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Teichmüller spaces for pointed Fuchsian groups and their modular groups

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Abstract. It is known that the modular group $\mathrm{Mod}(X)$ acts discontinuously (but not freely) on the Teichmüller space $T(X)$ for a finite type Riemann surface X, while it does not necessarily act discontinuously on $T(X)$ when X is of infinite type. The primary purpose of the paper is to discuss those subgroups of $\mathrm{Mod}(X)$ acting discontinuously and freely on $T(X)$ and to discuss the properties of the corresponding quotient complex manifolds as well. Actually, we will discuss some generalized Teichmüller spaces, the Teichmüller spaces for pointed Riemann surfaces and pointed Fuchsian groups, and their modular groups, generalizing and completing some results of Bers [**Be1**], Kra [**Kr1**] and Nag ([**Na1**], [**Na3**], [**Na4**]).

1. Introduction.

A basic question in the theory of Riemann surfaces is to investigate the moduli spaces of the complex structures on a Riemann surface. The Teichmüller space $T(X)$ of a Riemann surface X is the biggest of such spaces. It is a contractible complex manifold and is also a branched covering of the classical Riemann moduli space $R(X)$ when the surface X is of finite type. Actually, when the surface X is of finite type, that is, X is a compact surface with at most finitely many points removed, the Teichmüller modular group $\mathrm{Mod}(X)$ acts discontinuously on $T(X)$ as a group of biholomorphic automorphisms and gives $R(X)$ as the quotient space, namely, $R(X) = T(X)/\mathrm{Mod}(X)$. However, this action is not fixed point free and so the Riemann space $R(X)$ is a normal complex space but in general not a complex manifold. Now, as pointed out by Nag (see [**Na4**, p. 167]), it is an important problem to classify all the subgroups of the modular group $\mathrm{Mod}(X)$ acting freely on $T(X)$, and correspondingly all the complex quotient manifolds which are intermediate moduli spaces between the Teichmüller space $T(X)$ and the Riemann space $R(X)$. Both Kra [**Kr1**] and Nag ([**Na1**], [**Na3**], [**Na4**]) have introduced some classes of such subgroups and discussed the corresponding quotient manifolds.

When the Riemann surface X is of infinite type, the Teichmüller modular group $\mathrm{Mod}(X)$ does not necessarily act discontinuously on $T(X)$ (see [**Fu1**], [**Fu2**], [**FST**]). Thus, in this case it is a much complicated problem to classify the intermediate complex manifolds between the Teichmüller space $T(X)$ and the Riemann space $R(X)$. We need to find the subgroups of the Teichmüller modular group $\mathrm{Mod}(X)$ which act discontinuously and freely on the Teichmüller space $T(X)$. The primary purpose of this paper is to investigate such a problem for all Riemann surfaces, not necessarily of finite type. In fact,

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we will discuss a somewhat general problem in the setting of Fuchsian groups. Actually, Kra [**Kr1**] (see also [**Be1**], [**Na1**], [**Na3**], [**Na4**]) has introduced and discussed some moduli spaces of deformations of Fuchsian groups. He outlined his approach without proofs for Fuchsian groups of finite type. We will investigate these spaces and complete the proofs for general Fuchsian groups, not necessarily of finite type, along the lines of Kra [**Kr1**].

2. Preliminaries.

In this section, we shall review some basic definitions, notations and fundamental results from Teichmüller theory. For references, see the papers [**Be2**], [**Be3**], [**Be4**] and the books [**Ga1**], [**Le**], [**Na4**].

2.1. Teichmüller spaces for Fuchsian groups.

Let G be a Fuchsian group acting on the upper half plane $\boldsymbol{H}$ and also on the lower half plane $\boldsymbol{L}$ in the complex plane $\boldsymbol{C}$, and $\boldsymbol{H}_G$ be $\boldsymbol{H}$ with all of the fixed points of elliptic elements of G removed. Then G is finitely generated and of the first kind if and only if $\boldsymbol{H}_G/G$ is of finite type, namely, it is a compact Riemann surface with at most finitely many points removed. G is of type (g,n) if $\boldsymbol{H}_G/G$ is a compact surface of genus g with n points removed. G is said to be exceptional if it has type (g,n) with $2g+n\le 4$.

Let $L^\infty(G)$ denote the set of all Beltrami differentials for G on the upper half plane $\boldsymbol{H}$, namely,

$$L^\infty(G)=\big\{\mu\in L^\infty(\boldsymbol{H}):(\mu\circ g)\overline{g'}/g'=\mu,\quad \text{for all}\quad g\in G\big\}. \tag{2.1}$$

The open unit ball $M(G)$ of $L^\infty(G)$ is the set of all Beltrami coefficients for G. The Teichmüller distance between two points μ_1 and μ_2 in $M(G)$ is defined as

$$\sigma_G(\mu_1,\mu_2)=\frac{1}{2}\log\frac{1+\left\|\frac{\mu_1-\mu_2}{1-\overline{\mu}_1\mu_2}\right\|_\infty}{1-\left\|\frac{\mu_1-\mu_2}{1-\overline{\mu}_1\mu_2}\right\|_\infty}. \tag{2.2}$$

For any $\mu\in M(G)$, let w^μ denote the unique quasiconformal mapping of the plane $\boldsymbol{C}$ onto itself which fixes 0, 1 and ∞, is conformal in $\boldsymbol{L}$, and satisfies the Beltrami equation $\partial_{\bar z}w=\mu\partial_z w$ in $\boldsymbol{H}$. Two elements μ and ν in $M(G)$ are said to be equivalent if w^μ and w^ν coincide on the real axis $\boldsymbol{R}$. $[\mu]$ will denote the equivalence class of μ.

The Teichmüller space $T(G)$ is the set of all the equivalence classes $[\mu]$ of the Beltrami coefficients μ in $M(G)$. $T(G)$ is finite dimensional if and only if G is of finite type. We let Φ_G denote the natural projection of $M(G)$ onto $T(G)$, so that $\Phi_G(\mu)$ is the equivalence class of μ. The Teichmüller distance between two points $[\mu_1]$ and $[\mu_2]$ in $T(G)$ is defined as

$$\tau_G([\mu_1],[\mu_2])=\inf\{\sigma_G(\nu_1,\nu_2):[\nu_1]=[\mu_1],[\nu_2]=[\mu_2]\}. \tag{2.3}$$

Since it is an open set in the complex Banach space $L^\infty(G)$, $M(G)$ is a complex manifold, and the Teichmüller distance is precisely the Kobayashi distance on $M(G)$.

Fundamental work of Ahlfors and Bers shows that $T(G)$ is also a complex manifold. Precisely, $T(G)$ has a unique complex manifold structure so that the natural projection $\Phi_G : M(G) \to T(G)$ is a holomorphic split submersion. It is also known that the Teichmüller distance is precisely the Kobayashi distance on the Teichmüller space $T(G)$.

2.2. Allowable mappings and modular groups for Fuchsian groups. For any Fuchsian group G, let $Q(G)$ denote the set of all quasiconformal mappings w of $\boldsymbol{H}$ onto itself such that wGw^{-1} is again a Fuchsian group. Two elements w_1 and w_2 are said to be equivalent if they coincide on the real line $\boldsymbol{R}$. The equivalence class of w will be denoted by $[w]$. Let $\Sigma_0(G)$ denote the set of all quasiconformal self-mappings of $\boldsymbol{H}$ which are equivalent to the identity mapping.

For any $\mu \in M(G)$, let w_μ denote the unique quasiconformal mapping of $\boldsymbol{H}$ onto itself which fixes 0, 1 and ∞, and satisfies the Beltrami equation $\partial_{\bar z} w = \mu \partial_z w$. Then w_μ and w_ν are equivalent if and only if $[\mu] = [\nu]$. The point $[\mu]$ will also be denoted by $[w_\mu]$ later. For any $\mu \in M(G)$, we denote $G_\mu = w_\mu G w_\mu^{-1} = \{g_\mu = w_\mu g w_\mu^{-1} : g \in G\}$, $G^\mu = w^\mu G (w^\mu)^{-1} = \{g^\mu = w^\mu g (w^\mu)^{-1} : g \in G\}$. Then G_μ is again a Fuchsian group, while G^μ is a quasi-Fuchsian group.

Let $w \in Q(G)$ be given. We consider the mapping

$$w^*(w_\mu) = \alpha \circ w_\mu \circ w^{-1}, \tag{2.4}$$

where $\mu \in M(G)$, α is a Möbius transformation of $\boldsymbol{H}$ onto itself such that $\alpha \circ w_\mu \circ w^{-1}$ fixes 0, 1 and ∞. Since $[w^*(w_\mu)]$ depends only on $[w]$ and $[w_\mu]$, w^* induces a biholomorphic isomorphism $\chi([w])$ between $T(G)$ and $T(wGw^{-1})$.

For each $\mu \in M(G)$, the domain $w^\mu(\boldsymbol{L})$, hence also $w^\mu(\boldsymbol{H})$, depends only on $\Phi_G(\mu)$. We may form the Bers fiber space

$$F(G) = \big\{(\Phi_G(\mu), \zeta) \in T(G) \times \boldsymbol{C} : \mu \in M(G), \zeta \in w^\mu(\boldsymbol{H})\big\}. \tag{2.5}$$

It is known that $F(G)$ is a complex manifold. Now $\chi([w])$ can be extended to a biholomorphic isomorphism between the fiber spaces $F(G)$ and $F(wGw^{-1})$:

$$\rho([w])([w_\mu], z) = \big([w_\nu], w^\nu \circ w \circ (w^\mu)^{-1}(z)\big), \tag{2.6}$$

where $\nu \in M(wGw^{-1})$ satisfies $w^*(w_\mu) = w_\nu$. $\chi([w]) : T(G) \to T(wGw^{-1})$ and $\rho([w]) : F(G) \to F(wGw^{-1})$ are called allowable mappings.

Let $\Sigma(G)$ denote the set of all mappings w in $Q(G)$ such that $wGw^{-1} = G$. The extended modular group $\mathrm{mod}(G) = \Sigma(G)/\Sigma_0(G)$ for G is the set of all equivalence classes $[w]$ of all elements w in $\Sigma(G)$. Then each element $[w]$ in $\mathrm{mod}(G)$ acts on $F(G)$ by $\rho([w])$ as a biholomorphic fiber-preserving automorphism, and the action of $\mathrm{mod}(G)$ on $F(G)$ is always effective. The normal subgroup G of $\Sigma(G)$ can be considered as a normal subgroup of $\mathrm{mod}(G)$. Since the action $\chi([g])$ on $T(G)$ is trivial for each $g \in G$, we define naturally the modular group $\mathrm{Mod}(G) = \mathrm{mod}(G)/G$ for G. The element of $\mathrm{Mod}(G)$ induced by $w \in \Sigma(G)$ will be denoted by $\langle w \rangle$. Then each element $\langle w \rangle$ of $\mathrm{Mod}(G)$ acts on $T(G)$ by $\chi(\langle w \rangle)$ as a biholomorphic automorphism. However, the action of $\mathrm{Mod}(G)$ on $T(G)$ is not always effective. Mod(G) acts on $T(G)$ non-effectively if and only if G is exceptional (see [**EGL**], [**Ep1**], [**Mat**]).

