+\beta-q+1)}{\Gamma(\beta)\Gamma(\alpha-q+1)}\int_0^1 t^\beta(1-t)^{\alpha-q}\left(1-\frac{t}{u}\right)^{-\alpha}\mathrm{d}t,$$

which is valid if $\mathrm{Re}(\alpha-q+1) > 0$ and $\mathrm{Re}(\beta) > 0$ ([**11**, p. 54]), we obtain

$$B_m(s;u) = u^{-\rho_0/2}s^{2m}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q+1)}\int_0^1 t^{\beta-1}\left(1-\frac{t}{u}\right)^{-q}\left(\frac{1-t}{1-\dfrac{t}{u}}\right)^{\alpha-q}\mathrm{d}t, \qquad \mathrm{Re}(s) > \sigma_d.$$

Since $0 < 1-t < 1-t/u < 1$ for $0 < t < 1$ and $u > 1$, we have

$$0 < t^{\mathrm{Re}(\beta)-1}\left(1-\frac{t}{u}\right)^{-q}\left(\frac{1-t}{1-\dfrac{t}{u}}\right)^{\mathrm{Re}(\alpha)-q} < t^{-1/2}\left(1-\frac{1}{u_0}\right)^{-q}, \qquad t \in (0,1),\ u \in (u_0,+\infty)$$

provided $\mathrm{Re}(\alpha) - q \geq 0$ and $\mathrm{Re}(\beta) > 1/2$, or equivalently $\mathrm{Re}(s) > \sigma_d + 1$. Thus,

$$\begin{aligned}|B_m(s;u)| &\prec u^{-\rho_0/2}|s^{2m}\Gamma(\alpha)\Gamma(\alpha-q+1)^{-1}|\int_0^1 t^{-1/2}\mathrm{d}t \\ &= 2u^{-\rho_0/2}\left|s^{2m}\prod_{j=1}^{q-1}(\alpha-j)\right|\end{aligned}$$

for any $u>1$ and for $\mathrm{Re}(s)>\sigma_d+1$. This proves the estimation (5.10). $\square$

Lemma 20. *Given $u_0>1$ and $A>0$, there exists $T_0>0$ and $L_0>0$ such that*

$$|Q_m(T;u)| \prec T^{-(3N_0+1/2)}u^{-(A+\rho_0)/2}\exp\Big(-\frac{L_0^2}{8T}\Big),$$
$$T\in(0,T_0),\ u\in[u_0,+\infty). \tag{5.11}$$

Here $N_0=[m+(q+1)/2]$.

Proof. For $u>1$, set $L(u)=\log u^{1/2}$. We show that the numbers $L_0=L(u_0)$ and $T_0=L_0/\max(2\sigma_d+2,16A)$ meet the requirement. If $u\geq u_0$ and $T<T_0$, then the number $c=(2T)^{-1}L(u)$ satisfies $c>\sigma_d+1$. With this choice of c, we have

$$\begin{aligned}
Q_m(T;u) &= \frac{1}{2\pi}\int_{\boldsymbol{R}} B_m(c+iy;u)\exp(-(c+iy)L(u))\exp((c+iy)^2T)\mathrm{d}y\\
&= \int_{\boldsymbol{R}} B_m\Big(\frac{L(u)}{2T}+iy;u\Big)\exp\Big(-\frac{L(u)^2}{4T}-Ty^2\Big)\mathrm{d}y\\
&= \int_{\boldsymbol{R}} B_m\Big(\frac{L(u)}{2T}+\frac{iy}{\sqrt{T}};u\Big)\exp\Big(-\frac{L(u)^2}{4T}-y^2\Big)\frac{\mathrm{d}y}{\sqrt{T}}.
\end{aligned}$$

Set $N_0=[m+(q+1)/2]$. Then, by applying the estimate (5.10), we have

$$\begin{aligned}
&|Q_m(T;u)|\\
&\prec \exp\Big(-\frac{L(u)^2}{4T}\Big)u^{-\rho_0/2}\int_{\boldsymbol{R}}\Big(1+\Big|\frac{L(u)}{2T}+\frac{iy}{\sqrt{T}}\Big|^2\Big)^{N_0}\frac{e^{-y^2}\mathrm{d}y}{\sqrt{T}}\\
&\leq \exp\Big(-\frac{L(u)^2}{4T}\Big)u^{-\rho_0/2}\Big\{1+\frac{L(u)^2}{4T^2}\Big\}^{N_0}\int_{\boldsymbol{R}}\Big(1+\frac{y^2}{T}\Big)^{N_0}\frac{e^{-y^2}\mathrm{d}y}{\sqrt{T}}\\
&= \exp\Big(-\frac{L(u)^2}{4T}\Big)u^{-\rho_0/2}\Big\{1+\frac{L(u)^2}{4T^2}\Big\}^{N_0}\sum_{j=0}^{N_0}\binom{N_0}{j}T^{-(j+1/2)}\int_{\boldsymbol{R}}y^{2j}e^{-y^2}\mathrm{d}y.
\end{aligned}$$

Hence,

$$|Q_m(T;u)| \prec T^{-(N_0+1/2)}\exp\Big(-\frac{L(u)^2}{4T}\Big)u^{-\rho_0/2}\Big\{1+\frac{L(u)^2}{4T^2}\Big\}^{N_0},$$
$$u>1,\ T\in(0,T_0). \tag{5.12}$$

Furthermore, if $u \in [u_0, \infty)$ and $T \in (0, T_0)$, then $0 < L(u) \le L(u_0)$ and $T < T_0 < L(u_0)/16A$; thus the exponential factor in the majorant of (5.12) is estimated as

$$\begin{aligned}\exp\Big(-\frac{L(u)^2}{4T}\Big) &= \exp\Big(-\frac{L(u)^2}{8T}\Big)\exp\Big(-\frac{L(u)^2}{8T}\Big)\\ &\le \exp\Big(-\frac{L(u_0)^2}{8T}\Big)\exp\Big(-\frac{L(u_0)L(u)}{8T_0}\Big)\\ &\le \exp\Big(-\frac{L_0^2}{8T}\Big)u^{-A}. \end{aligned}\tag{5.13}$$

From (5.12), together with (5.13) and the obvious estimation

$$\Big\{1+\frac{L(u)^2}{4T^2}\Big\}^{N_0} \prec T^{-2N_0}u^{A/2}, \quad T \in (0, T_0),\ u \in [u_0, \infty),$$

we have the conclusion. □

LEMMA 21. *For $m \in \boldsymbol{N}$, there exist positive constants L_0, T_0, $N_0 \in \boldsymbol{N}$ and $\lambda > 2\rho_0$ such that*

$$\left|\frac{\mathrm{d}^m}{\mathrm{d}T^m}\hat{I}_\gamma(T)\right| \prec T^{-(N_0+1/2)}\exp\Big(\frac{-L_0^2}{8T}\Big)(\sinh t_\gamma)^{-\lambda}, \quad \gamma \in \Gamma - \Gamma_H,\ T \in (0, T_0).$$

PROOF. By (5.5), we can apply Lemma 20 to obtain the estimation

$$\begin{aligned}&\big|Q_m(T; 1+\sinh^2 t_\gamma \cosh^2 r)\big|\\ &\quad \prec T^{-(3N_0+1/2)}\exp\Big(-\frac{L_0^2}{8T}\Big)(1+\sinh^2 t_\gamma\cosh^2 r)^{-(A+\rho_0)/2}\end{aligned}$$

for $T \in (0, T_0)$, $\gamma \in \Gamma - \Gamma_H$ and $r \in (0, \infty)$. By this, together with the inequality $|(\theta_d|\tau_d(k_r)\theta_d)| \le \|\theta_d\|^2$, we have, from Lemma 18,

$$\begin{aligned}&\left|\frac{\mathrm{d}^m}{\mathrm{d}T^m}\hat{I}_\gamma(T)\right|\\ &\quad \prec \int_0^\infty |Q_m(T; 1+\sinh^2 t_\gamma\cosh^2 r)||(\theta_d \mid \tau_d(k_r)\theta_d)|\rho^H(r)\mathrm{d}r\end{aligned}$$

$$\begin{aligned}&\prec T^{-(3N_0+1/2)}\exp\Big(-\frac{L_0^2}{8T}\Big)\bigg\{\int_0^\infty (1+\sinh^2 t_\gamma \cosh^2 r)^{-(A+\rho_0)/2}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\cdot(\sinh r)^{2p-3}(\cosh r)^{2q-1}\mathrm{d}r\bigg\}\\ &\prec T^{-(3N_0+1/2)}\exp\Big(-\frac{L_0^2}{8T}\Big)(\sinh t_\gamma)^{-(A+\rho_0)},\quad T\in(0,T_0),\ \gamma\in\Gamma-\Gamma_H,\end{aligned}$$

provided the integral $\int_0^\infty (\sinh r)^{2p-3}(\cosh r)^{-A-\rho_0+2q-1}\mathrm{d}r$ is convergent, or equivalently $A>\rho_0-2$. Since A can be taken arbitrary, we are done. $\square$

Lemma 22. *For any $\lambda>2\rho_0$,*

$$\sum_{\gamma\in\Gamma-\Gamma_H}\mathrm{vol}\big(\Gamma_H\cap\Gamma_H^\gamma\backslash H\cap H^\gamma\big)(\sinh t_\gamma)^{-\lambda}<+\infty.$$

Proof. Let $\Xi:G\to\boldsymbol{R}^+$ be the function defined in [**12**, 6.1]. From definition, $\Xi(ka_tk)=(\cosh 2t)^{1/2}\le 2^{1/2}\cosh t$ for any $(h,k)\in H\times K$ and for any $t\in\boldsymbol{R}$.

By [**12**, Lemmas 21 and 22], the series

$$\mathscr{P}_\lambda(g)=\sum_{\gamma\in\Gamma_H\backslash\Gamma}\Xi(\gamma g)^{-2\lambda},\qquad g\in G$$

converges absolutely and locally uniformly on G if $\lambda>\rho_0$, defining a continuous function in g. Thus, the integral $\int_{\Gamma_H\backslash H}\mathscr{P}_\lambda(h)\mathrm{d}h$ is finite. By a standard computation, we have

$$\begin{aligned}&\int_{\Gamma_H\backslash H}\mathscr{P}_\lambda(h)\mathrm{d}h\\ &\quad=\sum_{\gamma\in\Gamma_H\backslash\Gamma/\Gamma_H}\mathrm{vol}\big(\Gamma_H\cap\Gamma_H^\gamma\backslash H\cap H^\gamma\big)\int_{H\cap H^\gamma\backslash H}\Xi(\gamma h)^{-2\lambda}\mathrm{d}h\\ &\quad\ge\sum_{\gamma\in\Gamma_H\backslash(\Gamma-\Gamma_H)/\Gamma_H}\mathrm{vol}\big(\Gamma_H\cap\Gamma_H^\gamma\backslash H\cap H^\gamma\big)\int_{H\cap H^\gamma\backslash H}\Xi(\gamma h)^{-2\lambda}\mathrm{d}h.\end{aligned}$$

For each $\gamma\in\Gamma-\Gamma_H$, by (5.6) and by Lemma 17,

$$\begin{aligned}
&\int_{H\cap H^{\gamma}\backslash H}\Xi(\gamma h)^{-2\lambda}\mathrm{d}h\\
&=\int_{H\cap H^{a_{t_\gamma}}\backslash H}\Xi(a_{t_\gamma}h)^{-2\lambda}\mathrm{d}h\\
&=\int_0^\infty \Xi\bigl(a_{t_\gamma}a_r^H\bigr)^{-2\lambda}\varrho^H(r)\mathrm{d}r\\
&\geq \int_0^\infty (2^{1/2}\cosh v)^{-2\lambda}\varrho^H(r)\mathrm{d}r\\
&=2^{-\lambda}\int_0^\infty \bigl(1+\sinh^2 t_\gamma\cosh^2 r\bigr)^{-\lambda}(\sinh r)^{2p-3}(\cosh r)^{2q-1}\mathrm{d}r\\
&\geq 2^{-\lambda}(1+\sinh^2 t_\gamma)^{-\lambda}\int_0^\infty (\cosh^2 r)^{-\lambda}(\sinh r)^{2p-3}(\sinh r)^{2q-1}\mathrm{d}r\\
&=(1+\sinh^2 t_\gamma)^{-\lambda}J(\lambda).
\end{aligned}$$

