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On the principal symbols of $K_{\boldsymbol{C}}$-invariant differential operators on Hermitian symmetric spaces

By Takashi HASHIMOTO

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Abstract. Let (G, K) be one of the following Hermitian symmetric pair: $(SU(p,q), S(U(p) \times U(q)))$, $(Sp(n, \boldsymbol{R}), U(n))$, or $(SO^*(2n), U(n))$. Let $G_{\boldsymbol{C}}$ and $K_{\boldsymbol{C}}$ be the complexifications of G and K, respectively, Q the maximal parabolic subgroup of $G_{\boldsymbol{C}}$ whose Levi part is $K_{\boldsymbol{C}}$, and V the holomorphic tangent space at the origin of G/K. It is known that the ring of $K_{\boldsymbol{C}}$-invariant differential operators on V has a generating system $\{\varGamma_k\}$ given in terms of determinant or Pfaffian that plays an essential role in the Capelli identities. Our main result is that determinant or Pfaffian of a deformation of the twisted moment map on the holomorphic cotangent bundle of $G_{\boldsymbol{C}}/Q$ provides a generating function for the principal symbols of $\varGamma_k$'s.

1. Introduction.

Let $V := \mathrm{Alt}_n$ be the $\boldsymbol{C}$-vector space consisting of all alternating $n \times n$ complex matrices, and $\boldsymbol{C}[V]$ the $\boldsymbol{C}$-algebra consisting of all polynomial functions on V. Then the complex general linear group GL_n acts on V by

$$g.Z := gZ\,{}^t g \quad (g \in GL_n,\ Z \in V), \tag{1.1}$$

where ${}^t g$ denotes the transpose of g, and one can define a representation π of GL_n on $\boldsymbol{C}[V]$ by

$$\pi(g)f(Z) := f(g^{-1}.Z) \quad (g \in GL_n,\ f \in \boldsymbol{C}[V]). \tag{1.2}$$

For $Z = (z_{i,j})_{i,j=1,\dots,n} \in V$, with $z_{j,i} = -z_{i,j}$, let $M := (z_{i,j})_{i,j}$ and $D := (\partial_{i,j})_{i,j}$ be the alternating $n \times n$ matrices whose (i,j)-th entries are given by the multiplication operator $z_{i,j}$ and the derivation $\partial_{i,j} := \partial/\partial z_{i,j}$, respectively. Then the representation $\mathrm{d}\pi$ of $\mathfrak{gl}_n$, the Lie algebra of GL_n, induced from π is given by

$$\mathrm{d}\pi(E_{i,j}) = -\sum_{k=1}^{n} z_{k,j}\partial_{k,i} \quad (i,j = 1,2,\dots,n) \tag{1.3}$$

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where $E_{i,j}$ denotes the matrix unit of size $n \times n$ which is a basis for $\mathfrak{gl}_n$.

Let us denote by $\mathrm{U}(\mathfrak{gl}_n)$ and $\mathrm{U}(\mathfrak{gl}_n)^{GL_n}$ the universal enveloping algebra of $\mathfrak{gl}_n$, and its subring consisting of GL_n-invariant elements, respectively. Also, let us denote by $\mathscr{PD}(V)$ and $\mathscr{PD}(V)^{GL_n}$ the ring of differential operators on V with polynomial coefficients, and its subring consisting of GL_n-invariant differential operators, respectively. Then the following fact is known:

Theorem ([**4**]).

(1) *The ring homomorphism* $\mathrm{U}(\mathfrak{gl}_n)^{GL_n} \to \mathscr{PD}(V)^{GL_n}$ *induced canonically from* $\mathrm{d}\pi$ *is surjective.*

(2) *For* $k = 1, 2, \dots, \lfloor n/2 \rfloor$[1], *let*

$$\varGamma_k := \sum_{I \subset [n],\, |I| = 2k} \mathrm{Pf}(z_I)\mathrm{Pf}(\partial_I), \tag{1.4}$$

where the sum is taken over all subsets $I \subset [n] := \{1, 2, \dots, n\}$ *such that its cardinality is* $2k$, *and* z_I, ∂_I *denote submatrices of* M, D *consisting of* $z_{i,j}, \partial_{i,j}$ *with* $i, j \in I$.[2] *Then* $\{\varGamma_k\}_{k=1,2,\dots,\lfloor n/2 \rfloor}$ *forms a generating system for* $\mathscr{PD}(V)^{GL_n}$.

In particular, for $k = 1, 2, \dots, \lfloor n/2 \rfloor$, there exist elements of $\mathrm{U}(\mathfrak{gl}_n)^{GL_n}$, called *skew Capelli elements*, that correspond to $\varGamma_k$ under the homomorphism. As the names show, they play an essential role in the skew Capelli identity.

Now, following [**6**], [**8**], let us consider a $2n \times 2n$ matrix $\mathbf{\Phi}$ alternating along the anti-diagonal with entries in $\mathscr{PD}(V)$:

$$\mathbf{\Phi} := \left[\begin{array}{cccc|cccc}
u & & & & z_{1,n} & \cdots & z_{1,2} & 0 \\
 & u & & & \vdots & \iddots & 0 & -z_{1,2} \\
 & & \ddots & & z_{n-1,n} & 0 & \iddots & \vdots \\
 & & & u & 0 & -z_{n-1,n} & \cdots & -z_{1,n} \\
\hline
\partial_{1,n} & \cdots & \partial_{n-1,n} & 0 & -u & & & \\
\vdots & \iddots & 0 & -\partial_{n-1,n} & & \ddots & & \\
\partial_{1,2} & 0 & \iddots & \vdots & & & -u & \\
0 & -\partial_{1,2} & \cdots & -\partial_{1,n} & & & & -u
\end{array}\right], \tag{1.5}$$

[1] For $x \in \boldsymbol{R}$, $\lfloor x \rfloor$ stands for the greatest integer not exceeding x.

[2] For an alternating matrix A, $\mathrm{Pf}(A)$ denotes the Pfaffian of A; see [**7**], or (A.1) and (A.2) for its definition.

where u is a parameter that commutes with $x_{i,j}$ and $\partial_{i,j}$ for all i,j. Our original motivation of this work is to understand the skew Capelli elements more deeply, but let us focus on the corresponding *commutative* objects i.e. the principal symbols of Γ_k, which we denote by γ_k, before we enter the *noncommutative* world. So, it is immediate from the minor summation formula of Pfaffian (see [**5**], or (A.4) below) that the symbol $\sigma(\mathrm{Pf}(\boldsymbol{\Phi}))$ of $\mathrm{Pf}(\boldsymbol{\Phi})$ provides a generating function for $\{\gamma_k\}$:

$$\sigma(\mathrm{Pf}(\boldsymbol{\Phi})) = \sum_{k=0}^{\lfloor n/2 \rfloor} u^{n-2k}\gamma_k, \tag{1.6}$$

where we put $\gamma_0 = 1$ for convenience.

As for the noncommutative counterpart, we can show that $\mathrm{Pf}(\boldsymbol{\Phi})$ can be expanded in the same way as (1.6), but with the coefficient u^{n-2k} of γ_k replaced by certain monic polynomial in u of degree $n-2k$ which is essentially the Hermite polynomial (see [**2**]). Thus, if $\boldsymbol{\Phi}$ came from $\mathrm{U}(\mathfrak{gl}_n) \otimes \mathrm{Mat}_n(\boldsymbol{C})$, it would follow immediately from the property of Pfaffian that all Γ_k belong to $\mathrm{U}(\mathfrak{gl}_n)^{GL_n}$. However, it follows from (1.3) that there exist no elements of $\mathrm{U}(\mathfrak{gl}_n)$ that correspond to the multiplication operators $z_{i,j}$, nor the derivations $\partial_{i,j}$.

What is the natural reason for considering the matrix $\boldsymbol{\Phi}$ above (or, its commutative counterpart)?

We observe that the action (1.1) of GL_n on $V = \mathrm{Alt}_n$ is the holomorphic part of the complexification of isotropy representation at the origin of Hermitian symmetric space $SO^*(2n)/U(n)$. So, we embed $\mathrm{U}(\mathfrak{gl}_n)^{GL_n}$ into $\mathrm{U}(\mathfrak{so}_{2n})^{GL_n}$ and seek for a generating function for $\{\gamma_k\}$ in the latter, in order to find an answer to the question raised above. Here $\mathfrak{so}_{2n}$ denotes the complexification of the Lie algebra of $SO^*(2n)$.

$$\begin{tikzcd}
\mathrm{U}(\mathfrak{gl}_n)^{GL_n} \arrow[dr, hook] \arrow[dd, twoheadrightarrow] & \\
& \mathrm{U}(\mathfrak{so}_{2n})^{GL_n} \arrow[dl] \\
\mathscr{P}\mathscr{D}(V)^{GL_n} &
\end{tikzcd}$$

The real linear Lie group $SO^*(2n)$ has irreducible unitary representations, called holomorphic discrete series representations. We construct one of them, say π_λ, via Borel-Weil theory, which we induce from a holomorphic character

$\lambda : Q \to \boldsymbol{C}^\times$ whose differential is given by a multiple of the half sum of the noncompact positive roots, where Q is the maximal parabolic subgroup of the complex special orthogonal group SO_{2n} whose Levi part is isomorphic to GL_n. Note that π_λ is represented on the Hilbert space consisting of square-integrable holomorphic functions defined on an open subset of V, and that the restriction of π_λ to K-finite part coincides with π given in (1.2) when λ is trivial.

Let $\mathrm{d}\pi_\lambda$ be the differential representation induced from π_λ, which we extend to the representation of $\mathfrak{so}_{2n}$ by linearity. Take a basis $\{X_i\}$ for $\mathfrak{so}_{2n}$, and its dual basis $\{X_i^\vee\}$, i.e. the basis for $\mathfrak{so}_{2n}$ satisfying that

$$B\big(X_i, X_j^\vee\big) = \delta_{i,j},$$

where B is the nondegenerate bilinear form on $\mathfrak{so}_{2n}$ given by $B(X,Y) := (1/2)\mathrm{tr}(XY)$. For $X \in \mathfrak{so}_{2n}$ given, we denote by $\sigma_\lambda(X)$ the symbol of the differential operator $\mathrm{d}\pi_\lambda(X)$ substituted $\xi_{i,j}$ for $\partial_{i,j}$. Now we define an element $\sigma_\lambda(\boldsymbol{X}) \in \boldsymbol{C}[z_{i,j}, \xi_{i,j}; 1 \le i < j \le n] \otimes \mathfrak{so}_{2n}$ by

$$\sigma_\lambda(\boldsymbol{X}) := \sum_i \sigma_\lambda(X_i^\vee) \otimes X_i.$$

Note that $\sigma_\lambda(\boldsymbol{X})$ is independent of the basis $\{X_i\}$ chosen and that it can be considered to be a variant of the twisted moment map on the holomorphic cotangent bundle $T^*(SO_{2n}/Q)$ (see [**11**]). Then we deform $\sigma_\lambda(\boldsymbol{X})$ by shifting the parameter of λ by γ_1, and replace the shifted parameter with a newly introduced indeterminate, say τ; we denote by $\widetilde{\sigma}_\lambda(\boldsymbol{X})$ the deformed symbol with τ (see Section 3 for details).