2.3. Teichmüller spaces and modular groups for Riemann surfaces. Let X be a Riemann surface with possibly empty ideal boundary ∂X. We denote by $Q(X)$ the set of all quasiconformal mappings defined on X. Two mappings f and g from X to Y are said to be homotopic (rel ∂X) if there exists a homotopy $f_t : (X \cup \partial X) \times [0,1] \to Y \cup \partial Y$ between f and g such that $f_t = f = g$ at each point of ∂X for all $t \in [0,1]$. The homotopy class of a mapping f is denoted by $\langle f \rangle$. Now let $\Sigma(X)$ denote the set of all quasiconformal self-mappings of X, and $\Sigma_0(X)$ the set of all quasiconformal self-mappings of X homotopic to the identity rel ∂X. Then, $\mathrm{Mod}(X) = \Sigma(X)/\Sigma_0(X)$, which is the group of homotopy classes of all quasiconformal self-mappings of X, is known as the modular group of X.

Two mappings f and g are said to be equivalent if there exists a conformal mapping c from $f(X)$ onto $g(X)$ such that $g^{-1} \circ c \circ f \in \Sigma_0(X)$. The Teichmüller space $T(X)$ of X is the set of all equivalence classes $[f]$ of all mappings f on X. The Teichmüller distance between two points $[f_1]$ and $[f_2]$ in $T(X)$ is defined as

$$\tau_X([f_1],[f_2]) = \inf\left\{\frac{1}{2}\log K[f] : f_2^{-1} \circ f \circ f_1 \in \Sigma_0(X)\right\}, \tag{2.7}$$

where $K[f]$ is the maximal dilatation of a mapping f from $f_1(X)$ onto $f_2(X)$.

Now let X be a Riemann surface of hyperbolic type, namely, there exists a torsion free Fuchsian group G such that $\boldsymbol{H}/G = X$. Let $\pi : \boldsymbol{H} \to X$ denote the natural projection. It is known that $f \in \Sigma_0(X)$ if and only if there exists some $w \in \Sigma_0(G)$ such that $\pi \circ w = f \circ \pi$, which implies that Mod$(X)$ is isomorphic to Mod(G). It is also known that $T(X)$ has a natural complex manifold structure so that $T(X)$ is biholomorphically and isometrically equivalent to $T(G)$.

A homotopy class $\langle g \rangle$ of a mapping $g : X \to Y$ induces a biholomorphic isomorphism $\chi(\langle g \rangle)$ sending $[f]$ to $[f \circ g^{-1}]$ from $T(X)$ onto $T(Y)$. In particular, an element $\langle g \rangle$ of the modular group Mod(X) induces a biholomorphic automorphism $\chi(\langle g \rangle)$ of $T(X)$. An important fact is that except in some special cases the converse is also true. This is a combination of results in a series of papers (see [**EG**], [**EK1**], [**EMa**], [**La**], [**Mar**], [**Ro**]). We state it in the setting of Fuchsian groups. Recall that the classical Riemann moduli space is $R(X) = T(X)/\mathrm{Mod}(X)$.

THEOREM A. *Let G and G' be two Fuchsian groups, each of which is torsion free and not exceptional, and $F : T(G) \to T(G')$ be a biholomorphic isomorphism. Then there exists some $w \in Q(G)$ such that $G' = wGw^{-1}$ and $F = \chi([w])$. Particularly, each biholomorphic automorphism of $T(G)$ is induced by an element of the modular group.*

3. Subgroups of modular groups.

In this section, we will introduce some classes of subgroups of modular groups which act freely and discontinuously on corresponding Teichmüller spaces. The corresponding quotient manifolds will be discussed and be generalized to the setting of Fuchsian groups in the following sections.

We first recall some basic definitions and facts on the action of a group on some metric space. Suppose H acts on a metric space S as a group of isometric homeomorphisms. H is said to act freely on S if for any $p \in S$, $H_p = \{h \in H : h(p) = p\}$ consists only of the identity element. H is said to act discontinuously on S if for any $p \in S$, there exists some $r > 0$ such that the set $\{h \in H : h(B(p,r)) \cap B(p,r) \neq \varnothing\}$ consists of only finitely many elements, where $B(p,r)$ is the open ball centered at p with radius r. Now a classical result of Cartan [**Ca**] says that if H acts discontinuously on a complex manifold S as a group of biholomorphic automorphisms, then S/H is a normal complex space; furthermore, if H also acts freely on S, then S/H is a complex manifold with S as a normal covering space.

Now let X be a hyperbolic Riemann surface, and $x_1, x_2, \dots, x_n$ be n ($n \geq 1$) distinct points on X. Set $X_n = X - \{x_i : 1 \leq i \leq n\}$. We introduce three classes of quasiconformal mappings of X_n onto itself:

$\Sigma_1(X_n)$ is the set of restrictions to X_n of all quasiconformal mappings f of X which are homotopic (rel ∂X) to the identity map by a homotopy $f_t : (X \cup \partial X) \times [0,1] \to X \cup \partial X$ such that $f_t(x_i) = x_i$ for all $t \in [0,1]$ and $1 \leq i \leq n$;

$\Sigma_2(X_n)$ is the set of restrictions to X_n of all quasiconformal mappings f of X which are homotopic (rel ∂X) to the identity map such that $f(x_i) = x_i$ for all $1 \leq i \leq n$;

$\Sigma_3(X_n)$ is the set of restrictions to X_n of all quasiconformal mappings f of X which are homotopic (rel ∂X) to the identity map such that f keep the set of points $\{x_i : 1 \leq i \leq n\}$ fixed.

Clearly, $\Sigma_0(X_n) \subset \Sigma_1(X_n) \subset \Sigma_2(X_n) \subset \Sigma_3(X_n) \subset \Sigma(X_n)$. For $j = 1,2,3$, set $\mathrm{Mod}_j(X_n) = \Sigma_j(X_n)/\Sigma_0(X_n)$, then $\mathrm{Mod}_1(X_n) \subset \mathrm{Mod}_2(X_n) \subset \mathrm{Mod}_3(X_n) \subset \mathrm{Mod}(X_n)$. Recall that $\Sigma_1(X_n)$ has been introduced and investigated by Kra [**Kr1**] and Nag [**Na3**] when X is of finite type. Note also that when $n = 1$, a well known result of Epstein [**Ep2**] implies that $\Sigma_1(X_1) = \Sigma_0(X_1)$ and so $\mathrm{Mod}_1(X_1)$ is the trivial group, while it trivially holds that $\Sigma_2(X_1) = \Sigma_3(X_1)$ and so $\mathrm{Mod}_2(X_1) = \mathrm{Mod}_3(X_1)$. We will show that each subgroup $\mathrm{Mod}_j(X_n)$ of $\mathrm{Mod}(X_n)$ acts freely and discontinuously on the Teichmüller space $T(X_n)$.

LEMMA 3.1. *Each group* $\mathrm{Mod}_j(X_n)$ $(j = 1,2,3)$ *acts freely on* $T(X_n)$.

PROOF. It is sufficient to show that if $\chi(\langle g \rangle)([f]) = [f \circ g^{-1}] = [f]$ for some $\langle g \rangle \in \mathrm{Mod}_3(X_n)$ and $[f] \in T(X_n)$, then $\langle g \rangle = \mathrm{id}$. In fact, from $[f \circ g^{-1}] = [f]$ we conclude that there exists some conformal map c from $f(X_n)$ onto itself such that $f^{-1} \circ c \circ f \circ g^{-1} \in \Sigma_0(X_n)$. Since $g \in \Sigma_3(X_n)$, g keeps the set of points $\{x_i : 1 \leq i \leq n\}$ fixed. Thus c keeps the set of points $\{f(x_i) : 1 \leq i \leq n\}$ fixed and can be extended to a conformal mapping of $f(X)$ onto itself, and $f^{-1} \circ c \circ f \circ g^{-1} \in \Sigma_0(X)$. Since $g \in \Sigma_3(X_n)$, so $g \in \Sigma_0(X)$, we obtain that $f^{-1} \circ c \circ f \in \Sigma_0(X)$ and consequently that $c \in \Sigma_0(f(X))$, which implies that $c = \mathrm{id}$. Consequently, $g \in \Sigma_0(X_n)$, that is, $\langle g \rangle = \mathrm{id}$. □

In order to prove the discontinuity of the groups $\mathrm{Mod}_j(X_n)$ on $T(X_n)$, we will make use of an isomorphism theorem of Bers [**Be3**]. Bers isomorphism theorem establishes biholomorphic isomorphisms between the Bers fiber spaces and Teichmüller spaces for torsion free Fuchsian groups. Let G and Γ be torsion free Fuchsian groups such that $\boldsymbol{H}/G = X$, $\boldsymbol{H}/\Gamma = X_1 = X - \{x_1\}$, $\pi : \boldsymbol{H} \to \boldsymbol{H}/G$ and $\pi_1 : \boldsymbol{H} \to \boldsymbol{H}/\Gamma$ the natural projections. Choose $z_1 \in \boldsymbol{H}$ with $\pi(z_1) = x_1$ and a universal covering mapping $h : \boldsymbol{H} \to \boldsymbol{H} - \pi^{-1}(x_1)$ with $\pi_1 = \pi \circ h$. Then h induces a norm-preserving isomorphism $h^* : M(\Gamma) \to M(G)$ by

$$(h^*\mu) \circ h = \mu h'/\overline{h'}, \qquad \mu \in M(\Gamma). \tag{3.1}$$

$h^* : M(\Gamma) \to M(G)$ induces a biholomorphic isomorphism between $T(\Gamma)$ and $F(G)$ sending $\Phi_\Gamma(\mu)$ to $(\Phi_G(h^*\mu), w^{h^*\mu}(z_1))$, known as the Bers isomorphism. We denote by B the Bers isomorphism.

Bers isomorphism establishes a biholomorphic isomorphism between $T(\Gamma)$ and $F(G)$, meanwhile it conjugates the action of $\operatorname{mod} G$ on $F(G)$ to (a subgroup of) $\mathrm{Mod}(\Gamma)$ on $T(\Gamma)$. Precisely, we denote by $\Sigma(\Gamma, z_1)$ the class of all mappings $w \in \Sigma(\Gamma)$ whose projections to X_1 can be completed to quasiconformal self-mappings f of X, or equivalently, w can be projected to a mapping $w_* : \boldsymbol{H} - \pi^{-1}(x_1) \to \boldsymbol{H} - \pi^{-1}(x_1)$ such that $h \circ w = w_* \circ h$. We also denote by $\Sigma_0(\Gamma, z_1)$ the class of all mappings $w \in \Sigma(\Gamma, z_1)$ such that f are homotopic to the identity rel ∂X, that is, $f \in \Sigma_2(X_1)$. Set $\mathrm{Mod}_0(\Gamma, z_1) = (\Sigma_0(\Gamma, z_1)/\Sigma_0(\Gamma))/\Gamma$, $\mathrm{Mod}(\Gamma, z_1) = (\Sigma(\Gamma, z_1)/\Sigma_0(\Gamma))/\Gamma$. Clearly, $\mathrm{Mod}_0(\Gamma, z_1) \simeq \mathrm{Mod}_2(X_1)$. Then there is an isomorphism I from $\operatorname{mod}(G)$ onto $\mathrm{Mod}(\Gamma, z_1)$ such that for any $[w] \in \operatorname{mod}(G)$, $B \circ \chi(I([w])) = \rho([w]) \circ B$, and $I(G) = \mathrm{Mod}_0(\Gamma, z_1)$. For more details, see [**Be3**], [**Kr3**], [**Ri**] and [**Sh1**]. Since $T(X_1)$ is biholomorphically equivalent to $T(\Gamma)$, Bers isomorphism implies that $F(G)$ is also biholomorphically equivalent to $T(X_1)$. We denote by $\tilde{B}$ the isomorphism from $T(X_1)$ to $F(G)$. Then there is an isomorphism $\tilde{I}$ from $\operatorname{mod}(G)$ into $\mathrm{Mod}(X_1)$ such that for any $[w] \in \operatorname{mod}(G)$, $\tilde{B} \circ \chi(\tilde{I}([w])) = \rho([w]) \circ \tilde{B}$, and $\tilde{I}(G) = \mathrm{Mod}_2(X_1)$.

