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Jørgensen's inequality for classical Schottky groups of real type II

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Abstract. In this paper we consider Jørgensen's inequality for classical Schottky groups of real types, that is, the third, sixth and eighth types. The infimum of Jørgensen's numbers for the groups of the third, sixth and eighth types are 4, 16 and 16, respectively.

0. Introduction.

Jørgensen's inequality gives a necessary condition for a non-elementary Möbius transformation group $G = \langle A_1, A_2 \rangle$ to be discrete (Jørgensen [**2**]): If $G = \langle A_1, A_2 \rangle$ is a non-elementary discrete group, then

$$J(\langle A_1, A_2 \rangle) := |\mathrm{tr}^2(A_1) - 4| + |\mathrm{tr}(A_1 A_2 A_1^{-1} A_2^{-1}) - 2| \geq 1,$$

where tr is the trace. The lower bound is the best possible. We call $J(\langle A_1, A_2 \rangle)$ *Jørgensen's number* for the marked two-generator group $G = \langle A_1, A_2 \rangle$.

Here we will consider Jørgensen's numbers for classical Schottky groups of real type of genus two. In Sato [**4**] we classified the groups into eight types (see §1). In [**7**] we announced that Jørgensen's inequalities were given for all such groups of eight types and that the lower bounds were all best possible (see Appendices). Gilman [**1**] and Sato [**6**] gave Jørgensen's inequalities and the best lower bounds for the groups of the first and fourth types, that is, for Fuchsian Schottky groups. Jørgensen's inequalities for the groups of the second, fifth and seventh types were considered in [**9**]. Here we will give Jørgensen's inequalities and the lower bounds for the groups of the third, sixth and eighth types.

In §1 we will state notation and terminology. In §2 we will state Schottky modular groups acting on the Schottky spaces of the third and sixth types and represent their fundamental regions for the Schottky modular groups which are given in Sato [**8**]. In §3 we will state the main results in this paper. In §4 we will list properties of Jørgensen's numbers in a series of lemmas, which play

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important roles in the proofs of the main theorems. In §5 we will give a proof of Theorem 1 on Jørgensen's numbers for classical Schottky groups of the third type. In §6 we will prove Theorems 2 and 3 on Jørgensen's numbers for classical Schottky groups of the sixth and eighth types. In §7 we will give some examples which guarantee that all of the lower bounds in the inequalities in Theorems 1, 2 and 3 are the best possible. In §8 we will make a summary of Jørgensen's inequalities for eight kinds of classical Schottky groups of real type.

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1. Notation and terminology.

We denote by Möb the group of all Möbius transformations. We say two marked subgroups $G = \langle A_1, \dots, A_g \rangle$ and $\hat{G} = \langle \hat{A}_1, \dots, \hat{A}_g \rangle$ of Möb to be *equivalent* if there exists a Möbius transformation T such that $\hat{A}_j = TA_jT^{-1}$ for $j = 1, 2 \dots, g$. The *Schottky space* (resp. the *classical Schottky space*) of genus g, denoted by $\boldsymbol{S}_g$ (resp. $\boldsymbol{S}_g^0$), is the set of all equivalence classes of marked Schottky groups (resp. marked classical Schottky groups) of genus $g \geq 1$.

We denote by $\boldsymbol{M}_2$ the set of all equivalence classes $[\langle A_1, A_2 \rangle]$ of marked groups $\langle A_1, A_2 \rangle$ generated by loxodromic transformations A_1 and A_2 whose fixed points are all distinct. Let $[\langle A_1, A_2 \rangle] \in \boldsymbol{M}_2$. For $j = 1, 2$, let λ_j ($|\lambda_j| > 1$), p_j and p_{2+j} be the multipliers, the repelling and the attracting fixed points of A_j, respectively. We define t_j by setting $t_j = 1/\lambda_j$. Thus $t_j \in D^* = \{z \mid 0 < |z| < 1\}$. We define ρ by setting $\rho = 1/(p_1, p_3, p_2, p_4)$, where (p_1, p_3, p_2, p_4) is the cross-ratio of p_1, p_3, p_2 and p_4, that is, $(p_1, p_3, p_2, p_4) = ((p_1 - p_2)(p_3 - p_4))/((p_1 - p_4) \cdot (p_3 - p_2))$. Thus $\rho \in \boldsymbol{C} - \{0, 1\}$. We can define a mapping α of the space $\boldsymbol{M}_2$ into $(D^*)^2 \times (\boldsymbol{C} - \{0, 1\})$ by setting $\alpha([\langle A_1, A_2 \rangle]) = (t_1, t_2, \rho)$. Then we say that $[\langle A_1, A_2 \rangle]$ or $\langle A_1, A_2 \rangle$ represents (t_1, t_2, ρ) and (t_1, t_2, ρ) corresponds to $[\langle A_1, A_2 \rangle]$ or $\langle A_1, A_2 \rangle$. Conversely, λ_1, λ_2 and p_4 are uniquely determined from a given point $\tau = (t_1, t_2, \rho) \in (D^*)^2 \times (\boldsymbol{C} - \{0, 1\})$ under the normalization condition $p_1 = 0$, $p_3 = \infty$ and $p_2 = 1$; we define λ_j $(j = 1, 2)$ and p_4 by setting $\lambda_j = 1/t_j$ and $p_4 = \rho$, respectively. We determine $A_1(z), A_2(z) \in$ Möb from τ as follows: the multiplier, the repelling and the attracting fixed points of $A_j(z)$ are λ_j, p_j and p_{2+j}, respectively. Thus we obtain a mapping β of $(D^*)^2 \times (\boldsymbol{C} - \{0, 1\})$ into $\boldsymbol{M}_2$ by setting $\beta(\tau) = [\langle A_1(z), A_2(z) \rangle]$. Then we note that $\beta\alpha = \alpha\beta = id$. Therefore we identify $\boldsymbol{M}_2$ with $\alpha(\boldsymbol{M}_2)$. Similarly we can define the mapping α^* of $\boldsymbol{S}_2$ or $\boldsymbol{S}_2^0$ into $(D^*)^2 \times (\boldsymbol{C} - \{0, 1\})$ by restricting α to this space, and identify $\boldsymbol{S}_2$ (resp. $\boldsymbol{S}_2^0$) with $\alpha^*(\boldsymbol{S}_2)$ (resp. $\alpha^*(\boldsymbol{S}_2^0)$). From now on we denote $\alpha(\boldsymbol{M}_2), \alpha^*(\boldsymbol{S}_2)$ and $\alpha^*(\boldsymbol{S}_2^0)$ by $\boldsymbol{M}_2$, $\boldsymbol{S}_2$ and $\boldsymbol{S}_2^0$, respectively.

We call $G = \langle A_1, A_2 \rangle$ a marked group of *real type* if $(t_1, t_2, \rho) \in \boldsymbol{R}^3 \cap \boldsymbol{M}_2$, that is, t_1, t_2 and ρ are all real numbers in $\boldsymbol{M}_2$, where (t_1, t_2, ρ) corresponds to

$G = \langle A_1, A_2 \rangle$. Then there are eight kinds of marked groups of real type as follows.

DEFINITION 1.1 (cf. [**4**]). Let $G = \langle A_1, A_2 \rangle$ be a marked two-generator group in $\boldsymbol{M}_2$ and let (t_1, t_2, ρ) correspond to $\langle A_1, A_2 \rangle$.

(1) G is of the first type (Type I) if $t_1 > 0$, $t_2 > 0$, $\rho > 0$.
(2) G is of the second type (Type II) if $t_1 > 0$, $t_2 < 0$, $\rho > 0$.
(3) G is of the third type (Type III) if $t_1 > 0$, $t_2 < 0$, $\rho < 0$.
(4) G is of the fourth type (Type IV) if $t_1 > 0$, $t_2 > 0$, $\rho < 0$.
(5) G is of the fifth type (Type V) if $t_1 < 0$, $t_2 > 0$, $\rho > 0$.
(6) G is of the sixth type (Type VI) if $t_1 < 0$, $t_2 < 0$, $\rho > 0$.
(7) G is of the seventh type (Type VII) if $t_1 < 0$, $t_2 < 0$, $\rho < 0$.
(8) G is of the eighth type (type VIII) if $t_1 < 0$, $t_2 > 0$, $\rho < 0$.

For each $k = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VIII}$, we call the set of all equivalence classes of marked Schottky groups (resp. marked classical Schottky groups) of Type k the *real Schottky space* (resp. the *real classical Schottky space*) *of Type k*, and denote them by $R_k\boldsymbol{S}_2$ (resp. $R_k\boldsymbol{S}_2^0$).

2. Fundamental regions and Schottky modular groups.

In this section we will state Schottky modular groups acting on the real Schottky spaces of the third and sixth types and represent their fundamental regions for the Schottky modular groups, which are obtained in [**8**].

THEOREM A (Neumann [**3**]). *The group Φ_2 of automorphisms of a marked two-generator group $G = \langle A_1, A_2 \rangle$ has the following presentation*:

$$\Phi_2 = \langle N_1, N_2, N_3 \,|\, (N_2N_1N_2N_3)^2 = 1,$$

$$N_3^{-1}N_2N_3N_2N_1N_3N_1N_2N_1 = 1, N_1N_3N_1N_3 = N_3N_1N_3N_1 \rangle,$$

where $N_1 : (A_1, A_2) \mapsto (A_1, A_2^{-1})$, $N_2 : (A_1, A_2) \mapsto (A_2, A_1)$ *and* $N_3 : (A_1, A_2) \mapsto (A_1, A_1A_2)$.

We call the mappings N_1, N_2 and N_3 the *Nielsen transformations*.