One of our main results is that Pfaffian of $\widetilde{\sigma}_\lambda(\boldsymbol{X})$ provides a generating function for $\{\gamma_k\}$ (Theorem 3.1 (3)). On the way of proof, we will find that there naturally appears a matrix which looks like $\boldsymbol{\Phi}$ given in (1.5). Namely, if we regard $\widetilde{\sigma}_\lambda(\boldsymbol{X})$ as the value at (x, ξ) of the map

$$T^*(SO_{2n}/Q) \to \boldsymbol{C}[\tau, z_{i,j}, \xi_{i,j}] \otimes \mathfrak{so}_{2n}, \quad (x,\xi) \mapsto \widetilde{\sigma}_\lambda(\boldsymbol{X}),$$

and denote by u_x the element of U_Ω (see Section 2 for the notation) that translates the origin of SO_{2n}/Q to x, we see that $\mathrm{Ad}(u_x^{-1})\widetilde{\sigma}_\lambda(\boldsymbol{X})$ is of the form

$$\left[\begin{array}{cccc|cccc}
\tau+\gamma_1 & & & & -\vartheta z_{1,n-1} & \cdots & -\vartheta z_{1,2} & 0 \\
 & \tau+\gamma_1 & & & -\vartheta z_{2,n-1} & \cdots & 0 & \vartheta z_{1,2} \\
 & & \ddots & & \vdots & \iddots & \vdots & \vdots \\
 & & & \tau+\gamma_1 & 0 & \cdots & \vartheta z_{2,n-1} & \vartheta z_{1,n-1} \\
\hline
-\xi_{1,n-1} & -\xi_{2,n-1} & \cdots & 0 & -\tau-\gamma_1 & & & \\
\vdots & \vdots & \iddots & \vdots & & \ddots & & \\
-\xi_{1,2} & 0 & \cdots & \xi_{2,n-1} & & & -\tau-\gamma_1 & \\
0 & \xi_{1,2} & \cdots & \xi_{1,n-1} & & & & -\tau-\gamma_1
\end{array}\right],$$

where $(z_{i,j};\xi_{i,j})$ are coordinates around $(x,\xi)\in T^*(SO_{2n}/Q)$, and we set $\vartheta := s-(\tau+\gamma_1)$ for brevity, with s the integral parameter of λ (Theorem 4.4).

All the setup so far also applies to the other two classical irreducible Hermitian symmetric pairs of noncompact type $(SU(p,q),S(U(p)\times U(q)))$ and $(Sp(n,\boldsymbol{R}),U(n))$. Let (G,K) be one of the two. Then the differential operators Γ_k given above have analogous objects, i.e. $K_{\boldsymbol{C}}$-invariant differential operators acting on the space of polynomial functions on V, the holomorphic tangent space at the origin of G/K, which generate $\mathscr{P}\mathscr{D}(V)^{K_{\boldsymbol{C}}}$ and play an essential role in the corresponding Capelli identity, where $K_{\boldsymbol{C}}$ denotes a complexification of K (cf. [**4**]). In these cases, they are given in terms of (minor-)determinants defined analogously to M and D above (see (3.1) for details). If we define $\widetilde{\sigma}_\lambda(\boldsymbol{X})$ in the same fashion as in the case of $\mathfrak{so}_{2n}$, we can show that the determinant of $\widetilde{\sigma}_\lambda(\boldsymbol{X})$ provides a generating function for their principal symbols (Theorem 3.1 (1) and (2)).

The contents of this paper are as follows: In Section 2, we give explicit realizations of $G=SU(p,q),Sp(n,\boldsymbol{R})$ and $SO^*(2n)$, and their complexification $G_{\boldsymbol{C}}$. Then we make a brief review of construction of the holomorphic discrete series representations π_λ via Borel-Weil theory, which we induce from a character λ of a maximal parabolic subgroup of $G_{\boldsymbol{C}}$ whose Levi part is $K_{\boldsymbol{C}}$. In Section 3, we recall the definition of the $K_{\boldsymbol{C}}$-invariant differential operators $\{\Gamma_k\}$ for each G, and introduce our main objects $\sigma_\lambda(\boldsymbol{X})$ and $\widetilde{\sigma}_\lambda(\boldsymbol{X})$. Then we state our main result that the determinant or Pfaffian of $\widetilde{\sigma}_\lambda(\boldsymbol{X})$ provides a generating function for the principal symbols of $\{\Gamma_k\}$ (Theorem 3.1). The proof is done case-by-case in Section 4 according as $G=SO^*(2n),Sp(n,\boldsymbol{R})$, or $SU(p,q)$ $(p\geq q)$, which is based on explicit forms of the differential operators $\mathrm{d}\pi_\lambda(X_i)$, with $\{X_i\}$ a basis for $\mathfrak{g}:=\mathrm{Lie}(G)\otimes_{\boldsymbol{R}}\boldsymbol{C}$. In Section 5, we show that $\sigma_\lambda(\boldsymbol{X})$ can be considered to be a variant of the twisted moment map. In the appendix, we recall the definition of Pfaffian, and collect a couple of minor summation formulae of Pfaffian and determinant without proof, which we make use of in proving the main results.

2. Holomorphic discrete series.

In what follows, let G denote one of $SU(p,q)$ $(p \geq q)$, $Sp(n,\boldsymbol{R})$, or $SO^*(2n)$, which we realize as follows:

$$\begin{aligned} SU(p,q) &= \{g \in SL_{p+q}(\boldsymbol{C});{}^t\bar{g}I_{p,q}g = I_{p,q}\}, \\ Sp(n,\boldsymbol{R}) &= \{g \in SU(n,n);{}^tgJ_{n,n}g = J_{n,n}\}, \\ SO^*(2n) &= \{g \in SU(n,n);{}^tgJ_{2n}g = J_{2n}\}. \end{aligned} \tag{2.1}$$

Here, for positive integers $p,q,n = 1,2,\dots$, the matrices $I_{p,q}$, J_n, and $J_{n,n}$ are given by

$$I_{p,q} = \begin{bmatrix} 1_p & \\ & -1_q \end{bmatrix}, \quad J_n = \begin{bmatrix} & & 1 \\ & \begin{smallmatrix}\cdot^{\cdot^{\cdot}}\end{smallmatrix} & \\ 1 & & \end{bmatrix}, \quad J_{n,n} = \begin{bmatrix} & J_n \\ -J_n & \end{bmatrix}.$$

Let K be a maximal compact subgroup of G given by $K := \left\{\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right] \in G\right\}$, where, for an element $\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$ in K, the submatrices a and d are of size $p \times p$ and $q \times q$, respectively, when $G = SU(p,q)$, or, both of size $n \times n$ when $G = SO^*(2n)$ and $Sp(n,\boldsymbol{R})$. Let $G_{\boldsymbol{C}}$ and $K_{\boldsymbol{C}}$ be the complexifications of G and K, respectively, and Q the maximal parabolic subgroup of $G_{\boldsymbol{C}}$ given by $Q := \left\{\left[\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right] \in G_{\boldsymbol{C}}\right\}$ so that its Levi part is $K_{\boldsymbol{C}}$. Define a holomorphic character $\lambda : Q \to \boldsymbol{C}^\times$ by

$$\lambda\left(\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}\right) = (\det d)^{-s} \tag{2.2}$$

for $s \in \boldsymbol{Z}$. Denote by $\boldsymbol{C}_\lambda$ the one-dimensional Q-module defined by $p.v = \lambda(p)v$ $(p \in Q, v \in \boldsymbol{C})$, and by L_λ the pull-back of the holomorphic line bundle $G_{\boldsymbol{C}} \times_Q \boldsymbol{C}_\lambda$ by the embedding $G/K = GQ/Q \hookrightarrow G_{\boldsymbol{C}}/Q$.

The space $\Gamma(L_\lambda)$ of all holomorphic sections for L_λ are identified with the space of all holomorphic functions f on the open subset $GQ \subset G_{\boldsymbol{C}}$ that satisfy

$$f(xp) = \lambda(p)^{-1}f(x) \quad (x \in GQ,\ p \in Q). \tag{2.3}$$

Define $\Gamma_{L^2}(L_\lambda)$ to be the subspace of $\Gamma(L_\lambda)$ consisting of square integrable holomorphic sections with respect to Haar measure on G:

$$\Gamma_{L^2}(L_\lambda) := \left\{f \in \Gamma(L_\lambda); \int_G |f(g)|^2 \mathrm{d}g < \infty\right\}, \tag{2.4}$$

and let G act on $\Gamma_{L^2}(L_\lambda)$ by

$$(T_\lambda(g)f)(x) := f(g^{-1}x) \quad (g \in G,\ x \in GQ).$$

Then $(T_\lambda, \Gamma_{L^2}(L_\lambda))$ is an irreducible unitary representation of G if it is not zero, called *a holomorphic discrete series*. One must impose some condition on s in order to require that $\Gamma_{L^2}(L_\lambda)$ be nonzero, which we do not consider here since we will be concerned with the representations of Lie algebras in this paper (see e.g. [**9**] for the condition).

Now, according as $G = SU(p,q)$ $(p \geq q)$, $Sp(n, \boldsymbol{R})$, or $SO^*(2n)$, let us realize the bounded symmetric domain Ω as follows:

$$\begin{aligned}
&\text{if } G = SU(p,q), && \Omega := \{z \in \mathrm{Mat}_{p,q}(\boldsymbol{C}); 1_q - {}^t\bar{z}z > 0\};\\
&\text{if } G = Sp(n,\boldsymbol{R}), && \Omega := \{z \in \mathrm{Mat}_n(\boldsymbol{C}); 1_n - {}^t\bar{z}z > 0, J_n{}^tzJ_n = z\};\\
&\text{if } G = SO^*(2n), && \Omega := \{z \in \mathrm{Mat}_n(\boldsymbol{C}); 1_n - {}^t\bar{z}z > 0, J_n{}^tzJ_n = -z\},
\end{aligned}$$

and let G act on Ω by linear fractional transformation:

$$g.z = (az+b)(cz+d)^{-1} \quad \left(g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G,\ z \in \Omega\right). \tag{2.5}$$

Then Ω is isomorphic to G/K in each case. If we denote by $\mathscr{O}(\Omega)$ the space of all holomorphic functions on Ω, we define a map

$$\Phi : \Gamma(L_\lambda) \to \mathscr{O}(\Omega), \quad f \mapsto F$$

by

$$F(z) = f\left(\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}\right).$$

Then Φ is a bijection. Let $\mathscr{H}_\lambda := \Phi(\Gamma_{L^2}(L_\lambda))$, and define an action π_λ of G on $\mathscr{H}_\lambda$ so that the diagram (2.6) commutes for all $g \in G$:

$$\begin{tikzcd}
\Gamma_{L^2}(L_\lambda) \arrow[r, "\Phi"] \arrow[d, "T_\lambda(g)"'] & \mathscr{H}_\lambda \arrow[d, "\pi_\lambda(g)"] \\
\Gamma_{L^2}(L_\lambda) \arrow[r, "\Phi"] & \mathscr{H}_\lambda\,.
\end{tikzcd} \tag{2.6}$$

Explicitly, it is given by

$$(\pi_\lambda(g)F)(z) = \det(cz+d)^s\, F((az+b)(cz+d)^{-1}) \tag{2.7}$$

for $g \in G$ and $F \in \mathscr{H}_\lambda$, where $g^{-1} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.

Let us introduce a few more notations. Let $\mathfrak{g} := \mathrm{Lie}(G) \otimes_{\boldsymbol{R}} \boldsymbol{C}$, $\mathfrak{q} := \mathrm{Lie}(Q)$, $\bar{\mathfrak{u}}$ the nilradical of $\mathfrak{q}$, and $\mathfrak{u}$ its opposite. Then we have $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{q}$ and $\mathfrak{q} = \mathfrak{k} \oplus \bar{\mathfrak{u}}$, where $\mathfrak{k} := \mathrm{Lie}(K) \otimes_{\boldsymbol{R}} \boldsymbol{C}$. We denote by $\mathfrak{t}$ the Cartan subalgebra of $\mathfrak{g}$ that consists of diagonal matrices in $\mathfrak{g}$. Let $\mathrm{d}\pi_\lambda$ be the differential representation of $\mathrm{Lie}(G)$ induced from π_λ, which we extend to the representation of the complex Lie algebra $\mathfrak{g}$ by linearity. Furthermore, identifying Ω with an open subset of $\mathfrak{u}$ by $z \leftrightarrow \left[\begin{smallmatrix} 0 & z \\ 0 & 0 \end{smallmatrix}\right]$, let U_Ω be the image of $\Omega \subset \mathfrak{u}$ by the exponential map. Note that if $t \in \boldsymbol{R}$ is sufficiently small, then $\exp tX$ acts on Ω for any $X \in \mathfrak{g}$ by (2.5), hence on $\mathscr{H}_\lambda$ by (2.7). Thus its differential at $t = 0$ coincides with $\mathrm{d}\pi_\lambda$.