LEMMA 3.2. *$\mathrm{Mod}_2(X_1)$ acts freely and discontinuously on the Teichmüller space $T(X_1)$.*

PROOF. It is well known that G acts freely and discontinuously on the Bers fiber space $F(G)$. Since the Bers isomorphism $\tilde{B}$ conjugates the action of $\mathrm{Mod}_2(X_1)$ on the Teichmüller space $T(X_1)$ to the action of G on $F(G)$, we conclude that $\mathrm{Mod}_2(X_1)$ acts freely and discontinuously on the Teichmüller space $T(X_1)$. $\square$

Now we can prove our main result in this section. It will play an important role in our later discussion.

THEOREM 3.1. *Each group $\mathrm{Mod}_j(X_n)$ $(j = 1, 2, 3)$ acts freely and discontinuously on the Teichmüller space $T(X_n)$.*

PROOF. By Lemma 3.1, we only need to show that $\mathrm{Mod}_j(X_n)$ acts discontinuously on $T(X_n)$. Noting that $\mathrm{Mod}_1(X_n) \subset \mathrm{Mod}_2(X_n) \subset \mathrm{Mod}_3(X_n)$, and $\mathrm{Mod}_2(X_n)$ is a subgroup of $\mathrm{Mod}_3(X_n)$ with index $n!$, we need to show that $\mathrm{Mod}_2(X_n)$ acts discontinuously on $T(X_n)$. By definition, we need to show that for any point $p \in T(X_n)$, there exists some $r > 0$ such that the set $\{\langle f\rangle \in \mathrm{Mod}_2(X_n) : \chi(\langle f\rangle)(B(p, r)) \cap B(p, r) \neq \varnothing\}$ consists of only finitely many elements, where $B(p, r)$ is the open ball centered at p with radius r.

Suppose to the contrary that for some point $[f] \in T(X_n)$ and for any $r > 0$ the set $\{\langle g\rangle \in \mathrm{Mod}_2(X_n) : \chi(\langle g\rangle)(B([f],r)) \cap B([f],r) \neq \varnothing\}$ consists of infinitely many elements. Then there exists some sequence of quasiconformal mappings f_j $(j \geq 1)$ such that $f_j \in \Sigma_0(X)$, $f_j(x_i) = x_i$ $(1 \leq i \leq n)$, f_j represent distinct elements $\langle f_j\rangle$ in $\mathrm{Mod}_2(X_n)$, and $\tau_{X_n}([f \circ f_j^{-1}], [f]) \to 0$ as $j \to \infty$. Now for $1 \leq i \leq n$, f_j also represent a sequence of elements $\langle f_j\rangle$ in $\mathrm{Mod}_2(X_i)$, f represents a point $[f]$ in $T(X_i)$, and $\tau_{X_i}([f \circ f_j^{-1}], [f]) \leq \tau_{X_n}([f \circ f_j^{-1}], [f]) \to 0$ as $j \to \infty$.

By Lemma 3.2, $\mathrm{Mod}_2(X_1)$ acts freely and discontinuously on the Teichmüller space $T(X_1)$. With $i = 1$ we conclude that when j is sufficiently large all mappings f_j represent the same identity element in $\mathrm{Mod}_2(X_1)$, that is, $f_j \in \Sigma_0(X_1)$. Repeating this procedure n times, we conclude that when j is sufficiently large all mappings f_j represent the same identity element in $\mathrm{Mod}_2(X_n)$. This is a contradiction, however. □

An immediate consequence of Theorem 3.1 is

Corollary 3.1. *Each quotient space $T(X_n)/\mathrm{Mod}_j(X_n)$ is a complex manifold with $T(X_n)$ as a universal covering space.*

Now we point out that the quotient manifolds $T(X_n)/\mathrm{Mod}_j(X_n)$ $(j = 1, 2)$ can be identified with the Teichmüller spaces of pointed Riemann surfaces in the sense of Kra [**Kr1**]. Recall that for a hyperbolic Riemann surface X and n $(n \geq 1)$ distinct points $x_1, x_2, \ldots, x_n$ on X, $X_n = X - \{x_i : 1 \leq i \leq n\}$. We say $\mathscr{X} = \{X; x_1, \ldots, x_n\}$ is an n-pointed Riemann surface. Two quasiconformal mappings f and g on X are respectively said to be Teichmüller equivalent, weakly Teichmüller equivalent and most weakly Teichmüller equivalent if there exists some conformal mapping c from $f(X_n)$ to $g(X_n)$ such that $g^{-1} \circ c \circ f \in \Sigma_1(X_n)$, $g^{-1} \circ c \circ f \in \Sigma_2(X_n)$ or $g^{-1} \circ c \circ f \in \Sigma_3(X_n)$. We denote respectively by $T_1(\mathscr{X})$, $T_2(\mathscr{X})$ and $T_3(\mathscr{X})$ to be the set of Teichmüller equivalence classes $[f]_1$, weakly Teichmüller equivalence classes $[f]_2$ or most weakly Teichmüller equivalence classes $[f]_3$ of quasiconformal mappings f of X. Kra [**Kr1**] called $T_1(\mathscr{X})$ and $T_2(\mathscr{X})$ the Teichmüller space and the weak Teichmüller space of the n-pointed Riemann surface $\mathscr{X}$, respectively. We can define the Teichmüller distance on $T_j(\mathscr{X})$ in a way similar to (2.7). Namely, for any two points $[f_1]_j$ and $[f_2]_j$ in $T_j(\mathscr{X})$, the Teichmüller distance between them is

$$\tau_{\mathscr{X},j}([f_1]_j, [f_2]_j) = \inf\left\{\frac{1}{2}\log K[f] | f : f_1(X_n) \to f_2(X_n), f_2^{-1} \circ f \circ f_1 \in \Sigma_j(X_n)\right\}. \quad (3.2)$$

On the other hand, the Kobayashi-Teichmüller distance τ_{X_n} on $T(X_n)$ induces the quotient metric on $T(X_n)/\mathrm{Mod}_j(X_n)$, which is precisely its Kobayashi metric. We have the following obvious result.

Proposition 3.1. *For $1 \leq j \leq 3$, $T_j(\mathscr{X})$ is isometrically isomorphic to the quotient manifold $T(X_n)/\mathrm{Mod}_j(X_n)$ and thus is a complex manifold.*

4. Teichmüller spaces and modular groups for pointed Fuchsian groups.

In a fundamental paper [**Kr1**], Kra introduced and discussed some some new kinds of Teichmüller spaces for Fuchsian groups of finite type, the Teichmüller spaces for pointed

Fuchsian groups. In the rest part of the paper, we will continue to discuss these Teichmüller spaces for general Fuchsian groups, not necessarily of finite type. As will be seen, these Teichmüller spaces contain as specific examples the complex manifolds we discussed in section 3, and Theorem 3.1 in the last section will play an important role in our discussion. For completeness, in this section we recall the basic definitions of these Teichmüller spaces and their modular groups following Kra [**Kr1**]. However, we will define the modular groups in a different manner from Kra's paper.

We say that $\mathscr{G} = \{G; z_1, \dots, z_n\}$ is an n-pointed Fuchsian group, if G is a Fuchsian group acting on the upper half plane $\boldsymbol{H}$ and $z_1, \dots, z_n$ are n-inequivalent points of $\boldsymbol{H}_G$, $n \geq 1$. Let $\boldsymbol{H}_{\mathscr{G}} = \{z \in \boldsymbol{H}_G : z \neq g(z_i), g \in G, 1 \leq i \leq n\}$. We say that two n-pointed Fuchsian groups $\mathscr{G} = \{G; z_1, \dots, z_n\}$ and $\mathscr{G}' = \{G'; z_1', \dots, z_n'\}$ conjugate if there exist some Möbius transformation α of $\boldsymbol{H}$ onto itself and some permutation σ of $\{1, \dots, n\}$ such that $G' = \alpha G \alpha^{-1}$, and for $1 \leq i \leq n$, $z_i' = \alpha(g_i z_{\sigma(i)})$ for some $g_i \in G$.

Let K be a normal subgroup of G. Two elements μ and ν in $M(G)$ are said to be $(\mathscr{G}, K)$-equivalent if they are equivalent (in $M(G)$) and for each $i = 1, \dots, n$ there exists some $k_i \in K$ such that $w^\mu(z_i) = w^\nu(k_i(z_i))$. Of course, the pair $(\mathscr{G}, \{\mathrm{id}\})$ will be abbreviated as $\mathscr{G}$. The set of all $(\mathscr{G}, K)$-equivalence classes $[\mu]_{\mathscr{G},K}$ of elements μ of $M(G)$ is called the Teichmüller space of $\mathscr{G}$ modulo K, $T(\mathscr{G}, K)$. We denote by $\Phi_{\mathscr{G},K}$ the natural projection from $M(G)$ to $T(\mathscr{G}, K)$. Note that if $\mathscr{G}$ and $\mathscr{G}'$ conjugate with $\alpha = \mathrm{id}$, then two pairs $(\mathscr{G}, K)$ and $(\mathscr{G}', K)$ give the same Teichmüller space $T(\mathscr{G}, K) = T(\mathscr{G}', K)$. Clearly, we have the natural projections:

$$M(G) \to T(\mathscr{G}) \to T(\mathscr{G}, K) \to T(\mathscr{G}, G) \to T(G). \tag{4.1}$$

The Teichmüller distance between two points $[\mu_1]_{\mathscr{G},K}$ and $[\mu_2]_{\mathscr{G},K}$ in $T(\mathscr{G}, K)$ is defined as

$$\tau_{\mathscr{G},K}([\mu_1]_{\mathscr{G},K}, [\mu_2]_{\mathscr{G},K}) = \inf\big\{\sigma_G(\nu_1, \nu_2) : [\nu_i]_{\mathscr{G},K} = [\mu_i]_{\mathscr{G},K}, i = 1, 2\big\}. \tag{4.2}$$

Two elements w_1 and w_2 in $Q(G)$ are said to be $(\mathscr{G}, K)$-equivalent if they coincide on the real line $\boldsymbol{R}$, and for each $i = 1, \dots, n$ there exists some $k_i \in K$ such that $w_1(z_i) = w_2(k_i(z_i))$. The $(\mathscr{G}, K)$-equivalence class of w will be denoted by $[w]_{\mathscr{G},K}$. Let $\Sigma_0(\mathscr{G}, K)$ denote the set of all quasiconformal self-mappings of $\boldsymbol{H}$ which are $(\mathscr{G}, K)$-equivalent to the identity mapping. It is not difficult to see that two elements μ and ν in $M(G)$ are $(\mathscr{G}, K)$-equivalent if and only if w_μ and w_ν are $(\mathscr{G}, K)$-equivalent. The point $\Phi_{\mathscr{G},K}(\mu)$ will also be denoted by $[w_\mu]_{\mathscr{G},K}$ later.