Let (t_1, t_2, ρ) be the point corresponding to a marked Schottky group $G = \langle A_1, A_2 \rangle$. Let $(t_1(j), t_2(j), \rho(j))$ be the images of (t_1, t_2, ρ) under the Nielsen transformations N_j $(j = 1, 2, 3)$. We set $X = \rho - t_2 - \rho t_1 t_2 + t_1$ and $Y = \rho - t_2 + \rho t_1 t_2 - t_1$. Then by straightforward calculations, we have the following.

LEMMA 2.1 (Sato [**4**, Lemma 2.1]). (i) $t_1(1) = t_1$, $t_2(1) = t_2$ *and* $\rho(1) = 1/\rho$. (ii) $t_1(2) = t_2$, $t_2(2) = t_1$ *and* $\rho(2) = \rho$. (iii) $t_1(3) = t_1$, $t_2(3) + (1/t_2(3)) = Y^2/t_1t_2(\rho - 1)^2 - 2$ *and* $\rho(3) + (1/\rho(3)) = X^2/(t_1\rho(1 - t_2)^2) - 2$.

DEFINITION 2.1. Let Φ_2 be the group of automorphisms of $G = \langle A_1, A_2 \rangle$.

Let $\phi_1, \phi_2 \in \Phi_2$. We say ϕ_1 and ϕ_2 are *equivalent* if $\phi_1(G)$ is equivalent to $\phi_2(G)$ and we denote it by $\phi_1 \sim \phi_2$. We denote by $[\phi]$ the equivalence class of ϕ in Φ_2.

REMARKS. (1) We can regard N_j $(j = 1, 2, 3)$ and so $\phi \in \Phi_2$ as automorphisms of the Schottky space of genus two.

(2) From the above (1) and Definition 2.1, we have the following: If $\langle A_1, A_2 \rangle \sim \langle \hat{A}_1, \hat{A}_2 \rangle$ and $\phi_1 \sim \phi_2$ $(\phi_1, \phi_2 \in \Phi_2)$, then $\phi_1(\langle A_1, A_2 \rangle) \sim \phi_2(\langle \hat{A}_1, \hat{A}_2 \rangle)$.

The *Schottky modular group* of genus two, which is denoted by $\mathrm{Mod}(\boldsymbol{S}_2)$, is the set of all equivalence classes of orientation preserving automorphisms of the Schottky space of genus two. We denote by $\mathrm{Mod}(R_k\boldsymbol{S}_2^0)$ the restriction of $\mathrm{Mod}(\boldsymbol{S}_2)$ to the classical Schottky space of real type $R_k\boldsymbol{S}_2^0$ for $k = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VIII}$. We denote by $F_k(\mathrm{Mod}(\boldsymbol{S}_2^0))$ fundamental regions in $R_k\boldsymbol{S}_2^0$ for $\mathrm{Mod}(R_k\boldsymbol{S}_2^0)$.

We define functions $t_2 = t_2(t_1, \rho : k)$ $(k = \mathrm{III}, \mathrm{VI})$ as follows:

(i) $t_2(t_1, \rho : \mathrm{III})$ is t_2 satisfying the equation

$$(1 + t_1)((-\rho)^{1/2} + 1/(-\rho)^{1/2}) = (1 - t_1)((-t_2)^{1/2} + 1/(-t_2)^{1/2}) \quad (0 < t_1 < 1).$$

(ii) $t_2(t_1, \rho : \mathrm{VI}) = -(1 + t_1\rho^{1/2})/(\rho^{1/2} + t_1)$ $(1 < \rho < 1/t_1^2,\ -1 < t_1 < 0)$.

PROPOSITION 2.1 (Sato [**5**]).

(i)

$$F_{\mathrm{III}}(\mathrm{Mod}(\boldsymbol{S}_2^0)) = \{(t_1, t_2, \rho) \in R_{\mathrm{III}}\boldsymbol{S}_2^0 \mid \rho^*(T_1, T_2) < \rho < -1,$$
$$t_2(t_1, \rho : \mathrm{III}) < t_2 < 0,\ 0 < t_1 < 1\},$$

where $\rho^*(T_1, T_2) = \{4 - T_1T_2 + ((4 - T_1^2)(4 - T_2^2))^{1/2}\}/2(T_2 - T_1)$, $T_1 = t_1 + 1/t_1$, $T_2 = t_2 + 1/t_2$.

(ii)

$$F_{\mathrm{VI}}(\mathrm{Mod}(\boldsymbol{S}_2^0)) = \{(t_1, t_2, \rho) \in R_{\mathrm{VI}}\boldsymbol{S}_2^0 \mid t_2(t_1, \rho : \mathrm{VI}) < t_2 < 0,$$
$$1 < \rho < 1/t_1^2,\ t_2 < t_1,\ -1 < t_1 < 0\}.$$

PROPOSITION 2.2 (Sato [**8**]). *The modular group* $\mathrm{Mod}(R_{\mathrm{VI}}\boldsymbol{S}_2^0)$ *is generated by* $[N_3^2]$ *and* $[N_1N_2]$, *where* N_1, N_2 *and* N_3 *are the Nielsen transformations defined in Theorem A*.

PROPOSITION 2.3 (Sato [**8**, Theorem 1]). *Let* N_3 *be the Nielsen transformation defined in Theorem A. Then* $N_3(R_{\mathrm{VI}}\boldsymbol{S}_2^0) = R_{\mathrm{VIII}}\boldsymbol{S}_2^0$ *and* $N_3(R_{\mathrm{VIII}}\boldsymbol{S}_2^0) = R_{\mathrm{VI}}\boldsymbol{S}_2^0$.

3. Theorems.

In this section the main theorems in this paper will be stated. Let $G = \langle A_1, A_2 \rangle$ be a marked two-generator group generated by Möbius transformations

A_1 and A_2. We remember that the number

$$J(\langle A_1, A_2\rangle) := |\mathrm{tr}^2(A_1) - 4| + |\mathrm{tr}(A_1 A_2 A_1^{-1} A_2^{-1}) - 2|$$

is called *Jørgensen's number* for $\langle A_1, A_2\rangle$.

THEOREM 1. *If* $G = \langle A_1, A_2\rangle \in R_{\mathrm{III}}\boldsymbol{S}_2^0$, *then* $J(\langle A_1, A_2\rangle) > 4$. *The lower bound is the best possible.*

THEOREM 2. *If* $G = \langle A_1, A_2\rangle \in R_{\mathrm{VI}}\boldsymbol{S}_2^0$, *then* $J(\langle A_1, A_2\rangle) > 16$. *The lower bound is the best possible.*

THEOREM 3. *If* $G = \langle A_1, A_2\rangle \in R_{\mathrm{VIII}}\boldsymbol{S}_2^0$, *then* $J(\langle A_1, A_2\rangle) > 16$. *The lower bound is the best possible.*

4. Lemmas.

In this section we will give some lemmas which are necessary to prove the theorems stated in the previous section. We introduce two regions as follows:

$$M_{\mathrm{III}} := \{\tau = (t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid t_2(t_1, \rho : \mathrm{III}) < t_2 < 0, 0 < t_1 < 1\},$$

$$M_{\mathrm{VI}} := \{\tau = (t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid t_2(t_1, \rho : \mathrm{VI}) < t_2 < 0, 1 < \rho < 1/t_1^2, -1 < t_1 < 0\},$$

where $t_2(t_1, \rho : k)$ $(k = \mathrm{III}, \mathrm{VI})$ are the functions defined in §2.

We can easily see the following lemma by Theorem 3 in [**8**] and we omit the proof.

LEMMA 4.1. *For each* $k = \mathrm{III}, \mathrm{VI}$

$$F_k(\mathrm{Mod}(\boldsymbol{S}_2^0)) \subseteq M_k \subseteq R_k\boldsymbol{S}_2^0.$$

THEOREM B (Jørgensen [**2**]). *Suppose that the Möbius transformations* A_1 *and* A_2 *generate a non-elementary discrete group* $G = \langle A_1, A_2\rangle$. *Then*

$$J(\langle A_1, A_2\rangle) := |\mathrm{tr}^2(A_1) - 4| + |\mathrm{tr}(A_1 A_2 A_1^{-1} A_2^{-1}) - 2| \geq 1.$$

The lower bound is the best possible.

Let $\tau = (t_1, t_2, \rho)$ correspond to $G = \langle A_1, A_2\rangle$. Since $|\mathrm{tr}^2(A_1) - 4| = |1 - t_1|^2/|t_1|$ and $|\mathrm{tr}(A_1 A_2 A_1^{-1} A_2^{-1}) - 2| = (|1 - t_1|^2 |1 - t_2|^2 |\rho|)/(|t_1||t_2||\rho - 1|^2)$, we have the following proposition from Theorem B.

PROPOSITION 4.1. *Let* $G = \langle A_1, A_2\rangle$ *be a non-elementary discrete group and let* $\tau = (t_1, t_2, \rho)$ *be the point corresponding to* $\langle A_1, A_2\rangle$. *Then*

$$J(\langle A_1, A_2\rangle) = \frac{|1 - t_1|^2}{|t_1|} + \frac{|1 - t_1|^2 |1 - t_2|^2 |\rho|}{|t_1||t_2||\rho - 1|^2} \geq 1.$$

Let $\tau = (t_1, t_2, \rho)$ correspond to $G = \langle A_1, A_2 \rangle$. We set

$$J_1(\langle A_1, A_2 \rangle) := |\mathrm{tr}^2(A_1) - 4| = |1 - t_1|^2/|t_1|,$$

and

$$J_2(\langle A_1, A_2 \rangle) := |\mathrm{tr}(A_1 A_2 A_1^{-1} A_2^{-1}) - 2| = \frac{|1 - t_1|^2 |1 - t_2|^2 |\rho|}{|t_1||t_2||\rho - 1|^2}.$$

Lemma 4.2 ([**9**, Lemma 5.2]). *$J_2(\langle A_1, A_2 \rangle)$ is Φ_2-invariant, that is, $J_2(\phi(\langle A_1, A_2 \rangle)) = J_2(\langle A_1, A_2 \rangle)$ for any $\phi \in \Phi_2$.*

We easily see the following lemma and so we omit the proof.