Since we realize the real linear Lie groups G as in (2.1), the corresponding complex Lie algebras $\mathfrak{g}$ are given by

$$\begin{aligned} \mathfrak{sl}_{p+q} &= \{X \in \mathrm{Mat}_{p+q}(\boldsymbol{C}); \mathrm{tr}(X) = 0\}, \\ \mathfrak{sp}_n &= \{X \in \mathrm{Mat}_{2n}(\boldsymbol{C}); {}^tXJ_{n,n} + XJ_{n,n} = O\}, \\ \mathfrak{so}_{2n} &= \{X \in \mathrm{Mat}_{2n}(\boldsymbol{C}); {}^tXJ_{2n} + XJ_{2n} = O\}, \end{aligned} \tag{2.8}$$

respectively.

Remark 2.1. If we denote the character of $\mathfrak{q}$ induced from $\lambda : Q \to \boldsymbol{C}^\times$ by the same letter λ, then it is proportional to ρ_n, the half sum of the noncompact positive roots; more precisely,

$$\lambda = \frac{2s}{\ell + \varepsilon}\, \rho_n,$$

where ℓ is the rank of $\mathfrak{g}$, and $\varepsilon = +1$ if $\mathfrak{g} = \mathfrak{sl}_{p+q}$ or $\mathfrak{sp}_n$, and $\varepsilon = -1$ if $\mathfrak{g} = \mathfrak{so}_{2n}$.

3. Principal symbols of $K_{\boldsymbol{C}}$-invariant differential operators.

Let V denote the holomorphic tangent space at the origin of G/K. Then one can identify V with $\mathfrak{u}$ and construct a representation of $K_{\boldsymbol{C}}$ on the $\boldsymbol{C}$-algebra $\boldsymbol{C}[V]$ of all polynomial function on V through the action of $K_{\boldsymbol{C}}$ on $V = \mathfrak{u}$. Let $\mathscr{P}\mathscr{D}(V)^{K_{\boldsymbol{C}}}$ be the ring of $K_{\boldsymbol{C}}$-invariant differential operators with polynomial coefficient and $r := \boldsymbol{R}\text{-rank}\, G$, the real rank of G. Then it is known that there exists a generating system $\{\Gamma_k\}_{k=1,2,\dots,r}$ for $\mathscr{P}\mathscr{D}(V)^{K_{\boldsymbol{C}}}$ which is given in terms of determinant or Pfaffian as follows ([**4**]):

3.1. The case $\boldsymbol{G = SU(p,q)}$ $\boldsymbol{(p \geq q)}$.

In this case, $V = \mathfrak{u}$ is isomorphic to $\mathrm{Mat}_{p,q}(\boldsymbol{C})$. Let M and D be $p \times q$ matrices whose (i,j)-th entries are given by the multiplication operators $z_{i,j}$ and the differential operators $\partial_{i,j} := \partial/\partial z_{i,j}$ for $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$, and z^I_J and ∂^I_J submatrices of M and D consisting of $z_{i,j}$ and $\partial_{i,j}$, respectively, with $i \in I \subset [p]$ and $j \in J \subset [q]$. Then we define

$$\Gamma_k = \sum_{\substack{I \subset [p], J \subset [q] \\ |I| = |J| = k}} \det(z^I_J) \det(\partial^I_J) \quad (k = 1, 2, \dots, q), \tag{3.1a}$$

which indeed form a generating system for $\mathscr{P}\mathscr{D}(V)^{\tilde{K}_{\boldsymbol{C}}}$ with $\tilde{K}_{\boldsymbol{C}} = GL_p \times GL_q \supset K_{\boldsymbol{C}}$.

3.2. The case $\boldsymbol{G = Sp(n, R)}$.

In this case, $V = \mathfrak{u}$ is isomorphic to the vector space of all symmetric $n \times n$ complex matrices $z = (z_{i,j})$, $z_{j,i} = z_{i,j}$. Let

$$\tilde{\partial}_{i,j} = \begin{cases} 2\partial_{i,j} & (i = j), \\ \partial_{i,j} & (i \neq j), \end{cases}$$

and let M and D be symmetric $n \times n$ matrices whose (i,j)-th entries are given by the multiplication operators $z_{i,j}$ and the differential operators $\tilde{\partial}_{i,j}$ for $i, j = 1, 2, \dots, n$. Then we define

$$\Gamma_k = \sum_{\substack{I, J \subset [n] \\ |I| = |J| = k}} \det(z^I_J) \det(\tilde{\partial}^I_J) \quad (k = 1, 2, \dots, n), \tag{3.1b}$$

where z^I_J, ∂^I_J are as above, with $I, J \subset [n]$.

3.3. The case $\boldsymbol{G = SO^*(2n)}$.

In this case, $V = \mathfrak{u}$ is isomorphic to the vector space of all alternating $n \times n$ complex matrices $z = (z_{j,i})$, $z_{j,i} = -z_{i,j}$. Let M and D be alternating $n \times n$ matrices whose (i,j)-th entries are given by the multiplication operators $z_{i,j}$ and the differential operators $\partial_{i,j}$ for $i, j = 1, 2, \dots, n$. Then we define

$$\Gamma_k = \sum_{I \subset [n], |I| = 2k} \mathrm{Pf}(z_I) \mathrm{Pf}(\partial_I), \quad (k = 1, 2, \dots, \lfloor n/2 \rfloor) \tag{3.1c}$$

where z_I, ∂_I are as in the Introduction, with $I \subset [n]$.

Let $\mathfrak{g}$ denote one of $\mathfrak{sl}_{p+q}, \mathfrak{sp}_n$, or $\mathfrak{so}_{2n}$ again. Define a nondegenerate $G_{\boldsymbol{C}}$-invariant bilinear form B on $\mathfrak{g}$ by

$$B(X,Y) = \begin{cases} \mathrm{tr}(XY) & \text{if } \mathfrak{g} = \mathfrak{sl}_{p+q}, \\ \dfrac{1}{2}\mathrm{tr}(XY) & \text{if } \mathfrak{g} = \mathfrak{so}_{2n}, \text{or } \mathfrak{sp}_n. \end{cases} \tag{3.2}$$

Given a basis $\{X_i\}_{i=1,\ldots,\dim\mathfrak{g}}$ for $\mathfrak{g}$, let us denote by $\{X_i^\vee\}$ the dual basis with respect to B i.e. the basis for $\mathfrak{g}$ satisfying

$$B\bigl(X_i, X_j^\vee\bigr) = \delta_{i,j}$$

for $i, j = 1, \ldots, \dim\mathfrak{g}$. Then we define an element $\boldsymbol{X}$ of $\mathrm{U}(\mathfrak{g}) \otimes \mathrm{Mat}_N(\boldsymbol{C})$ by

$$\boldsymbol{X} := \sum_{i=1}^{\dim\mathfrak{g}} X_i^\vee \otimes X_i, \tag{3.3}$$

where $\mathrm{U}(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$, and N the size of matrices when $\mathfrak{g}$ is realized as in (2.8). We regard the former and the latter factors in (3.3) as elements of $\mathrm{U}(\mathfrak{g})$ and $\mathrm{Mat}_N(\boldsymbol{C})$ respectively.

Denoting by $\sigma_\lambda(X)$ the symbol of the differential operator $\mathrm{d}\pi_\lambda(X)$ for $X \in \mathfrak{g}$, let us define a $\mathfrak{g}$-valued polynomial function on the holomorphic cotangent bundle $T^*(G_{\boldsymbol{C}}/Q)$ by

$$\sigma_\lambda(\boldsymbol{X}) := \sum_{i=1}^{\dim\mathfrak{g}} \sigma_\lambda\bigl(X_i^\vee\bigr) \otimes X_i. \tag{3.4}$$

By definition, $\boldsymbol{X}$ and $\sigma_\lambda(\boldsymbol{X})$ are independent of the basis $\{X_i\}$ chosen.

Note that $\sigma_\lambda(X_i^\vee)$ involves the parameter s if and only if $X_i^\vee$ is in $\mathfrak{t}$, or in $\mathfrak{u}^-$. We will see that $\sigma_\lambda(X_i^\vee)$ takes the following form:

$$\sigma_\lambda(X_i^\vee) = \begin{cases} \kappa_i s + (\text{linear terms in } \xi_\alpha\text{'s}) & \text{if } X_i^\vee \in \mathfrak{t}, \\ \kappa_i s\, z_i + (\text{linear terms in } \xi_\alpha\text{'s}) & \text{if } X_i^\vee \in \mathfrak{u}^-, \end{cases} \tag{3.5}$$

for some $\kappa_i \neq 0 \in \boldsymbol{Q}$, where $(z_\alpha; \xi_\alpha)$ are coordinates around $(z, \xi) \in T^*(G_{\boldsymbol{C}}/Q)$ (see Propositions 4.1, 4.5 and 4.8 below). Now, we shift the parameter s by γ_1 and replace $s - \gamma_1$ by a new parameter, say τ, as follows:

- If the symbol $\sigma_\lambda(X_i^\vee)$ involves s, we rewrite $\kappa_i s$ in (3.5) as

$$\kappa_i s = \frac{\kappa_i}{|\kappa_i|}(s - (1 - |\kappa_i|)s),$$

and substitute $\tau + \gamma_1$ into the former s:

$$\frac{\kappa_i}{|\kappa_i|}(\tau + \gamma_1 - (1 - |\kappa_i|)s).$$

Let us denote by $\widetilde{\sigma}_\lambda(X_i^\vee)$ the deformed symbol with τ obtained in this way.

- If the symbol $\sigma_\lambda(X_i^\vee)$ does not involve s, we let $\widetilde{\sigma}_\lambda(X_i^\vee) := \sigma_\lambda(X_i^\vee)$.

Finally, we define

$$\widetilde{\sigma}_\lambda(\boldsymbol{X}) := \sum_{i=1}^{\dim \mathfrak{g}} \widetilde{\sigma}_\lambda\big(X_i^\vee\big) \otimes X_i \in \boldsymbol{C}[\tau, z_\alpha, \xi_\alpha] \otimes \mathfrak{g}. \tag{3.6}$$

Denoting the principal symbol of $\varGamma_k$ by γ_k for each $k = 1, 2, \ldots, r$, our main result is the following.

THEOREM 3.1. *The determinant or Pfaffian of $\widetilde{\sigma}_\lambda(\boldsymbol{X})$ provides a generating function for $\{\gamma_k\}_{k=1,2,\ldots,r}$. More precisely, we have the following identities*:

1. *For $G = SU(p,q)$ with $p \geq q$,*

$$\det(\widetilde{\sigma}_\lambda(\boldsymbol{X})) = (-1)^q \sum_{k=0}^{q} \left(\tau + \gamma_1 - \frac{p}{p+q}s\right)^{p-k} \cdot \left(\tau + \gamma_1 - \frac{q}{p+q}s\right)^{q-k} (s - (\tau + \gamma_1))^k \gamma_k; \tag{3.7a}$$

2. *For $G = Sp(n, \boldsymbol{R})$,*

$$\det(\widetilde{\sigma}_\lambda(\boldsymbol{X})) = (-1)^n \sum_{k=0}^{n} (\tau + \gamma_1)^{2n-2k} (s - (\tau + \gamma_1))^k \gamma_k; \tag{3.7b}$$

3. *For $G = SO^*(2n)$,*

$$\mathrm{Pf}(\widetilde{\sigma}_\lambda(\boldsymbol{X})) = \sum_{k=0}^{\lfloor n/2 \rfloor} (\tau + \gamma_1)^{n-2k} (s - (\tau + \gamma_1))^k \gamma_k; \tag{3.7c}$$

where we set $\gamma_0 = 1$.

We will give the proof of these identities in the next section case-by-case.