Here,

$$J(\lambda)=2^{-\lambda}\int_0^\infty(\cosh r)^{-2\lambda+2q-1}(\sinh r)^{2p-3}\mathrm{d}r,$$

which is convergent if $\lambda>\rho_0-1$. Thus,

$$\sum_{\gamma\in\Gamma_H\backslash(\Gamma-\Gamma_H)/\Gamma_H}\mathrm{vol}\bigl(\Gamma_H\cap\Gamma_H^{\gamma}\backslash H\cap H^{\gamma}\bigr)(1+\sinh^2 t_\gamma)^{-\lambda}<+\infty$$

if $\lambda>\rho_0$. By (5.5), we are done. □

LEMMA 23. *For $m\in\boldsymbol{N}$, there exist positive constants L_0, T_0, N_0 such that*

$$\left|\frac{\mathrm{d}^m}{\mathrm{d}T^m}R_d(\Gamma,1;T)\right|\prec T^{-N_0}\exp\left(\frac{-L_0^2}{8T}\right),\quad T\in(0,T_0).$$

PROOF. This follows from Lemmas 21 and 22. □

COROLLARY 24. *For any $m\in\boldsymbol{N}$,*

$$\lim_{T\to+0}\frac{\mathrm{d}^m}{\mathrm{d}T^m}R_d(\Gamma,1;T)=0.$$

PROOF. Obvious by Lemma 23. □

5.1.2. The identity term.

By [**12**, Proposition 13],

$$\|\theta_d\|^{-2}\big(\hat{\varphi}^{(d)}(T;e)\mid\theta_d\big)=\frac{1}{4(q-1)\pi}\int_{\boldsymbol{R}}e^{-y^2T}\frac{\mathrm{d}y}{|\boldsymbol{c}_d(iy)|^2}+r(T),\quad T>0 \tag{5.14}$$

with some C^∞-function $r(T)$ on $\boldsymbol{R}$, whose exact form is unimportant for our purpose here.

Lemma 25. *For $y\in\boldsymbol{R}-\{0\}$,*

$$\big|\boldsymbol{c}_d(\sqrt{-1}y)\big|^2=\frac{\Gamma(q-1)^2}{2\pi y}\prod_{j=1}^{q-1}\left\{\left(\frac{y}{2}\right)^2+\left(\frac{\rho_0}{2}+d-j\right)^2\right\}^{-1}$$
$$\cdot\begin{cases}\tanh\left(\dfrac{\pi y}{2}\right), & (\nu_0=0),\\[2ex] \coth\left(\dfrac{\pi y}{2}\right), & (\nu_0=1).\end{cases}$$

Proof. This is deduced from $\boldsymbol{c}_d(s)=\Gamma(q-1)\Gamma(s)\Gamma((s+\sigma_d)/2+q)^{-1}\Gamma((s-\sigma_d)/2)^{-1}$ by a direct computation. □

For $T>0$ and for $n\in\boldsymbol{N}$, set

$$I_n^+(T)=\int_{\boldsymbol{R}}e^{-Tx^2}x^{2n+1}\tanh\left(\frac{\pi x}{2}\right)\mathrm{d}x,$$
$$I_n^-(T)=\int_{\boldsymbol{R}}e^{-Tx^2}x^{2n+1}\coth\left(\frac{\pi x}{2}\right)\mathrm{d}x.$$

Let us introduce a convenient notation. For given functions $a(T)$ and $b(T)$ on $T>0$, if the difference $a(T)-b(T)$ has a C^∞ extension to a neighborhood of $T=0$, we write

$$a(T)\equiv b(T)\qquad\mathrm{mod}\ C^\infty(T=0).$$

Lemma 26. *Let $n\in\boldsymbol{N}$. Then,*

$$I_n^\pm(T)\equiv n!\,T^{-(n+1)}\qquad\mathrm{mod}\ C^\infty(T=0).$$

PROOF. We have $I_n^+(T) = J_0^1(T) + J_1^\infty(T)$ with

$$J_a^b(T) = 2\int_a^b e^{-Tx^2} x^{2n+1} \tanh\left(\frac{\pi x}{2}\right) \mathrm{d}x.$$

It is obvious that the function $J_0^1(T)$ is of class C^∞ on $\boldsymbol{R}$. A computation shows that the function

$$F(x) = \frac{-1}{2} T^{-(n+1)} \sum_{j=0}^{n} \frac{n!}{j!} T^j x^{2j} e^{-Tx^2}$$

is a primitive function of $x \mapsto x^{2n+1}e^{-Tx^2}$, i.e., $F'(x) = x^{2n+1}e^{-Tx^2}$. Hence, applying integration by parts to the integral $J_1^\infty(T)$, we have

$$\begin{aligned}
J_1^\infty(T) &= 2[F(x)]_1^\infty - 2\int_1^\infty F(x)\frac{\pi}{2}\frac{1}{\cosh^2\left(\frac{\pi x}{2}\right)}\mathrm{d}x \\
&= T^{-(n+1)}\sum_{j=0}^{n}\frac{n!}{j!}T^j e^{-T}\tanh\left(\frac{\pi}{2}\right) + \frac{\pi}{2}T^{-(n+1)}\sum_{j=0}^{n}\frac{n!}{j!}T^j\int_1^\infty \frac{x^{2j}e^{-Tx^2}}{\cosh^2\left(\frac{\pi x}{2}\right)}\mathrm{d}x \\
&= T^{-(n+1)}\sum_{j=0}^{n}\frac{n!}{j!}\left\{\tanh\left(\frac{\pi}{2}\right)\sum_{m=0}^{\infty}\frac{(-T)^m}{m!}\right. \\
&\qquad\qquad \left. + \frac{\pi}{2}\int_1^\infty \frac{x^{2j}}{\cosh^2\left(\frac{\pi x}{2}\right)}\sum_{m=0}^{\infty}\frac{(-T)^m}{m!}x^{2m}\mathrm{d}x\right\}T^j \\
&= T^{-(n+1)}\sum_{m=0}^{\infty}(-1)^m\sum_{j=0}^{n}\frac{n!}{j!}\left\{\tanh\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\int_1^\infty\frac{x^{2(j+m)}}{\cosh^2\left(\frac{\pi x}{2}\right)}\mathrm{d}x\right\}\frac{T^{m+j}}{m!} \\
&= T^{-(n+1)}\sum_{l=0}^{\infty}\sum_{\substack{j+m=l\\ j,m\in\boldsymbol{N}, j\le n}}\frac{(-1)^m n!}{j!\,m!}\left\{\tanh\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\int_1^\infty\frac{x^{2l}}{\cosh^2\left(\frac{\pi x}{2}\right)}\mathrm{d}x\right\}T^l.
\end{aligned}$$

Apply the obvious relation

$$\sum_{\substack{j+m=l\\ j,m\in\boldsymbol{N}, j\le n}}\frac{(-1)^m n!}{j!\,m!} = \frac{n!}{l!}\sum_{m=0}^{l}(-1)^m\binom{l}{m} = \delta_{l,0}\, n!, \quad 0\le l\le n$$

to the last formula of $J_1^{\infty}(T)$, we have

$$J_1^{\infty}(T) \equiv T^{-(n+1)} n! \left\{ \tanh\left(\frac{\pi}{2}\right) + \frac{\pi}{2} \int_1^{\infty} \frac{\mathrm{d}x}{\cosh^2\left(\dfrac{\pi x}{2}\right)} \right\} = T^{-(n+1)}\, n! \qquad (\mathrm{mod}\ C^{\infty}(T=0)).$$

Thus, $I_n^+(T) = J_0^1(T) + J_1^{\infty}(T) \equiv n!\, T^{-(n+1)} \pmod{C^{\infty}(T=0)}$. The integral $I_n^-(T)$ is treated in a similar way. □

Proposition 27. *We have*

$$\|\theta_d\|^{-2} \big(\hat{\varphi}^{(d)}(T;e) \mid \theta_d\big) \equiv \frac{1}{2\Gamma(q)\Gamma(q-1)} \sum_{j=0}^{q-1} b_j\, j!\, T^{-j} \qquad \mathrm{mod}\ C^{\infty}(T=0).$$

Here, b_j $(0 \le j \le q-1)$ *is the family of constants defined by*

$$\prod_{j=1}^{q-1} \left\{ \left(\frac{s}{2}\right)^2 + \left(\frac{\rho_0}{2} + d - j\right)^2 \right\} = \sum_{j=0}^{q-1} b_j\, s^{2j}.$$

Proof. This follows from (5.14) by Lemmas 25 and 26. □

5.1.3. Proof of Theorem 9.

The small-time asymptotic (3.5) follows from (5.2) combined with Corollary 24 and Lemma 26. The large time estimation is easier to prove. Indeed, let $\nu_1 \in S_\tau^H(\Gamma)_{\mathrm{ct}} - \{0\}$ be the element with smallest norm; then,

$$\begin{aligned} \hat{\boldsymbol{P}}_\tau^H(\Gamma;T) &= \sum_{\nu \in S_\tau^H(\Gamma)_{\mathrm{ct}} - \{0\}} \boldsymbol{P}_\tau^H(\Gamma;\nu) + \boldsymbol{P}_\tau^H(\Gamma;0) \\ &\le e^{-|\nu_1|^2 T/2}\, \hat{\boldsymbol{P}}_\tau^H\left(\Gamma;\frac{T}{2}\right) + \boldsymbol{P}_\tau^H(\Gamma;0) \end{aligned}$$

for any $T > 0$. □

5.2. Proof of Theorem 8.

Let $\{\mathscr{I}_n\}_{n \in \boldsymbol{N}}$ be as in Theorem 8. Then, by Lemma 7, there exists $n_0 \in \boldsymbol{N}$ such that $\Gamma_n = \Gamma_{\mathscr{L}}(\mathscr{I}_n)$ with $n \ge n_0$ satisfies the condition ♡ in Theorem 9. We have a uniform bound of the magnitude of the H-hyperbolic term for such a Γ_n.

LEMMA 28. *There exists a constant T_0 such that the estimate*

$$\frac{|R_d(\Gamma_n;1,T)|}{\mathrm{vol}(\Gamma_n\cap H\backslash H)}\prec 1,\quad n\geq n_0,\, T\in(0,T_0)$$

holds.