Lemma 4.3. *$J_1(\langle A_1, A_2 \rangle)$ and $J(\langle A_1, A_2 \rangle)$ are invariant under the Nielsen transformations N_1 and N_3, that is,*

(i) $J_1(N_1(\langle A_1, A_2 \rangle)) = J_1(\langle A_1, A_2 \rangle)$ *and* $J_1(N_3(\langle A_1, A_2 \rangle)) = J_1(\langle A_1, A_2 \rangle)$.

(ii) $J(N_1(\langle A_1, A_2 \rangle)) = J(\langle A_1, A_2 \rangle)$ *and* $J(N_3(\langle A_1, A_2 \rangle)) = J(\langle A_1, A_2 \rangle)$.

The following lemma follows from Lemma 4.1.

Lemma 4.4. *For $k = \mathrm{III}, \mathrm{VI}$*

$$\begin{aligned}
&\inf\{J(\langle A_1, A_2 \rangle) \mid \langle A_1, A_2 \rangle \in F_k(\mathrm{Mod}\, \boldsymbol{S}_2^0)\} \\
&\quad \geq \inf\{J(\langle A_1, A_2 \rangle) \mid \langle A_1, A_2 \rangle \in M_k\} \\
&\quad \geq \inf\{J(\langle A_1, A_2 \rangle) \mid \langle A_1, A_2 \rangle \in R_k \boldsymbol{S}_2^0\}.
\end{aligned}$$

Lemma 4.5. *For each $k = \mathrm{III}, \mathrm{VI}$, if $\tau = (t_1, t_2, \rho) \in M_k$ and $\tau_0 = (t_1, t_{20}, \rho) \in \partial M_k$ $(t_{20} \neq 0)$, then $J(\langle A_{10}, A_{20} \rangle) < J(\langle A_1, A_2 \rangle)$ and $J_2(\langle A_{10}, A_{20} \rangle) < J_2(\langle A_1, A_2 \rangle)$ where $\langle A_{10}, A_{20} \rangle$ and $\langle A_1, A_2 \rangle$ represent τ_0 and τ, respectively, and M_k are the regions defined in the beginning of this section.*

Proof. Since the function $(1 - t_2)^2/t_2$ is negative and monotonously decreasing in the interval $-1 < t_2 < 0$, we have the desired result by the definition of the space M_k. $\square$

5. Proof of Theorem 1.

In this section we will prove Theorem 1. Let $G = \langle A_1, A_2 \rangle$ be a marked two-generator group and let $\tau = (t_1, t_2, \rho)$ correspond to $G = \langle A_1, A_2 \rangle$.

Lemma 5.1. *If $\tau = (t_1, t_2, \rho)$ is a point on the boundary surface of M_{III} defined by the equation*

$$(1+t_1)((-\rho)^{1/2}+1/(-\rho)^{1/2})=(1-t_1)((-t_2)^{1/2}+1/(-t_2)^{1/2}) \qquad (*)$$
$$(\rho<0,\ 0<t_1<1),$$

then $J_2(\langle A_1,A_2\rangle)=(1+t_1)^2/t_1>4$, *where* $\langle A_1,A_2\rangle$ *represents* τ.

Proof. By substituting the identity $(*)$ for $J_2(\langle A_1,A_2\rangle)=\{|1-t_1|^2|1-t_2|^2|\rho|\}/\{|t_1||t_2||\rho-1|^2\}$, we have the desired result $J_2(\langle A_1,A_2\rangle)=(1+t_1)^2/t_1>4$. □

Proof of Theorem 1. Let $G=\langle A_1,A_2\rangle\in R_{\mathrm{III}}\boldsymbol{S}_2^0$ and let $\tau=(t_1,t_2,\rho)$ correspond to $\langle A_1,A_2\rangle$. By Proposition 2 (i) there exists $\phi\in\mathrm{Mod}(R_{\mathrm{III}}\boldsymbol{S}_2^0)$ such that $\phi(\tau)\in F_{\mathrm{III}}(\mathrm{Mod}(\boldsymbol{S}_2^0))$. By Lemma 4.1 we have $\phi(\tau)\in M_{\mathrm{III}}$. Let $\langle B_1,B_2\rangle$ represent $\phi(\tau)$. By Lemma 4.2 we have

$$J(\langle A_1,A_2\rangle)=J_1(\langle A_1,A_2\rangle)+J_2(\langle A_1,A_2\rangle)\geq J_2(\langle A_1,A_2\rangle)=J_2(\langle B_1,B_2\rangle).$$

By Lemmas 4.5 and 5.1 we have

$$J_2(\langle B_1,B_2\rangle)>4.$$

It is seen by Example 1 in §7 that the lower bound is the best possible. □

6. Proofs of Theorems 2 and 3.

In this section we will prove Theorems 2 and 3. Let N_1,N_2 and N_3 be the Nielsen transformations defined in §2. We set $\varphi=N_3^2$ and $\chi=N_1N_2$. We set $M_{\mathrm{VI}}(1):=M_{\mathrm{VI}}$ and $M_{\mathrm{VI}}(-1):=N_1(M_{\mathrm{VI}})$. We remember that $\phi_1\sim\phi_2$ ($\phi_1,\phi_2\in\Phi_2$) means ϕ_1 is equivalent to ϕ_2. We easily see the following Lemmas 6.1, 6.2 and 6.3 (cf. [**8**]). We omit the proofs.

Lemma 6.1. *Let* N_j $(j=1,2,3)$ *be the Nielsen transformations defined in* §2 *and let* $\varphi=N_3^2$ *and* $\chi=N_1N_2$. *Then*

(i) $N_1^2=1$, $N_2^2=1$, $N_1N_2\sim N_2N_1$ *and* $N_1N_2N_1N_2\sim 1$.
(ii)

$$\chi^n\sim\begin{cases}N_1N_2 & \text{if } n \text{ is odd.}\\ 1 & \text{if } n \text{ is even.}\end{cases}$$

(iii) $\chi^{-1}\sim\chi$.
(iv) $\chi N_1=N_1\chi^{-1}$.
(v) $\varphi N_1\sim N_1\varphi^{-1}$.

Lemma 6.2. *Let* φ *and* χ *be the transformations in Lemma* 6.1 *and let* $M_{\mathrm{VI}}(1)$ *and* $M_{\mathrm{VI}}(-1)$ *be the regions defined in the above. Then*

(i) $\varphi\chi(M_{\mathrm{VI}}(1))=M_{\mathrm{VI}}(1)$.

(ii) $\chi\varphi^{-1}(M_{\mathrm{VI}}(1)) = M_{\mathrm{VI}}(1)$.

(iii) $N_2(M_{\mathrm{VI}}(1)) = M_{\mathrm{VI}}(1)$.

(iv) $\varphi N_1(M_{\mathrm{VI}}(1)) = M_{\mathrm{VI}}(1)$.

(v)

$$\chi^n(M_{\mathrm{VI}}(1)) = \begin{cases} M_{\mathrm{VI}}(-1) & \textit{if } n \textit{ is odd.} \\ M_{\mathrm{VI}}(1) & \textit{if } n \textit{ is even.} \end{cases}$$

We can easily see the following lemma and we omit the proof.

LEMMA 6.3. *Let* $\phi, \psi \in \Phi_2$. *If* $\phi \sim \psi$, *then* $J(\phi(\langle A_1, A_2\rangle)) = J(\psi(\langle A_1, A_2\rangle))$.

By Lemmas 4.2 and 4.3 we have the following lemma.

LEMMA 6.4. *Let* $G = \langle A_1, A_2\rangle \in R_{\mathrm{VI}}\boldsymbol{S}_2^0$ *be a marked two-generator group and* $G^* = \phi(G) = \langle A_1^*, A_2^*\rangle$ $(\phi \in \mathrm{Mod}(R_{\mathrm{VI}}\boldsymbol{S}_2^0))$. *Let* $(t_1(G), t_2(G), \rho(G))$ *and* $(t_1(G^*), t_2(G^*), \rho(G^*))$ *correspond to* G *and* G^*, *respectively. Then the following four inequalities are equivalent.*

(i) $J(\langle A_1^*, A_2^*\rangle) > J(\langle A_1, A_2\rangle)$.

(ii) $J_1(\langle A_1^*, A_2^*\rangle) > J_1(\langle A_1, A_2\rangle)$.

(iii) $\mathrm{tr}^2(A_1^*) < \mathrm{tr}^2(A_1)$.

(iv) $-t_1(G^*) < -t_1(G)$.

We easily see the following lemma by Lemma 4.3.

LEMMA 6.5. *Let* φ *be the transformation in Lemma* 6.1. *Then* $J(\varphi^m(\langle A_1, A_2\rangle)) = J(\langle A_1, A_2\rangle)$ $(m \in \boldsymbol{N})$ *for* $\langle A_1, A_2\rangle \in R_{\mathrm{VI}}\boldsymbol{S}_2^0$.