4. Proof of main result.

Let us denote the $m \times n$ matrix whose (i,j)-th entry is 1 and all others are 0 by $E_{i,j}^{(m,n)}$, which, if $m = n$, we abbreviate to $E_{i,j}^{(n)}$; moreover, if the size of $E_{i,j}^{(n)}$ is equal to the size of the linear Lie algebra under consideration, we suppress the superscript.

4.1. The case $G = SO^*(2n)$.

First let us prove the case where $G = SO^*(2n)$, or $\mathfrak{g} = \mathfrak{so}_{2n}$, since this is our starting point of this project. Take a basis $\{X_{i,j}^{\epsilon}\}_{\epsilon=0,\pm;i,j\in[n]}$ for $\mathfrak{so}_{2n}$ as follows:

$$\begin{aligned} X_{i,j}^0 &:= E_{i,j} - E_{-j,-i} \quad (1 \le i, j \le n), \\ X_{i,j}^+ &:= E_{i,-j} - E_{j,-i} \quad (1 \le i < j \le n), \\ X_{i,j}^- &:= E_{-j,i} - E_{-i,j} \quad (1 \le i < j \le n), \end{aligned} \tag{4.1}$$

where we agree that $-i$ stands for $2n+1-i$ in what follows.

Proposition 4.1. *The differential operators* $\mathrm{d}\pi_\lambda(X_{i,j}^{\epsilon})$ $(\epsilon = 0, \pm; i, j \in [n])$ *are given by*

$$\mathrm{d}\pi_\lambda\big(X_{i,j}^0\big) = s\delta_{i,j} - \sum_{k\in[n]} z_{k,j}\partial_{k,i}, \tag{4.2}$$

$$\mathrm{d}\pi_\lambda\big(X_{i,j}^+\big) = -\partial_{i,j}, \tag{4.3}$$

$$\mathrm{d}\pi_\lambda\big(X_{i,j}^-\big) = -2sz_{i,j} - \sum_{1\le k<l\le n} (z_{k,i}z_{j,l} - z_{k,j}z_{i,l})\partial_{k,l}. \tag{4.4}$$

Proof. Let $\tilde{z} = \sum_{k,l\in[n]} z_{k,l}E_{k,l}^{(n)}$ with $z_{l,k} = -z_{k,l}$. Then $z := \tilde{z}J_n$ belongs to Ω if it is positive definite.

(I) First we calculate $\mathrm{d}\pi_\lambda(X_{i,j}^0)$. Writing $\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right] := \exp(-tX_{i,j}^0)$, we see that

$$\begin{aligned} &\exp\big(-tX_{i,j}^0\big).z \\ &\quad = azd^{-1} \\ &\quad = \Big(1_n - tE_{i,j}^{(n)} + O(t^2)\Big)z\Big(1_n + tE_{n+1-j,n+1-i}^{(n)} + O(t^2)\Big)^{-1} \end{aligned}$$

$$
\begin{aligned}
&= \Bigl(\tilde{z} - t\Bigl(E^{(n)}_{i,j}\tilde{z} + \tilde{z}E^{(n)}_{j,i}\Bigr)\Bigr)J_n + O(t^2) \\
&= \Biggl(\sum_{k,l\in[n]} z_{k,l}E^{(n)}_{k,l} - t\sum_{k,l\in[n]} z_{k,l}\Bigl(E^{(n)}_{i,j}E^{(n)}_{k,l} + E^{(n)}_{k,l}E^{(n)}_{j,i}\Bigr)\Biggr)J_n + O(t^2) \\
&= \sum_{k,l\in[n]} \bigl(z_{k,l} - t(\delta_{i,k}z_{j,l} + \delta_{i,l}z_{k,j})\bigr)E^{(n)}_{k,l}J_n + O(t^2)
\end{aligned}
$$

and that

$$
(\det d)^s = \Bigl(\det\bigl(1_n + tE^{(n)}_{n+1-j,n+1-i} + O(t^2)\bigr)\Bigr)^s = e^{st\operatorname{tr}(E^{(n)}_{n+1-j,n+1-i}) + O(t^2)}.
$$

Therefore, for $F \in \mathscr{H}_\lambda$, we obtain that

$$
\begin{aligned}
&\bigl(\mathrm{d}\pi_\lambda(X^0_{i,j})F\bigr)(z) \\
&\quad= \frac{d}{dt}\bigg|_{t=0}\bigl(\pi_\lambda(\exp(tX^0_{i,j}))F\bigr)(z) \\
&\quad= \frac{d}{dt}\bigg|_{t=0} e^{st\operatorname{tr}(E^{(n)}_{n+1-j,n+1-i})}F\Biggl(\sum_{k,l\in[n]}\bigl(z_{k,l} - t(\delta_{i,k}z_{j,l} + \delta_{i,l}z_{k,j})\bigr)E^{(n)}_{k,l}J_n\Biggr) \\
&\quad= \Biggl(s\delta_{i,j} - \sum_{k<l}(\delta_{i,k}z_{j,l} + \delta_{i,l}z_{k,j})\partial_{k,l}\Biggr)F(z) \\
&\quad= \Biggl(s\delta_{i,j} - \sum_{i<l} z_{j,l}\partial_{i,l} - \sum_{k<i} z_{k,j}\partial_{k,i}\Biggr)F(z) \\
&\quad= \Biggl(s\delta_{i,j} - \sum_{k\in[n]} z_{j,k}\partial_{i,k}\Biggr)F(z).
\end{aligned}
$$

(II) Next we calculate $\mathrm{d}\pi_\lambda(X^+_{i,j})$. Writing $\left[\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right] := \exp(-tX^+_{i,j})$, we see that

$$
\begin{aligned}
\exp\bigl(-tX^+_{i,j}\bigr).z = z + b& \\
&= \Bigl(\tilde{z} - t\Bigl(E^{(n)}_{i,j} - E^{(n)}_{j,i}\Bigr)\Bigr)J_n \\
&= \sum_{k,l\in[n]} (z_{k,l} - t\delta_{i,k}\delta_{j,l} + t\delta_{j,k}\delta_{i,l})E^{(n)}_{k,l}J_n,
\end{aligned}
$$

hence we obtain

$$\begin{aligned}
\mathrm{d}\pi_\lambda(X^+_{i,j}) &= -\sum_{k<l}(\delta_{i,k}\delta_{j,l} - \delta_{i,l}\delta_{j,k})\partial_{k,l} \\
&= -\partial_{i,j}.
\end{aligned}$$

(III) Finally, let us calculate $\mathrm{d}\pi_\lambda(X^-_{i,j})$. Writing $\left[\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right] := \exp(-tX^-_{i,j})$, we see that

$$\begin{aligned}
&\exp\big(-tX^-_{i,j}\big).z \\
&\quad = z(1_n + cz)^{-1} \\
&\quad = z\Big(1_n + tJ_n\Big(E^{(n)}_{i,j} - E^{(n)}_{j,i}\Big)z\Big)^{-1} \\
&\quad = \Big(\tilde{z} - t\tilde{z}\Big(E^{(n)}_{i,j} - E^{(n)}_{j,i}\Big)\tilde{z}\Big)J_n + O(t^2) \\
&\quad = \bigg(\sum_{k,l} z_{k,l}E^{(n)}_{k,l} - t\sum_{a,b,c,d} z_{a,b}z_{c,d}E^{(n)}_{a,b}\Big(E^{(n)}_{i,j} - E^{(n)}_{j,i}\Big)E^{(n)}_{c,d}\bigg)J_n + O(t^2) \\
&\quad = \sum_{k,l}\big(z_{k,l} - t(z_{k,i}z_{j,l} - z_{k,j}z_{i,l})\big)E^{(n)}_{k,l}J_n + O(t^2),
\end{aligned}$$

and that

$$\begin{aligned}
\det(1_n + cz)^s &= \Big(\det\Big(1_n + tJ_n\Big(E^{(n)}_{i,j} - E^{(n)}_{j,i}\Big)\tilde{z}J_n\Big)\Big)^s \\
&= e^{st\mathrm{tr}((E^{(n)}_{i,j} - E^{(n)}_{j,i})\tilde{z}) + O(t^2)} \\
&= e^{st(z_{j,i} - z_{i,j}) + O(t^2)} = e^{-2stz_{i,j} + O(t^2)},
\end{aligned}$$

from which we obtain

$$\mathrm{d}\pi_\lambda\big(X^-_{i,j}\big) = -2sz_{i,j} - \sum_{k<l}(z_{k,i}z_{j,l} - z_{k,j}z_{i,l})\partial_{k,l}.$$

This completes the proof. □

Noting that the dual basis of (4.1) is given by

$$(X^0_{i,j})^\vee = X^0_{j,i}, \qquad (X^\pm_{i,j})^\vee = X^\mp_{i,j},$$

let us define the element $\boldsymbol{X}$ of $\mathrm{U}(\mathfrak{g}) \otimes \mathrm{Mat}_{2n}(\boldsymbol{C})$ by (3.3); it looks like

$$
\boldsymbol{X} = \left[\begin{array}{cccc|cccc}
X^0_{1,1} & X^0_{2,1} & \cdots & X^0_{n,1} & X^-_{1,n} & \cdots & X^-_{1,2} & 0 \\
X^0_{1,2} & X^0_{2,2} & \cdots & X^0_{n,2} & \vdots & \iddots & 0 & -X^-_{1,2} \\
\vdots & \vdots & & \vdots & X^-_{n-1,n} & 0 & \iddots & \vdots \\
X^0_{1,n} & X^0_{2,n} & \cdots & X^0_{n,n} & 0 & -X^-_{n-1,n} & \cdots & -X^-_{1,n} \\
\hline
X^+_{1,n} & \cdots & X^+_{n-1,n} & 0 & -X^0_{n,n} & \cdots & -X^0_{n,2} & -X^0_{n,1} \\
\vdots & \iddots & 0 & -X^+_{n-1,n} & \vdots & & \vdots & \vdots \\
X^+_{1,2} & 0 & \iddots & \vdots & -X^0_{2,n} & \cdots & -X^0_{2,2} & -X^0_{2,1} \\
0 & -X^+_{1,2} & \cdots & -X^+_{1,n} & -X^0_{1,n} & \cdots & -X^0_{1,2} & -X^0_{1,1}
\end{array}\right].
$$

Remark 4.2. It is well known that Pfaffian $\mathrm{Pf}(\boldsymbol{X})$ of the matrix $\boldsymbol{X}$ given above[3] is a central element of the universal enveloping algebra $\mathrm{U}(\mathfrak{so}_{2n})$ (see [**7**] for the definition and the properties of Pfaffian with noncommutative entries).

Following the prescription (3.4), let us define $\sigma_\lambda(\boldsymbol{X})$ by substituting $\xi_{i,j}$ into $\partial_{i,j}$:

$$
\sigma_\lambda(\boldsymbol{X}) := \sum_{\epsilon;i,j} \sigma_\lambda\big((X^\epsilon_{i,j})^\vee\big) \otimes X^\epsilon_{i,j}. \tag{4.5}
$$

Theorem 4.3. *Let $u(z) := \exp \sum_{i<j} z_{i,j}(X^-_{i,j})^\vee \in U_\Omega$. Then we have*

$$
\mathrm{Ad}(u(z)^{-1})\sigma_\lambda(\boldsymbol{X}) = s\sum_i X^0_{i,i} - \sum_{i<j} \xi_{i,j} X^-_{i,j} \tag{4.6}
$$

$$
= \left[\begin{array}{cccc|cccc}
s & & & & & & & \\
 & s & & & & & & \\
 & & \ddots & & & & & \\
 & & & s & & & & \\
\hline
-\xi_{1,n-1} & -\xi_{2,n-1} & \cdots & 0 & -s & & & \\
\vdots & \vdots & \iddots & \vdots & & \ddots & & \\
-\xi_{1,2} & 0 & \cdots & \xi_{2,n-1} & & & -s & \\
0 & \xi_{1,2} & \cdots & \xi_{1,n-1} & & & & -s
\end{array}\right]. \tag{4.7}
$$

[3] Throughout the paper, for a $2n \times 2n$ matrix A alternating along the antidiagonal, we denote $\mathrm{Pf}(AJ_{2n})$ by $\mathrm{Pf}(A)$ for brevity.