Let $w \in Q(G)$ be given. Then w induces an isometric isomorphism w^* by (2.4). For $\mathscr{G} = \{G; z_1, \dots, z_n\}$, set $w\mathscr{G}w^{-1} = \{wGw^{-1}; w(z_1), \dots, w(z_n)\}$. It is not difficult to prove that $[w^*(w_\mu)]_{w\mathscr{G}w^{-1}, wKw^{-1}}$ depends only on $[w]_{\mathscr{G},K}$ and $[w_\mu]_{\mathscr{G},K}$. So w^* induces an isometric isomorphism $\chi([w]_{\mathscr{G},K})$ between $T(\mathscr{G}, K)$ and $T(w\mathscr{G}w^{-1}, wKw^{-1})$. $\chi([w]_{\mathscr{G},K})$ are called allowable mappings.

Let $\Sigma(\mathscr{G}, K)$ denote the set of all elements $w \in \Sigma(G)$ such that $wKw^{-1} = K$, $w(\boldsymbol{H}_{\mathscr{G}}) = \boldsymbol{H}_{\mathscr{G}}$. $\mathrm{Mod}(\mathscr{G}, K) = (\Sigma(\mathscr{G}, K)/\Sigma_0(\mathscr{G}, K))/G$ is called the modular group of $(\mathscr{G}, K)$. The element of $\mathrm{Mod}(\mathscr{G}, K)$ induced by w will be denoted by $\langle w\rangle_{\mathscr{G},K}$. For each $w \in \Sigma(\mathscr{G}, K)$, the pairs $(\mathscr{G}, K)$ and $(w\mathscr{G}w^{-1}, wKw^{-1})$ determine the same Teichmüller space, so w^* induces a self-mapping $\chi(\langle w\rangle_{\mathscr{G},K})$ of the Teichmüller space $T(\mathscr{G}, K)$.

Finally, we point out that the (weak) Teichmüller spaces for pointed Riemann surfaces are special kinds of Teichmüller spaces for pointed Fuchsian groups. Let $\mathscr{G} = \{G; z_1, \dots, z_n\}$ be an n-pointed Fuchsian group with G torsion free. Let $\pi : \boldsymbol{H} \to X = \boldsymbol{H}/G$ be the natural projection. Set $x_i = \pi(z_i)$ $(1 \le i \le n)$, $X_n = X - \{x_1, \dots, x_n\}$, and $\mathscr{X} = \{X; x_1, \dots, x_n\}$.

Proposition 4.1. *$T_1(\mathscr{X})$ is isometrically isomorphic to $T(\mathscr{G})$, and $T_2(\mathscr{X})$ is isometrically isomorphic to $T(\mathscr{G}, G)$.*

Proposition 4.1 follows immediately from the following lemma.

Lemma 4.1. *Let $f \in \Sigma(X)$. Then $f \in \Sigma_1(X_n)$ if and only if there exists some $w \in \Sigma_0(\mathscr{G})$ with $\pi \circ w = f \circ \pi$; $f \in \Sigma_2(X_n)$ if and only if there exists some $w \in \Sigma_0(\mathscr{G}, G)$ with $\pi \circ w = f \circ \pi$.*

Proof. Let $f \in \Sigma_1(X_n)$. Then there exists some homotopy $f_t : (X \cup \partial X) \times [0, 1] \to X \cup \partial X$ between $f_1 = f$ and $f_0 = \mathrm{id}$ such that $f_t(x_i) = x_i$ for all $t \in [0, 1]$ and $1 \le i \le n$. Lifting the homotopy f_t to $\boldsymbol{H}$ to obtain a homotopy w_t with $w_0 = \mathrm{id}$ and $\pi \circ w_t = f_t \circ \pi$. It is easy to see that $w = w_1 \in \Sigma_0(\mathscr{G})$. Conversely, let $w \in \Sigma_0(\mathscr{G})$ with $\pi \circ w = f \circ \pi$. Consider the Ahlfors homotopy w_t between $w_1 = w$ and $w_0 = \mathrm{id}$ (see [**Ah**]). Then w_t can be projected to a homotopy f_t between $f_1 = f$ and $f_0 = \mathrm{id}$. For each $1 \le i \le n$, since $w_t(z_i) = z_i$, $f_t(x_i) = x_i$. Thus $f \in \Sigma_1(X_n)$.

The second assertion follows from the facts that $f \in \Sigma_0(X)$ if and only if there exists some $w \in \Sigma_0(G)$ such that $\pi \circ w = f \circ \pi$, and that $f(x_i) = x_i$ if and only if there exists some $g_i \in G$ such that $w(z_i) = g_i(z_i)$. $\square$

Remark 4.1. Propositions 3.1 and 4.1 imply that both $T(\mathscr{G})$ and $T(\mathscr{G}, G)$ are complex manifolds when G is torsion free. In the following sections, we will show that $T(\mathscr{G}, K)$ is always a complex manifold for every Fuchsian group G and every normal subgroup K.

5. Complex structures on Teichmüller spaces for pointed Fuchsian groups: I.

In this section, we will prove the following theorem, which says that for any n-pointed Fuchsian group $\mathscr{G} = \{G; z_1, \dots, z_n\}$, the Teichmüller space $T(\mathscr{G})$ carries a natural complex structure.

Theorem 5.1. *For any n-pointed Fuchsian group $\mathscr{G} = \{G; z_1, \dots, z_n\}$, there exists a unique complex manifold structure on the Teichmüller space $T(\mathscr{G})$ so that the natural projection $\Phi_{\mathscr{G}} : M(G) \to T(\mathscr{G})$ is a holomorphic split submersion.*

Proof. Choose some torsion free Fuchsian group Γ such that $\boldsymbol{H}/\Gamma = \boldsymbol{H}_{\mathscr{G}}/G$. Let $\pi : \boldsymbol{H} \to \boldsymbol{H}/G$ and $\pi_1 : \boldsymbol{H} \to \boldsymbol{H}/\Gamma$ be the natural projections. Then there exists some holomorphic universal covering mapping $h : \boldsymbol{H} \to \boldsymbol{H}_{\mathscr{G}}$ such that $\pi_1 = \pi \circ h$. Hence there exists some homomorphism $\theta : \Gamma \to G$ such that

$$h \circ \gamma = \theta(\gamma) \circ h, \quad \gamma \in \Gamma. \tag{5.1}$$

Lemma 5.1. *$\theta : \Gamma \to G$ is surjective.*

Proof. Any $g \in G$ fixes $\boldsymbol{H}_{\mathscr{G}}$ and so $g|\boldsymbol{H}_{\mathscr{G}}$ can be lifted to a Möbius transformation $\gamma : \boldsymbol{H} \to \boldsymbol{H}$, that is, $h \circ \gamma = g \circ h$. Thus we obtain $g = \theta(\gamma)$. □

Using h, we may define the norm-preserving isomorphism $h^* : M(\Gamma) \to M(G)$ by

$$(h^*\mu) \circ h = \mu h'/\overline{h'}, \qquad \mu \in M(\Gamma). \tag{5.2}$$

We will show that h^* can project to a mapping from $T(\Gamma)$ onto $T(\mathscr{G})$.

A direct computation will show

Lemma 5.2. *Let $\mu \in M(\Gamma)$ and $\sigma \in M(G)$. Then $\sigma = h^*(\mu)$ if and only if there exists some holomorphic universal covering mapping $\tilde{h} : \boldsymbol{H} \to w_\sigma(\boldsymbol{H}_{\mathscr{G}})$ such that $w_\sigma \circ h = \tilde{h} \circ w_\mu$.*

For $\mu \in M(\Gamma)$ and $\sigma = h^*(\mu)$, set $h_\mu = w_\sigma \circ h \circ w_\mu^{-1}$. Lemma 5.2 implies that h_μ is a holomorphic universal covering mapping from $\boldsymbol{H}$ to $w_\sigma(\boldsymbol{H}_{\mathscr{G}})$.

Lemma 5.3. *For any $w \in \Sigma_0(\Gamma)$, there is a unique map $w_* \in \Sigma_0(\mathscr{G})$ such that $w_* \circ h = h \circ w$.*

Proof. For any Fuchsian group Γ, let $\Lambda(\Gamma)$ denote the limit set of Γ, and $D(\Gamma) = \overline{\boldsymbol{R}} - \Lambda(\Gamma)$. Since $w \in \Sigma_0(\Gamma)$, it can be projected to a quasiconformal mapping $w_* : \boldsymbol{H}_{\mathscr{G}} \to \boldsymbol{H}_{\mathscr{G}}$ via the universal covering $h : \boldsymbol{H} \to \boldsymbol{H}_{\mathscr{G}}$ such that w_* is homotopy to the identity rel the ideal boundary $\partial \boldsymbol{H}_{\mathscr{G}} = D(G)$. w_* can be completed to a mapping by the identity on $\boldsymbol{H} - \boldsymbol{H}_{\mathscr{G}}$, which is still denoted by $w_* : \boldsymbol{H} \to \boldsymbol{H}$, such that $w_* \circ h = h \circ w$ and w_* is identity on $\boldsymbol{H}_{\mathscr{G}}$ and $D(G)$. We need to show that $w_* \in \Sigma_0(G)$. It is sufficient to show that $w_* \circ g = g \circ w_*$ for all $g \in G$, which will imply that w_* is also the identity on $\Lambda(G)$.

Since $w \in \Sigma_0(\Gamma)$, $w \circ \gamma = \gamma \circ w$ for all $\gamma \in \Gamma$. So

$$w_* \circ \theta(\gamma) \circ h = w_* \circ h \circ \gamma = h \circ w \circ \gamma = h \circ \gamma \circ w = \theta(\gamma) \circ h \circ w = \theta(\gamma) \circ w_* \circ h.$$

Since $\theta : \Gamma \to G$ is surjective, we conclude that $w_* \circ g = g \circ w_*$ for all $g \in G$. □

Corollary 5.1. *If μ and ν are equivalent in $M(\Gamma)$, then $\sigma = h^*(\mu)$ and $\tau = h^*(\nu)$ are $\mathscr{G}$-equivalent in $M(G)$.*

Proof. Since μ and ν are equivalent in $M(\Gamma)$, there exists some $w \in \Sigma_0(\Gamma)$ such that $w_\nu = w_\mu \circ w$. By Lemma 5.3, there exists some $w_* \in \Sigma_0(\mathscr{G})$ such that $w_* \circ h = h \circ w$. Since $\sigma = h^*(\mu)$, Lemma 5.2 implies that $h_\mu = w_\sigma \circ h \circ w_\mu^{-1}$ is a holomorphic universal covering mapping from $\boldsymbol{H}$ to $w_\sigma(\boldsymbol{H}_{\mathscr{G}})$. Since

$$h_\mu \circ w_\nu = h_\mu \circ w_\mu \circ w = w_\sigma \circ h \circ w = w_\sigma \circ w_* \circ h,$$

we conclude by Lemma 5.2 that $w_\tau = w_\sigma \circ w_*$, so $\sigma = h_*(\mu)$ and $\tau = h_*(\nu)$ are $\mathscr{G}$-equivalent in $M(G)$. We also obtain

$$h_\nu = w_\tau \circ h \circ (w_\nu)^{-1} = w_\sigma \circ w_* \circ h \circ w^{-1} \circ (w_\mu)^{-1} = w_\sigma \circ h \circ (w_\mu)^{-1} = h_\mu. \qquad \square$$