LEMMA 6.6. *Let* χ *be the transformation in Lemma* 6.1. *Then*

$$\inf\{J(\chi(\langle A_1, A_2\rangle)) \mid \langle A_1, A_2\rangle \in M_{\mathrm{VI}}(1)\} = \inf\{J(\langle A_1, A_2\rangle) \mid \langle A_1, A_2\rangle \in M_{\mathrm{VI}}(1)\}.$$

PROOF. By Lemmas 4.3 and 6.2 (iii) we have

$$\begin{aligned} &\inf\{J(\chi(\langle A_1, A_2\rangle)) \mid \langle A_1, A_2\rangle \in M_{\mathrm{VI}}(1)\} \\ &\quad = \inf\{J(N_2(\langle A_1, A_2\rangle)) \mid \langle A_1, A_2\rangle \in M_{\mathrm{VI}}(1)\} \\ &\quad = \inf\{J(\langle A_1, A_2\rangle) \mid \langle A_1, A_2\rangle \in M_{\mathrm{VI}}(1)\}. \end{aligned}$$ □

LEMMA 6.7. *Let* φ *and* χ *be the transformations in Lemma* 6.1. *Let* $G = \langle A_1, A_2\rangle \in R_{\mathrm{VI}}\boldsymbol{S}_2^0$ *and let* (t_1, t_2, ρ) *correspond to* $G = \langle A_1, A_2\rangle$. *If* $\rho > 1$ *and* $m \geq 1$, *then* $J(\chi\varphi^m(\langle A_1, A_2\rangle)) > J(\langle A_1, A_2\rangle)$.

In particular, if $G = \langle A_1, A_2 \rangle \in M_{\mathrm{VI}}(1)$ *and* $m \geq 1$, *then* $J(\chi\varphi^m(\langle A_1, A_2 \rangle)) > J(\langle A_1, A_2 \rangle)$.

Proof. We set $G_2 = \chi\varphi^m(G)$. Then we have $G_2 = \langle A_1^{2m}A_2, A_1^{-1} \rangle$. By Lemma 6.4 $J(\chi\varphi^m(\langle A_1, A_2 \rangle)) > J(\langle A_1, A_2 \rangle)$ if and only if $0 > \mathrm{tr}^2(A_1) > \mathrm{tr}^2(A_1^{2m}A_2)$.

We set

$$A_1 = 1/t_1^{1/2}\begin{pmatrix} 1 & 0 \\ 0 & t_1 \end{pmatrix} \quad \text{and} \quad A_2 = 1/t_2^{1/2}(\rho - 1)\begin{pmatrix} \rho - t_2 & \rho(t_2 - 1) \\ 1 - t_2 & t_2\rho - 1 \end{pmatrix}.$$

Then

$$A_1^{2m}A_2 = 1/t_1^m t_2^{1/2}(\rho - 1)\begin{pmatrix} \rho - t_2 & \rho(t_2 - 1) \\ t_1^{2m}(1 - t_2) & t_1^{2m}(t_2\rho - 1) \end{pmatrix}.$$

Hence we have $\mathrm{tr}^2(A_1) = (1 + t_1)^2/t_1$ and $\mathrm{tr}^2(A_1^{2m}A_2) = \{\rho - t_2 + t_1^{2m}(t_2\rho - 1)\}^2/t_1^{2m}t_2(\rho - 1)^2$. Therefore $\mathrm{tr}^2(A_1) > \mathrm{tr}^2(A_1^{2m}A_2)$ if and only if $\{\rho - t_2 + t_1^{2m}(t_2\rho - 1)\}^2 - t_1^{2m-1}t_2(\rho - 1)^2(1 + t_1)^2 > 0$. By easy calculations we have

$$\begin{aligned}
&\{\rho - t_2 + t_1^{2m}(t_2\rho - 1)\}^2 - t_1^{2m-1}t_2(\rho - 1)^2(1 + t_1)^2 \\
&\quad = (\rho - t_2 + t_1^{2m}t_2\rho - t_1^{2m})^2 - \{(-t_1)^{m-1/2}(-t_2)^{1/2}(\rho - 1)(1 + t_1)\}^2 \\
&\quad = \{\rho - t_2 + t_1^{2m}t_2\rho - t_1^{2m} - (-t_1)^{m-1/2}(-t_2)^{1/2}(\rho - 1)(1 + t_1)\} \\
&\qquad \times \{\rho - t_2 + t_1^{2m}t_2\rho - t_1^{2m} + (-t_1)^{m-1/2}(-t_2)^{1/2}(\rho - 1)(1 + t_1)\}.
\end{aligned}$$

Since $1 + t_1^{2m}t_2 - (-t_1)^{m-1/2}(-t_2)^{1/2} + (-t_1)^{m+1/2}(-t_2)^{1/2} > 0$, $\rho > 1$ and $m \geq 1$, we have

$$\begin{aligned}
&\rho - t_2 + t_1^{2m}t_2\rho - t_1^{2m} - (-t_1)^{m-1/2}(-t_2)^{1/2}(\rho - 1)(1 - (-t_1)) \\
&\quad = \rho\{1 + t_1^{2m}t_2 - (-t_1)^{m-1/2}(-t_2)^{1/2} + (-t_1)^{m+1/2}(-t_2)^{1/2}\} \\
&\qquad + \{(-t_2) - t_1^{2m} + (-t_1)^{m-1/2}(-t_2)^{1/2} - (-t_1)^{m+1/2}(-t_2)^{1/2}\} \\
&\quad > \{1 + t_1^{2m}t_2 - (-t_1)^{m-1/2}(-t_2)^{1/2} + (-t_1)^{m+1/2}(-t_2)^{1/2}\} \\
&\qquad + \{(-t_2) - t_1^{2m} + (-t_1)^{m-1/2}(-t_2)^{1/2} - (-t_1)^{m+1/2}(-t_2)^{1/2}\} \\
&\quad = 1 - t_2 + t_1^{2m}t_2 - t_1^{2m} \\
&\quad = (1 - t_2)(1 - t_1^{2m}) > 0.
\end{aligned}$$

Furthermore we have the following:

$$\begin{aligned}\rho - t_2 + t_1^{2m} t_2 \rho - t_1^{2m} + (-t_1)^{m-1/2}(-t_2)^{1/2}(\rho - 1)(1 - (-t_1)) \\ > \rho - t_2 + t_1^{2m} t_2 \rho - t_1^{2m} \\ > (1 - t_2)(1 - t_1^{2m}) > 0.\end{aligned}$$

Therefore we have $\mathrm{tr}^2(A_1) > \mathrm{tr}^2(A_1^{2m} A_2)$, which implies $J(\chi\varphi^m(\langle A_1, A_2 \rangle)) > J(\langle A_1, A_2 \rangle)$. $\square$

COROLLARY 1. *Let* $G = \langle A_1, A_2 \rangle \in M_{\mathrm{VI}}(1)$ *and* $m \geq 1$. *Then*

$$\inf\{J(\chi\varphi^m(G)) \mid G \in M_{\mathrm{VI}}(1)\} \geq \inf\{J(G) \mid G \in M_{\mathrm{VI}}(1)\}.$$

COROLLARY 2. *Let* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{VI}} \boldsymbol{S}_2^0$ *and let* $G_2 = \chi\varphi^m(\langle A_1, A_2 \rangle)$. *Let* (t_1, t_2, ρ) *and* $(t_1(G_2), t_2(G_2), \rho(G_2))$ *correspond to* G *and* G_2, *respectively. If* $\rho > 1$ *and* $m \geq 1$, *then* $-t_1(G_2) < -t_2(G_2)$.
In particular, if $G = \langle A_1, A_2 \rangle \in M_{\mathrm{VI}}(1)$ *and* $m \geq 1$, *then* $-t_1(G_2) < -t_2(G_2)$.

PROOF. By Lemmas 6.4 and 6.7 we have $-t_1(G_2) < -t_1$. Since $t_2(G_2) = t_1$, we have the desired result $-t_1(G_2) < -t_2(G_2)$. $\square$

LEMMA 6.8. *Let* φ *and* χ *be the transformations in Lemma* 6.1. *Let* $G = \langle A_1, A_2 \rangle \in M_{\mathrm{VI}}(1)$. *Then*

$$\inf\{J(\chi\varphi^{-1}(G)) \mid G \in M_{\mathrm{VI}}(1)\} = \inf\{J(G) \mid G \in M_{\mathrm{VI}}(1)\}.$$

PROOF. This follows from Lemma 6.2 (ii). $\square$

LEMMA 6.9. *Let* φ *and* χ *be the transformations in Lemma* 6.1. *Let* $G = \langle A_1, A_2 \rangle \in M_{\mathrm{VI}}(1)$ *and* $G_2 = \chi\varphi^m(G)$ $(m = -2, -3, -4, \ldots)$. *Let* $(t_1(G_2), t_2(G_2), \rho(G_2))$ *correspond to* G_2. *Then* $-t_1(G_2) < -t_2(G_2)$.

PROOF. By Lemma 6.1 (iv) and (v) we have

$$\begin{aligned}G_2 = \chi\varphi^m(G) = \chi\varphi^m N_1 N_1(G) = N_1 \chi^{-1} \varphi^{-m} N_1(G) \\ = N_1 \chi^{-1} \varphi^{-m-1} \varphi N_1(G).\end{aligned}$$

By Lemma 6.2 (iv) we have $G' := \varphi N_1(G) \in M_{\mathrm{VI}}(1)$. Thus $G_2 = N_1 \chi^{-1} \varphi^{-m-1}(G')$. We set $G_2' = N_1(G_2)$. Then $G_2' = \chi^{-1}\varphi^{-m-1}(G') \sim \chi\varphi^{-m-1}(G')$ by Lemma 6.1 (iii). Since $(-m) - 1 \geq 1$ and $G' \in M_{\mathrm{VI}}(1)$, we have $-t_1(G_2') < -t_2(G_2')$ by Corollary 2 to Lemma 6.7. Since $t_1(G_2) = t_1(G_2')$ and $t_2(G_2) = t_2(G_2')$, we have the desired result, $-t_1(G_2) < -t_2(G_2)$. $\square$

LEMMA 6.10. *Let φ and χ be the transformations in Lemma* 6.1. *Let* $G = \langle A_1, A_2 \rangle \in M_{\mathrm{VI}}(1)$. *Then*

$$\inf\{J(\chi\varphi^m(G)) \mid G \in M_{\mathrm{VI}}(1)\} \geq \inf\{J(G) \mid G \in M_{\mathrm{VI}}(1)\}$$

$$\text{for } m = -2, -3, -4, \dots.$$

PROOF. Noting that $J(N_1\chi\varphi^m(G)) = J(\chi\varphi^m(G))$, we can see this corollary by the same method as in the proof of Lemma 6.9. □

By Corollary 1 to Lemma 6.7, Lemmas 6.8 and 6.10 we have the following.