PROOF. By [**12**, Lemmas 40 and 41], we have the expression

$$\frac{R_d(\Gamma_n;1,T)}{\mathrm{vol}(\Gamma_n\cap H\backslash H)}\leq\sum_{[\xi]}v\big(\Gamma_{n_0}/\Gamma_n;[\xi]\big)\frac{\mathrm{vol}(\Gamma_{n_0,H}\cap\Gamma^{\xi}_{n_0,H}\backslash H\cap H^{\xi})}{\mathrm{vol}(\Gamma_{n_0,H}\backslash H)}\,\hat{I}_{\xi}(T)$$

with

$$v\big(\Gamma_{n_0}/\Gamma_n;[\xi]\big)=\sum_{[\gamma]\in j_n^{-1}[\xi]}\frac{[\Gamma_{n_0,H}\cap\Gamma^{\gamma}_{n_0,H}:\Gamma_{n,H}\cap\Gamma^{\gamma}_{n,H}]}{[\Gamma_{n_0,H}:\Gamma_{n,H}]}$$

for any $[\xi]$ belonging to the image of the natural map

$$j_n:\Gamma_{n,H}\backslash(\Gamma_n-\Gamma_{n,H})/\Gamma_{n,H}\longrightarrow\Gamma_{n_0,H}\backslash(\Gamma_{n_0}-\Gamma_{n_0,H})/\Gamma_{n_0,H}.$$

Since $v(\Gamma_{n_0}/\Gamma_n;[\xi])\leq 1$ by [**12**, Lemma 43], we obtain the inequality

$$\frac{|R_d(\Gamma_n;1,T)|}{\mathrm{vol}(\Gamma_n\cap H\backslash H)}\leq\frac{1}{\mathrm{vol}(\Gamma_{n_0}\cap H\backslash H)}\cdot\sum_{[\xi]\in\Gamma_{n_0,H}\backslash(\Gamma_{n_0}-\Gamma_{n_0,H})/\Gamma_{n_0,H}}\mathrm{vol}\big(\Gamma_{n_0,H}\cap\Gamma^{\xi}_{n_0,H}\backslash H\cap H^{\xi}\big)\big|\hat{I}_{\xi}(T)\big|,$$

for any $n\geq n_0$, whose right-hand-side is independent of n. By using Lemmas 21 and 22 to bound the right-hand-side, we have the conclusion. □

PROOF OF THEOREM 8. The estimate (3.3) results from (5.2) combined with Lemmas 27 and 28 immediately. Let $x>0$. For $\nu\in S^H_{\tau}(\Gamma)_{\mathrm{ct}}$, the inequality $|\nu|^2\leq x$ gives us

$$\frac{\nu^2}{1+x}\geq\frac{\nu^2}{x}\geq -1$$

since ν^2 is a negative real number. By this, we have

$$
\begin{aligned}
e^{-1}\sum_{\nu\in S_\tau^H(\Gamma_n)_{\mathrm{ct}};|\nu|^2\le x}\boldsymbol{P}_\tau^H(\Gamma_n;\nu) &\le \sum_{\nu\in S_\tau^H(\Gamma_n)_{\mathrm{ct}};|\nu|^2\le x}\boldsymbol{P}_\tau^H(\Gamma_n;\nu)e^{\nu^2(1+x)^{-1}}\\
&\le \hat{\boldsymbol{P}}_\tau^H(\Gamma_n;(1+x)^{-1})\\
&\le C\mathrm{vol}(\Gamma_n\cap H\backslash H)(1+x)^q
\end{aligned}
$$

for any $n\ge n_0$ using (3.3). This completes the proof of (3.4). $\square$

5.3. Proof of Theorem 11.

In this section, we prove Theorem 11 using Theorem 8.

Lemma 29. *Let $f\in\mathscr{S}(\boldsymbol{R}^+)$ be of the form $f(x)=e^{-xT}\beta(x)$ with some $T>0$ and $\beta(x)\in\boldsymbol{C}[x]$. Suppose $\spadesuit(\tau_d)$ when ρ_0 is odd. Then, (3.7) is true.*

Proof. Set $\alpha(s)=\beta(s^2)$. Then, the estimation given in [**12**, Lemma 39] is valid for $\varphi^{(d)}(\alpha,T;g)$ as it is. Hence by the same reasoning as [**12**, Proposition 49], we have

$$\lim_{n\to\infty}\frac{R_d(\Gamma_n;\alpha,T)}{\mathrm{vol}(\Gamma_n\cap H\backslash H)}=0.$$

In combination with this, (5.2) yields that the limit of

$$\sum_{\nu\in S_\tau^H(\Gamma)}e^{-(\nu_0^2-\nu^2)T}\alpha(\nu)\frac{\boldsymbol{P}_\tau^H(\Gamma_n;\nu)}{\mathrm{vol}(\Gamma_n\cap H\backslash H)} \tag{5.15}$$

as $n\to\infty$ exists and is equal to the number

$$\frac{\Gamma(q-1)}{2\pi^q}\|\theta_d\|^{-2}\big(\hat{\varphi}^{(d)}(\alpha,T;e)\mid\theta_d\big)e^{-\nu_0^2T},\quad (n\to\infty). \tag{5.16}$$

By [**12**, Proposition 4], (5.15) is expressed as a sum of $\langle\mu_\tau^H(\Gamma_n),f\rangle$ and the terms $\lambda_k(\Gamma_n)\alpha(\sigma_d-2k)$ for $k\in\boldsymbol{Z}$, $\inf(0,d-q)\le k\le[\sigma_d/2]$, where

$$\lambda_k(\Gamma_n)=\frac{\boldsymbol{P}_\tau^H(\Gamma_n;\sigma_d-2k)}{\mathrm{vol}(\Gamma_n\cap H\backslash H)}e^{\{(\sigma_d-2k)^2-\nu_0^2\}T}.$$

Consider the case when ρ_0 is even. Then, by [**12**, Theorem 5], we have

$$\lim_{n\to\infty}\lambda_k(\Gamma_n)=\delta(k>0)\frac{\Gamma(\sigma_d-k+q)\Gamma(q+k)(\sigma_d-2k)}{\Gamma(\sigma_d-k+1)\Gamma(q)\Gamma(k+1)\pi^q}e^{\{(\sigma_d-2k)^2-\nu_0^2\}T} \tag{5.17}$$

for $\inf(0, d-q) \le k \le [\sigma_d/2]$. Here, $\delta(k>0)$ denotes an element of $\{0,1\}$ which equals 1 if and only if $k>0$. Hence we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \langle \mu_\tau^H(\Gamma_n), f\rangle \\
&\quad = e^{-\nu_0^2 T}\Bigg\{\frac{\Gamma(q-1)}{2\pi^q}\|\theta_d\|^{-2}\big(\hat{\varphi}^{(d)}(\alpha, T; e) \mid \theta_d\big) \\
&\qquad - \sum_{k=0}^{[\sigma_d/2]} \frac{\Gamma(\sigma_d-k+q)\Gamma(q+k)(\sigma_d-2k)}{\Gamma(\sigma_d-k+1)\Gamma(q)\Gamma(k+1)\pi^q} e^{(\sigma_d-2k)^2 T}\alpha(\sigma_d-2k)\Bigg\}.
\end{aligned}$$

By [**12**, Proposition 13], the right-hand-side of the equality is

$$\frac{\Gamma(q-1)}{8(q-1)\pi^{q+1}\sqrt{-1}}\int_{\sqrt{-1}\boldsymbol{R}} \alpha(\nu) e^{-(\nu_0^2-\nu^2)T}\frac{\mathrm{d}\nu}{|\boldsymbol{c}_d(\nu)|^2} = \mu_\tau^H(f).$$

This complete the proof.

When ρ_0 is odd, [**12**, Theorem 5] gives us the formula (5.17) only for $\inf(0, d-q) \le k < [\sigma_d/2]$. However, by using the assumption $\spadesuit(\tau_d)$, we can extend the formula to $k = [\sigma_d/2]$ as follows. By [**12**, Corollary 38], we have the inequality

$$\limsup_{n\to\infty} \lambda_k(\Gamma_n) \le \frac{\Gamma(\sigma_d-k+q)\Gamma(q+k)(\sigma_d-2k)}{\Gamma(\sigma_d-k+1)\Gamma(q)\Gamma(k+1)\pi^q} \tag{5.18}$$

for $k = [\sigma_d/2]$ without using $\spadesuit(\tau_d)$. Let us show the converse inequality. The assumption $\spadesuit(\tau_d)$ yields the inclusion

$$S_\tau^H(\Gamma_n)_{\mathrm{ct}} \subset \{\nu \in \boldsymbol{C} \mid 0 \le \mathrm{Re}(\nu) < 1-\epsilon\},$$

which ensures that the proof [**12**, Proposition 51] goes through with $c_0 = 1-\epsilon/2$ in the argument. Consequently, we have the same conclusion as [**12**, Proposition 51] with $c = 1-\epsilon$. Then, in the same way as [**12**, Proposition 52], we deduce

$$\liminf_{n\to\infty} \lambda_k(\Gamma_n) \ge \frac{\Gamma(\sigma_d-k+q)\Gamma(q+k)(\sigma_d-2k)}{\Gamma(\sigma_d-k+1)\Gamma(q)\Gamma(k+1)\pi^q} \tag{5.19}$$

for $k = [\sigma_d/2]$. Now, the formula (5.17) for $k = [\sigma_d/2]$ follows from (5.18) and (5.19).

Having established (5.17) for $\inf(0, d-q) \le k \le [\sigma_d/2]$, we have the conclusion by the same argument as in the case when ρ_0 is even. $\square$

Lemma 30. *There exists a constant $A>0$ and $N_1>0$ such that*

$$\sup_{n\in\boldsymbol{N}}\left|\langle\mu_\tau^H(\Gamma_n),f\rangle\right|\le A\sup_{x\in\boldsymbol{R}^+}\left|f(x)(1+x^{N_1})\right|,\quad\forall f\in\mathscr{S}(\boldsymbol{R}^+).\tag{5.20}$$

Proof. Since

$$\begin{aligned}&\left|\langle\mu_\tau^H(\Gamma_n),f\rangle\right|\\&\quad\le\sum_{\nu\in S_\tau^H(\Gamma_n)_{\mathrm{ct}}}\frac{\boldsymbol{P}_\tau^H(\Gamma_n;\nu)}{\mathrm{vol}(\Gamma_n\cap H\backslash H)}\left(1+|\nu_0^2-\nu^2|^{N_1}\right)^{-1}\cdot\sup_{x\in\boldsymbol{R}^+}\left|f(x)(1+x^{N_1})\right|,\end{aligned}$$

it suffices to show that there exists $N_1\in\boldsymbol{N}$ such that $\sup_{n\in\boldsymbol{N}}Q_n^{(N_1)}<+\infty$, where

$$Q_n^{(N_1)}=\sum_{\nu\in S_\tau^H(\Gamma_n)_{\mathrm{ct}}}\frac{\boldsymbol{P}_\tau^H(\Gamma_n;\nu)}{\mathrm{vol}(\Gamma_n\cap H\backslash H)}(1+|\nu^2|^{N_1})^{-1}.$$

We start from the expression of $Q_n^{(N_1)}$ as a Stieltjes integral:

$$Q_n^{(N_1)}=\mathrm{vol}(\Gamma_n\cap H\backslash H)^{-1}\int_0^{+\infty}(1+x^{2N_1})^{-1}\mathrm{d}\mathrm{N}_\tau^H(\Gamma_n;x).$$

We have $\lim_{x\to\infty}(1+x^{2N_1})^{-1}\mathrm{N}_\tau^H(\Gamma_n;x)=0$ if $2N_1>q$ by (3.3). Then, by integration-by-parts,

$$\begin{aligned}&\mathrm{vol}(\Gamma_n\cap H\backslash H)Q_n^{(N_1)}\\&\quad=\left[(1+x^{2N_1})^{-1}\mathrm{N}_\tau^H(\Gamma_n;x)\right]_0^\infty+\int_0^{+\infty}\mathrm{N}_\tau^H(\Gamma_n;x)\frac{2N_1x^{2N_1-1}}{(1+x^{2N_1})^2}\mathrm{d}x\\&\quad=2N_1\int_0^{+\infty}\mathrm{N}_\tau^H(\Gamma_n;x)\frac{x^{2N_1-1}}{(1+x^{2N_1})^2}\mathrm{d}x\\&\quad\le2N_1\,e\,C\mathrm{vol}(\Gamma_n\cap H\backslash H)\int_0^\infty(1+x^q)\frac{x^{2N_1-1}}{(1+x^{2N_1})^2}\mathrm{d}x.\end{aligned}$$

Here, we use (3.3) to obtain the last inequality. Thus,

$$\sup_{n\in\boldsymbol{N}}Q_n^{(N_1)}\le2N_1\,C\int_0^\infty(1+x^q)\frac{x^{2N_1-1}}{(1+x^{2N_1})^2}\mathrm{d}x,$$

which implies $\sup_{n\in\boldsymbol{N}} Q_n^{(N_1)} < +\infty$ if $2N_1 > q$. □

LEMMA 31. *There exists a constant $A > 0$ and $N_1 > 0$ such that*

$$\left|\langle \mu_\tau^H, f\rangle\right| \le A \sup_{x\in\boldsymbol{R}^+} \left|f(x)(1+x^{N_1})\right|, \quad \forall f \in \mathscr{S}(\boldsymbol{R}^+). \tag{5.21}$$