Proof. This is just a simple matrix calculation, but we give a rather detailed one, which we will need in proving the theorem stated below. In what follows, for a matrix A given, let us denote its (i,j)-th entry by A_{ij}.

Writing $\left[\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right] := u(z)$ and $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] := \sigma_\lambda(\boldsymbol{X})$, we have

$$\mathrm{Ad}(u(z)^{-1})\sigma_\lambda(\boldsymbol{X}) = \begin{bmatrix} A - zC & Az - zCz + B - zD \\ C & Cz + D \end{bmatrix}. \tag{4.8}$$

(I) First, we calculate the $(1,1)$ and $(2,2)$-blocks. Let $\tilde{C} := J_n C$. Note that, by definition, A_{ij} is $\sigma_\lambda(X^0_{j,i})$. Since $(zC)_{ij} = (\tilde{z}\tilde{C})_{ij}$ equals $\sum_{k=1}^n z_{i,k}\xi_{k,j} = -\sum_{k=1}^n z_{i,k}\xi_{j,k}$, it follows from (4.2) that

$$A = s1_n + zC. \tag{4.9}$$

Then the fact that (4.8) is an element of $\mathfrak{so}_{2n}$ implies that

$$Cz + D = -J_n {}^t(A - zC)J_n = -s1_n. \tag{4.10}$$

(II) Next we show that the $(1,2)$-block equals 0. Let $\tilde{B} := BJ_n$. Since $(\tilde{z}\tilde{C}\tilde{z})_{ij}$ equals

$$\sum_{k,l\in[n]} z_{i,k}\xi_{k,l}z_{l,j} = \left(\sum_{k<l} + \sum_{k>l}\right) z_{i,k}z_{l,j}\xi_{k,l} = \sum_{1\le k<l\le n} (z_{k,i}z_{j,l} - z_{k,j}z_{i,l})\xi_{k,l},$$

it follows from (4.4) that $\tilde{B} = -2s\tilde{z} - \tilde{z}\tilde{C}\tilde{z}$ and

$$B = -2sz - zCz. \tag{4.11}$$

Therefore we obtain that

$$\begin{aligned} Az - zCz + B - zD &= (s1_n + zC)z - zCz - 2sz - zCz - z(-s1_n - Cz) \\ &= 0. \end{aligned}$$

This completes the proof. □

Now, using the principal symbol $\gamma_1 = \sum_{k<l} z_{k,l}\xi_{k,l}$ of the Euler operator Γ_1 on V and a new parameter τ, we deform the symbols $\sigma_\lambda(X^0_{i,i})$ and $\sigma_\lambda(X^-_{i,j})$, following the prescription given in Section 3 (see the paragraph just before Theorem 3.1). Note that they can be written as

$$\sigma_\lambda\big(X^0_{i,i}\big) = s - \gamma_1 + \sum_{\substack{k<l\\ k,l\neq i}} z_{k,l}\xi_{k,l},$$

$$\sigma_\lambda(X^-_{i,j}) = -(s-\gamma_1+s)z_{i,j} + \gamma_1 z_{i,j} - \sum_{k<l}(z_{k,i}z_{j,l} - z_{k,j}z_{i,l})\xi_{k,l}.$$

We replace $s-\gamma_1$ by τ in the above formulae, and denote the deformed symbols with τ by $\widetilde{\sigma}_\lambda(X^0_{i,i})$ and $\widetilde{\sigma}_\lambda(X^-_{i,j})$, respectively:

$$\widetilde{\sigma}_\lambda\big(X^0_{i,i}\big) := \tau + \sum_{\substack{k<l\\ k,l\neq i}} z_{k,l}\xi_{k,l}, \tag{4.12}$$

$$\widetilde{\sigma}_\lambda\big(X^-_{i,j}\big) := -(\tau+s+\gamma_1)z_{i,j} - \sum_{k<l}(z_{k,i}z_{j,l} - z_{k,j}z_{i,l})\xi_{k,l}; \tag{4.13}$$

as for the other symbols, we set $\widetilde{\sigma}_\lambda(X^\epsilon_{i,j}) := \sigma_\lambda(X^\epsilon_{i,j})$. Then we define $\widetilde{\sigma}_\lambda(\boldsymbol{X})$ by

$$\widetilde{\sigma}_\lambda(\boldsymbol{X}) := \sum_{\epsilon;i,j} \widetilde{\sigma}_\lambda\big((X^\epsilon_{i,j})^\vee\big) \otimes X^\epsilon_{i,j}.$$

THEOREM 4.4. *Let $u(z) \in U_\Omega$ be as in Theorem 4.3. Then we have*

$$\begin{aligned}
&\mathrm{Ad}(u(z)^{-1})\widetilde{\sigma}_\lambda(\boldsymbol{X})\\
&\quad = (\tau+\gamma_1)\sum_i X^0_{i,i} - \sum_{i<j}\xi_{i,j}X^-_{i,j} - (s-(\tau+\gamma_1))\sum_{i<j} z_{i,j}X^+_{i,j}
\end{aligned} \tag{4.14}$$

$$= \left[\begin{array}{cccc|cccc}
\tau+\gamma_1 & & & & -\vartheta z_{1,n-1} & \cdots & -\vartheta z_{1,2} & 0\\
 & \tau+\gamma_1 & & & -\vartheta z_{2,n-1} & \cdots & 0 & \vartheta z_{1,2}\\
 & & \ddots & & \vdots & \iddots & \vdots & \vdots\\
 & & & \tau+\gamma_1 & 0 & \cdots & \vartheta z_{2,n-1} & \vartheta z_{1,n-1}\\
\hline
-\xi_{1,n-1} & -\xi_{2,n-1} & \cdots & 0 & -\tau-\gamma_1 & & & \\
\vdots & \vdots & \iddots & \vdots & & \ddots & & \\
-\xi_{1,2} & 0 & \cdots & \xi_{2,n-1} & & & -\tau-\gamma_1 & \\
0 & \xi_{1,2} & \cdots & \xi_{1,n-1} & & & & -\tau-\gamma_1
\end{array}\right]. \tag{4.15}$$

Here, we set $\vartheta := s-(\tau+\gamma_1)$ in (4.15) for brevity.

PROOF. Let $\left[\begin{smallmatrix} A(\tau) & B(\tau)\\ C & D(\tau)\end{smallmatrix}\right] := \widetilde{\sigma}_\lambda(\boldsymbol{X})$. (Note that the submatrix C is the same as in the proof of Theorem 4.3). Then it suffices to show that

$$A(\tau) - zC = (\tau + \gamma_1)1_n, \tag{4.16}$$

$$A(\tau)z - zCz + B(\tau) - zD(\tau) = (\tau + \gamma_1 - s)z, \tag{4.17}$$

$$Cz + D(\tau) = -(\tau + \gamma_1)1_n \tag{4.18}$$

(see (4.8)).

(I) First we show (4.16) and (4.18). But the latter follows from the former, as in the proof of Theorem 4.3. Note that $A(\tau)_{ij} = \widetilde{\sigma}_\lambda(X_{j,i})$ can be written as

$$A(\tau)_{ij} = \left(\tau + \sum_{k<l} z_{k,l}\xi_{k,l}\right)\delta_{i,j} - \sum_{k=1}^{n} z_{k,i}\xi_{k,j}. \tag{4.19}$$

Since the second summation in (4.19) is equal to $-(\tilde{z}\tilde{C})_{ij} = -(zC)_{ij}$ as shown in (I) of the proof of Theorem 4.3, we obtain that

$$A(\tau) = (\tau + \gamma_1)1_n + zC.$$

(II) Next we show (4.17). Let $\tilde{B}(\tau) := B(\tau)J_n$. Since the summation part of $\widetilde{\sigma}_\lambda(X^-_{i,j}) = \tilde{B}(\tau)_{ij}$ equals $(\tilde{z}\tilde{C}\tilde{z})_{ij}$ as in (II) of the proof of Theorem 4.3, we obtain that

$$B(\tau) = -(\tau + s + \gamma_1)z - zCz. \tag{4.20}$$

Now, combining (4.20) with (4.16) and (4.18), we obtain that

$$A(\tau)z - zCz + B(\tau) - zD(\tau) = (\tau + \gamma_1 - s)z.$$

This completes the proof. □

Note the similarity between the matrix $\boldsymbol{\Phi}$ given in (1.5) and $\widetilde{\sigma}_\lambda(\boldsymbol{X})$ conjugated by $u(z)^{-1} \in U_\Omega$ given in (4.15).

It follows immediately from Theorem 4.4 and the minor summation formula of Pfaffian (Theorem A.1) that $\mathrm{Pf}(\widetilde{\sigma}_\lambda(\boldsymbol{X}))$ yields a generating function for $\{\gamma_k\}$.

4.2. The case $G = Sp(n, \boldsymbol{R})$.

Next we prove the case where $G = Sp(n, \boldsymbol{R})$, or $\mathfrak{g} = \mathfrak{sp}_n$. Take a basis $\{X^\epsilon_{i,j}\}_{\epsilon=0,\pm;i,j\in[n]}$ for $\mathfrak{sp}_n$ as follows:

$$\begin{aligned} X^0_{i,j} &:= E_{i,j} - E_{-j,-i} \quad (1 \le i, j \le n), \\ X^+_{i,j} &:= E_{i,-j} + E_{j,-i} \quad (1 \le i \le j \le n), \\ X^-_{i,j} &:= E_{-j,i} + E_{-i,j} \quad (1 \le i \le j \le n), \end{aligned} \tag{4.21}$$

where $-i$ stands for $2n+1-i$ as in Section 4.1.

PROPOSITION 4.5. *The differential operators* $\mathrm{d}\pi_\lambda(X^\epsilon_{i,j})$ $(\epsilon = 0, \pm; i, j \in [n])$ *are given by*

$$\mathrm{d}\pi_\lambda(X^0_{i,j}) = s\delta_{i,j} - \sum_{k=1}^{n} z_{j,k}\tilde{\partial}_{i,k}, \tag{4.22}$$

$$\mathrm{d}\pi_\lambda(X^+_{i,j}) = -\tilde{\partial}_{i,j}, \tag{4.23}$$

$$\mathrm{d}\pi_\lambda(X^-_{i,j}) = -2sz_{i,j} + \sum_{1 \le k,l \le n} z_{k,i}z_{j,l}\tilde{\partial}_{k,l}. \tag{4.24}$$

PROOF. Let $E^{(n)}_{i,j}$ be as in the proof of Proposition 4.1, and let $\tilde{z} := \sum_{k,l} z_{k,l}E^{(n)}_{k,l}$ with $z_{l,k} = z_{k,l}$. Then $z := \tilde{z}J_n$ belongs to Ω if it is positive definite. Then, we can calculate the differential operators $\mathrm{d}\pi_\lambda(X^\epsilon_{i,j})$ as in the case of $\mathfrak{so}_{2n}$.