LEMMA 6.11. *Let φ and χ be the transformations in Lemma* 6.1. *Let* $G = \langle A_1, A_2 \rangle \in M_{\mathrm{VI}}(1)$. *Then for* $m \in \boldsymbol{Z}$,

$$\inf\{J(\chi\varphi^m(G)) \mid G \in M_{\mathrm{VI}}(1)\} \geq \inf\{J(G) \mid G \in M_{\mathrm{VI}}(1)\}.$$

Let $G_0 = \langle A_1, A_2 \rangle$ be a marked Schottky group in $M_{\mathrm{VI}}(1)$. We remember that $\varphi = N_3^2$ and $\chi = N_1N_2$, where N_j $(j = 1, 2, 3)$ be the Nielsen transformations defined in §2. We introduce some marked Schottky groups $G_j(G_0)$ and $G_j^*(G_0)$ $(j = 1, 2, 3, \dots)$ in $R_{\mathrm{VI}}\boldsymbol{S}_2^0$ as follows:

$$G_{2k-1}(G_0) := \varphi^{m(k)}\chi\varphi^{m(k-1)} \cdots \chi\varphi^{m(2)}\chi\varphi^{m(1)}(G_0),$$

$$G_{2k}^*(G_0) := \varphi^{m(k)}\chi\varphi^{m(k-1)} \cdots \chi\varphi^{m(1)}\chi(G_0),$$

$G_{2k}(G_0) := \chi(G_{2k-1}(G_0))$ and $G_{2k+1}^*(G_0) := \chi(G_{2k}^*(G_0))$, where $m(j) \in \boldsymbol{Z} - \{0\}$ $(j = 1, 2, 3, \dots)$. We call $G_j(G_0)$ (resp. $G_j^*(G_0)$) *a group of length j* (resp. *a group of length j with an asterisk*) *for G_0*.

Noting that $G_{2k+1}(G_0) = \varphi^{m(k+1)}(G_{2k}(G_0))$ and $G_{2k}^*(G_0) = \varphi^{m(k)}(G_{2k-1}^*(G_0))$, we have the following lemmas by Lemma 4.3.

LEMMA 6.12. $J(G_{2k+1}(G_0)) = J(G_{2k}(G_0))$ $(k = 0, 1, 2, 3, \dots)$.

LEMMA 6.13. $J(G_{2k}^*(G_0)) = J(G_{2k-1}^*(G_0))$ $(k = 1, 2, 3, \dots)$.

By Lemma 6.2 (i), we have the following.

LEMMA 6.14. *Let* $G_{2k+1}^*(G_0) = \chi\varphi^{m(k)}\chi \cdots \varphi^{m(2)}\chi\varphi^{m(1)}\chi(G_0)$ *be a group of length* $2k+1$ *with an asterisk for* G_0, *where* $G_0 \in M_{\mathrm{VI}}(1)$.

(i) *If* $m(1) \neq 1$, *then*

$$\inf\{J(G_{2k+1}^*(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\}$$
$$= \inf\{J(\chi\varphi^{m(k)}\chi \cdots \varphi^{m(2)}\chi\varphi^{m(1)-1}(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\}.$$

(ii) *If* $m(1) = 1$, *then*

$$\begin{aligned}&\inf\{J(G^*_{2k+1}(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\}\\&\quad = \inf\{J(\chi\varphi^{m(k)}\chi\cdots\varphi^{m(2)}\chi(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\}.\end{aligned}$$

Since by Lemmas 6.12 and 6.13

$$\inf\{J(G^*_\ell(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\} = \inf\{J(G_{2j}(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\}$$

for some integer j with $0 \le 2j < \ell$, we have the following.

Corollary. *Let* G_k (*resp.* G^*_k) *be groups of length* k (*resp. groups of length* k *with an asterisk*) *for* G_0. *Then for each* $k = 1, 2, 3, \ldots,$ *if*

$$\inf\{J(G_{2j}(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\} \ge \inf\{J(G_0) \mid G_0 \in M_{\mathrm{VI}}(1)\}$$

holds for $0 \le j \le k$, *then*

$$\inf\{J(G^*_\ell(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\} \ge \inf\{J(G_0) \mid G_0 \in M_{\mathrm{VI}}(1)\}$$

holds for each $\ell = 2k+1, 2k+2$, *where* $G_0(G_0) = G_0$.

Thus it suffices to consider $G_j(G_0)$. For simplicity, we write G_j for $G_j(G_0)$ if there is no confusion.

Let $(t_1(G_j), t_2(G_j), \rho(G_j))$ correspond to $G_j = \langle A_{1j}, A_{2j}\rangle$. Then by properties of the Nielsen transformations N_j $(j = 1, 2, 3)$, we easily see the following.

Lemma 6.15. *Let* $G_j = \langle A_{1j}, A_{2j}\rangle$ *be the marked group defined in the above, and let* $(t_1(G_j), t_2(G_j), \rho(G_j))$ *correspond to* G_j. *Then*

(i) $t_1(G_{2k}) = t_2(G_{2k-1})$.

(ii) $t_2(G_{2k}) = t_1(G_{2k-1})$.

(iii) $t_1(G_{2k}) = t_1(G_{2k+1})$.

Lemma 6.16. *Let* $G_{2k+1}(G_0) = \varphi^{m(k+1)}\chi\varphi^{m(k)}\cdots\chi\varphi^{m(1)}(G_0)$ *be a group of length* $2k+1$ *for* $G_0 \in M_{\mathrm{VI}}(1)$. *Set* $G_{2k} = \chi\varphi^{m(k)}\cdots\chi\varphi^{m(1)}(G_0)$. *If* $m(1) \ne -1$, *then* $\varphi^{-m(k+1)}(N_1(G_{2k}))$ *is equivalent to a group of length* $2k+1$ *for* $G'_0 = \varphi N_1(G_0) \in M_{\mathrm{VI}}(1)$, *where* N_1 *is the Nielsen transformation defined in* §2.

Proof. By Lemma 6.1 (iii), (iv) and (v) we have

$$\begin{aligned}\varphi^{-m(k+1)}(N_1(G_{2k})) &= \varphi^{-m(k+1)}N_1\chi\varphi^{m(k)}\cdots\chi\varphi^{m(1)}(G_0)\\&\sim \varphi^{-m(k+1)}\chi^{-1}\varphi^{-m(k)}\cdots\chi^{-1}\varphi^{-m(1)}N_1(G_0)\\&\sim \varphi^{-m(k+1)}\chi\varphi^{-m(k)}\cdots\chi\varphi^{-m(1)-1}(\varphi N_1(G_0))\\&= \varphi^{-m(k+1)}\chi\varphi^{-m(k)}\cdots\chi\varphi^{-m(1)-1}(G'_0).\end{aligned}$$

By Lemma 6.2(iv) we have $G_0' = \varphi N_1(G_0) \in M_{\mathrm{VI}}(1)$. Since $m(1) \neq -1$, we have the desired result. $\square$

LEMMA 6.17. *Assume that* $-t_1(G_{2k+1}(G_0)) > -t_2(G_{2k+1}(G_0))$ *holds for every* $G_0 \in M_{\mathrm{VI}}(1)$ *and for every group* $G_{2k+1}(G_0) = \varphi^{m(k+1)}\chi \cdots \chi\varphi^{m(1)}(G_0)$ *of length* $2k+1$ *with* $\rho(G_{2k}(G_0)) > 1$ *and* $m(1) \neq -1$. *Then* $-t_1(G_{2k+1}(G_0')) > -t_2(G_{2k+1}(G_0'))$ *holds for every* $G_0' \in M_{\mathrm{VI}}(1)$ *and every group* $G_{2k+1}(G_0') = \varphi^{n(k+1)}\chi \cdots \chi\varphi^{n(1)}(G_0')$ *of length* $2k+1$ *with* $0 < \rho(G_{2k}(G_0')) < 1$ *and* $n(1) \neq -1$.

PROOF. Let $G_{2k+1}(G_0') = \varphi^{n(k+1)}\chi\varphi^{n(k)} \cdots \chi\varphi^{n(1)}(G_0')$ be a group of length $2k+1$ such that $0 < \rho(G_{2k}(G_0')) < 1$, $n(1) \neq -1$ and $G_0' \in M_{\mathrm{VI}}(1)$. We easily see that

$$G_{2k+1}(G_0') \sim N_1\varphi^{-n(k+1)}(\chi\varphi^{-n(k)} \cdots \chi\varphi^{-n(1)-1}(\varphi N_1(G_0'))).$$

If we set $G_0 = \varphi N_1(G_0')$, then $G_0 \in M_{\mathrm{VI}}(1)$ by Lemma 6.2 (iv). Since $\varphi^{-n(k+1)}(\chi\varphi^{-n(k)} \cdots \chi\varphi^{-n(1)-1}(G_0)) \sim \varphi^{-n(k+1)}(N_1\chi\varphi^{n(k)} \cdots \chi\varphi^{n(1)}(G_0'))$ is a group of length $2k+1$ by Lemma 6.16 and

$$\begin{aligned}
\rho(\chi\varphi^{-n(k)} \cdots \chi\varphi^{-n(1)-1}(G_0)) &= \rho(\chi\varphi^{-n(k)} \cdots \chi\varphi^{-n(1)-1}\varphi N_1(G_0')) \\
&= \rho(\chi\varphi^{-n(k)} \cdots \chi\varphi^{-n(1)} N_1(G_0')) \\
&= \rho(N_1\chi\varphi^{n(k)} \cdots \chi\varphi^{n(1)}(G_0')) \\
&> 1,
\end{aligned}$$

we have

$$-t_1(\varphi^{-n(k+1)}(\chi\varphi^{-n(k)} \cdots \chi\varphi^{-n(1)-1}(G_0))) > -t_2(\varphi^{-n(k+1)}(\chi\varphi^{-n(k)} \cdots \chi\varphi^{-n(1)-1}(G_0)))$$

by the assumption.