PROOF. This is obvious by definition (3.6) of $\mu_\tau^H(f)$ and by Lemma 25. □

PROOF OF THEOREM 11. We follow the proof of [**5**, Theorem 9.1]. We choose sufficiently large $A > 0$ and $N_1 > 0$ such that the estimations (5.20) and (5.21) are valid. Let $f \in \mathscr{S}(\boldsymbol{R}^+)$ and take an arbitrary $\epsilon > 0$. Then, by [**5**, Lemma 9.3], there exists $T > 0$ and $\beta(x) \in \boldsymbol{C}[x]$ such that

$$\sup_{x\in\boldsymbol{R}^+} |f(x) - f_0(x)|(1+x^{N_1}) < \frac{\epsilon}{3A}$$

with $f_0(x) = e^{-xT}\beta(x)$. By applying Lemma 29 to $f_0(x)$, we have n_0 such that

$$\left|\left\langle \mu_\tau^H(\Gamma_n) - \mu_\tau^H, f_0\right\rangle\right| \le \frac{\epsilon}{3}$$

for any $n \ge n_0$. By the estimations (5.20) for the function $f - f_0$,

$$\sup_{n\in\boldsymbol{N}} \left|\left\langle \mu_\tau^H(\Gamma_n), f - f_0\right\rangle\right| \le A \sup_{x\in\boldsymbol{R}^+} \left\{|f(x) - f_0(x)|(1+x^{N_1})\right\} \le A\cdot\frac{\epsilon}{3A} = \frac{\epsilon}{3}.$$

Similarly, from (5.21), we have

$$\left|\left\langle \mu_\tau^H, f - f_0\right\rangle\right| \le \frac{\epsilon}{3}.$$

Therefore,

$$\begin{aligned}
&\left|\left\langle \mu_\tau^H(\Gamma_n) - \mu_\tau^H, f\right\rangle\right| \\
&\quad \le \left|\left\langle \mu_\tau^H(\Gamma_n), f - f_0\right\rangle\right| + \left|\left\langle \mu_\tau^H(\Gamma_n) - \mu_\tau^H, f_0\right\rangle\right| + \left|\left\langle \mu_\tau^H, f - f_0\right\rangle\right| \\
&\quad \le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

if $n \ge n_0$. This completes the proof. □

6. Application to geometry.

The symmetric spaces $\mathfrak{D}_G = G/K$ and $\mathfrak{D}_H = H/K_H$ have invariant Kähler structures such that the natural inclusion $\mathfrak{D}_H \hookrightarrow \mathfrak{D}_G$ is holomorphic isometry ([**10**, 2.1]). Let Γ be an H-admissible uniform lattice in G. Since Γ acts on $\mathfrak{D}_G = G/K$ properly discontinuously without fixed points, the G-invariant Kähler structure on $\mathfrak{D}_G$ is pushed down to the discrete quotient $\mathfrak{D}_G^\Gamma = \Gamma\backslash\mathfrak{D}_G$. Similar remark is applied to the action of Γ_H on $\mathfrak{D}_H = H/K_H$ to make the quotient $\mathfrak{D}_H^\Gamma = \Gamma_H\backslash\mathfrak{D}_H$ a Kähler manifold. Then, by Lemma 1, we have an holomorphic embedding of Kähler manifolds

$$j : \mathfrak{D}_H^\Gamma \hookrightarrow \mathfrak{D}_G^\Gamma$$

compatible with the inclusion $\mathfrak{D}_H \hookrightarrow \mathfrak{D}_G$ such that $\mathrm{codim}_{\boldsymbol{C}}(\mathfrak{D}_G^\Gamma/\mathfrak{D}_H^\Gamma) = q$. Let $E[\lambda]$ be the λ-eigenspace of the Hodge-Laplace operator Δ acting on $\mathscr{A}^{q,q}(\mathfrak{D}_G^\Gamma)$, the space of $\boldsymbol{C}$-valued C^∞-differential forms on $\mathfrak{D}_G^\Gamma$ of type (q,q). Let Λ_Γ be the set of eigenvalues of Δ; then, Λ_Γ is a countable discrete subset of non-negative real numbers. Let us introduce a counting function $\mathrm{N}(\mathfrak{D}_G^\Gamma/\mathfrak{D}_H^\Gamma; x)$ by setting

$$\mathrm{N}\big(\mathfrak{D}_G^\Gamma/\mathfrak{D}_H^\Gamma; x\big) = \sum_{\lambda\in\Lambda_\Gamma;\lambda\le x} \bigg\{ \sum_{\beta\in\mathscr{B}(\lambda)} \bigg| \int_{\mathfrak{D}_H^\Gamma} j^*(\star\beta) \bigg|^2 \bigg\}, \qquad x > 0,$$

where $\mathscr{B}(\lambda)$ is an orthonormal basis of the finite dimensional Hilbert space $E[\lambda]$ for each $\lambda \in \Lambda_\Gamma$, and $\star : \mathscr{A}^{q,q}(\mathfrak{D}_G^\Gamma) \longrightarrow \mathscr{A}^{pq-q,pq-q}(\mathfrak{D}_G^\Gamma)$ is the Hodge star operator. We have the base-free expression:

$$\mathrm{N}\big(\mathfrak{D}_G^\Gamma/\mathfrak{D}_H^\Gamma; x\big) = \sum_{\lambda\in\Lambda_\Gamma;\lambda\le x} \|\boldsymbol{P}_\lambda\|^2,$$

where $\|\boldsymbol{P}_\lambda\|^2$ is the norm square of the linear form

$$\boldsymbol{P}_\lambda : \beta \mapsto \int_{\mathfrak{D}_H^\Gamma} j^*(\star\beta)$$

on $E[\lambda]$. Corollary 10 yields an asymptotic formula of the counting function $\mathrm{N}(\mathfrak{D}_G^\Gamma/\mathfrak{D}_H^\Gamma; x)$:

Theorem 32. *Suppose Γ satisfies the condition $\heartsuit$ in Theorem 9. Then,*

$$\mathrm{N}\big(\mathfrak{D}_G^\Gamma/\mathfrak{D}_H^\Gamma;x\big)\sim\frac{\mathrm{vol}(\mathfrak{D}_H^\Gamma)}{(4\pi)^q\,\Gamma(q+1)}\,x^q,\qquad x\to+\infty.\tag{6.1}$$

Proof. Let (τ,V) denotes the finite dimensional representation of K on the space of tensors $\bigwedge^q\mathfrak{p}_+^*\otimes_{\boldsymbol{C}}\bigwedge^q\mathfrak{p}_-^*$ (see [**10**, 3.2]). It contains an $H\cap K$-invariant unit vector $\eta_\tau=\star\mathrm{vol}_{\mathfrak{p}\cap\mathfrak{h}}$ (see [**10**, Lemma 3]). Take a family $\{(\rho_i,F_i)\mid i\in I\}$ of irreducible representations of K such that $\tau\cong\bigoplus_{i\in I}\rho_i$. Then, $\mathscr{A}^{q,q}(\mathfrak{D}_G^\Gamma)$ is identified with $C^\infty(G/K;\tau)^\Gamma$ ([**10**, 4.1]), and the latter space is decomposed as

$$C^\infty(G/K;\tau)^\Gamma\cong\bigoplus_{i\in I}C^\infty(G/K;\rho_i)^\Gamma\tag{6.2}$$

according to $\tau\cong\bigoplus_{i\in I}\rho_i$. Since the action of the Casimir element has a decomposition $\Delta=\bigoplus_{i\in I}\Delta_{\rho_i}$ compatible with (6.2), we have

$$E[\lambda]=\bigoplus_{i\in I}\mathscr{A}_{\rho_i}\big(\Gamma;\sqrt{\rho_0^2-\lambda}\big)\quad(\lambda\in\boldsymbol{R}^+)\quad\text{and}$$
$$\Lambda_\Gamma=\bigsqcup_{i\in I}\big\{\rho_0^2-\nu^2\mid\nu\in S_{\rho_i}(\Gamma)\big\}.\tag{6.3}$$

Let I_0 be the set of indexes $i\in I$ such that η_τ has a non-zero projection $\eta_\tau^{(i)}$ to F_i. Then, from (6.3), it is proved that

$$\mathrm{N}\big(\mathfrak{D}_G^\Gamma/\mathfrak{D}_H^\Gamma;x\big)=\sum_{i\in I_0}\big\|\eta_\tau^{(i)}\big\|^2\,\mathrm{N}_{\rho_i}^H\big(\Gamma;x-\rho_0^2\big),\qquad x>\rho_0^2.$$

Applying Corollary 10 to each ρ_i, we obtain

$$\mathrm{N}\big(\mathfrak{D}_G^\Gamma/\mathfrak{D}_H^\Gamma;x\big)\sim\sum_{i\in I_0}\big\|\eta_\tau^{(i)}\big\|^2\,\frac{\mathrm{vol}(\Gamma_H\backslash H)}{(4\pi)^q\Gamma(q+1)}x^q,\qquad x\to+\infty.$$

Since $1=\|\eta_\tau\|^2=\sum_{i\in I_0}\|\eta_\tau^{(i)}\|^2$ and $\mathrm{vol}(\Gamma_H\backslash H)=\mathrm{vol}(\mathfrak{D}_H^\Gamma)$, we obtain the desired asymptotic formula (6.1). □

Remark. The resemblance of the formula (6.1) to Weyl's law for the Hodge-Laplacian on forms is evident.

7. Limit formulas.

7.1. *H*-period invariants.

7.1.1.

For any continuous representation π on a separable Hilbert space $\mathscr{H}_\pi$ of a Lie group G, let $\mathscr{H}_\pi^\infty$ be the space of smooth vectors of π; it is a G-invariant dense subspace of $\mathscr{H}_\pi$ carrying a Frechet space structure with respect to which the G-action is differentiable. Then, the space of distribution vectors of π, denoted by $\mathscr{H}_\pi^{-\infty}$, is defined to be the topological dual of $\mathscr{H}_\pi^\infty$. The contragredient action of G on $\mathscr{H}_\pi^{-\infty}$ is denoted by π', i.e.,

$$\langle \pi'(g)\, l, v\rangle = \langle l, \pi(g)^{-1}\, v\rangle, \qquad l \in \mathscr{H}_\pi^{-\infty},\ v \in \mathscr{H}_\pi^\infty.$$

7.1.2.

Let G be a connected reductive Lie group with compact center and Γ a uniform lattice in G. Fix a Haar measure $\mathrm{d}g$ of G and consider the right regular representation R_Γ of G on $L^2(\Gamma\backslash G)$. For an irreducible unitary representation $(\pi, \mathscr{H}_\pi)$ of G, let $\mathscr{V}_{\Gamma,\pi}$ be the π-isotypic component of $L^2(\Gamma\backslash G)$. Then, by a fundamental theorem of Gelfand, Graev and Piatetsuki-Shapiro, (a) $L^2(\Gamma\backslash G)$ is decomposed as a Hilbert direct sum of closed G-subspaces $\mathscr{V}_{\Gamma,\pi}$ for varying $\pi \in \hat{G}$, and, (b) for each π, $\mathscr{V}_{\Gamma,\pi}$ is a direct sum of finite, say $m_\Gamma(\pi)$, copies of π. The latter fact (b) is equivalently said this way: For each $\pi \in \hat{G}$, the space of all the bounded G-intertwining linear operators from $\mathscr{H}_\pi$ to $L^2(\Gamma\backslash G)$, denoted by $\mathscr{I}_G(\pi \mid R_\Gamma)$, is of finite dimension $m_\Gamma(\pi)$. This is because, by definition, $\mathscr{V}_{\Gamma,\pi}$ is the image of the natural G-isomorphism

$$\mathscr{I}_G(\pi \mid R_\Gamma) \otimes_{\boldsymbol{C}} \mathscr{H}_\pi \ni T \otimes v \longrightarrow T(v) \in \mathscr{V}_{\Gamma,\pi}.$$

Lemma 33. *There exists a hermitian inner product* $\langle\,|\,\rangle$ *on* $\mathscr{I}_G(\pi \mid R_\Gamma)$ *such that* $T^* \circ T' = \langle T' \mid T\rangle\, \mathrm{id}_{\mathscr{H}_\pi}$ *for any pair* (T, T') *of elements of* $\mathscr{I}_G(\pi \mid R_\Gamma)$*, where* T^* *denotes the adjoint of* T*. For* $T, T' \in \mathscr{I}_G(\pi \mid R_\Gamma)$*,* $\langle T' \mid T\rangle = 0$ *if and only if* $\mathrm{Im}(T)$ *and* $\mathrm{Im}(T')$ *are orthogonal in* $L^2(\Gamma\backslash G)$*.*