(I) Writing $\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right] := \exp(-tX^0_{i,j})$, we see that

$$\begin{aligned} \exp\bigl(-tX^0_{i,j}\bigr).z &= azd^{-1} \\ &= \sum_{k,l \in [n]} (z_{k,l} - t(\delta_{i,k}z_{j,l} + \delta_{i,l}z_{k,j}))E^{(n)}_{k,l}J_n + O(t^2) \end{aligned}$$

and

$$(\det d)^s = e^{st\mathrm{tr}(E^{(n)}_{n+1-j,n+1-i}) + O(t^2)}.$$

Hence

$$\begin{aligned} \mathrm{d}\pi_\lambda\bigl(X^0_{i,j}\bigr) &= s\delta_{i,j} - \sum_{i \le l} z_{j,l}\partial_{i,l} - \sum_{k \le i} z_{k,j}\partial_{k,i} \\ &= s\delta_{i,j} - \sum_{k \in [n]} z_{j,k}\tilde{\partial}_{i,k}. \end{aligned}$$

(II) Writing $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right] := \exp(-tX^+_{i,j})$, we see that

$$\begin{aligned}\exp\big(-tX^+_{i,j}\big).z &= z+b\\ &= \Big(\tilde{z} - t\Big(E^{(n)}_{i,j} + E^{(n)}_{j,i}\Big)\Big)J_n\\ &= \sum_{k,l\in[n]} (z_{k,l} - t\delta_{i,k}\delta_{j,l} - t\delta_{j,k}\delta_{i,l})E^{(n)}_{k,l}J_n.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{d}\pi_\lambda(X^+_{i,j}) &= -\sum_{k\le l}(\delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k})\partial_{k,l}\\ &= -\tilde{\partial}_{i,j}.\end{aligned}$$

(III) Writing $\left[\begin{smallmatrix}1 & 0\\ c & 1\end{smallmatrix}\right] := \exp(-tX^-_{i,j})$, we see that

$$\begin{aligned}\exp\big(-tX^-_{i,j}\big).z &= z(1_n + cz)^{-1}\\ &= z\Big(1_n - tJ_n\Big(E^{(n)}_{i,j} + E^{(n)}_{j,i}\Big)z\Big)^{-1}\\ &= \Big(\tilde{z} + t\tilde{z}\Big(E^{(n)}_{i,j} + E^{(n)}_{j,i}\Big)\tilde{z}\Big)J_n + O(t^2)\\ &= \sum_{k,l}(z_{k,l} + t(z_{k,i}z_{j,l} + z_{k,j}z_{i,l}))E^{(n)}_{k,l}J_n + O(t^2),\end{aligned}$$

and

$$\begin{aligned}\det(1_n + cz)^s &= \Big(\det\Big(1_n - tJ_n\Big(E^{(n)}_{i,j} + E^{(n)}_{j,i}\Big)\tilde{z}J_n\Big)\Big)^s\\ &= e^{-st\mathrm{tr}((E^{(n)}_{i,j} + E^{(n)}_{j,i})\tilde{z}) + O(t^2)}\\ &= e^{-st(z_{j,i}+z_{i,j}) + O(t^2)} = e^{-2stz_{i,j} + O(t^2)}.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{d}\pi_\lambda\big(X^-_{i,j}\big) &= -2sz_{i,j} + \sum_{k\le l}(z_{k,i}z_{j,l} + z_{k,j}z_{i,l})\partial_{k,l}\\ &= -2sz_{i,j} + \sum_{k,l\in[n]} z_{k,i}z_{j,l}\tilde{\partial}_{k,l}.\end{aligned}$$

This completes the proof. □

Noting that the dual basis of (4.21) is given by

$$\left(X^0_{i,j}\right)^\vee = X^0_{j,i}, \qquad \left(X^\pm_{i,j}\right)^\vee = \begin{cases} X^\mp_{i,j} & (i \neq j), \\ \dfrac{1}{2} X^\mp_{i,j} & (i = j), \end{cases}$$

let us define $\sigma_\lambda(\boldsymbol{X})$ by substituting $\tilde{\xi}_{i,j}$ into $\tilde{\partial}_{i,j}$ following the prescription (3.4) as above.

THEOREM 4.6. *Let $u(z) := \exp \sum_{i \le j} z_{i,j} (X^-_{i,j})^\vee \in U_\Omega$. Then we have*

$$\operatorname{Ad}(u(z)^{-1}) \sigma_\lambda(\boldsymbol{X}) = s \sum_i X^0_{i,i} - \sum_{i \le j} \tilde{\xi}_{i,j} X^-_{i,j} \tag{4.25}$$

$$= \left[\begin{array}{cccc|cccc} s & & & & & & & \\ & s & & & & & & \\ & & \ddots & & & & & \\ & & & s & & & & \\ \hline -\xi_{1,n-1} & -\xi_{2,n-1} & \cdots & -2\xi_{n,n} & -s & & & \\ \vdots & \vdots & \iddots & \vdots & & \ddots & & \\ -\xi_{1,2} & -2\xi_{2,2} & \cdots & -\xi_{2,n-1} & & & -s & \\ -2\xi_{1,1} & -\xi_{1,2} & \cdots & -\xi_{1,n-1} & & & & -s \end{array}\right]. \tag{4.26}$$

PROOF. Again, this is just a simple matrix calculation, and can be shown in the same way as in the case of $\mathfrak{so}_{2n}$. In fact, if we write $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] := \sigma_\lambda(\boldsymbol{X})$, then the summation $\sum_k z_{i,k} \tilde{\xi}_{j,k}$ in $\sigma_\lambda(X^0_{j,i}) = A_{ij}$ equals $-(zC)_{ij}$, from which it follows that

$$A = s1_n + zC.$$

Similarly, the summation $\sum_{k,l} z_{i,k} z_{l,j} \tilde{\xi}_{k,l}$ in $\sigma_\lambda(X^-_{i,j}) = (BJ_n)_{ij}$ equals $(zCzJ_n)_{ij}$, from which it follows that

$$B = -2sz - zCz.$$

Now exactly the same calculation as in the proof of Theorem 4.3 yields the identity to be shown. $\square$

Now as above, following the prescription given in Section 3, we deform the symbols $\sigma_\lambda(X^0_{i,i})$ and $\sigma_\lambda(X^-_{i,j})$ using $\gamma_1 = \sum_{k,l\in[n]} z_{k,l}\tilde{\xi}_{k,l}$ and a new parameter τ. The results, which we denote by $\widetilde{\sigma}_\lambda(X^0_{i,i})$ and $\widetilde{\sigma}_\lambda(X^-_{i,j})$, are given by

$$\widetilde{\sigma}_\lambda\big(X^0_{i,i}\big) := \tau + \sum_{\substack{k,l\in[n] \\ l\neq i}} z_{k,l}\tilde{\xi}_{k,l}, \tag{4.27}$$

$$\widetilde{\sigma}_\lambda(X^-_{i,j}) := -(\tau + s + \gamma_1)z_{i,j} + \sum_{k,l\in[n]} z_{k,i}z_{j,l}\tilde{\xi}_{k,l}, \tag{4.28}$$

respectively; as for the others, we set $\widetilde{\sigma}_\lambda(X^\epsilon_{i,j}) := \sigma_\lambda(X^\epsilon_{i,j})$. Then we define $\widetilde{\sigma}_\lambda(\boldsymbol{X})$ by

$$\widetilde{\sigma}_\lambda(\boldsymbol{X}) := \sum_{\epsilon;i,j} \widetilde{\sigma}_\lambda\big((X^\epsilon_{i,j})^\vee\big) \otimes X^\epsilon_{i,j}.$$

Theorem 4.7. *Let $u(z) \in U_\Omega$ be as in Theorem 4.6. Then we have*

$$\begin{aligned} &\mathrm{Ad}(u(z)^{-1})\widetilde{\sigma}_\lambda(\boldsymbol{X}) \\ &\quad = (\tau+\gamma_1)\sum_i X^0_{i,i} - \sum_{i\leq j}\tilde{\xi}_{i,j}X^-_{i,j} - (s-(\tau+\gamma_1))\sum_{i\leq j} z_{i,j}X^+_{i,j} \end{aligned} \tag{4.29}$$

$$= \left[\begin{array}{cccc|cccc} \tau+\gamma_1 & & & & -\vartheta z_{1,n} & \cdots & -\vartheta z_{1,2} & -\vartheta z_{1,1} \\ & \tau+\gamma_1 & & & -\vartheta z_{2,n} & \cdots & -\vartheta z_{2,2} & -\vartheta z_{1,2} \\ & & \ddots & & \vdots & \iddots & \vdots & \vdots \\ & & & \tau+\gamma_1 & -\vartheta z_{1,n} & \cdots & -\vartheta z_{2,n} & -\vartheta z_{1,n} \\ \hline -\xi_{1,n} & -\xi_{2,n} & \cdots & -2\xi_{n,n} & -\tau-\gamma_1 & & & \\ \vdots & \vdots & \iddots & \vdots & & \ddots & & \\ -\xi_{1,2} & -2\xi_{2,2} & \cdots & -\xi_{2,n} & & & -\tau-\gamma_1 & \\ -2\xi_{1,1} & -\xi_{1,2} & \cdots & -\xi_{1,n} & & & & -\tau-\gamma_1 \end{array}\right]. \tag{4.30}$$

Here, we set $\vartheta := s - (\tau+\gamma_1)$ in (4.30) *for brevity.*

Proof. The theorem follows from matrix calculation parallel to that in the proof of Theorem 4.4.

In fact, if we write $\left[\begin{smallmatrix} A(\tau) & B(\tau) \\ C & D(\tau) \end{smallmatrix}\right] := \widetilde{\sigma}_\lambda(\boldsymbol{X})$, then we see that

$$A(\tau)_{ij} = \left(\tau + \sum_{k,l} z_{k,l}\tilde{\xi}_{k,l}\right)\delta_{i,j} - \sum_{k=1}^{n} z_{k,i}\tilde{\xi}_{k,j} \tag{4.31}$$

and the second summation in (4.31) equals $-(\tilde{z}\tilde{C})_{ij} = -(zC)_{ij}$, hence we obtain that

$$A(\tau) = (\tau + \gamma_1)1_n + zC.$$

Similarly, we see that the summation in $\widetilde{\sigma}_\lambda(X^-_{i,j}) = \tilde{B}(\tau)_{ij}$ equals $(\tilde{z}\tilde{C}\tilde{z})_{ij}$ as shown in the proof of Theorem 4.6. Thus we obtain that

$$B(\tau) = -(\tau + s + \gamma_1)z - zCz. \tag{4.32}$$

Again, exactly the same matrix calculation as in the proof of Theorem 4.4 yields the identity to be shown. $\square$

It follows immediately from Theorem 4.7 and Proposition A.2 that determinant of $\widetilde{\sigma}_\lambda(\boldsymbol{X})$ yields a generating function for $\{\gamma_k\}$.

4.3. The case $G = SU(p,q)$ $(p \geq q)$.

Finally, we prove the case where $G = SU(p,q)$, or $\mathfrak{g} = \mathfrak{sl}_{p+q}$, with $p \geq q$. Take a basis $\{H_i, E^\pm_{i,j}, X^\pm_{i,j}\}$ for $\mathfrak{sl}_{p+q}$ as follows:

$$\begin{aligned}
H_i &:= E_{i,i} - E_{p+q,p+q} && (1 \leq i \leq p+q-1),\\
E^+_{i,j} &:= E_{i,j} && (1 \leq i \neq j \leq p),\\
E^-_{i,j} &:= E_{p+i,p+j} && (1 \leq i \neq j \leq q),\\
X^+_{i,j} &:= E_{i,p+j} && (1 \leq i \leq p, 1 \leq j \leq q),\\
X^-_{i,j} &:= E_{p+j,i} && (1 \leq i \leq p, 1 \leq j \leq q).
\end{aligned} \tag{4.33}$$

Proposition 4.8. *The differential operators* $\mathrm{d}\pi_\lambda(H_i)$, $\mathrm{d}\pi_\lambda(E^\pm_{i,j})$, $\mathrm{d}\pi_\lambda(X^\pm_{i,j})$ *are given by*

$$\mathrm{d}\pi_\lambda(H_i) = s - \sum_{l=1}^{q} z_{i,l}\partial_{i,l} - \sum_{k=1}^{p} z_{k,q}\partial_{k,q} \qquad (1 \leq i \leq p), \tag{4.34}$$

$$\mathrm{d}\pi_\lambda(H_{p+j}) = \sum_{k=1}^{p}(z_{k,j}\partial_{k,j} - z_{k,q}\partial_{k,q}) \qquad (1 \leq j \leq q-1), \tag{4.35}$$

$$\mathrm{d}\pi_\lambda(E^+_{i,j}) = -\sum_{l=1}^{q} z_{j,l}\partial_{i,l} \qquad (1 \le i \ne j \le p), \tag{4.36}$$

$$\mathrm{d}\pi_\lambda(E^-_{i,j}) = \sum_{k=1}^{p} z_{k,i}\partial_{k,j} \qquad (1 \le i \ne j \le q), \tag{4.37}$$

$$\mathrm{d}\pi_\lambda(X^+_{i,j}) = -\partial_{i,j} \qquad (1 \le i \le p, 1 \le j \le q), \tag{4.38}$$

$$\mathrm{d}\pi_\lambda(X^-_{i,j}) = -sz_{i,j} + \sum_{\substack{1\le k\le p,\\ 1\le l\le q}} z_{k,j}z_{i,l}\partial_{k,l} \qquad (1 \le i \le p, 1 \le j \le q). \tag{4.39}$$

Proof. Let $z := \sum_{i\in[p],j\in[q]} z_{i,j}E^{(p,q)}_{i,j}$. Then z belongs to Ω if it is positive definite.