By Lemma 2.1 (i) we have

$$-t_1(N_1\varphi^{-n(k+1)}\chi\varphi^{-n(k)} \cdots \chi\varphi^{-n(1)-1}(G_0)) > -t_2(N_1\varphi^{-n(k+1)}\chi\varphi^{-n(k)} \cdots \chi\varphi^{-n(1)-1}(G_0)).$$

Thus $-t_1(G_{2k+1}(G_0')) > -t_2(G_{2k+1}(G_0'))$ holds for $0 < \rho(G_{2k}(G_0')) < 1$. $\square$

LEMMA 6.18. *Let* $G_j = \langle A_{1j}, A_{2j} \rangle \in M_{\mathrm{VI}}(1)$ *be the marked groups of length* j *defined in the above, and let* $(t_1(G_j), t_2(G_j), \rho(G_j))$ *correspond to* G_j. *Then*

(i) $-t_1(G_{2k-1}) > -t_2(G_{2k-1})$ $(k = 1, 2, 3, \ldots)$ *if* $m(1) \neq -1$.

(ii) $-t_1(G_{2k}) < -t_2(G_{2k})$ $(k = 1, 2, 3, \ldots)$ *if* $m(1) \neq -1$.

PROOF. First we will show that (i) holds if and only if (ii) holds. Since $\langle A_{1,2k}, A_{2,2k} \rangle = \langle A_{2,2k-1}, A_{1,2k-1}^{-1} \rangle$, we have $(t_1(G_{2k}), t_2(G_{2k}), \rho(G_{2k})) = (t_2(G_{2k-1}), t_1(G_{2k-1}), 1/\rho(G_{2k-1}))$. Hence we have the desired result.

Now we will prove (i) by induction. We consider two cases: (1) $k = 1$ and (2) $k \geq 2$.

(1) The case of $k = 1$. By Corollary 2 to Lemmas 6.7 and 6.9 we have $-t_1(G_2) < -t_2(G_2)$ for $m(1) \neq -1$. Since $t_1(G_1) = t_2(G_2)$ and $t_2(G_1) = t_1(G_2)$, we have $-t_2(G_1) < -t_1(G_1)$ for $m(1) \neq -1$.

(2) The case of $k \geq 2$. Assume that $-t_1(G_{2k-1}) > -t_2(G_{2k-1})$ holds for every $G_0 \in M_{\mathrm{VI}}(1)$. Then it suffices to show that $-t_1(G_{2k+1}) > -t_2(G_{2k+1})$ holds. We only prove $-t_1(G_{2k+1}) > -t_2(G_{2k+1})$ for the case of $\rho(G_{2k}) > 1$ by Lemma 6.17. Therefore we only prove that if $\rho(G_{2k}) > 1$ and $-t_1(G_{2k-1}) > -t_2(G_{2k-1})$, then $-t_1(G_{2k+1}) > -t_2(G_{2k+1})$. We will show this for the following two cases: case 1. $m(k+1) \geq 1$; case 2. $m(k+1) \leq -1$. For simplicity, we set $G_{2k} := \langle B_1, B_2 \rangle$ and $t_1 := t_1(G_{2k})$, $t_2 := t_2(G_{2k})$ and $\rho := \rho(G_{2k})$. Then $G_{2k+1} = \langle B_1, B_1^{2m(k+1)} B_2 \rangle$.

Case 1: $m(k+1) \geq 1$. By the same method as in the proof of Lemma 6.7, we have $J(\chi\varphi^{m(k+1)}(G_{2k})) > J(G_{2k})$. Let $(t_1(G_j), t_2(G_j), \rho(G_j))$ correspond to G_j $(j = 2k, 2k+1, 2k+2)$. By Lemma 6.4 we have $-t_1(\chi(G_{2k+1})) < -t_1(G_{2k})$ where $G_{2k+1} = \varphi^{m(k+1)}(G_{2k})$. Since $t_1(\chi(G_{2k+1})) = t_2(G_{2k+1})$ and $t_1(G_{2k}) = t_1(G_{2k+1})$ by Lemma 6.15, we have the desired result, $-t_2(G_{2k+1}) < -t_1(G_{2k+1})$.

Case 2: $m(k+1) \leq -1$. We set

$$B_1 = 1/t_1^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & t_1 \end{pmatrix} \quad \text{and} \quad B_2 = 1/t_2^{1/2}(\rho - 1) \begin{pmatrix} \rho - t_2 & \rho(t_2 - 1) \\ 1 - t_2 & t_2\rho - 1 \end{pmatrix}.$$

For simplicity, we set $m = -m(k+1)$. Then $m \geq 1$. We have

$$B_1^{2m(k+1)} B_2 = B_1^{-2m} B_2 = 1/t_1^m t_2^{1/2}(\rho - 1) \begin{pmatrix} t_1^{2m}(\rho - t_2) & t_1^{2m}\rho(t_2 - 1) \\ 1 - t_2 & t_2\rho - 1 \end{pmatrix}.$$

We note that the following two inequalities are equivalent: (i) $-t_1(G_{2k+1}) > -t_2(G_{2k+1})$, (ii) $0 > \mathrm{tr}^2(B_1) > \mathrm{tr}^2(B_1^{-2m} B_2)$. Therefore we only show that the following inequality holds:

$$\{t_1^{2m}(\rho - t_2) + t_2\rho - 1\}^2 > t_1^{2m-1} t_2 (\rho - 1)^2 (1 + t_1)^2.$$

We set

$$f(\rho) = (t_1^{2m}\rho - t_1^{2m} t_2 + t_2\rho - 1)^2 - t_1^{2m-1} t_2 (\rho - 1)^2 (1 + t_1)^2. \qquad (*)$$

By the assumption $-t_1 < -t_2$, the coefficient of ρ^2 in the above equation $(*)$ is positive, that is

$$(t_1^{2m} + t_2)^2 - t_1^{2m-1} t_2 (1 + t_1)^2 > 0.$$

Thus it suffices to show that the axis of the quadratic equation $(*)$ in ρ is less than one and $f(1) > 0$. The axis of the equation $(*)$ is less than one if and only if $f'(1) > 0$. By simple calculations we have

$$f'(1) = 2(1 - t_2)(1 - t_1^{2m})(-t_2 - t_1^{2m})$$
$$> 2(1 - t_2)(1 - t_1^{2m})(-t_2 + t_1) > 0.$$

Furthermore substituting $\rho = 1$ for the equation $(*)$, we have

$$f(1) = (t_1^{2m} - t_1^{2m} t_2 + t_2 - 1)^2 > 0.$$

Thus we have $\mathrm{tr}^2(B_1) > \mathrm{tr}^2(B_1^{-2m} B_2)$. Therefore we have the desired result, $-t_1(G_{2k+1}) > -t_2(G_{2k+1})$. □

Corollary. *If* $G_0 \in M_{\mathrm{VI}}(1)$ *and* $m(1) \neq -1$, *then*

$$J(G_{2k}(G_0)) > J(G_{2k-2}(G_0))$$

for $k = 1, 2, 3, \ldots$.

Proof. This corollary follows from Lemmas 6.4, 6.15 and 6.18. □

By this corollary we have the following.

Lemma 6.19. *If* $G_0 \in M_{\mathrm{VI}}(1)$ *and* $m(1) \neq -1$, *then* $J(G_{2k}(G_0)) > J(G_0)$ *for* $k = 1, 2, 3, \ldots$.

Next we consider the case of $m(1) = -1$. Let

$$G_{2k}(G_0) = \chi\varphi^{m(k)}\chi \cdots \varphi^{m(2)}\chi\varphi^{m(1)}(G_0)$$

be a group of length $2k$ for $G_0 \in M_{\mathrm{VI}}(1)$. Since $m(1) = -1$, we have $G_{2k}(G_0) = \chi\varphi^{m(k)}\chi \cdots \varphi^{m(2)}\chi\varphi^{-1}(G_0)$. By Lemma 6.2 (ii) we have $G_0' := \chi\varphi^{-1}(G_0) \in M_{\mathrm{VI}}(1)$. Thus

$$\inf\{J(G_{2k}(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\} = \inf\{J(G_{2(k-1)}(G_0')) \mid G_0' \in M_{\mathrm{VI}}(1)\}.$$

If $m(1) = -1$ and $m(2) \neq -1$, then by the same method as the proof in the above lemmas we have

$$\inf\{J(G_{2k}(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\} \geq \inf\{J(G_0) \mid G_0 \in M_{\mathrm{VI}}(1)\}$$

for $k = 1, 2, 3, \ldots$.

If $m(1) = -1$ and $m(2) = -1$, then by the same reason as above we have

$$\inf\{J(G_{2k}(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\} = \inf\{J(G_{2(k-2)}(G_0) \mid G_0 \in M_{\mathrm{VI}}(1)\}.$$

By continuing this procedure we have the following.