Corollary 5.1 implies that $h^* : M(\Gamma) \to M(G)$ can project to a mapping from $T(\Gamma)$ to $T(\mathscr{G})$, which is denoted by P. Then $\Phi_{\mathscr{G}} \circ h^* = P \circ \Phi_\Gamma$, so P is continuous and surjective. Next we will show that P is a universal covering mapping. To do so, we denote by $\Sigma_0(\Gamma, \mathscr{G})$ the set of all quasiconformal mappings w in $\Sigma(\Gamma)$ such that there exists some quasiconformal mapping $w_* \in \Sigma_0(\mathscr{G})$ with $h \circ w = w_* \circ h$. Lemma 5.3 implies that $\Sigma_0(\Gamma) \subset \Sigma_0(\Gamma, \mathscr{G})$. Set $\mathrm{Mod}_0(\Gamma, \mathscr{G}) = (\Sigma_0(\Gamma, \mathscr{G})/\Sigma_0(\Gamma))/\Gamma$. Then we have

Lemma 5.4. *$P(\Phi_\Gamma(\mu_1)) = P(\Phi_\Gamma(\mu_2))$ if and only if there exists some $\langle w \rangle \in \mathrm{Mod}_0(\Gamma, \mathscr{G})$ such that $\chi(\langle w \rangle)(\Phi_\Gamma(\mu_1)) = \Phi_\Gamma(\mu_2)$.*

Proof. Suppose $P(\Phi_\Gamma(\mu_1)) = P(\Phi_\Gamma(\mu_2))$, then $\Phi_{\mathscr{G}} \circ h^*(\mu_1) = \Phi_{\mathscr{G}} \circ h^*(\mu_1)$. So there exists some $w_* \in \Sigma_0(\mathscr{G})$ such that $w_{\sigma_2} = w_{\sigma_1} \circ w_*$, where $\sigma_i = h^*(\mu_i)$ $(i = 1, 2)$. w_* can be lifted to a mapping $w \in \Sigma_0(\Gamma, \mathscr{G})$ such that $h \circ w = w_* \circ h$. Thus, $w_{\sigma_2} \circ h = w_{\sigma_1} \circ w_* \circ h = w_{\sigma_1} \circ h \circ w$, which implies that $h_{\mu_2} \circ w_{\mu_2} = h_{\mu_1} \circ w_{\mu_1} \circ w$. Comparing the Beltrami coefficients of both sides of the equation, we conclude that $w_{\mu_1} = w^*(w_{\mu_2})$, that is, $\chi(\langle w \rangle)(\Phi_\Gamma(\mu_2)) = \Phi_\Gamma(\mu_1)$.

Reversing the procedure above we can obtain the proof of the other direction. $\square$

Now let P_1 denote the natural projection from $T(\Gamma)$ to $T(\Gamma)/\mathrm{Mod}_0(\Gamma, \mathscr{G})$. Lemma 5.4 implies that $P : T(\Gamma) \to T(\mathscr{G})$ can project to a mapping from $T(\Gamma)/\mathrm{Mod}_0(\Gamma, \mathscr{G})$ to $T(\mathscr{G})$. We denote this map by P_2, then $P = P_2 \circ P_1$.

Lemma 5.5. *$P_2 : T(\Gamma)/\mathrm{Mod}_0(\Gamma, \mathscr{G}) \to T(\mathscr{G})$ is a homeomorphism.*

Proof. Lemma 5.4 implies that P_2 is injective. Since $P_2 \circ P_1 = P$ is continuous and surjective, P_2 is continuous and surjective. Since $P_2^{-1} \circ \Phi_{\mathscr{G}} = P_1 \circ \Phi_\Gamma \circ h^{*-1}$ is continuous, P_2^{-1} is continuous. So $P_2 : T(\Gamma)/\mathrm{Mod}_0(\Gamma, \mathscr{G}) \to T(\mathscr{G})$ is a homeomorphism. $\square$

Now we consider the action of the subgroup $\mathrm{Mod}_0(\Gamma, \mathscr{G})$ of $\mathrm{Mod}(\Gamma)$ on the Teichmüller space $T(\Gamma)$. Set $X_G = \boldsymbol{H}_G/G$, $X_{\mathscr{G}} = \boldsymbol{H}_{\mathscr{G}}/G = \boldsymbol{H}/\Gamma$, $x_i = \pi(z_i)$ $(1 \leq i \leq n)$. Then $X_{\mathscr{G}} = X_G - \{x_1, \ldots, x_n\}$, or under the notations in section 3, $X_{\mathscr{G}} = (X_G)_n$. Let $w \in \Sigma_0(\Gamma, \mathscr{G})$. Then there exists some quasiconformal mapping $w_* \in \Sigma_0(\mathscr{G})$ with $h \circ w = w_* \circ h$. Now w_* can be restricted to a self-mapping of $\boldsymbol{H}_{\mathscr{G}}$ and then projects to a quasiconformal mapping f of $X_{\mathscr{G}}$ onto itself under the covering $\pi : \boldsymbol{H}_{\mathscr{G}} \to X_{\mathscr{G}}$. It is easy to see that f is also the projection of w under the covering mapping $\pi_1 : \boldsymbol{H} \to \boldsymbol{H}/\Gamma$. $w_* \in \Sigma_0(\mathscr{G})$ implies that $w_*(z_i) = z_i$, which implies that $f(x_i) = x_i$. $w_* \in \Sigma_0(\mathscr{G})$ also implies that $w_* \in \Sigma_0(G)$. Now an important result in Teichmüller theory, known as Bers-Greenberg Theorem (see [**BG**], [**EK1**], [**EMc**], [**Ga2**], [**HS**], [**Kr1**], [**Ma**]), implies that $f \in \Sigma_0(X_G)$. Consequently, under the notations in section 3, $f \in \Sigma_2((X_G)_n)$. Now, since Γ is torsion free, $T(\Gamma)$ is biholomorphically equivalent to $T(X_{\mathscr{G}}) = T((X_G)_n)$, and $\mathrm{Mod}(\Gamma)$ is isomorphic to $\mathrm{Mod}(X_{\mathscr{G}}) = \mathrm{Mod}((X_G)_n)$. We have already obtained that $\mathrm{Mod}_0(\Gamma, \mathscr{G})$ is isomorphic to some subgroup of $\mathrm{Mod}_2((X_G)_n)$. By Theorem 3.1, $\mathrm{Mod}_2((X_G)_n)$ acts freely and discontinuously on $T((X_G)_n)$. We conclude that $\mathrm{Mod}_0(\Gamma, \mathscr{G})$ acts freely and discontinuously on $T(\Gamma)$. We state this as a lemma.

LEMMA 5.6. *$\mathrm{Mod}_0(\Gamma, \mathscr{G})$ acts freely and discontinuously on $T(\Gamma)$.*

An immediate consequence of Lemma 5.6 is

COROLLARY 5.2. *$T(\Gamma)/\mathrm{Mod}_0(\Gamma, \mathscr{G})$ is a complex manifold, and the natural projection $P_1 : T(\Gamma) \to T(\Gamma)/\mathrm{Mod}_0(\Gamma, \mathscr{G})$ is a holomorphic universal covering.*

COROLLARY 5.3. *$T(\mathscr{G})$ has a complex structure such that $P_2 : T(\Gamma)/\mathrm{Mod}_0(\Gamma, \mathscr{G}) \to T(\mathscr{G})$ is a biholomorphic homeomorphism, and $P : T(\Gamma) \to T(\mathscr{G})$ is a holomorphic universal covering with covering group $\chi(\mathrm{Mod}_0(\Gamma, \mathscr{G}))$.*

PROOF. We can push the complex structure on $T(\Gamma)/\mathrm{Mod}_0(\Gamma, \mathscr{G})$ by the homeomorphism P_2 to obtain a complex structure on $T(\mathscr{G})$. The rest of the Corollary follows immediately. $\square$

LEMMA 5.7. *Under the complex structure of $T(\mathscr{G})$ obtained in Corollary 5.3, $\Phi_{\mathscr{G}} : M(G) \to T(\mathscr{G})$ is a holomorphic split submersion.*

PROOF. Under the complex structure of $T(\mathscr{G})$ in Corollary 5.3, $P : T(\Gamma) \to T(\mathscr{G})$ is a holomorphic universal covering. Since $h^* : M(\Gamma) \to M(G)$ is an isometric isomorphism, and $\Phi_\Gamma : M(\Gamma) \to T(\Gamma)$ is a holomorphic split submersion, $\Phi_{\mathscr{G}} = P \circ \Phi_\Gamma \circ h^{*-1}$ is also a holomorphic split submersion. $\square$

Finally, Lemma 5.7 says that there exists some complex manifold structure on the Teichmüller space $T(\mathscr{G})$ so that the natural projection $\Phi_{\mathscr{G}} : M(G) \to T(\mathscr{G})$ is a holomorphic split submersion. The uniqueness of such a complex structure is obvious. This completes the proof of Theorem 5.1. $\square$

In order to give a good description of $T(\mathscr{G})$, we introduce some fiber spaces. First note that the natural projections $\Phi_{\mathscr{G}} : M(G) \to T(\mathscr{G})$ and $\Phi_G : M(G) \to T(G)$ induce a natural projection $\Phi : T(\mathscr{G}) \to T(G)$. Since $\Phi \circ \Phi_{\mathscr{G}} = \Phi_G$ is a holomorphic split submersion, $\Phi : T(\mathscr{G}) \to T(G)$ is also a holomorphic split submersion. We state this as

PROPOSITION 5.1. *$\Phi : T(\mathscr{G}) \to T(G)$ is a holomorphic split submersion so that $T(\mathscr{G})$ is a holomorphic fiber space over $T(G)$.*

Now we introduce some fiber space over $T(G)$ which can be identified with $T(\mathscr{G})$. Let

$$F^n(G) = \left\{(\Phi_G(\mu), \zeta) \in T(G) \times \boldsymbol{C}^n : \mu \in M(G), \zeta = (\zeta_1, \ldots, \zeta_n) \in (w^\mu(\boldsymbol{H}))^n\right\}, \quad (5.3)$$

and

$$F_0^n(G) = \left\{(\Phi_G(\mu), \zeta) \in F^n(G) : \zeta = (\zeta_1, \ldots, \zeta_n) \in (w^\mu(\boldsymbol{H}_G))^n, \zeta_i \neq g^\mu(\zeta_j), i \neq j\right\}. \quad (5.4)$$

Clearly, both $F^n(G)$ and $F_0^n(G)$ are complex manifolds. Let $\Psi : F^n(G) \to T(G)$, $\Psi(\Phi_G(\mu), \zeta) = \Phi_G(\mu)$, be the natural projection. It is clear that Ψ is a holomorphic split submersion.

For an n-pointed Fuchsian group $\mathscr{G} = \{G; z_1, \ldots, z_n\}$ and for any $\mu \in M(G)$, set $R(\mu) = (\Phi_G(\mu), (w^\mu(z_1), \ldots, w^\mu(z_n)))$. Then R is a holomorphic mapping from $M(G)$ to $F_0^n(G)$. It is not difficult to verify that R is a surjective mapping. Since $\Phi_G : M(G) \to T(G)$ is a holomorphic split submersion, it is easy to see that R is also a holomorphic split submersion.