(I) Write $\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right] := \exp(-tH_i)$. For $i = 1, \dots, p$, we see that

$$\begin{aligned}\exp(-tH_i).z &= azd^{-1}\\ &= \Big(1_p - tE^{(p)}_{i,i} + O(t^2)\Big)z\Big(1_q + tE^{(q)}_{q,q} + O(t^2)\Big)^{-1}\\ &= \sum_{k\in[p],l\in[q]} (z_{k,l} - t(\delta_{i,k}z_{i,l} + \delta_{l,q}z_{k,q}))E^{(p,q)}_{k,l} + O(t^2)\end{aligned}$$

and

$$(\det d)^s = e^{st\mathrm{tr}(E^{(q)}_{p+q,p+q})+O(t^2)}.$$

Hence

$$\mathrm{d}\pi_\lambda(H_i) = s - \sum_{l\in[q]} z_{i,l}\partial_{i,l} - \sum_{k\in[p]} z_{k,q}\partial_{k,q} \quad (i = 1, \dots, p).$$

For $j = 1, \dots, q-1$, we see that

$$\begin{aligned}\exp(-tH_{p+j}).z &= azd^{-1}\\ &= z\Big(1_q - t\Big(E^{(q)}_{j,j} - E^{(q)}_{q,q}\Big) + O(t^2)\Big)^{-1}\\ &= \sum_{k\in[p],l\in[q]} (z_{k,l} + t(\delta_{l,j}z_{k,j} - \delta_{l,q}z_{k,q}))E^{(p,q)}_{k,l} + O(t^2)\end{aligned}$$

and $(\det d)^s = 1$. Hence

$$\mathrm{d}\pi_\lambda(H_{p+j}) = \sum_{k\in[p]} (z_{k,j}\partial_{k,j} - z_{k,q}\partial_{k,q}) \quad (j = 1, \dots, q-1).$$

(II) Writing $\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right] := \exp(-tE^+_{i,j})$, we see that

$$\begin{aligned}\exp\big(-tE^+_{i,j}\big).z &= az \\ &= \Big(1_p - tE^{(p)}_{i,j}\Big)z \\ &= \sum_{k\in[p],l\in[q]} (z_{k,l} - t\delta_{i,k}z_{j,l})E^{(p,q)}_{k,l} + O(t^2).\end{aligned}$$

Hence

$$\mathrm{d}\pi_\lambda(E^+_{i,j}) = -\sum_{l\in[q]} z_{j,l}\partial_{i,l}.$$

(III) Writing $\left[\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right] := \exp(-tE^-_{i,j})$, we see that

$$\begin{aligned}\exp\big(-tE^-_{i,j}\big).z &= zd^{-1} \\ &= z\Big(1_q - tE^{(q)}_{i,j}\Big)^{-1} \\ &= \sum_{k\in[p],l\in[q]} (z_{k,l} + t\delta_{j,l}z_{k,i})E^{(p,q)}_{k,l} + O(t^2)\end{aligned}$$

and $\det d = 1$. Hence

$$\mathrm{d}\pi_\lambda\big(E^-_{i,j}\big) = \sum_{k\in[p]} z_{k,i}\partial_{k,j}.$$

(IV) Writing $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right] := \exp(-tX^+_{i,j})$, we see that

$$\begin{aligned}\exp\big(-tX^+_{i,j}\big).z &= z + b \\ &= z - tE^{(p,q)}_{i,j}.\end{aligned}$$

Hence

$$\mathrm{d}\pi_\lambda\bigl(X^+_{i,j}\bigr) = -\partial_{i,j}.$$

(V) Writing $\left[\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right] := \exp(-tX^-_{i,j})$, we see that

$$\begin{aligned}
\exp\bigl(-tX^-_{i,j}\bigr).z &= z(1_q + cz)^{-1} \\
&= z\Bigl(1_q - tE^{(q,p)}_{j,i} z\Bigr)^{-1} \\
&= z + tzE^{(q,p)}_{j,i} z + O(t^2) \\
&= \sum_{k\in[p],l\in[q]} (z_{k,l} + tz_{k,j}z_{i,l})E^{(p,q)}_{k,l} + O(t^2),
\end{aligned}$$

and

$$\begin{aligned}
\det(1_q + cz)^s = \Bigl(\det\Bigl(1_q - tE^{(q,p)}_{j,i} z\Bigr)\Bigr)^s = e^{-st\operatorname{tr}(E^{(q,p)}_{j,i} z) + O(t^2)} \\
= e^{-stz_{i,j} + O(t^2)}.
\end{aligned}$$

Hence

$$\mathrm{d}\pi_\lambda(X^-_{i,j}) = -sz_{i,j} + \sum_{k\in[p],l\in[q]} z_{k,j}z_{i,l}\partial_{k,l}.$$

This completes the proof. $\square$

Noting that the dual basis of (4.33) is given by

$$H_i^\vee = H_i - \frac{1}{p+q}\sum_{k=1}^{p+q-1} H_k, \qquad \bigl(E^\pm_{i,j}\bigr)^\vee = E^\pm_{j,i}, \qquad \bigl(X^\pm_{i,j}\bigr)^\vee = X^\mp_{i,j},$$

let us define $\sigma_\lambda(\boldsymbol{X})$ following the prescription (3.4) as above.

Theorem 4.9. *Let $u(z) := \exp\sum_{i\in[p],j\in[q]} z_{i,j}(X^-_{i,j})^\vee \in U_\Omega$. Then we have*

$$\begin{aligned}
&\operatorname{Ad}(u(z)^{-1})\sigma_\lambda(\boldsymbol{X}) \\
&= \frac{q}{p+q}s\sum_{i=1}^{p} H_i - \frac{p}{p+q}s\sum_{j=1}^{q-1} H_{p+j} - \sum_{i\in[p],j\in[q]} \xi_{i,j}X^-_{i,j} \qquad (4.40)
\end{aligned}$$

$$= \left[\begin{array}{cccc|cccc} \frac{q}{p+q}s & & & & & & & \\ & \frac{q}{p+q}s & & & & & & \\ & & \ddots & & & & & \\ & & & \frac{q}{p+q}s & & & & \\ \hline -\xi_{1,1} & -\xi_{2,1} & \cdots & -\xi_{p,1} & -\frac{p}{p+q}s & & & \\ -\xi_{1,2} & -\xi_{2,2} & \cdots & -\xi_{p,2} & & -\frac{p}{p+q}s & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ -\xi_{1,q} & -\xi_{2,q} & \cdots & -\xi_{p,q} & & & & -\frac{p}{p+q}s \end{array}\right]. \tag{4.41}$$

Proof. It follows from (4.34) and (4.35) that

$$\frac{1}{p+q}\sum_{k=1}^{p+q-1}\sigma_\lambda(H_k) = \frac{p}{p+q}s - \sum_{k\in[p]} z_{k,q}\xi_{k,q}.$$

Thus we obtain that

$$\begin{aligned} \sigma_\lambda(H_i^\vee) &= \frac{q}{p+q}s - \sum_{l\in[q]} z_{i,l}\xi_{i,l} \qquad (i=1,\dots,p), \\ \sigma_\lambda(H_{p+j}^\vee) &= -\frac{p}{p+q}s + \sum_{k\in[p]} z_{k,j}\xi_{k,j} \quad (j=1,\dots,q-1). \end{aligned} \tag{4.42}$$

Now, if we write $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] := \sigma_\lambda(\boldsymbol{X})$, then

$$A_{ij} = \frac{q}{p+q}s\delta_{i,j} - \sum_{l\in[q]} z_{i,l}\xi_{j,l} \qquad (i,j\in[p]), \tag{4.43a}$$

$$B_{ij} = -sz_{i,j} + \sum_{k\in[p],j\in[q]} z_{k,j}z_{i,l}\xi_{k,l} \qquad (i\in[p],\ j\in[q]), \tag{4.43b}$$

$$C_{ij} = -\xi_{j,i} \qquad (i\in[q],\ j\in[p]), \tag{4.43c}$$

$$D_{ij} = -\frac{p}{p+q}s\delta_{i,j} + \sum_{k\in[p]} z_{k,j}\xi_{k,i} \qquad (i,j\in[q]) \tag{4.43d}$$

by (4.36), (4.37), (4.38), (4.39) and (4.42). Now exactly the same matrix calculation as in the proof of Theorem 4.6 yields the formula to be shown. □

As above, following the prescription given in Section 3, we deform the symbols

$\sigma_\lambda(H_i^\vee)$ and $\sigma_\lambda(X_{i,j}^-)$ using $\gamma_1 = \sum_{k,l\in[n]} z_{k,l}\tilde{\xi}_{k,l}$ and a new parameter τ. The results, which we denote by $\widetilde{\sigma}_\lambda(H_i^\vee)$ and $\widetilde{\sigma}_\lambda(X_{i,j}^-)$, are given by

$$\widetilde{\sigma}_\lambda\big(H_i^\vee\big) := \Big(\tau - \frac{p}{p+q}s\Big) + \sum_{\substack{k\in[p],l\in[q]\\ k\neq i}} z_{k,l}\xi_{k,l} \qquad (i=1,\dots,p), \tag{4.44}$$

$$\widetilde{\sigma}_\lambda\big(H_{p+j}^\vee\big) := \Big(-\tau + \frac{q}{p+q}s\Big) - \sum_{\substack{k\in[p],l\in[q]\\ l\neq j}} z_{k,l}\xi_{k,l} \quad (j=1,\dots,q-1), \tag{4.45}$$

$$\widetilde{\sigma}_\lambda\big(X_{i,j}^-\big) := -(\tau+\gamma_1)z_{i,j} + \sum_{k\in[p],l\in[q]} z_{k,j}z_{i,l}\xi_{k,l} \quad (i=1,\dots,p;\ j=1,\dots,q) \tag{4.46}$$

respectively; as for the others, we set $\widetilde{\sigma}_\lambda(\cdot) := \sigma_\lambda(\cdot)$. Then we define $\widetilde{\sigma}_\lambda(\boldsymbol{X})$ by

$$\widetilde{\sigma}_\lambda(\boldsymbol{X}) = \sum_i \widetilde{\sigma}_\lambda\big(H_i^\vee\big)\otimes H_i + \sum_{\epsilon;i,j}\big(\widetilde{\sigma}_\lambda\big((E_{i,j}^\epsilon)^\vee\big)\otimes E_{i,j}^\epsilon + \widetilde{\sigma}_\lambda\big((X_{i,j}^\epsilon)^\vee\big)\otimes X_{i,j}^\epsilon\big).$$