Lemma 6.20. *Let* $G_{2k}(G_0)$ $(k = 1, 2, 3, \ldots)$ *be a group of length* $2k$ *for* $G_0 \in M_{\mathrm{VI}}(1)$:

$$G_{2k}(G_0) = \chi\varphi^{m(k)}\chi \cdots \chi\varphi^{m(2)}\chi\varphi^{m(1)}(G_0).$$

Then

$$\inf\{J(G_{2k}(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\} \geq \inf\{J(G_0) \mid G_0 \in M_{\mathrm{VI}}(1)\}.$$

By Lemmas 6.19 and 6.20 we have the following.

Lemma 6.21. *Let G_j be a group of length j for $G_0 \in M_{\mathrm{VI}}(1)$. Then*

$$\inf\{J(G_j(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\} \geq \inf\{J(G_0) \mid G_0 \in M_{\mathrm{VI}}(1)\}$$
$$(j = 1, 2, 3, \ldots).$$

By Corollary to Lemma 6.14 and Lemma 6.21 we have the following.

Lemma 6.22. *Let G_j^* be a group of length j with an asterisk for $G_0 \in M_{\mathrm{VI}}(1)$. Then*

$$\inf\{J(G_j^*(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\} \geq \inf\{J(G_0) \mid G_0 \in M_{\mathrm{VI}}(1)\}$$
$$(j = 1, 2, 3, \ldots).$$

By Lemmas 6.21 and 6.22 we have the following.

Proposition 6.1. *Let $\phi \in \mathrm{Mod}(R_{\mathrm{VI}}\boldsymbol{S}_2^0)$. Then*

$$\inf\{J(\phi(G_0)) \mid G_0 \in M_{\mathrm{VI}}(1)\} \geq \inf\{J(G_0) \mid G_0 \in M_{\mathrm{VI}}(1)\}.$$

Corollary. $\inf\{J(G_0) \mid G_0 \in R_{\mathrm{VI}}\boldsymbol{S}_2^0\} \geq \inf\{J(G_0) \mid G_0 \in M_{\mathrm{VI}}(1)\}$.

Combining the above corollary with Lemma 4.4 we have the following.

Proposition 6.2.

$$\inf\{J(G_0) \mid G_0 \in R_{\mathrm{VI}}\boldsymbol{S}_2^0\} = \inf\{J(G_0) \mid G_0 \in M_{\mathrm{VI}}(1)\}.$$

Lemma 6.23. *Let (t_1, t_2, ρ) correspond to a marked group $G = \langle A_1, A_2 \rangle$. Let $M_{\mathrm{VI}}^*(1) = \{\tau = (t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid t_2 = -(1 + t_1\rho^{1/2})/(\rho^{1/2} + t_1),\ \ 1 < \rho < 1/t_1^2, -1 < t_1 < 0\}$. Then $J(\langle A_1, A_2 \rangle) \geq 16$ on $M_{\mathrm{VI}}^*(1)$. The lower bound is the best possible.*

Proof. Let $\tau = (t_1, t_2, \rho) \in M_{\mathrm{VI}}^*(1)$. We set $x = -t_1$ and $y = \rho^{1/2}$. Then the equation $t_2 = -(1 + t_1\rho^{1/2})/(\rho^{1/2} + t_1)$ turns into $t_2 = -(1 - xy)/(y - x)$. By substituting $t_1 = -x$, $\rho = y^2$ and $t_2 = -(1 - xy)/(y - x)$ for

$$J(\langle A_1, A_2 \rangle) = \frac{|1 - t_1|^2}{|t_1|} + \frac{|1 - t_1|^2 |1 - t_2|^2 |\rho|}{|t_1|\,|t_2|\,|\rho - 1|^2}$$

we have

$$J(\langle A_1, A_2 \rangle) = \frac{(1 + x)^2}{x} + \frac{(1 + x)^2 (1 - x)^2 y^2}{x(y - x)(1 - xy)(y - 1)^2}.$$

Setting $X = x + 1/x$ and $Y = y + 1/y$, we have

$$J(\langle A_1, A_2\rangle) = X + 2 + (X^2 - 4)/(X - Y)(Y - 2).$$

By calculus we have that $J(\langle A_1, A_2\rangle)$ attains the minimum value 16 at $X = 6$, $Y = 4$, that is, at the point $\tau = (t_1, t_2, \rho) = (-(3 - 2\sqrt{2}), -(5 - 2\sqrt{6}), (2 + \sqrt{3})^2)$ on $M_{\mathrm{VI}}^*(1)$. □

Since $M_{\mathrm{VI}}^* = \partial M_{\mathrm{VI}} \setminus \{\tau = (t_1, t_2, \rho) \in \partial M_{\mathrm{VI}} \mid t_2 = 0\}$, we have the following by Lemmas 4.5 and 6.23.

Proposition 6.3. *If* $\langle A_1, A_2\rangle \in M_{\mathrm{VI}}(1)$, *then* $J(\langle A_1, A_2\rangle) > 16$.

Proof of Theorem 2. We can prove Theorem 2 by Propositions 6.2 and 6.3. We can see by Example 2 in §7 that the lower bound 16 is the best possible. □

Proof of Theorem 3. Theorem 3 follows from Theorem 2, Proposition 2.3 and Lemma 4.3. Example 3 in §7 shows that the lower bound 16 is the best possible. □

7. Examples.

In this section we will give some examples which guarantee that all of the lower bounds in the inequalities in Theorems 1, 2 and 3 are the best possible. Let $\{\tau_n = (t_{1n}, t_{2n}, \rho_n)\}$ $(n = 1, 2, 3, \ldots)$ be a sequence of points in $\boldsymbol{R}^3$ and let $G_n = \langle A_{1n}, A_{2n}\rangle$ be the groups representing τ_n. Here we will give sequences of marked classical Schottky groups $\{G_n\}$ whose Jørgensen's numbers $J(G_n)$ tend to the lower bounds in the inequalities in Theorems 1, 2 and 3.

Example 1 (Type III). Let $t_{1n} = ((n-1)/(n+1))^2$, $t_{2n} = -1/(n+1)^2$ and $\rho_n = -1$ $(n = 2, 3, 4, \ldots)$. Then (i) $G_n \in R_{\mathrm{III}}\boldsymbol{S}_2^0$ and (ii) $\lim_{n\to\infty} J(G_n) = 4$.

We easily see that $\tau = (t_{1n}, t_{2n}, \rho_n) \in M_{\mathrm{III}}$ $(n = 1, 2, 3, \ldots)$ and so $G_n \in R_{\mathrm{III}}\boldsymbol{S}_2^0$, where M_{III} is the region defined in §4. Furthermor we can see $\lim_{n\to\infty} J(G_n) = 4$ by simple calculations.

Example 2 (Type VI). Let $t_{1n} = -(3 - 2\sqrt{2}) + 1/(n+10)$, $t_{2n} = -(5 - 2\sqrt{6}) + 1/(n+10)$ and $\rho_n = 7 + 4\sqrt{3}$ $(n = 1, 2, 3, \ldots)$. Then (i) $G_n \in R_{\mathrm{VI}}\boldsymbol{S}_2^0$ and (ii) $\lim_{n\to\infty} J(G_n) = 16$.

We easily see that $\tau = (t_{1n}, t_{2n}, \rho_n) \in M_{\mathrm{VI}}(1)$ $(n = 1, 2, 3, \ldots)$ and so $G_n \in R_{\mathrm{VI}}\boldsymbol{S}_2^0$. Furthermor we can see $\lim_{n\to\infty} J(G_n) = 16$ by simple calculations.

Example 3 (Type VIII). Let $t_{1n} = -(3 - 2\sqrt{2}) + 1/(n+10)$, $t_{2n} = 3 - 2\sqrt{2} - 1/(n+10)$ and $\rho_n = -1$ $(n = 1, 2, 3, \ldots)$. Then (i) $G_n \in R_{\mathrm{VIII}}\boldsymbol{S}_2^0$ and (ii) $\lim_{n\to\infty} J(G_n) = 16$.

We set

$$M_{\mathrm{VIII}}(0) = \left\{ (t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid 0 < t_2 < \frac{\{(-\rho)^{1/2} - (-t_1)^{1/2}\}\{1 - (-t_1)^{1/2}(-\rho)^{1/2}\}}{\{(-\rho)^{1/2} + (-t_1)^{1/2}\}\{1 + (-t_1)^{1/2}(-\rho)^{1/2}\}}, \right.$$
$$\left. 1/t_1 < \rho < t_1,\ -1 < t_1 < 0 \right\}.$$

Then we note that $M_{\mathrm{VIII}}(0) \subset R_{\mathrm{VIII}}\boldsymbol{S}_2^0$ (cf. [**8**]). We easily see that $\tau = (t_{1n}, t_{2n}, \rho_n) \in M_{\mathrm{VIII}}(0)$ $(n = 1, 2, 3, \dots)$ and so $G_n \in R_{\mathrm{VIII}}\boldsymbol{S}_2^0$. We can see (ii) in Example 3 by Proposition 2.3, Lemma 4.3 and Example 2.

Though we have just seen that the group $G_n = \langle A_{1n}, A_{2n} \rangle$ in Examples 1, 2 and 3 are classical Schottky group of types III, VI and VIII, respectively, we will show this fact by drawing explicitely defining circles for the groups $G_n = \langle A_{1n}, A_{2n} \rangle$. Namely, we draw four circles C_{jn} $(j = 1, 2, 3, 4)$ satisfying the following two conditions:

(1) $A_{1n}(C_{1n}) = C_{3n}$ and $A_{2n}(C_{2n}) = C_{4n}$.

(2) C_{1n}, C_{2n}, C_{3n} and C_{4n} comprise the boundary of a 4-ply connected region.