By definition of $T(\mathscr{G})$ and R we conclude that $\Phi_{\mathscr{G}}(\mu_1) = \Phi_{\mathscr{G}}(\mu_2)$ if and only if $R(\mu_1) = R(\mu_2)$. So R can project to an isomorphism from $T(\mathscr{G})$ onto $F_0^n(G)$, which is denoted by Q. Then $R = Q \circ \Phi_{\mathscr{G}}$. Thus Q is an injective holomorphic split submersion and consequently a biholomorphic isomorphism. We have proved

THEOREM 5.2. *$Q : T(\mathscr{G}) \to F_0^n(G)$, $Q(\Phi_{\mathscr{G}}(\mu)) = (\Phi_G(\mu), (w^\mu(z_1), \ldots, w^\mu(z_n)))$, is a biholomorphic isomorphism.*

REMARK 5.1. Without knowing the existence of a complex structure on $T(\mathscr{G})$, we conclude from the above discussion that there exists a unique complex manifold structure on the Teichmüller space $T(\mathscr{G})$ so that the natural projection $\Phi_{\mathscr{G}} : M(G) \to T(\mathscr{G})$ is a holomorphic split submersion. Actually, we have proved that there exists an isomorphism $Q : T(\mathscr{G}) \to F_0^n(G)$ such that $R = Q \circ \Phi_{\mathscr{G}}$. Since R is a holomorphic split submersion, Q is continuous with local continuous sections, which implies that Q is a homeomorphism. We can pull back the complex structure on $F_0^n(G)$ by Q to obtain a complex structure on $T(\mathscr{G})$. Then Q becomes a biholomorphic isomorphism, and $\Phi_{\mathscr{G}} = Q^{-1} \circ R$ becomes a holomorphic split submersion. This gives an alternative approach of the proof of Theorem 5.1.

An immediate consequence of Corollary 5.3 and Theorem 5.2 is

COROLLARY 5.4. *$Q \circ P : T(\Gamma) \to F_0^n(G)$ is a universal covering mapping with covering group $\chi(\mathrm{Mod}_0(\Gamma, \mathscr{G}))$.*

REMARK 5.2. Corollaries 5.3 and 5.4 imply that $P : T(\Gamma) \to T(\mathscr{G})$ and $Q \circ P : T(\Gamma) \to F_0^n(G)$ are biholomorphic isomorphisms if and only if $T(\mathscr{G}) = F_0^n(G)$ is simply connected, which happens precisely when $\mathscr{G}$ is a 1-pointed Fuchsian group $(G; z_1)$ with G torsion free. When $\mathscr{G}$ is actually a 1-pointed Fuchsian group $(G; z_1)$ with G torsion free, $F_0^1(G) = F^1(G)$ is precisely the Bers fiber space $F(G)$, and the biholomorphic isomorphism $Q \circ P$ is precisely the Bers isomorphism $B : T(\Gamma) \to F(G)$, which we have introduced in section 3. In this case, $\mathrm{Mod}_0(\Gamma, \mathscr{G})$ is the trivial group, which is precisely Epstein's result ([**Ep2**]) we stated before Lemma 3.1.

6. Complex structures on Teichmüller spaces for pointed Fuchsian groups: II.

In the last section, we showed that for any pointed Fuchsian group $\mathscr{G}$ the Teichmüller space $T(\mathscr{G})$ is a complex manifold. Now we show that for any pointed Fuchsian group $\mathscr{G}$ and any normal subgroup K of G the Teichmüller space $T(\mathscr{G}, K)$ is a complex manifold.

We begin with the isomorphism $Q : T(\mathscr{G}) \to F_0^n(G)$. Recall that each $\langle w \rangle_{\mathscr{G}} \in \mathrm{Mod}(\mathscr{G})$ acts on $T(\mathscr{G})$ by the biholomorphic isomorphism

$$\chi(\langle w \rangle_{\mathscr{G}})([w_\mu]_{\mathscr{G}}) = \big[w^*(w_\mu)\big]_{w\mathscr{G}w^{-1}}. \tag{6.1}$$

By the isomorphism Q we consider the action of $\mathrm{Mod}(\mathscr{G})$ on $F_0^n(G)$ by defining

$$\rho(\langle w\rangle_{\mathscr{G}}) = Q \circ \chi(\langle w\rangle_{\mathscr{G}}) \circ Q^{-1}. \tag{6.2}$$

First we determine the precise expression of $\rho(\langle w\rangle_{\mathscr{G}})$. We let $w^{-1}(z_i) = g_i(z_{\sigma^{-1}(i)})$, where σ is a permutation of $\{1, 2, \ldots, n\}$, $g_i \in G$ for $1 \le i \le n$. This follows from the condition $w(\boldsymbol{H}_{\mathscr{G}}) = \boldsymbol{H}_{\mathscr{G}}$. Now for any point $(\Phi_G(\mu), \zeta)$, $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)$, in $F_0^n(G)$, $\zeta_i = w^{\mu}(z_i)$. Then, with $w_\nu = w^*(w_\mu) = \alpha \circ w_\mu \circ w^{-1}$, $\rho(\langle w\rangle_{\mathscr{G}})(\Phi_G(\mu), \zeta) = (\Phi_G(\nu), (w^\nu(z_1), w^\nu(z_2), \ldots, w^\nu(z_n)))$. Now

$$\begin{aligned}
w^\nu(z_i) &= w^\nu \circ w_\nu^{-1} \circ w_\nu(z_i) = w^\nu \circ w_\nu^{-1} \circ \alpha \circ w_\mu \circ w^{-1}(z_i) \\
&= w^\nu \circ w_\nu^{-1} \circ \alpha \circ w_\mu \circ g_i(z_{\sigma^{-1}(i)}) \\
&= w^\nu \circ w_\nu^{-1} \circ \alpha \circ w_\mu \circ (w^\mu)^{-1} \circ w^\mu \circ g_i(z_{\sigma^{-1}(i)}) \\
&= w^\nu \circ w_\nu^{-1} \circ \alpha \circ w_\mu \circ (w^\mu)^{-1} \circ g_i^\mu \circ w^\mu(z_{\sigma^{-1}(i)}) \\
&= w^\nu \circ w_\nu^{-1} \circ \alpha \circ w_\mu \circ (w^\mu)^{-1} \circ g_i^\mu(\zeta_{\sigma^{-1}(i)}) \\
&= w^\nu \circ w \circ g_i \circ (w^\mu)^{-1}(\zeta_{\sigma^{-1}(i)}).
\end{aligned}$$

Consequently, when $w^{-1}(z_i) = g_i(z_{\sigma^{-1}(i)})$,

$$\rho(\langle w\rangle_{\mathscr{G}})(\Phi_G(\mu), (\zeta_i)_{1\le i\le n}) = \big(\Phi_G(\nu), (w^\nu \circ w \circ g_i \circ (w^\mu)^{-1}(\zeta_{\sigma^{-1}(i)}))_{1\le i\le n}\big). \tag{6.3}$$

We consider several special cases. Firstly, when $w(z_i) = z_i$ for all i, we have

$$\rho(\langle w\rangle_{\mathscr{G}})(\Phi_G(\mu), \zeta) = \big(\Phi_G(\nu), w^\nu \circ w \circ (w^\mu)^{-1}(\zeta)\big). \tag{6.4}$$

Secondly, when $w \in \Sigma_0(G)$, then $w_\nu = w_\mu \circ w^{-1}$, $w^\nu = w^\mu \circ w^{-1}$, so with $w^{-1}(z_i) = g_i(z_{\sigma^{-1}(i)})$,

$$\rho(\langle w\rangle_{\mathscr{G}})(\Phi_G(\mu), (\zeta_i)_{1\le i\le n}) = \big(\Phi_G(\mu), (g_i^\mu(\zeta_{\sigma^{-1}(i)}))_{1\le i\le n}\big). \tag{6.5}$$

Especially, if w satisfies the further condition that $w^{-1}(z_i) = g_i(z_i)$ for some $g_i \in G$, $1 \le i \le n$, then

$$\rho(\langle w\rangle_{\mathscr{G}})(\Phi_G(\mu), (\zeta_i)_{1\le i\le n}) = \big(\Phi_G(\mu), (g_i^\mu(\zeta_i))_{1\le i\le n}\big). \tag{6.6}$$

Now G^n may be considered as a subgroup of $\mathrm{Mod}(\mathscr{G})$. Actually, for any element $g = (g_1, g_2, \ldots, g_n) \in G^n$, choose $w \in \Sigma_0(G)$ such that $w(z_i) = g_i^{-1}(z_i)$, or equivalently, $w^{-1}(z_i) = g_i(z_i)$, $1 \le i \le n$. Then the mapping $g \to \langle w\rangle_{\mathscr{G}}$ determines a well-defined and injective homomorphism from G^n into $\mathrm{Mod}(\mathscr{G})$. The action of G^n on $F_0^n(G)$ is determined by (6.6). More generally, let P_n denote the permutation group on n letters, and $G(n)$ the twisted product of P_n and G^n, where for σ, $\tau \in P_n$ and $g = (g_1, g_2, \ldots, g_n)$,

$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ in G^n we define

$$(\sigma; g_1, g_2, \dots, g_n) \cdot (\tau; \gamma_1, \gamma_2, \dots, \gamma_n) = \big(\sigma\tau; g_1 \circ \gamma_{\sigma^{-1}(1)}, \dots, g_n \circ \gamma_{\sigma^{-1}(n)}\big).$$

G^n may be consider as a subgroup of $G(n)$ via the inclusion sending $g \in G^n$ to $(1, g)$ in $G(n)$. Now $G(n)$ may also be considered as a subgroup of $\mathrm{Mod}(\mathscr{G})$. In fact, for any $(\sigma; g_1, g_2, \dots, g_n) \in G(n)$, choose $w \in \Sigma_0(G)$ such that $w(z_i) = g^{-1}_{\sigma(i)}(z_{\sigma(i)})$, or equivalently, $w^{-1}(z_i) = g_i(z_{\sigma^{-1}(i)})$, $1 \le i \le n$. Then the mapping $(\sigma, g) \to \langle w\rangle_{\mathscr{G}}$ determines a well-defined and injective homomorphism from $G(n)$ into $\mathrm{Mod}(\mathscr{G})$. The action of $G(n)$ on $F^n_0(G)$ is determined by (6.5).

An immediate consequence of the expression (6.3) is the following result.

Proposition 6.1. *$\mathrm{Mod}(\mathscr{G})$ acts effectively on $T(\mathscr{G})$ and $F^n_0(G)$.*

Proof. We only need to show that $\mathrm{Mod}(\mathscr{G})$ acts effectively on $F^n_0(G)$. Let $w \in \Sigma(\mathscr{G})$ satisfy $\rho(\langle w\rangle_{\mathscr{G}}) = \mathrm{id}$. Then, with $w^{-1}(z_i) = g_i(z_{\sigma^{-1}(i)})$,

$$\big(\Phi_G(\nu), (w^\nu \circ w \circ g_i \circ (w^\mu)^{-1}(\zeta_{\sigma^{-1}(i)}))_{1\le i\le n}\big) = (\Phi_G(\mu), (\zeta_i)_{1\le i\le n}). \tag{6.7}$$

In particular, for any $\mu \in M(G)$, $[w_\nu] = [\alpha \circ w_\mu \circ w^{-1}] = [w_\mu]$. Letting $\mu = 0$ we get $[\alpha_0 \circ w^{-1}] = [\mathrm{id}]$. Thus $[w_\nu] = [\alpha \circ w_\mu \circ \alpha_0^{-1}] = [w_\mu]$. On the other hand, (6.7) also implies that $w^\nu \circ w \circ g_i \circ (w^\mu)^{-1}(\zeta_{\sigma^{-1}(i)}) = \zeta_i$ for $1 \le i \le n$. Letting $\mu = 0$ again we get that $\nu = 0$ and so $w \circ g_i(\zeta_{\sigma^{-1}(i)}) = \zeta_i$, $1 \le i \le n$. So $\sigma = \mathrm{id}$, $w \circ g_i = \mathrm{id}$. Consequently, $w = \alpha_0 = g_i^{-1}$ is the same element in G, that is, $\langle w\rangle_{\mathscr{G}} = \mathrm{id}$. □

Next we discuss the discontinuity on $T(\mathscr{G})$ and $F^n_0(G)$ of $G(n)$ as a subgroup of $\mathrm{Mod}(\mathscr{G})$. To do so, set

$$(\boldsymbol{H}^n_G)_0 = \big\{\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in (\boldsymbol{H}_G)^n : \zeta_i \ne g(\zeta_j) \quad \text{for all } g \in G, i \ne j\big\}. \tag{6.8}$$

$G(n)$ acts on $(\boldsymbol{H}^n_G)_0$ as a group of biholomorphic automorphisms by

$$(\sigma; g_1, g_2, \dots, g_n)(\zeta_1, \zeta_2, \dots, \zeta_n) = \big(g_1(\zeta_{\sigma^{-1}(1)}), g_2(\zeta_{\sigma^{-1}(2)}), \dots, g_n(\zeta_{\sigma^{-1}(n)})\big). \tag{6.9}$$

Clearly, the action is free and discontinuous.