Theorem 4.10. *Let $u(z)\in U_\Omega$ be as in Theorem 4.9. Then we have*

$$\begin{aligned}
&\mathrm{Ad}(u(z)^{-1})\widetilde{\sigma}_\lambda(\boldsymbol{X})\\
&\quad = \Big(\tau+\gamma_1 - \frac{p}{p+q}s\Big)\sum_{i=1}^p H_i + \Big(-\tau-\gamma_1+\frac{q}{p+q}s\Big)\sum_{j=1}^{q-1} H_{p+j}\\
&\qquad - \sum_{i\in[p],j\in[q]} \xi_{i,j}X_{i,j}^- - (s-(\tau+\gamma_1))\sum_{i\in[p],j\in[q]} z_{i,j}X_{i,j}^+
\end{aligned} \tag{4.47}$$

$$= \left[\begin{array}{cccc|cccc}
\tau_+ & & & & -\vartheta z_{1,1} & -\vartheta z_{1,2} & \cdots & -\vartheta z_{1,q}\\
 & \tau_+ & & & -\vartheta z_{2,1} & -\vartheta z_{2,2} & \cdots & -\vartheta z_{2,q}\\
 & & \ddots & & \vdots & \vdots & & \vdots\\
 & & & \tau_+ & -\vartheta z_{p,1} & -\vartheta z_{p,2} & \cdots & -\vartheta z_{p,q}\\ \hline
-\xi_{1,1} & -\xi_{2,1} & \cdots & -\xi_{p,1} & \tau_- & & & \\
-\xi_{1,2} & -\xi_{2,2} & \cdots & -\xi_{p,2} & & \tau_- & & \\
\vdots & \vdots & & \vdots & & & \ddots & \\
-\xi_{1,q} & -\xi_{2,q} & \cdots & -\xi_{p,q} & & & & \tau_-
\end{array}\right]. \tag{4.48}$$

Here, we set $\tau_+ := \tau + \gamma_1 - (p/(p+q))s$, $\tau_- := -\tau - \gamma_1 + (q/(p+q))s$ and

$\vartheta := s - (\tau + \gamma_1)$ *in* (4.48) *for brevity.*

PROOF. If we write $\left[\begin{smallmatrix} A(\tau) & B(\tau) \\ C & D(\tau) \end{smallmatrix}\right] := \widetilde{\sigma}_\lambda(\boldsymbol{X})$, then we can show that

$$A(\tau) = \left(\tau + \gamma_1 - \frac{p}{p+q} s\right) 1_p + zC, \tag{4.49}$$

$$B(\tau) = -(\tau + \gamma_1) z - zCz, \tag{4.50}$$

$$D(\tau) = \left(- \tau - \gamma_1 + \frac{q}{p+q} s\right) 1_q - Cz \tag{4.51}$$

exactly in the same way as in the proofs of Theorems 4.4 and 4.7. Thus we obtain the theorem. □

It follows immediately from Theorem 4.10 and Proposition A.2 that determinant of $\widetilde{\sigma}_\lambda(\boldsymbol{X})$ yields a generating function for $\{\gamma_k\}$.

REMARK 4.11. There is one more classical irreducible Hermitian symmetric pair of noncompact type, namely, $(SO_0(p,2), SO(p) \times SO(2))$. Let us denote by $\mathfrak{q}$ the maximal parabolic subalgebra of $\mathfrak{so}_{p+2}$ whose Levi part is $\mathfrak{k} \simeq \mathfrak{so}_p \oplus \boldsymbol{C}$. If we define the character $\lambda : \mathfrak{q} \to \boldsymbol{C}$ by

$$\lambda = \frac{2s}{p}\, \rho_n,$$

with $s \in \boldsymbol{Z}$ and ρ_n the half sum of the noncompact positive roots as above, then all formulae corresponding to those stated in the theorems in Section 4 also hold true.

5. Twisted moment map.

In general, let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. If M is a G-manifold, then the cotangent bundle T^*M is a symplectic G-manifold, and the moment map $\mu : T^*M \to \mathfrak{g}^*$ is given by

$$\langle \mu(x,\xi), X\rangle = \xi(X_M(x)) \qquad (x \in M, \xi \in T_x^*M) \tag{5.1}$$

for $X \in \mathfrak{g}$, where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, and $X_M(x) \in T_xM$ the tangent vector at $x \in M$ given by

$$X_M(x)f = \frac{d}{dt}\bigg|_{t=0} f(\exp(-tX).x) \tag{5.2}$$

for functions f defined around $x \in M$ (see e.g. [**10**]).

Returning to our case, it follows from (5.1) and (5.2) that the principal part of $\sigma_\lambda(\boldsymbol{X})$ (i.e., the linear part in ξ's) is identical to the moment map $\mu : T^*(G_{\boldsymbol{C}}/Q)|_{G/K} \to \mathfrak{g}^*$ composed by the isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ via the bilinear form B given in (3.2), which we also denote by μ. Furthermore, the total symbol $\sigma_\lambda(\boldsymbol{X})$ can be regarded as a *variant* of the twisted moment map $\mu_\lambda : T^*(G_{\boldsymbol{C}}/Q)|_{G/K} \to \mathfrak{g}^* \simeq \mathfrak{g}$ due to Rossmann, though we follow the notation by Schmid and Vilonen (see Section 7 of [**11**]). In fact, the difference $\mu_\lambda - \mu$, which they denote by λ_x with $x \in G_{\boldsymbol{C}}/Q$, can be expressed as $\mu_\lambda - \mu = \mathrm{Ad}(g)\lambda^\vee$, or

$$\mu_\lambda(x,\xi) = \mathrm{Ad}(g)\lambda^\vee + \mu(x,\xi), \tag{5.3}$$

where $\lambda^\vee \in \mathfrak{g}$ corresponds to $\lambda \in \mathfrak{g}^*$ under the isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ via the bilinear form B, and g is an element of a compact real form $U_{\boldsymbol{R}}$ of $G_{\boldsymbol{C}}$ such that $x = g\dot{e}$ with $\dot{e}$ the origin of $G_{\boldsymbol{C}}/Q$. Note that if x is in the open subset $G/K = GQ/Q = U_\Omega Q/Q \subset G_{\boldsymbol{C}}/Q$, one can choose a unique u_x from U_Ω (which we denoted by $u(z)$ in Section 4) so that $x = u_x\dot{e}$ instead of g from $U_{\boldsymbol{R}}$. Then, writing $\mu'_\lambda(x,\xi) := \sigma_\lambda(\boldsymbol{X})$ to make its dependence on x and ξ transparent, one can immediately verify that

$$\mu'_\lambda(x,\xi) = \mathrm{Ad}(u_x)\lambda^\vee + \mu(x,\xi). \tag{5.4}$$

Theorems 4.3, 4.6, and 4.9 state that an analogue of $U_{\boldsymbol{R}}$-equivariance of the twisted moment map holds:

$$\mathrm{Ad}(u_x^{-1})\mu'_\lambda(x,\xi) = \mu'_\lambda(u_x^{-1}.(x,\xi)), \tag{5.5}$$

though U_Ω is not a subgroup. Since $u_x^{-1}.(x,\xi) = (\dot{e},(u_x)^*\xi)$ and $(u_x)^*\xi \in T^*_{\dot{e}}(G_{\boldsymbol{C}}/Q)$, the relation (5.5) says that all the symbols of the representation operators $\mathrm{d}\pi_\lambda(X)$, with $X \in \mathfrak{g}$, are determined by those at the origin.

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A. Minor summation formulae.

In this appendix, we collect a couple of formulae concerning Pfaffian and determinant needed to show the identities in (3.7).

Recall from [**7**] that for an alternating $2n \times 2n$ matrix $A = (a_{i,j})$, $a_{j,i} = -a_{i,j}$, with entries in an associative algebra $\mathscr{A}$, the Pfaffian $\mathrm{Pf}(A)$ is defined by

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \mathrm{sgn}(\sigma)\, a_{\sigma(1),\sigma(2)} a_{\sigma(3),\sigma(4)} \cdots a_{\sigma(2n-1),\sigma(2n)}. \tag{A.1}$$

If $\mathscr{A}$ happens to be commutative, it reduces to

$$\mathrm{Pf}(A) = \sum_{\sigma} \mathrm{sgn}(\sigma)\, a_{\sigma(1),\sigma(2)} a_{\sigma(3),\sigma(4)} \cdots a_{\sigma(2n-1),\sigma(2n)}, \tag{A.2}$$

where the sum is taken over those σ satisfying

$$\sigma(2i-1) < \sigma(2i) \quad \text{for} \quad i = 1, 2, \ldots, n \quad \text{and} \quad \sigma(1) < \sigma(3) < \cdots < \sigma(2n-1).$$

Let X be a $2n \times 2n$ complex matrix alternating along the anti-diagonal. Since XJ_{2n} is an alternating matrix, one can define Pfaffian of XJ_{2n}, which we denote by $\mathrm{Pf}(X)$ as mentioned above. If we write X as $X = \left[\begin{smallmatrix} a & b \\ c & -J_n {}^t a J_n \end{smallmatrix}\right]$ with submatrices a, b, c all being of size $n \times n$, then b and c are also alternating along the anti-diagonal. Thus we parametrize the submatrices a, b, c as follows:

$$a = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}, \quad b = \begin{bmatrix} b_{1,n} & \cdots & b_{1,2} & 0 \\ \vdots & \iddots & 0 & -b_{1,2} \\ b_{n-1,n} & 0 & \iddots & \vdots \\ 0 & -b_{n-1,n} & \cdots & -b_{1,n} \end{bmatrix},$$
$$c = \begin{bmatrix} c_{1,n} & \cdots & c_{n-1,n} & 0 \\ \vdots & \iddots & 0 & -c_{n-1,n} \\ c_{1,2} & 0 & \iddots & \vdots \\ 0 & -c_{1,2} & \cdots & -c_{1,n} \end{bmatrix}. \tag{A.3}$$

Theorem A.1 ([**5**]). *Let $X = \left[\begin{smallmatrix} a & b \\ c & -J_n {}^t a J_n \end{smallmatrix}\right]$ be a $2n \times 2n$ complex matrix alternating along the anti-diagonal with submatrices a, b, c parametrized as in* (A.3). *Then we can expand Pfaffian* $\mathrm{Pf}(X)$ *in the following way.*

$$\mathrm{Pf}(X) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{I,J \subset [n] \\ |I|=|J|=2k}} \mathrm{sgn}(\bar{I}, I)\mathrm{sgn}(\bar{J}, J) \det \left(a_{\bar{J}}^{\bar{I}}\right) \mathrm{Pf}(b_I)\mathrm{Pf}(c_J). \tag{A.4}$$

Here, $\bar{I}$ denotes the complement of I in $[n]$, a_J^I the submatrix of a whose row- and column-indices are in I and J respectively, b_I, c_I the submatrices of b, c whose row- and column-indices are both in I, and $\mathrm{sgn}(I, J)$ the signature of the permutation $\binom{1,2,\cdots,n}{I,\ J}$ for $I, J \subset [n]$.

In order to show (3.7a) and (3.7b), it suffices to consider a square matrix X of the form $X = \left[\begin{smallmatrix} u1_p & b \\ c & v1_q \end{smallmatrix}\right]$ with $u, v \in \boldsymbol{C}$ and submatrices b, c of size $p \times q$, $q \times p$ respectively. Let us parametrize X as follows:

$$X = \left[\begin{array}{cccc|ccc} u & & & & b_{1,1} & \cdots & b_{1,q} \\ & u & & & b_{2,1} & \cdots & b_{2,q} \\ & & \ddots & & \vdots & & \vdots \\ & & & u & b_{p,1} & \cdots & b_{p,q} \\ \hline c_{1,1} & c_{2,1} & \cdots & c_{p,1} & v & & \\ \vdots & \vdots & & \vdots & & \ddots & \\ c_{1,q} & c_{2,q} & \cdots & c_{p,q} & & & v \end{array}\right].$$

Proposition A.2. *Let X be a square matrix given as above. Then we can expand $\det X$ in the following way*:

$$\det X = \sum_{k=0}^{q} \sum_{\substack{I \subset [p], J \subset [q] \\ |I|=|J|=k}} u^{p-k} v^{q-k} \det\left(b_J^I\right) \det\left(c_J^I\right). \tag{A.5}$$

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Takashi Hashimoto
Education Center
Tottori University
4-101, Koyama-Minami
Tottori, 680-8550, Japan
E-mail: thashi@uec.tottori-u.ac.jp