Example 1. If we can choose circles C_{1n}, C_{2n}, C_{3n} and C_{4n} satisfying the following two conditions, then we easily see that $G_n = \langle A_{1n}, A_{2n} \rangle \in M_{\mathrm{III}} \subset R_{\mathrm{III}}\boldsymbol{S}_2^0$.

(i) C_{jn} $(j = 1, 2, 3, 4)$ are the circles perpendicular to the real axis such that $A_{1n}(C_{1n}) = C_{3n}$ and $A_{2n}(C_{2n}) = C_{4n}$.

(ii) For $j = 1, 2, 3, 4$, the points a_{jn} and b_{jn} satisfy the inequality

$$a_{3n} < a_{4n} < \rho_n < b_{4n} < a_{1n} < 0 < b_{1n} < a_{2n} < 1 < b_{2n} < b_{3n},$$

where a_{jn} and b_{jn} $(j = 1, 2, 3, 4)$ are the intersection points of the circles C_{jn} with the real axis.

Now we take a_{jn} and b_{jn} $(j = 1, 2, 3, 4)$ as follows: $a_{1n} = -(n^2 - 3)/(n^2 + 2n + 2)$, $b_{1n} = (n^2 - 3)/(n^2 + 2n + 2)$; $a_{2n} = (n^2 - 2)/(n^2 + 2n + 2)$, $b_{2n} = (n^2 + 4n + 2)/(n^2 + 2n + 2)$; $a_{3n} = -(n^2 - 3)(n + 1)^2/(n^2 + 2n + 2)(n - 1)^2$, $b_{3n} = (n^2 - 3)(n + 1)^2/(n^2 + 2n + 2)(n - 1)^2$; $a_{4n} = -(n^2 + 4n + 2)/(n^2 + 2n + 2)$, $b_{4n} = -(n^2 - 2)/(n^2 + 2n + 2)$.

Then we can easily see that these points a_{jn} and b_{jn} and the circles C_{jn} $(j = 1, 2, 3, 4)$, which have a_{jn} and b_{jn} as the intersection points of the circles C_{jn} with the real axis, satisfy the above conditions (i) and (ii) for sufficiently large n. Thus $G_n = \langle A_{1n}, A_{2n} \rangle$ are classical Schottky groups of type III.

Example 2. If we can choose circles C_{1n}, C_{2n}, C_{3n} and C_{4n} satisfying the following two conditions, then we easily see that $G_n = \langle A_{1n}, A_{2n} \rangle \in M_{\mathrm{VI}}(1) \subset R_{\mathrm{VI}}\boldsymbol{S}_2^0$.

(i) C_{jn} $(j = 1, 2, 3, 4)$ are the circles perpendicular to the real axis such that $A_{1n}(C_{1n}) = C_{3n}$ and $A_{2n}(C_{2n}) = C_{4n}$.

(ii) For $j = 1, 2, 3, 4$, the points a_{jn} and b_{jn} satisfy the inequalities

$$a_{3n} < a_{1n} < 0 < b_{1n} < a_{2n} < 1 < b_{2n} < a_{4n} < \rho_n < b_{4n} < b_{3n},$$

where a_{jn} and b_{jn} $(j = 1, 2, 3, 4)$ are the intersection points of the circles C_{jn} with the real axis.

We take t_1, t_2, ρ and t_{1n}, t_{2n}, ρ_n as follows:

$$t_1 = -(3 - 2\sqrt{2}), \quad t_2 = -(5 - 2\sqrt{6}), \quad \rho = 7 + 4\sqrt{3};$$

$$t_{1n} = -(3 - 2\sqrt{2}) + 1/(n + 10), \quad t_{2n} = -(5 - 2\sqrt{6}) + 1/(n + 10), \quad \rho_n = 7 + 4\sqrt{3}.$$

Let $G = \langle A_1, A_2 \rangle$ and $G_n = \langle A_{1n}, A_{2n} \rangle$ be the groups representing $\tau = (t_1, t_2, \rho)$ and $\tau_n = (t_{1n}, t_{2n}, \rho_n)$, respectively. We take a_{jn} and b_{jn} $(j = 1, 2, 3, 4)$ as follows:

$a_{1n} = -(2+\sqrt{3})$, $b_{3n} = A_{1n}(a_{1n})$, $b_{4n} = A_1(a_{1n})$; $b_{2n} = A_{2n}^{-1}(b_{4n})$, $a_{4n} = A_2^{-1}(b_{4n})$; $a_{2n} = A_{2n}^{-1}(a_{4n})$, $b_{1n} = A_2^{-1}(a_{4n})$; $a_{3n} = A_{1n}(b_{1n})$.

Then we can easily see that these points a_{jn}, b_{jn} and the circles C_{jn} $(j = 1, 2, 3, 4)$, which have a_{jn} and b_{jn} as the intersection points of the circles C_{jn} with the real axis, satisfy the above conditions (i) and (ii). Thus $G_n = \langle A_{1n}, A_{2n} \rangle$ are classical Schottky groups of type VI.

Example 3. If we can choose circles C_{1n}, C_{2n}, C_{3n} and C_{4n} satisfying the following two conditions, then we easily see that $G_n = \langle A_{1n}, A_{2n} \rangle \in M_{\mathrm{VIII}}(0) \subset R_{\mathrm{VIII}}\boldsymbol{S}_2^0$.

(i) C_{jn} $(j = 1, 2, 3, 4)$ are the circles perpendicular to the real axis such that $A_{1n}(C_{1n}) = C_{3n}$ and $A_{2n}(C_{2n}) = C_{4n}$.

(ii) For $j = 1, 2, 3, 4$, the points a_{jn} and b_{jn} satisfy the inequalities

$$a_{3n} < a_{4n} < \rho_n < b_{4n} < a_{1n} < 0 < b_{1n} < a_{2n} < 1 < b_{2n} < b_{3n},$$

where a_{jn} and b_{jn} $(j = 1, 2, 3, 4)$ are the intersection points of the circles C_{jn} with the real axis.

We take t_1, t_2, ρ and t_{1n}, t_{2n}, ρ_n as follows:

$$t_1 = -(3 - 2\sqrt{2}), \quad t_2 = 3 - 2\sqrt{2}, \quad \rho = -1;$$

$$t_{1n} = -(3 - 2\sqrt{2}) + 1/(n + 10), \quad t_{2n} = (3 - 2\sqrt{2}) - 1/(n + 10), \quad \rho_n = -1.$$

Let $G = \langle A_1, A_2 \rangle$ and $G_n = \langle A_{1n}, A_{2n} \rangle$ be the groups representing $\tau = (t_1, t_2, \rho)$ and $\tau_n = (t_{1n}, t_{2n}, \rho_n)$, respectively. We take a_{jn} and b_{jn} $(j = 1, 2, 3, 4)$ as follows:

$a_{1n} = -(-t_1)^{1/2}(-\rho)^{1/2} = 1 - \sqrt{2}$, $b_{3n} = A_{1n}(a_{1n})$, $b_{2n} = A_1(a_{1n}) = \sqrt{2} + 1$; $a_{4n} = A_{2n}(b_{2n})$, $a_{3n} = A_2(b_{2n}) = -(\sqrt{2}+1)$; $b_{1n} = A_{1n}^{-1}(a_{3n})$, $a_{2n} = A_1^{-1}(a_{3n}) = \sqrt{2} - 1$; $b_{4n} = A_{2n}(a_{2n})$.

Then we can easily see that these points a_{jn}, b_{jn} and the circles C_{jn} $(j = 1, 2, 3, 4)$, which have a_{jn} and b_{jn} as the intersection points of the circles C_{jn} with the real axis, satisfy the above conditions (i) and (ii). Thus $G_n = \langle A_{1n}, A_{2n} \rangle$ are classical Schottky groups of type VIII.

8. Appendices.

We will make a summary of Jørgensen's inequalities for classical Schottky groups of real type obtained in [**1**], [**6**], [**9**] and the present paper.

THEOREM I (Gilman [**1**], Sato [**6**]). *If* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{I}}\boldsymbol{S}_2^0$, *then* $J(\langle A_1, A_2 \rangle) > 16$. *The lower bound is the best possible.*

THEOREM II (Sato [**9**]). *If* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{II}}\boldsymbol{S}_2^0$, *then* $J(\langle A_1, A_2 \rangle) > 16$. *The lower bound is the best possible.*

THEOREM III (Theorem 1 in the present paper). *If* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{III}}\boldsymbol{S}_2^0$, *then* $J(\langle A_1, A_2 \rangle) > 4$. *The lower bound is the best possible.*

THEOREM IV (Gilman [**1**], Sato [**6**]). *If* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{IV}}\boldsymbol{S}_2^0$, *then* $J(\langle A_1, A_2 \rangle) > 4$. *The lower bound is the best possible.*

THEOREM V (Sato [**9**]). *If* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{V}}\boldsymbol{S}_2^0$, *then* $J(\langle A_1, A_2 \rangle) > 4(1 + \sqrt{2})^2$. *The lower bound is the best possible.*

THEOREM VI (Theorem 2 in the present paper). *If* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{VI}}\boldsymbol{S}_2^0$, *then* $J(\langle A_1, A_2 \rangle) > 16$. *The lower bound is the best possible.*

THEOREM VII (Sato [**9**]). *If* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{VII}}\boldsymbol{S}_2^0$, *then* $J(\langle A_1, A_2 \rangle) > 4(1 + \sqrt{2})^2$. *The lower bound is the best possible.*

THEOREM VIII (Theorem 3 in the present paper). *If* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{VIII}}\boldsymbol{S}_2^0$, *then* $J(\langle A_1, A_2 \rangle) > 16$. *The lower bound is the best possible.*

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