Since $\Phi_G : M(G) \to T(G)$ is a holomorphic split submersion, for any point $\Phi_G(\mu) \in T(G)$ there is a holomorphic map ϕ from a neighborhood U of $\Phi_G(\mu)$ to $M(G)$ such that $\Phi_G \circ \phi = \mathrm{id}$. Consider the mapping

$$\psi(u, \zeta) = \big(u, w^{\phi(u)}(\zeta)\big), \quad (u, \zeta) \in U \times (\boldsymbol{H}^n_G)_0 \tag{6.10}$$

from $U \times (\boldsymbol{H}^n_G)_0$ to $\Psi^{-1}(U)$. Recall $\Psi : F^n_0(G) \to T(G)$ is the natural projection. Clearly, $\psi : U \times (\boldsymbol{H}^n_G)_0 \to \Psi^{-1}(U)$ is a bijective mapping. From the well known results of Ahlfors-Bers [**AB**], ψ is continuous and is holomorphic for u for fixed ζ. It is also a homeomorphism for ζ for fixed u. Consequently, ψ is a homeomorphism.

Now we consider the action of $G(n)$ on $(\boldsymbol{H}_G^n)_0$ and the one on $F_0^n(G)$ as a subgroup of $\operatorname{Mod}(\mathscr{G})$. Let $(\sigma, g) = (\sigma; g_1, g_2, \ldots, g_n) \in G(n)$ correspond to $\langle w \rangle_{\mathscr{G}} \in \operatorname{Mod}(\mathscr{G})$. For any $(u, \zeta) \in U \times (\boldsymbol{H}_G^n)_0$, from (6.5), (6.9) and (6.10) we get

$$\begin{aligned}\rho(\langle w \rangle_{\mathscr{G}})(\psi(u,\zeta)) &= \rho(\langle w \rangle_{\mathscr{G}})\big(u, w^{\phi(u)}(\zeta)\big) = \big(u, g_i^{\phi(u)}(w^{\phi(u)}(\zeta_{\sigma^{-1}(i)}))\big) \\ &= \big(u, w^{\phi(u)}(g_i(\zeta_{\sigma^{-1}(i)}))\big) = \big(u, w^{\phi(u)}((\sigma, g)(\zeta)\big) = \psi(u, (\sigma, g)(\zeta)\big).\end{aligned} \tag{6.11}$$

Since $G(n)$ acts freely and discontinuously on $(\boldsymbol{H}_G^n)_0$, we conclude from (6.11) that $G(n)$ also acts freely and discontinuously on $F_0^n(G)$. We state it as

PROPOSITION 6.2. *$G(n)$ acts freely and discontinuously on $T(\mathscr{G})$ and $F_0^n(G)$.*

REMARK 6.1. Proposition 6.2 implies that $T(\mathscr{G})/G(n)$ is a complex manifold. Let $\mathscr{G} = \{G; z_1, \ldots, z_n\}$ be an n-pointed Fuchsian group with G torsion free. Let $\pi : \boldsymbol{H} \to X = \boldsymbol{H}/G$ be the natural projection. Set $x_i = \pi(z_i)$ $(1 \leq i \leq n)$, $X_n = X - \{x_1, \ldots, x_n\}$, and $\mathscr{X} = \{X; x_1, \ldots, x_n\}$. Then, under the notations in section 3, $T(\mathscr{G})/G(n)$ is biholomorphically isomorphic to $T_3(\mathscr{X}) \simeq T(X_n)/\operatorname{Mod}_3(X_n)$, a supplement to Propositions 3.1 and 4.1.

Now we can prove

THEOREM 6.1. *For any pointed Fuchsian group $\mathscr{G} = \{G; z_1, z_2, \ldots, z_n\}$ and any normal subgroup K of G, the Teichmüller space $T(\mathscr{G}, K)$ has a unique complex manifold structure such that the natural projection $\Phi_{\mathscr{G},K} : M(G) \to T(\mathscr{G}, K)$ is a holomorphic split submersion. Furthermore, under this natural complex structure, $T(\mathscr{G}, K)$ is biholomorphically isomorphic to $T(\mathscr{G})/K^n$.*

PROOF. As a subgroup of G^n, K^n is also a subgroup of $G(n)$. Proposition 6.2 implies that K^n acts freely and discontinuously on $T(\mathscr{G})$ so that $T(\mathscr{G})/K^n$ is a complex manifold, and $T(\mathscr{G}) \to T(\mathscr{G})/K^n$ is a holomorphic covering. We denote this covering by Φ_1.

By definition, it is obvious that there is a bijective map from $T(\mathscr{G}, K)$ to $T(\mathscr{G})/K^n$, say Φ_2, such that $\Phi_1 \circ \Phi_{\mathscr{G}} = \Phi_2 \circ \Phi_{\mathscr{G},K}$. Since $\Phi_{\mathscr{G}} : M(G) \to T(\mathscr{G})$ is a holomorphic split submersion, Φ_2 is a homeomorphism. Now we can pull back the complex structure on $T(\mathscr{G})/K^n$ by Φ_2 to obtain a complex structure on $T(\mathscr{G}, K)$. Then Φ_2 becomes a biholomorphic isomorphism, and $\Phi_{\mathscr{G},K}$ becomes a holomorphic split submersion. $\square$

REMARK 6.2. We also have $T(\mathscr{G}, K) \simeq F_0^n(G)/K^n$. When $\mathscr{G} = \{G; z_1\}$ is a 1-pointed Fuchsian group, $T(\mathscr{G}) \simeq F_0^1(G)$ is the punctured fiber space $F_0(G)$ of G, and $T(\mathscr{G}, G) \simeq F_0^1(G)/G$ is the punctured Teichmüller curve $F_0(G)/G = V_0(G)$. If G is also torsion free, $T(\mathscr{G}) \simeq F_0(G)$ is the Bers fiber space $F(G)$, and $T(\mathscr{G}, G) \simeq V_0(G)$ is the Teichmüller curve $V(G)$. These spaces are very important in the moduli theory of Riemann surfaces and in the theory of holomorphic families of Riemann surfaces. They have been much investigated in the literature (see [**Be3**], [**CS**], [**EF**], [**EK1**], [**EK2**], [**Gr**], [**HS**], [**Kr2**], [**Kr3**], [**Na2**], [**Ri**], [**Sh1**], [**Sh2**], [**Zh**]).

REMARK 6.3. As usual, let T_g denote the Teichmüller space of marked closed Riemann surfaces of genus g, $g \geq 2$, and $T_{g,n}$ the Teichmüller space of marked closed Riemann surfaces of genus g with n points removed. In a fundamental paper [**Be1**], Bers introduced two other spaces of Riemann surfaces, one is $T_g^{(n)}$, the space of marked closed Riemann surfaces of genus g on each of which one has distinguished an ordered n-tuple of points, another is $\hat{T}_g^{(n)}$, the set of points of $T_g^{(n)}$ corresponding to the choice of n distinct points on a surface. He also proved, among other things, $T_{g,n}$ is the universal covering of $\hat{T}_g^{(n)}$. We have generalized these spaces and results in our discussion. Actually, let $\mathscr{G} = \{G; z_1, \ldots, z_n\}$ be an n-pointed Fuchsian group with G torsion free such that $\boldsymbol{H}/G$ is a closed Riemann surface of genus $g \geq 2$. Choose Γ as before such that $\boldsymbol{H}/\Gamma = \boldsymbol{H}_{\mathscr{G}}/G$. It is known that $T_g \cong T(G)$, $T_{g,n} \cong T(\Gamma)$. We also have $T_g^{(n)} \cong F^n(G)/G^n$, $\hat{T}_g^{(n)} \cong F_0^n(G)/G^n \cong T(\mathscr{G}, G)$. Note that we have proved in the last section that $P : T(\Gamma) \to T(\mathscr{G})$ is a universal covering mapping. From the proof of Theorem 6.1 we find that $\Phi_2^{-1} \circ \Phi_1 \circ P : T(\Gamma) \to T(\mathscr{G}, G)$ is a universal covering mapping.

7. Modular groups for pointed Fuchsian groups.

We have obtained that $T(\Gamma)$ is a universal covering of $T(\mathscr{G})$, while $T(\mathscr{G})$ is a holomorphic fiber space over $T(G)$. In this section, we will discuss the modular group $\mathrm{Mod}(\mathscr{G})$ and show how it is related to the modular groups $\mathrm{Mod}(\Gamma)$ and $\mathrm{Mod}(G)$.

To show how $\mathrm{Mod}(\mathscr{G})$ is related to $\mathrm{Mod}(\Gamma)$, we consider the set $\Sigma(\Gamma, \mathscr{G})$ consisting of all mappings $w \in \Sigma(\Gamma)$ for which there exist $w_* \in \Sigma(\mathscr{G})$ such that $h \circ w = w_* \circ h$. Recall $P : T(\Gamma) \to T(\mathscr{G})$ is the universal covering mapping.

LEMMA 7.1. $P \circ \chi(\langle w \rangle) = \chi(\langle w_* \rangle_{\mathscr{G}}) \circ P$.