Proof. For $T, T' \in \mathscr{I}_G(\pi \mid R_\Gamma)$, the composite $T^* \circ T'$ belongs to $\mathscr{I}_G(\pi \mid \pi)$, which coincides with $\boldsymbol{C}\, \mathrm{id}_{\mathscr{H}_\pi}$ by Schur's lemma. Thus, there corresponds the unique scalar $\langle T' \mid T\rangle$ such that $T^* \circ T' = \langle T' \mid T\rangle\, \mathrm{id}_{\mathscr{H}_\pi}$. It is obvious that $\langle T \mid T'\rangle$ defines a hermitian form on $\mathscr{I}_G(\pi \mid R_\Gamma)$. Fix a non-zero unit vector $\xi_0 \in \mathscr{H}_\pi$. Then, $\langle T \mid T\rangle = \|T(\xi_0)\|^2$ is a non-negative number, which is zero if and only if $T(\xi_0) = 0$. Since $\mathscr{H}_\pi$ is irreducible, $T(\xi_0) = 0$ if and only if $T = 0$. Thus, the hermitian form is positive definite. The remaining assertion is obvious by the formula

$$\langle T' \mid T \rangle \langle \xi \mid \eta \rangle_\pi = \langle T(\xi) \mid T(\eta) \rangle_{R_\Gamma}, \qquad \xi, \eta \in \mathscr{H}_\pi. \qquad \square$$

Let H be a unimodular closed subgroup of G such that $\Gamma_H \backslash H \hookrightarrow \Gamma \backslash G$ has the compact image. Fix a Haar measure $\mathrm{d}h$ on H. Then, by assigning the integral over the compact subset $\Gamma_H \backslash H \hookrightarrow \Gamma \backslash G$ to each smooth function on $\Gamma \backslash G$, we obtain a linear functional $\mathscr{P}_\Gamma^H$ on $C^\infty(\Gamma \backslash G) = L^2(\Gamma \backslash G)^\infty$, which, indeed, is an H-invariant distribution vector of the unitary representation R_Γ:

$$\left\langle \mathscr{P}_\Gamma^H, \phi \right\rangle = \int_{\Gamma_H \backslash H} \phi(h) \mathrm{d}h, \qquad \phi \in L^2(\Gamma \backslash G)^\infty.$$

For each $\pi \in \hat{G}$, let $\mathscr{P}_\Gamma^H(\pi)$ be the image of $\mathscr{P}_\Gamma^H$ by the restriction map $L^2(\Gamma \backslash G)^{-\infty} \longrightarrow \mathscr{V}_{\Gamma,\pi}^{-\infty}$. Thus,

$$\mathscr{P}_\Gamma^H(\pi) \in \left(\mathscr{V}_{\Gamma,\pi}^{-\infty}\right)^H \cong \mathscr{I}_G(\pi \mid R_\Gamma)^* \otimes_{\boldsymbol{C}} (\mathscr{H}_\pi^{-\infty})^H.$$

Suppose π satisfies the condition

$$\clubsuit : \qquad \dim_{\boldsymbol{C}} \left(\mathscr{H}_\pi^{-\infty}\right)^H = 1.$$

For a non-zero element $l_\pi^0 \in (\mathscr{H}_\pi^{-\infty})^H$, there exists a unique element $\mathscr{P}_\Gamma^H(\pi)/l_\pi^0 \in \mathscr{I}_G(\pi \mid R_\Gamma)^*$ such that

$$\mathscr{P}_\Gamma^H(\pi) = \left(\mathscr{P}_\Gamma^H(\pi)/l_\pi^0\right) \otimes l_\pi^0,$$

when regarded in $\mathscr{I}_G(\pi \mid R_\Gamma)^* \otimes_{\boldsymbol{C}} (\mathscr{H}_\pi^{-\infty})^H$. Let $\|\mathscr{P}_\Gamma^H(\pi)/l_\pi^0\|^2$ be the norm square of the element $\mathscr{P}_\Gamma^H(\pi)/l_\pi^0$ in the Hilbert space dual to $\mathscr{I}_G(\pi \mid R_\Gamma)$.

Lemma 34. *Let $\mathscr{V}_{\Gamma,\pi} = \bigoplus_{j=1}^{m_\Gamma(\pi)} \mathscr{H}_\pi^{(j)}$ be an irreducible decomposition to closed G-invariant subspaces $\mathscr{H}_\pi^{(j)}$, and choose a G-isometry $T_j : \mathscr{H}_\pi \to \mathscr{H}_\pi^{(j)}$ for each j. Then, there exists a unique system of scalars c_j $(1 \leq j \leq m_\Gamma(\pi))$ such that*

$$\mathscr{P}_\Gamma^H \circ \left(T_j \mid \mathscr{H}_\pi^\infty\right) = c_j \, l_\pi^0 \tag{7.1}$$

for any j, and we have

$$\left\|\mathscr{P}_\Gamma^H(\pi)/l_\pi^0\right\|^2 = \sum_{j=1}^{m_\Gamma(\pi)} |c_j|^2.$$

Proof. The existence of $\{c_j\}$ results from the condition ♣. The system $\{T_j\}$ is an orthonormal basis of $\mathscr{I}_G(\pi \mid R_\Gamma)$; let $\{\check{T}_j\}$ be the dual basis of $\mathscr{I}_G(\pi \mid R_\Gamma)^*$. Then, by chasing definitions, we have

$$\mathscr{P}_\Gamma^H(\pi) = \sum_{j=1}^{m_\Gamma(\pi)} c_j \check{T}_j \otimes l_\pi^0,$$

or equivalently $\mathscr{P}_\Gamma^H(\pi)/l_\pi^0 = \sum_j c_j \check{T}_j$. Since $\{\check{T}_j\}$ is an orthonormal basis of $\mathscr{I}_G(\pi \mid R_\Gamma)^*$, we are done. □

Lemma 35.

(1) *If we replace* $(\mathrm{d}h, \mathrm{d}g)$ *by* $(c\mathrm{d}h, c'\mathrm{d}g)$ *with positive numbers* c *and* c', *then* $\|\mathscr{P}_\Gamma^H(\pi)/l_\pi^0\|^2$ *is multiplied by* c^2/c'. *If we replace the* G*-invariant inner product on* $\mathscr{H}_\pi$ *with a positive multiple, then the number* $\|\mathscr{P}_\Gamma^H(\pi)/l_\pi^0\|^2$ *is multiplied by the same constant.*

(2) *If a pair* (π, l_π^0) *and* $(\pi', l_{\pi'}^0)$ *are unitary equivalent, that is, if there exists an* G*-isometry* $S : \mathscr{H}_\pi \to \mathscr{H}_{\pi'}$ *such that* $l_{\pi'}^0 \circ (S \mid \mathscr{H}_\pi^\infty) = l_\pi^0$, *then*

$$\left\|\mathscr{P}_\Gamma^H(\pi)/l_\pi^0\right\|^2 = \left\|\mathscr{P}_\Gamma^H(\pi')/l_{\pi'}^0\right\|^2.$$

7.1.3.

From now on, let H be a symmetric subgroup of G, i.e., there exists an involution σ of G such that $(G^\sigma)^\circ \subset H \subset G^\sigma$. Fix a maximal compact subgroup $K \subset G$, whose Cartan involution commutes with σ. Then, $H \cap K$ is a maximal compact subgroup of H. For any irreducible representation (τ, V) of K, the $H \cap K$-invariant subspace $(V^*)^{H\cap K}$ is at most one dimensional. Recall that, for an irreducible unitary representation π of G, we defined a positive number $\boldsymbol{P}_\tau^H(\Gamma)_\pi$ by

$$\boldsymbol{P}_\tau^H(\Gamma)_\pi = \sum_{\phi \in \mathscr{B}} \|\phi^H(e)\|^2$$

with $\mathscr{B}$ an orthonormal basis of the finite dimensional Hilbert space $\mathscr{A}_\tau(\Gamma)_\pi = (V \otimes_{\boldsymbol{C}} \mathscr{V}_{\Gamma,\pi})^K$ ([**12**, Section 1]). There is a case when the number $\|\mathscr{P}_\Gamma^H(\pi)/l_\pi^0\|^2$ is written in terms of $\boldsymbol{P}_\tau^H(\Gamma)_\pi$.

PROPOSITION 36. *Suppose the condition ♣ on π. Let (τ, V) be an irreducible unitary representation of K contained in $\pi \mid K$ with multiplicity 1. Let $\iota^0 : V^* \to \mathscr{H}_\pi$ be a K-intertwining operator satisfying $\|\iota^0\|_{\mathrm{HS}} = 1$, where $\| \ \|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm on $\mathscr{I}_K(\tau^* \mid \pi)$. If $l_\pi^0 \in \mathscr{H}_\pi^\infty$ and $\check{\theta}_\tau \in (V^*)^{H\cap K}$ satisfy the condition*

$$l_\pi^0 \circ \iota^0(\check{\theta}_\tau) \neq 0,$$

then

$$\left\| \mathscr{P}_\Gamma^H(\pi)/l_\pi^0 \right\|^2 = \frac{\|\check{\theta}_\tau\|^2}{|l_\pi^0 \circ \iota^0(\check{\theta}_\tau)|^2} \, \boldsymbol{P}_\tau^H(\Gamma)_\pi.$$

PROOF. We set up an isomorphism

$$\mathscr{A}_\tau(\Gamma)_\pi \cong \mathscr{I}_G(\pi \mid R_\Gamma) \otimes \mathscr{I}_K(\tau^* \mid \pi)$$

by identifying $T \otimes \iota$ belonging to the space in right-hand-side with the function ϕ belonging to the space in the left-hand side so that

$$\langle \check{v}, \phi(g) \rangle = (T \circ \iota(\check{v}))\,(g), \qquad g \in G,\ \check{v} \in V^*.$$

When this holds, we write $\phi = [T \otimes \iota]$. It is a straightforward matter to check that, if $\phi = [T \otimes \iota]$ and $\phi' = [T' \otimes \iota']$, then

$$\langle \phi \mid \phi' \rangle = \langle T \mid T' \rangle \, (\iota \mid \iota')_{\mathrm{HS}},$$

where $(\iota \mid \iota')_{\mathrm{HS}}$ is the inner product associated with the Hilbert-Schmidt norm on $\mathscr{I}_K(\tau^* \mid \pi)$. Let $\{T_j\}$ and $\{\iota_\alpha\}$ be orthonormal basis of $\mathscr{I}_G(\pi \mid R_\Gamma)$ and $\mathscr{I}_K(\tau^* \mid \pi)$, respectively. Then, the associated functions $\phi_{j\alpha} = [T_j \otimes \iota_\alpha]$ afford an orthonormal basis of $\mathscr{A}_\tau(\Gamma)_\pi$. Let $\theta_\tau \in V$ be the element defined by the relation $(v \mid \theta_\tau) = \langle \check{\theta}_\tau, v \rangle$ for any $v \in V$. By Lemma 34, there exists a system of scalars $\{c_j\}$ satisfying (7.1). Now,

$$\begin{aligned} \int_{\Gamma_H \backslash H} (\phi_{j\alpha}(h) \mid \theta_\tau) \mathrm{d}h &= \int_{\Gamma_H \backslash H} \langle \check{\theta}_\tau, \phi_{j\alpha}(h) \rangle \mathrm{d}h \\ &= \int_{\Gamma_H \backslash H} (T_j \circ \iota_\alpha(\check{\theta}_\tau))(h) \mathrm{d}h \\ &= \mathscr{P}_\Gamma^H \circ T_j \circ \iota_\alpha(\check{\theta}_\tau) = c_j \, l_\pi^0 \circ \iota_\alpha(\check{\theta}_\tau), \end{aligned}$$

from which we have

$$|c_j|^2 = \left| \int_{\Gamma_H \backslash H} (\phi_{j\alpha}(h) \mid \theta_\tau) \mathrm{d}h \right|^2 \left| l_\pi^0 \circ \iota_\alpha(\check{\theta}_\tau) \right|^{-2}$$

provided $l_\pi^0 \circ \iota_\alpha(\check{\theta}_\tau) \neq 0$. By assumption, the singleton $\{\iota^0\}$ can be served as the system $\{\iota_\alpha\}$. Then, by Lemma 34 and by the definition of $\boldsymbol{P}_\tau^H(\Gamma)_\pi$,

$$\begin{aligned} \left\| \mathscr{P}_\Gamma^H(\pi)/l_\pi^0 \right\|^2 &= \sum_j \left| \int_{\Gamma_H \backslash H} (\phi_{j\alpha}(h) \mid \theta_\tau) \mathrm{d}h \right|^2 \left| l_\pi^0 \circ \iota^0(\check{\theta}_\tau) \right|^{-2} \\ &= \left| l_\pi^0 \circ \iota^0(\check{\theta}_\tau) \right|^{-2} \|\check{\theta}_\tau\|^2 \sum_j \left| \int_{\Gamma_H \backslash H} (\phi_{j\alpha}(h) \mid \theta_\tau) \mathrm{d}h \right|^2 \|\theta_\tau\|^{-2} \\ &= \frac{\|\check{\theta}_\tau\|^2}{|l_\pi^0 \circ \iota^0(\check{\theta}_\tau)|^2} \boldsymbol{P}_\tau^H(\Gamma)_\pi \end{aligned}$$

as desired. $\square$

7.2. Plancherel formula.

Now, we return to our special setting in Section 2 and recall the Plancherel formula of the symmetric space $H\backslash G$ proved by Faraut [**8**].

Let us first describe the H-spherical irreducible unitary representations π of G that enter in the decomposition of $L^2(H\backslash G)$, and for such a π, let us fix a basis element of $(\mathscr{H}_\pi^{-\infty})^H$ and recall the Fourier transform $\hat{f}(\pi)$ of $f \in C_{\mathrm{c}}^\infty(H\backslash G)$ at π.

7.2.1. *H*-spherical distribution vectors.

Let $s \in \boldsymbol{C}$. Recall the space of C^∞-vectors of π_s, denoted by $\mathscr{H}_s^\infty$, consists of all the complex valued C^∞-functions φ on the cone $\mathscr{C} = \{\boldsymbol{v} \in W - \{0\} \mid \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0\}$ satisfying $\varphi(t\boldsymbol{v}) = |t|^{-(s+\rho_0)}\varphi(\boldsymbol{v})$ for any $t \in \boldsymbol{C}^\times$ and for any $\boldsymbol{v} \in \mathscr{C}$; any $g \in G$ acts on $\mathscr{H}_s^\infty$ by the rule $\pi_s(g)\varphi(\boldsymbol{v}) = \varphi(g^{-1}\boldsymbol{v})$. If $\mathrm{Re}(s) > \rho_0$, let us define $u(s) : \mathscr{H}_s^\infty \to \boldsymbol{C}$ by the convergent integral

$$\langle u(s), \varphi \rangle = \frac{1}{\Gamma\left(\dfrac{s-\rho_0}{2} + 1\right)} \int_{\mathscr{C}_0} |\langle \boldsymbol{v}, \boldsymbol{u}_\ell \rangle|^{s-\rho_0}\, \varphi(\boldsymbol{v})\, \mathrm{d}\omega(\boldsymbol{v}), \qquad \varphi \in \mathscr{H}_s^\infty$$

([**8**, p. 395]), where $\boldsymbol{u}_\ell$ is a unit vector of ℓ, $\mathscr{C}_0 = \{\boldsymbol{v} = (\boldsymbol{x}; \boldsymbol{y}) \in W^+ \oplus W^- \mid \|\boldsymbol{x}\| = \|\boldsymbol{y}\| = 1\}$ and $\mathrm{d}\omega(\boldsymbol{v})$ is the K-invariant measure on $\mathscr{C}_0$ of total mass 1. Then, it is known that the function $s \mapsto \langle u(s), \varphi \rangle$ has a holomorphic continuation to the whole complex plane ([**8**, Proposition 5.3]), and defines an H-invariant distribution vector $u(s)$ of π_s for any $s \in \boldsymbol{C}$.

There constructed a meromorphic family of intertwining operators $\mathscr{A}_s^0 : \mathscr{H}_s^\infty \longrightarrow \mathscr{H}_{-s}^\infty$ such that

$$u(-s) \circ \mathscr{A}_s^0 = u(s) \tag{7.2}$$

in an appropriate sense ([**8**, Theorems 6.2 and 7.4]). ($\mathscr{A}_s^0$ coincides with $\gamma(-s)\mathscr{A}_s$ in the notation of [**8**].) From its explicit construction, we know that $\mathscr{A}_s^0$ is holomorphic on the set $\sqrt{-1}\boldsymbol{R} \cup (0, \nu_0) \cup \{\sigma_d \mid d \in \boldsymbol{N},\ \sigma_d > 0\}$.

- *Unitary principal series*: π_ν ($\nu \in \sqrt{-1}\boldsymbol{R}$). For $\nu \in \sqrt{-1}\boldsymbol{R}$, the representation $(\pi_\nu, \mathscr{H}_\pi^\infty)$ is unitarizable by the inner product

$$\langle \varphi \mid \varphi' \rangle_{\pi_\nu} = \int_{\mathscr{C}_0} \varphi(\boldsymbol{v})\bar{\varphi}'(\boldsymbol{v}) \mathrm{d}\omega(\boldsymbol{v})$$

([**8**, Proposition 5.1]). Thus, we obtain an irreducible unitary representation π_ν on the completion of $\mathscr{H}_\nu^\infty$ by the inner product above.

Let $l_{\pi_\nu}^0$ to be $u(\nu)$ for any $\nu \in \sqrt{-1}\boldsymbol{R}$. Then, $(\mathscr{H}_\nu^{-\infty})^H = \boldsymbol{C}\, l_{\pi_\nu}^0$.

- *Complementary series*: π_s ($s \in (0, \nu_0)$). The representations π_s with $0 < s < \nu_0$ is unitarizable by the inner product

$$\langle \varphi \mid \varphi' \rangle_{\pi_s} = \int_{\mathscr{C}_0} (\mathscr{A}_s^0 \varphi)(\boldsymbol{v})\bar{\varphi}'(\boldsymbol{v}) \mathrm{d}\omega(\boldsymbol{v}) \tag{7.3}$$

on $\mathscr{H}_s^\infty$ ([**8**, p. 416]). By completion $\mathscr{H}_s^\infty$ yields an irreducible unitary representation π_s; the representations π_s's with $0 < s < \nu_0$ afford the complementary series of $H\backslash G$, which is empty if ρ_0 is even. Let $l_{\pi_s}^0$ to be $u(s)$ for any $s \in (0, \nu_0)$. Then, $(\mathscr{H}_s^{-\infty})^H = \boldsymbol{C}\, l_{\pi_s}^0$.

- *Relative discrete series*: δ_d ($d \in \boldsymbol{N}$, $\sigma_d = \rho_0 + 2(d - q) > 0$) (see [**12**, A.7]).

Recall that the representation (τ_d, V_d) occurs in $(\mathscr{H}_{\sigma_d}^\infty)_K$ with multiplicity one; let $\mathscr{V}_d$ be the smallest $(\mathfrak{g}, K)$-submodule of $(\mathscr{H}_{\sigma_d}^\infty)_K$ containing τ_d. Then, as a $(\mathfrak{g}, K)$-module, $\mathscr{V}_d$ is irreducible. Let $\mathscr{V}_d^\infty$ be the closure of $\mathscr{V}_d$ in $\mathscr{H}_{\sigma_d}^\infty$ and δ_d the action of G on $\mathscr{V}_d^\infty$.

  - The case $0 \leq d < q$. In this case, $\mathscr{A}_s^0$ is holomorphic and is non-zero at $s = \sigma_d$. We fix a G-invariant inner product on $\mathscr{V}_d^\infty$ by

$$\langle \varphi \mid \varphi' \rangle_{\delta_d} = \int_{\mathscr{C}_0} (\mathscr{A}_{\sigma_d}^0 \varphi)(\boldsymbol{v})\bar{\varphi}'(\boldsymbol{v}) \mathrm{d}\omega(\boldsymbol{v}).$$

Let us define $l^0_{\delta_d} : \mathscr{V}_d^\infty \to \boldsymbol{C}$ by

$$\langle l^0_{\delta_d}, \varphi\rangle = \langle u(\sigma_d), \varphi\rangle, \quad \varphi \in \mathscr{V}_d^\infty.$$

– The case $q \leq d$. In this case, $(s-\sigma_d)^{-1}\mathscr{A}_s^0$ is holomorphic and is not zero at $s = \sigma_d$. Set $\tilde{\mathscr{A}}^0_{\sigma_d} = \lim_{s\to\sigma_d}(s-\sigma_d)^{-1}\mathscr{A}^0_{\sigma_d}$. Then, we fix a G-invariant inner product on $\mathscr{V}_d^\infty$ by

$$\langle \varphi \mid \varphi'\rangle_{\delta_d} = \int_{\mathscr{C}_0} (\tilde{\mathscr{A}}^0_{\sigma_d}\varphi)(\boldsymbol{v})\bar{\varphi}'(\boldsymbol{v})\mathrm{d}\omega(\boldsymbol{v}). \tag{7.4}$$

Let us define $l^0_{\delta_d} : \mathscr{V}_d^\infty \to \boldsymbol{C}$ by

$$\left\langle l^0_{\delta_d}, \varphi\right\rangle = \frac{\mathrm{d}}{\mathrm{d}\nu}\bigg|_{\nu=\sigma_d} \langle u(\nu), \varphi\rangle, \quad \varphi \in \mathscr{V}_d^\infty. \tag{7.5}$$

Let $\mathscr{V}_d$ be the completion of $\mathscr{V}_d^\infty$ by the inner products fixed above and δ_d the action of G on $\mathscr{V}_d$. Then, $(\delta_d, \mathscr{V}_d)$'s comprise a class of irreducible unitary representations of G, called the relative discrete series of $H\backslash G$. For these representations, $(\mathscr{V}_d^{-\infty})^H = \boldsymbol{C}l^0_{\delta_d}$.

Now, the representations listed above, together with the trivial representation $\boldsymbol{C}$, exhaust all the H-spherical irreducible unitary representations of G up to equivalence ([**8**, Section IX]). For such a π, the multiplicity free condition ♣ in the paragraph 7.1.2 is satisfied.

7.2.2. Fourier transforms.
Set $S^H = S^H_{\mathrm{ct}} \cup S^H_{\mathrm{dis}}$ with $S^H_{\mathrm{ct}} = \sqrt{-1}\boldsymbol{R}^+ \cup (0, \nu_0)$ and $S^H_{\mathrm{dis}} = \{\sigma_d = \rho_0 - 2q + 2d \mid d \in \boldsymbol{N},\ \rho_0 - 2q + 2d > 0\}$. For $\nu \in S^H$, set

$$\Pi_\nu = \begin{cases} \pi_\nu, & \nu \in S^H_{\mathrm{ct}}, \\ \delta_d, & \nu = \sigma_d \in S^H_{\mathrm{dis}}. \end{cases}$$

Then the Fourier transform $f \mapsto \hat{f}(\nu)$, as a mapping from $C_{\mathrm{c}}^\infty(H\backslash G)$ to the space of functions on the set S^H, is characterized by

$$\hat{\varphi^0}(\nu) = \left\langle l^0_{\Pi_\nu}, \Pi'_\nu(\varphi) l^0_{\Pi_\nu}\right\rangle, \quad \nu \in S^H,\ \varphi \in C_{\mathrm{c}}^\infty(G).$$

Here, $\varphi^0(g) = \int_H \varphi(hg)\mathrm{d}h$ for any $\varphi \in C_{\mathrm{c}}^\infty(G)$ ([**8**, p. 396, p. 412]). We should note that $\Pi'_\nu(\varphi) l^0_{\Pi_\nu}$ is a smooth vector of Π_ν for any $\varphi \in C_{\mathrm{c}}^\infty(G)$.