PROOF. For any $\mu \in M(\Gamma)$, let $\sigma = h^*(\mu)$, $\nu \in M(\Gamma)$ be such that $w_\nu = w^*(w_\mu)$, and $\tau \in M(G)$ be such that $w_\tau = w_*^*(w_\sigma)$. Then $h_\mu \circ w_\mu = w_\sigma \circ h$, $w_\nu = \alpha \circ w_\mu \circ w^{-1}$, and $w_\tau = \beta \circ w_\sigma \circ w_*^{-1}$, where α and β are Möbius transformations of $\boldsymbol{H}$ onto itself. So

$$w_\tau \circ h = \beta \circ w_\sigma \circ w_*^{-1} \circ h = \beta \circ w_\sigma \circ h \circ w^{-1} = \beta \circ h_\mu \circ w_\mu \circ w^{-1} = \beta \circ h_\mu \circ \alpha^{-1} \circ w_\nu.$$

Lemma 5.2 implies that $\tau = h^*(\nu)$. Then,

$$\begin{aligned} P \circ \chi(\langle w \rangle)(\Phi_\Gamma(\mu)) &= P \circ \Phi_\Gamma(\nu) = \Phi_{\mathscr{G}} \circ h^*(\nu) = \Phi_{\mathscr{G}}(\tau) \\ &= \chi(\langle w_* \rangle_{\mathscr{G}})(\Phi_{\mathscr{G}}(\sigma)) = \chi(\langle w_* \rangle_{\mathscr{G}})\big(\Phi_{\mathscr{G}}(h^*(\mu))\big) \\ &= \chi(\langle w_* \rangle_{\mathscr{G}}) \circ P(\Phi_\Gamma(\mu)). \end{aligned}$$

Consequently, $P \circ \chi(\langle w \rangle) = \chi(\langle w_* \rangle_{\mathscr{G}}) \circ P$. □

Now the mapping $w \to w_*$ related by $h \circ w = w_* \circ h$ determines a surjective homomorphism from $\Sigma(\Gamma, \mathscr{G})$ to $\Sigma(\mathscr{G})$. If $\langle w \rangle = \langle \mathrm{id} \rangle$, Lemma 7.1 implies that $\chi(\langle w_* \rangle_{\mathscr{G}}) = \mathrm{id}$. Since $\mathrm{Mod}(\mathscr{G})$ acts effectively on $T(\mathscr{G})$, $\langle w_* \rangle_{\mathscr{G}} = \langle \mathrm{id} \rangle_{\mathscr{G}}$. Set $\mathrm{Mod}(\Gamma, \mathscr{G}) = (\Sigma(\Gamma, \mathscr{G})/\Sigma_0(\Gamma))/\Gamma$. Then, we have a well-defined surjective homomorphism from $\mathrm{Mod}(\Gamma, \mathscr{G})$ to $\mathrm{Mod}(\mathscr{G})$, which we denote by Θ_1, such that $P \circ \chi(\langle w \rangle) = \chi(\Theta_1(\langle w \rangle)) \circ P$ for any $\langle w \rangle \in \mathrm{Mod}(\Gamma, \mathscr{G})$.

Clearly, the kernel of the homomorphism Θ_1 is precisely the subgroup $\mathrm{Mod}_0(\Gamma, \mathscr{G})$, which plays an important role in section 5. We state these as the following theorem.

Theorem 7.1. *There is a surjective homomorphism* $\Theta_1 : \mathrm{Mod}(\Gamma, \mathscr{G}) \to \mathrm{Mod}(\mathscr{G})$ *with kernel* $\mathrm{Mod}_0(\Gamma, \mathscr{G})$ *such that for any* $\langle w \rangle \in \mathrm{Mod}(\Gamma, \mathscr{G})$, $P \circ \chi(\langle w \rangle) = \chi(\Theta_1(\langle w \rangle)) \circ P$.

We also have the following counterpart of Theorem 7.1.

Theorem 7.2. *For the surjective homomorphism* $\Theta_1 : \mathrm{Mod}(\Gamma, \mathscr{G}) \to \mathrm{Mod}(\mathscr{G})$ *and for any* $\langle w \rangle \in \mathrm{Mod}(\Gamma, \mathscr{G})$, $Q \circ P \circ \chi(\langle w \rangle) = \rho(\Theta_1(\langle w \rangle)) \circ Q \circ P$.

Remark 7.1. $\Theta_1 : \mathrm{Mod}(\Gamma, \mathscr{G}) \to \mathrm{Mod}(\mathscr{G})$ is injective if and only if $\mathrm{Mod}_0(\Gamma, \mathscr{G})$ is the trivial group, which happens precisely when $\mathscr{G}$ is a 1-pointed Fuchsian group $(G; z_1)$ with G torsion free. When $\mathscr{G}$ is actually a 1-pointed Fuchsian group $(G; z_1)$ with G torsion free, $F_0^1(G) = F^1(G) = F(G)$, $Q \circ P = B : T(\Gamma) \to F(G)$ is the Bers isomorphism. In this case, $\mathrm{Mod}(\Gamma, \mathscr{G})$ is the group $\mathrm{Mod}(\Gamma, z_1)$ which we introduced in section 3. Now there is an isomorphism $I' : \mathrm{mod}(G) \to \mathrm{Mod}(\mathscr{G})$ such that $\Theta_1^{-1} \circ I'$ is the isomorphism $I : \mathrm{mod}(G) \to \mathrm{Mod}(\Gamma, z_1)$, which we also introduced in section 3.

We proceed to discuss the relation between $\mathrm{Mod}(\mathscr{G})$ and $\mathrm{Mod}(G)$. There is a natural mapping from $\mathrm{Mod}(\mathscr{G})$ to $\mathrm{Mod}(G)$ sending $\langle w \rangle_{\mathscr{G}}$ to $\langle w \rangle$. The mapping is surjective, since for any $w \in \Sigma(G)$ there is some $\tilde{w} \in \Sigma(G)$ such that $[\tilde{w}] = [w]$, and $w(z_i) = z_i$, $1 \le i \le n$. Consequently, we have a surjective homeomorphism $\Theta_2 : \mathrm{Mod}(\mathscr{G}) \to \mathrm{Mod}(G)$. Clearly, the kernel of Θ_2 is precisely $G(n)$ as a subgroup of $\mathrm{Mod}(\mathscr{G})$.

We consider the natural projection $\Phi : T(\mathscr{G}) \to T(G)$. For any $\Phi_{\mathscr{G}}(\mu) \in T(\mathscr{G})$, with $w_\nu = w^*(w_\mu)$,

$$\Phi \circ \chi(\langle w \rangle_{\mathscr{G}})(\Phi_{\mathscr{G}}(\mu)) = \Phi \circ \Phi_{\mathscr{G}}(\nu) = \Phi_G(\nu) = \chi(\langle w \rangle) \circ \Phi_G(\mu) = \chi(\langle w \rangle) \circ \Phi \circ \Phi_{\mathscr{G}}(\mu).$$

Thus, $\Phi \circ \chi(\langle w \rangle_{\mathscr{G}}) = \chi(\langle w \rangle) \circ \Phi$. We have proved

Theorem 7.3. *There is a surjective homomorphism* $\Theta_2 : \mathrm{Mod}(\mathscr{G}) \to \mathrm{Mod}(G)$ *with kernel* $G(n)$ *such that for any* $\langle w \rangle_{\mathscr{G}} \in \mathrm{Mod}(\mathscr{G})$, $\Phi \circ \chi(\langle w \rangle_{\mathscr{G}}) = \chi(\Theta_2(\langle w \rangle_{\mathscr{G}})) \circ \Phi$.

Similarly, we can prove the following result. We omit the details here.

Theorem 7.4. *There exists a surjective homomorphism* $\Theta_3 : (\Sigma(\mathscr{G}, K)/\Sigma_0(\mathscr{G}))/G \to \mathrm{Mod}(\mathscr{G}, K)$ *with kernel* K^n *such that* $\Phi_2^{-1} \circ \Phi_1 \circ \chi(\langle w \rangle_{\mathscr{G}}) = \chi(\Theta_3(\langle w \rangle_{\mathscr{G}})) \circ \Phi_2^{-1} \circ \Phi_1$ *for any* $\langle w \rangle_{\mathscr{G}} \in (\Sigma(\mathscr{G}, K)/\Sigma_0(\mathscr{G}))/G$. *In particular, when* $K = G$, Θ_3 *is a homomorphism from* $\mathrm{Mod}(\mathscr{G})$ *to* $\mathrm{Mod}(\mathscr{G}, G)$ *with kernel* G^n.

8. Kobayashi metric on $T(\mathscr{G}, K)$.

In this last section, we point out that the Kobayashi metric and the Teichmüller metric coincide on the Teichmüller space $T(\mathscr{G}, K)$, as it should. We give a complete proof for completeness.

Theorem 8.1. *The Kobayashi metric and the Teichmüller metric coincide on the Teichmüller space* $T(\mathscr{G}, K)$.

PROOF. Let $k_{M(G)}$, $k_{T(G)}$ and $k_{\mathscr{G},K}$ denote the Kobayashi distance on $M(G)$, $T(G)$ and $T(\mathscr{G},K)$, respectively. Then $k_{M(G)} = \sigma_G$, $k_{T(G)} = \tau_G$. Since $\Phi_{\mathscr{G},K} : M(G) \to T(\mathscr{G},K)$ is holomorphic, for any σ, $\tau \in M(G)$ we have

$$k_{\mathscr{G},K}([\sigma]_{\mathscr{G},K}, [\tau]_{\mathscr{G},K}) \le k_{M(G)}(\sigma,\tau) = \sigma_G(\sigma,\tau),$$

which implies

$$\begin{aligned} k_{\mathscr{G},K}([\sigma]_{\mathscr{G},K}, [\tau]_{\mathscr{G},K}) &\le \inf\left\{\sigma_G(\sigma',\tau') : [\sigma']_{\mathscr{G},K} = [\sigma]_{\mathscr{G},K}, [\tau']_{\mathscr{G},K} = [\tau]_{\mathscr{G},K}\right\} \\ &= \tau_{\mathscr{G},K}([\sigma]_{\mathscr{G},K}, [\tau]_{\mathscr{G},K}). \end{aligned}$$

On the other hand, since $\hat{\Phi} \doteq \Phi_2^{-1} \circ \Phi_1 \circ P : T(\Gamma) \to T(\mathscr{G},K)$ is a universal covering map, and $\hat{\Phi} \circ \Phi_\Gamma = \Phi_{\mathscr{G},K} \circ h^*$, we obtain

$$\begin{aligned} &k_{\mathscr{G},K}([\sigma]_{\mathscr{G},K}, [\tau]_{\mathscr{G},K}) = \inf\left\{k_{T(\Gamma)}([\mu],[\nu]) : \hat{\Phi}([\mu]) = [\sigma]_{\mathscr{G},K},\ \hat{\Phi}([\nu]) = [\tau]_{\mathscr{G},K}\right\} \\ &\quad = \inf\left\{\tau_\Gamma([\mu],[\nu]) : [h^*\mu]_{\mathscr{G},K} = [\sigma]_{\mathscr{G},K},\ [h^*\nu]_{\mathscr{G},K} = [\tau]_{\mathscr{G},K}\right\} \\ &\quad = \inf\left\{\ \inf\{\sigma_\Gamma([\mu'],[\nu']) : [\mu'] = [\mu], [\nu'] = [\nu]\} : [h^*\mu]_{\mathscr{G},K} = [\sigma]_{\mathscr{G},K},\ [h^*\nu]_{\mathscr{G},K} = [\tau]_{\mathscr{G},K}\right\} \\ &\quad \ge \inf\left\{\sigma_\Gamma([\mu'],[\nu']) : [h^*\mu']_{\mathscr{G},K} = [\sigma]_{\mathscr{G},K},\ [h^*\nu']_{\mathscr{G},K} = [\tau]_{\mathscr{G},K}\right\} \\ &\quad = \inf\left\{\sigma_G([h^*\mu'],[h^*\nu']) : [h^*\mu']_{\mathscr{G},K} = [\sigma]_{\mathscr{G},K},\ [h^*\nu']_{\mathscr{G},K} = [\tau]_{\mathscr{G},K}\right\} \\ &\quad = \tau_{\mathscr{G},K}([\sigma]_{\mathscr{G},K}, [\tau]_{\mathscr{G},K}). \end{aligned}$$

Consequently, $k_{\mathscr{G},K}([\sigma]_{\mathscr{G},K}, [\tau]_{\mathscr{G},K}) = \tau_{\mathscr{G},K}([\sigma]_{\mathscr{G},K}, [\tau]_{\mathscr{G},K})$. □

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