7.2.3. Inversion formula.

Let

$$\boldsymbol{c}(s) = 2\pi^{q/2}\Gamma(p)\Gamma(q)^{1/2}\Gamma(s)\Gamma\left(\frac{s+\rho}{2}\right)^{-1}\Gamma\left(\frac{s+2p-\rho_0}{2}\right)^{-1}\Gamma\left(\frac{s+2q-\rho_0}{2}\right)$$

and, for any $\sigma_d \in S_{\mathrm{dis}}$, let $C(\sigma_d)$ be the first non-zero coefficient of the Laurent expansion of $\boldsymbol{c}(s)^{-1}\boldsymbol{c}(-s)^{-1}$ at $s = \sigma_d$. Explicitly,

$$C(\sigma_d) = \frac{1}{\pi^q\Gamma(p)^2\Gamma(q)} \cdot \begin{cases} \sigma_d\Gamma\left(\frac{\rho_0-\sigma_d}{2}\right)\Gamma\left(\frac{\rho_0+\sigma_d}{2}\right), & \sigma_d < \rho_0, \\ 2(-1)^{(\sigma_d-\rho)/2}\sigma_d\Gamma\left(\frac{\sigma_d-\rho_0}{2}+1\right)^{-1}\Gamma\left(\frac{\rho_0+\sigma_d}{2}\right), & \\ & \sigma_d \geq \rho_0. \end{cases}$$

By regarding $S^H \subset \boldsymbol{C}$ naturally, we endow the set S^H with the induced topology from $\boldsymbol{C}$. Let $\mathscr{D}(S^H)$ be the space of functions $\beta : S^H \to \boldsymbol{C}$ such that $y \mapsto \beta(iy)$ is a Schwartz function on $\boldsymbol{R}^+$, such that $x \mapsto \beta(x)$ is C^∞ on $(0, \nu_0)$ and such that $\beta(\sigma_d) = 0$ except for finitely many σ_d's. Then, there exists a unique Radon measure $\mathrm{d}\mu^H$ on S^H, called the *Plancherel measure*, satisfying

$$\int_{S^H} \beta(\nu)\mathrm{d}\mu^H = \frac{1}{\pi}\int_0^\infty \beta(iy)\frac{\mathrm{d}y}{|\boldsymbol{c}(iy)|^2} + \sum_{\sigma_d \in S^H_{\mathrm{dis}}} C(\sigma_d)\beta(\sigma_d), \quad \beta \in \mathscr{D}(S^H).$$

Note that $\mathrm{supp}(\mathrm{d}\mu^H) \subset S^H - (0, \nu_0)$.

LEMMA 37. *For any right K-finite function $f \in C_{\mathrm{c}}^\infty(H\backslash G)$, the Fourier transform $\hat{f}$, regarded as a function on S^H, belongs to the space $\mathscr{D}(S^H)$.*

The inversion formula represents the Dirac distribution on $H\backslash G$ supported at the origin He as a superposition of Fourier transforms; in terms of the Plancherel measure, it can be stated simply as

$$f(He) = \int_{S^H} \hat{f}(\nu)\mathrm{d}\mu^H, \quad f \in C_{\mathrm{c}}^\infty(H\backslash G)_K.$$

7.3. Introduction of K-types.

For any $d \in \boldsymbol{N}$, set

$$\beta_d(s) = \frac{\Gamma(p)}{\Gamma\left(\dfrac{s+\sigma_d}{2}+1\right)} \prod_{j=0}^{d-1} \left(\frac{s-\rho_0}{2} - j\right).$$

Let $\nu \in \boldsymbol{C}$. For any $d \in \boldsymbol{N}$, the representation τ_d occurs in π_ν as a K-type with multiplicity one; we fix a K-embedding $\iota^{(\nu)}_{d,d;0,0} : V_d \hookrightarrow \mathscr{H}_\nu^\infty$ as in [**12**, Lemma A.3] and denote it by $\iota_d^{(\nu)}$ for simplicity. We endow V_d with the inner-product defined by

$$(v \mid w)_\tau = \int_{\mathscr{C}_0} \left[\iota_d^{(\nu)}(v)\right](\boldsymbol{v}) \overline{[\iota_d^{(\nu)}(w)](\boldsymbol{v})} \mathrm{d}\omega(\boldsymbol{v}),$$

which is independent of ν. By examining the highest weight, it turns out that τ_d is self-dual. Fix a K-isomorphism $\epsilon_d : V_d^* \to V_d$ and set $\check{\iota}_d^{(\nu)} = \iota_d^{(\nu)} \circ \epsilon_d$. Let θ'_d be the $H \cap K$-invariant tensor defined by [**12**, Lemma A.8] (and was denoted by θ_d there), and $\check{\theta}'_d \in (V_d^*)^{H\cap K}$ the element corresponding to θ'_d by the isomorphism ϵ_d. Then, [**12**, Proposition A.13] yields the identity

$$u(\nu) \circ \check{\iota}_d^{(\nu)}(\check{\theta}'_d) = \beta_d(\nu)\|\theta'_d\|^2. \tag{7.6}$$

We need the value $\|\theta'_d\|^2$.

Lemma 38.

$$\|\theta'_d\|^2 = \dim_{\boldsymbol{C}} V_d.$$

Proof. From [**12**, Lemma A.8], $(\tau_d(k)\theta'_d \mid \theta'_d) = [\iota_d^{(\nu)}(\theta'_d)](k^{-1}\boldsymbol{v}'_0)$ for any $k \in K$. Thus,

$$\begin{aligned}\|\theta'_d\|^2 &= \int_K \left[\iota_d^{(\nu)}(\theta'_d)\right](k\boldsymbol{v}'_0)\overline{[\iota_d^{(\nu)}(\theta'_d)](k\boldsymbol{v}'_0)}\mathrm{d}k \\ &= \int_K \left|(\tau_d(k)\theta'_d \mid \theta'_d)\right|^2 \mathrm{d}k = (\dim V_d)^{-1}\|\theta'_d\|^4\end{aligned}$$

by the orthogonal relation for matrix coefficients of τ_d. □

Lemma 39. *Let $\nu \in S^H_{\mathrm{ct}}$. Then, for any $d \in \boldsymbol{N}$,*

$$\left\|\mathscr{P}^H_\Gamma(\pi_\nu)/l^0_{\pi_\nu}\right\|^2 = \beta_d(\nu)^{-1}\beta_d(-\nu)^{-1}\boldsymbol{P}^H_{\tau_d}(\Gamma;\nu).$$

PROOF. First assume $\nu \in \sqrt{-1}\boldsymbol{R}^+$. Then, by definition of the inner product on $\mathscr{H}_\nu^\infty$ and that on V_d, the map $\iota^0 = (\dim V_d)^{-1/2}\check{\iota}_d^{(\nu)}$ satisfies $\|\iota^0\|_{\mathrm{HS}} = 1$. With this choice of ι_0, we apply Proposition 36 to have

$$\left\|\mathscr{P}_\Gamma^H(\pi_\nu)/l_{\pi_\nu}^0\right\|^2 = |\beta_d(\nu)^{-1}|^2 \boldsymbol{P}_{\tau_d}^H(\Gamma)_{\pi_\nu},$$

by using (7.6) and by Lemma 38. Note that $|\beta_d(\nu)^{-1}|^2 = \beta_d(\nu)^{-1}\beta_d(-\nu)^{-1}$ since $\nu \in \sqrt{-1}\boldsymbol{R}$.

Let $\nu \in (0, \nu_0)$. Then, from (7.2) and (7.6), we have

$$\mathscr{A}_\nu^0 \circ \iota_d^{(\nu)} = \left(\frac{\beta_d(\nu)}{\beta_d(-\nu)}\right)\iota_d^{(-\nu)}. \tag{7.7}$$

By this, we compute the inner product $\|\iota_d^{(\nu)}(\theta_d')\|_{\pi_\nu}^2$ following the definition (7.3) of the G-invariant inner product to show the identity $\|\iota_d^{(\nu)}(\theta_d')\|_{\pi_\nu}^2 = (\beta_d(\nu)/\beta_d(-\nu))\|\theta_d'\|^2$. Thus, the map

$$\iota^0 = (\dim V_d)^{-1/2}\left(\frac{\beta_d(-\nu)}{\beta_d(\nu)}\right)^{1/2}\check{\iota}_d^{(\nu)}$$

satisfies $\|\iota^0\|_{\mathrm{HS}} = 1$. With this choice of ι^0, we apply Proposition 36 to have

$$\left\|\mathscr{P}_\Gamma^H(\pi_\nu)/l_{\pi_\nu}^0\right\|^2 = \beta_d(\nu)^{-1}\beta_d(-\nu)^{-1}\boldsymbol{P}_{\tau_d}^H(\Gamma)_{\pi_\nu}$$

by using (7.6) and by Lemma 38.

It remains to show $\boldsymbol{P}_{\tau_d}^H(\Gamma)_{\pi_\nu} = \boldsymbol{P}_{\tau_d}^H(\Gamma;\nu)$. The inclusion $\mathscr{A}_{\tau_d}(\Gamma)_{\pi_\nu} \subset \mathscr{A}_{\tau_d}(\Gamma;\nu)$ follows from the fact that $\Omega_{\mathfrak{g}}$ acts on π_ν by the scalar $\nu^2 - \rho_0^2$. Thus, $\boldsymbol{P}_{\tau_d}^H(\Gamma)_{\pi_\nu} \le \boldsymbol{P}_{\tau_d}^H(\Gamma;\nu)$. Suppose $\boldsymbol{P}_{\tau_d}^H(\Gamma;\nu) \ne 0$ and fix an orthonormal basis $\{\phi_j\}$ of $\mathscr{A}_{\tau_d}(\Gamma;\nu)$. Then, for j such that $\phi_j^H(e) \ne 0$, it is proved that $\phi_j \in \mathscr{A}_{\tau_d}(\Gamma)_{\pi_\nu}$ in the proof of [**12**, Proposition 4]. Hence, $\boldsymbol{P}_{\tau_d}^H(\Gamma;\nu) = \boldsymbol{P}_{\tau_d}^H(\Gamma)_{\pi_\nu}$. □

LEMMA 40. *Let $\sigma_d \in S_{\mathrm{dis}}^H$. Then,*

$$\left\|\mathscr{P}_\Gamma^H(\delta_d)/l_{\delta_d}^0\right\|^2 = \boldsymbol{P}_{\tau_d}^H(\Gamma;\sigma_d)\cdot\begin{cases}\beta_d(\sigma_d)^{-1}\beta_d(-\sigma_d)^{-1}, & 0 < \sigma_d < \rho_0,\\ \beta_d'(\sigma_d)^{-1}\beta_d(-\sigma_d)^{-1}, & \rho_0 \le \sigma_d.\end{cases}$$

PROOF. It suffices to prove the following two formulas.

$$\left\|\mathscr{P}_\Gamma^H(\delta_d)/l_{\delta_d}^0\right\|^2 = \boldsymbol{P}_{\tau_d}^H(\Gamma)_{\delta_d} \cdot \begin{cases} \beta_d(\sigma_d)^{-1}\beta_d(-\sigma_d)^{-1}, & 0 < \sigma_d < \rho_0, \\ \beta_d'(\sigma_d)^{-1}\beta_d(-\sigma_d)^{-1}, & \rho_0 \le \sigma_d, \end{cases} \tag{7.8}$$

$$\boldsymbol{P}_{\tau_d}^H(\Gamma)_{\delta_d} = \boldsymbol{P}_{\tau_d}^H(\Gamma; \sigma_d). \tag{7.9}$$

We first consider the case $\rho_0 \le \sigma_d$. Then, $\beta_d(-\sigma_d) \neq 0$ and $\beta_d(s)$ has a simple zero at $s = \sigma_d$. From the relation (7.7), we have

$$\tilde{\mathscr{A}}_{\sigma_d}^0 \iota_d^{(\sigma_d)} = \left(\frac{\beta_d'(\sigma_d)}{\beta_d(-\sigma_d)}\right) \iota_d^{(-\sigma_d)}.$$

From (7.6) and by definition (7.5), we have

$$l_{\delta_d}^0 \circ \check{\iota}_d^{(\sigma_d)} = \beta_d'(\sigma_d)\|\theta_d'\|^2.$$

By these, we can confirm that the map

$$\iota^0 = (\dim V_d)^{-1/2} \left(\frac{\beta_d(-\sigma_d)}{\beta_d'(\sigma_d)}\right)^{1/2} \check{\iota}_d^{(\sigma_d)},$$

when regarded as the K-inclusion $V_d^* \hookrightarrow \mathscr{V}_d^\infty$, satisfies $\|\iota^0\|_{\mathrm{HS}} = 1$ by the same way as in the proof of Lemma 39. Having this choice of ι^0, we apply Proposition 36 to obtain (7.8) using (7.6) and by Lemma 38. The case $0 < \sigma_d < \rho_0$ is settled similarly. Indeed, we take

$$\iota^0 = (\dim V_d)^{-1/2} \left(\frac{\beta_d(\sigma_d)}{\beta_d(-\sigma_d)}\right)^{1/2} \check{\iota}_d^{(\sigma_d)}$$

and proceed the same way.

It remains to show (7.9). The inclusion $\mathscr{A}_{\tau_d}(\Gamma)_{\delta_d} \subset \mathscr{A}_{\tau_d}(\Gamma; \sigma_d)$ is obvious by the fact that $\Omega_{\mathfrak{g}}$ acts on δ_d by the scalar $\sigma_d^2 - \rho_0^2$. Thus, we have the inequality $\boldsymbol{P}_{\tau_d}^H(\Gamma)_{\delta_d} \le \boldsymbol{P}_{\tau_d}^H(\Gamma; \sigma_d)$. Fix an orthonormal basis $\{\phi_j\}$ of $\mathscr{A}_{\tau_d}(\Gamma; \sigma_d)$. For any index j such that $\phi_j^H(e) \neq 0$, let $\mathscr{H}(\phi_j)$ be the closed G-subspace of $L^2(\Gamma\backslash G)$ generated by ϕ_j. Then, any irreducible subspace $\mathscr{V}$ of $\mathscr{H}(\phi_j)$ is an H-spherical irreducible unitary representation on which $\Omega_{\mathfrak{g}}$ acts by the scalar $\sigma_d^2 - \rho_0^2$. From the list of equivalence classes of such representations recalled in 7.2.1, the representation of G on $\mathscr{V}$ has to be equivalent to δ_d. Thus, $\phi_j \in \mathscr{A}_{\tau_d}(\Gamma)_{\delta_d}$. This completes the proof of (7.9). $\square$

7.3.1. Spectral gap hypothesis.

LEMMA 41. *Suppose ρ_0 is odd, and let $\{\Gamma_n\}$ be a sequence of H-admissible lattices in G. Then, the following statements are equivalent to each other.*

(1) *The condition $\spadesuit(\tau_d)$ is true for any $d \in \boldsymbol{N}$.*
(2) *The condition $\spadesuit(\tau_d)$ is true for some $d \in \boldsymbol{N}$.*
(3) *The condition $\spadesuit(\tau_0)$ is true.*
(4) *The condition*

$$(\exists \epsilon \in (0,1))(\forall n \in \boldsymbol{N})(\forall \nu \in (1-\epsilon, 1))\big(\boldsymbol{P}^H_{\tau_0}(\Gamma_n)_{\pi_\nu} = 0\big)$$

is true.

PROOF. It is obvious that (1) implies (3), and that (3) implies (2). The equivalence of (3) and (4) follows from the identity $\boldsymbol{P}^H_{\tau_0}(\Gamma;\nu) = \boldsymbol{P}^H_{\tau_0}(\Gamma)_{\pi_\nu}$ for $\nu \in (0,\nu_0)$, which is proved in the proof of Lemma 39. It remains to show that (2) implies (1). By Lemma 39, we have

$$\beta_d(\nu)^{-1}\beta_d(-\nu)^{-1}\boldsymbol{P}^H_{\tau_d}(\Gamma_n;\nu) = \beta_{d'}(\nu)^{-1}\beta_{d'}(-\nu)^{-1}\boldsymbol{P}^H_{\tau_{d'}}(\Gamma_n;\nu).$$

for any $d, d' \in \boldsymbol{N}$. Since $\beta_d(\nu)^{-1}\beta_d(-\nu)^{-1}$, for any d, does not have zeros or poles on the interval $(0,1)$, we are done. □

REMARK. The equivalent conditions in Lemma 41 are obviously implied by the following 'spectral gap hypothesis' for the complementary series π_ν $(0 < \nu < \nu_0)$.

$$(\exists \epsilon \in (0,1))(\forall n \in \boldsymbol{N})(\forall \nu \in (1-\epsilon,1))(m_{\Gamma_n}(\pi_\nu) = 0).$$

It is shown that this follows from Arthur's conjecture ([**2**]).

7.4. Limit formula.

LEMMA 42. *Set $S^H(\Gamma) = \{\nu \in S^H \mid \|\mathscr{P}^H_\Gamma(\Pi_\nu)/l^0_{\Pi_\nu}\|^2 \neq 0\}$. Then,*

$$S^H(\Gamma) = S^H_{\tau_0}(\Gamma)_{\mathrm{ct}} \cup \big\{\sigma_d \in S^H_{\mathrm{dis}} \mid \boldsymbol{P}^H_{\tau_d}(\Gamma;\sigma_d) \neq 0\big\}.$$

In particular, $S^H(\Gamma)$ is discrete.

PROOF. This follows from Lemmas 39 and 40. □

DEFINITION. For an H-admissible uniform lattice Γ in G, define a measure $\mathrm{d}\mu^H_\Gamma$ on S^H by

$$\mathrm{d}\mu_{\Gamma}^{H} = \sum_{\nu \in S^{H}(\Gamma)} \frac{\left\| \mathscr{P}_{\Gamma}^{H}(\Pi_\nu)/l_{\Pi_\nu}^{0} \right\|^2}{\mathrm{vol}(\Gamma \cap H\backslash H)} \, \delta_\nu .$$

Theorem 43. *Let $\mathscr{L} = \bigoplus_{j=1}^{N} \mathscr{O}_E \, \boldsymbol{u}_j$ be an $\mathscr{O}_E$-lattice generated by a $\boldsymbol{C}$-basis $\{\boldsymbol{u}_j\}$ of W such that $\ell = \boldsymbol{C}\boldsymbol{u}_1$. Let $\{\mathscr{I}_n\}_{n\in\boldsymbol{N}}$ be a sequence of $\mathscr{O}_E$-ideals such that $\mathscr{I}_{n+1} \subset \mathscr{I}_n$ for any $n \in \boldsymbol{N}$ and $\lim_{n\to\infty} \delta(\mathscr{I}_n) = +\infty$. Suppose $\Gamma_{\mathscr{L}}(\mathscr{I}_0)$ is torsion free, and set $\Gamma_n = \Gamma_{\mathscr{L}}(\mathscr{I}_n)$ for $n \in \boldsymbol{N}$. Suppose the condition $\spadesuit(\tau_d)$ is satisfied for $d \in \boldsymbol{N}$ such that $\sigma_d = 1$ if ρ_0 is odd. Then,*

$$\lim_{n\to\infty} \int_{S^H} f(\nu) \mathrm{d}\mu_{\Gamma_n}^{H} = \int_{S^H} f(\nu) \mathrm{d}\mu^{H} \quad \textit{for any } f \in \mathscr{D}(S^H).$$

Proof. Let $f \in \mathscr{D}(S^H)$. It suffices to show the formula for the two cases: (a) $\mathrm{supp}(f) \subset S_{\mathrm{ct}}^{H} \cup \{\nu_0\}$; (b) $\mathrm{supp}(f) \subset S_{\mathrm{dis}}^{H}$.

Case (a): The trivial representation τ_0 occurs in the principal series representation π_ν as a K-type with multiplicity 1. For $x \in \boldsymbol{R}^{+}$, let $\sqrt{\nu_0 - x}$ denote the square root of $\nu_0 - x$ such that $\mathrm{Re}(\sqrt{\nu_0 - x}) \geq 0$, and set

$$\begin{aligned} B(x) &= \beta_0(\sqrt{\nu_0 - x})^{-1} \beta_0(-\sqrt{\nu_0 - x})^{-1}, \\ F(x) &= f(\sqrt{\nu_0 - x}). \end{aligned}$$

Then, by $f \in \mathscr{D}(S^H)$ and by Stirling's formula, $BF \in \mathscr{S}(\boldsymbol{R}^{+})$ is confirmed easily. Applying Theorem 11, we have

$$\lim_{n\to+\infty} \left\langle \mu_{\tau_0}^{H}(\Gamma_n), BF \right\rangle = \left\langle \mu_{\tau_0}^{H}, BF \right\rangle. \tag{7.10}$$

Now, by Lemma 39,

$$\begin{aligned} \left\langle \mu_{\tau_0}^{H}(\Gamma_n), BF \right\rangle &= \sum_{\nu \in S_{\tau_0}^{H}(\Gamma_n)_{\mathrm{ct}}} \frac{\boldsymbol{P}_{\tau_0}^{H}(\Gamma_n;\nu)}{\mathrm{vol}(\Gamma_n \cap H\backslash H)} \beta_0(\nu)^{-1} \beta_0(-\nu)^{-1} f(\nu) \\ &= \sum_{\nu \in S_{\tau_0}^{H}(\Gamma_n)_{\mathrm{ct}}} \frac{\left\| \mathscr{P}_{\Gamma}^{H}(\pi_\nu)/l_{\pi_\nu}^{0} \right\|^2}{\mathrm{vol}(\Gamma_n \cap H\backslash H)} f(\nu) \\ &= \int_{S^H} f(\nu) \, \mathrm{d}\mu_{\Gamma_n}^{H}. \end{aligned} \tag{7.11}$$

On the other hand, by Lemma 44, we have

$$\begin{aligned}\langle \mu^H_{\tau_0}, BF\rangle &= \frac{\Gamma(q-1)}{4(q-1)\pi^{q+1}} \int_0^\infty f(iy)\,|\beta_0(iy)|^2 \frac{\mathrm{d}y}{|\boldsymbol{c}_d(iy)|^2} \\ &= \frac{1}{\pi}\int_0^\infty f(iy)\,\frac{\mathrm{d}y}{|\boldsymbol{c}(iy)|^2} \\ &= \int_{S^H} f(\nu)\mathrm{d}\mu^H. \end{aligned} \tag{7.12}$$

By (7.10), (7.11) and (7.12), we are done.

Case (b): If the condition $\spadesuit(\tau_d)$ for $d \in \boldsymbol{N}$ such that $\sigma_d = 1$ is satisfied, then each point of S^H_{dis} is isolated in $\bigcup_n \mathrm{supp}(\mathrm{d}\mu^H_{\Gamma_n})$. Hence, it suffices to prove the formula point-wisely on S^H_{dis}. Let $\sigma_d \in S^H_{\mathrm{dis}}$. By [**12**, Theorem 55], we have the formula

$$\lim_{n\to\infty} \frac{\boldsymbol{P}^H_{\tau_d}(\Gamma_n;\sigma_d)}{\mathrm{vol}(\Gamma_n\cap H\backslash H)} = \frac{\Gamma(\sigma_d+q)}{\pi^q\Gamma(\sigma_d)} \tag{7.13}$$

unless $\sigma_d = 1$, in which case it is also true by the condition $\spadesuit(\tau_d)$ as was shown in the proof of Lemma 29. By Lemmas 40 and 45, the formula (7.13) is equivalent to the required limit formula with f being the characteristic function of the singleton $\{\sigma_d\}$.

This completes the proof. □

LEMMA 44.

$$\beta_d(s)\beta_d(-s)\cdot \boldsymbol{c}_d(s)\boldsymbol{c}_d(-s) = \frac{4\pi^q}{\Gamma(q-1)}\boldsymbol{c}(s)\boldsymbol{c}(-s).$$

PROOF. A direct computation. □

LEMMA 45. *For each* $\sigma_d \in S^H_{\mathrm{dis}}$,

$$C(\sigma_d) = \frac{\Gamma(\sigma_d+q)}{\pi^q\Gamma(\sigma_d)}\cdot \begin{cases} \beta_d(\sigma_d)^{-1}\beta_d(-\sigma_d)^{-1}, & 0<\sigma_d<\rho_0, \\ \beta_d'(\sigma_d)^{-1}\beta_d(-\sigma_d)^{-1}, & \rho_0 \le \sigma_d. \end{cases}$$

PROOF. A direct computation. □

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