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Resolutivity of ideal boundary for nonlinear Dirichlet problems

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Abstract. We consider a quasi-linear second order elliptic differential equation on a euclidean domain. After developing necessary potential theory for the equation which extends some part of the theories in the book by Heinonen–Kilpeläinen–Martio, we show that the ideal boundary of the Royden type compactification of the domain is resolutive with respect to the Dirichlet problem for the equation.

Introduction.

Resolutivity of ideal boundaries of Riemann surfaces was systematically developed in [**CC**]. An ideal boundary is the boundary of a compactification R^* of an open Riemann surface R and R^* is called a resolutive compactification if every continuous function on $\Gamma = R^* \setminus R$ is resolutive with respect to the Dirichlet problem for harmonic functions. Here, the Dirichlet problem is treated in the so called PWB (Perron-Wiener-Brelot) method.

Such a theory can be readily extended to the case R is a domain in $\boldsymbol{R}^N$ $(N \geq 3)$ or more generally a Riemannian manifold, and to the case the Dirichlet problem is considered with respect to a second order elliptic linear partial differential equation on R, or even to the case R is a harmonic space and the equation is semilinear ([**M**]).

The Dirichlet problem in the PWB-method has been discussed for nonlinear (or, more precisely, quasilinear) elliptic equation of the form

$$(\mathrm{E}_0) \qquad -\mathrm{div}\, \mathscr{A}(x, \nabla u) = 0$$

in a series of papers [**LM**], [**K**], [**KM**], etc., and collectively in [**HKM**]. In [**HKM**], it is assumed that $\mathscr{A} : \boldsymbol{R}^N \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ satisfies the following conditions for a fixed $1 < p < \infty$ and a "p-admissible weight" w:

(1) $x \mapsto \mathscr{A}(x, \xi)$ is measurable on $\boldsymbol{R}^N$ for every $\xi \in \boldsymbol{R}^N$ and $\xi \mapsto \mathscr{A}(x, \xi)$ is continuous for a.e. $x \in \boldsymbol{R}^N$;

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(2) $\mathscr{A}(x,\xi)\cdot\xi \geq \alpha_1 w(x)|\xi|^p$ for all $\xi \in \boldsymbol{R}^N$ and a.e. $x \in \boldsymbol{R}^N$ with a constant $\alpha_1 > 0$;

(3) $|\mathscr{A}(x,\xi)| \leq \alpha_2 w(x)|\xi|^{p-1}$ for all $\xi \in \boldsymbol{R}^N$ and a.e. $x \in \boldsymbol{R}^N$ with a constant $\alpha_2 > 0$;

(4) $(\mathscr{A}(x,\xi_1) - \mathscr{A}(x,\xi_2))\cdot(\xi_1 - \xi_2) > 0$ whenever $\xi_1, \xi_2 \in \boldsymbol{R}^N$, $\xi_1 \neq \xi_2$, for a.e. $x \in \boldsymbol{R}^N$;

(5) $\mathscr{A}(x,\lambda\xi) = \lambda|\lambda|^{p-2}\mathscr{A}(x,\xi)$ for $\lambda \in \boldsymbol{R}\backslash\{0\}$.

Further, resolutivity of the Royden and Kuramochi boundaries of a Riemannian manifold with respect to the p-Laplacian has been investigated in [**T**], and the resolutivity of the Royden boundary for the equation (E_0) where $\mathscr{A}$ satisfies the above conditions with $w = 1$, has been shown in [**N**].

The purpose of the present paper is to extend these resolutivity results to the equation of the form

$$(\mathrm{E}) \qquad -\mathrm{div}\,\mathscr{A}(x,\nabla u) + \mathscr{B}(x,u) = 0$$

on a domain Ω in $\boldsymbol{R}^N$ $(N \geq 2)$. We assume that $\mathscr{A} : \Omega \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ satisfies the conditions (1) to (4) above, but *not* the homogeneity condition (5). $\mathscr{B}(x,t)$ is a function $\Omega \times \boldsymbol{R} \to \boldsymbol{R}$ which we assume to be nondecreasing in t (see §1 for other conditions). A prototype of the equation (E) is

$$-\mathrm{div}\,(w(x)|\nabla u|^{p-2}\nabla u) + b(x)|u|^{p-2}u = 0$$

with a nonnegative measurable function b such that $b(x)/w(x)$ is locally bounded in Ω.

In the case where there is no weight, namely the case $w = 1$, quasilinear equations of the form (E) have been studied in the theory of partial differential equations, e.g., in [**S**], [**GZ**], and the basis to develop some part of potential theory for (E) has been established. In case $w \neq 1$, such basis for the equation (E_0) is given in [**HKM**], and that for (E) in [**O**]. Using the results in [**O**], we obtain potential theory for the equation (E) in §1 and §2 as an extension of a part of the theory given in [**HKM**].

We discuss the Dirichlet problem with respect to ideal boundaries for the equation (E) in §3 and state our main theorem as Theorem 3.2, which, generalizing the resolutivity results given in [**T**] and [**N**], asserts that a Royden type compactification of Ω is resolutive for the solutions of (E). For its proof, we use obstacle problems with respect to (E), which will be discussed in §4 and in the Appendix. Finally, the proof of Theorem 3.2 will be given in §5. One may see that the essential idea of the proof is already given in [**HKM**]. We need, however, some deviations from the arguments in [**HKM**] due to the existence of the term $\mathscr{B}(x,u)$ in (E).

Throughout this paper, we use some standard notation without explanation. One may refer to [**HKM**] for most of such notation.

§1. $(\mathscr{A},\mathscr{B})$-harmonic functions.

Let Ω be a domain in $\boldsymbol{R}^N$ $(N \geq 2)$ and we consider a quasi-linear elliptic differential equation

$$\text{(E)} \qquad -\operatorname{div} \mathscr{A}(x, \nabla u) + \mathscr{B}(x, u) = 0$$

on Ω. Here, $\mathscr{A} : \Omega \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ and $\mathscr{B} : \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ satisfy the following conditions for a fixed $1 < p < \infty$ and a *weight* w which is *p-admissible* in the sense of [**HKM**]:

(A.1) $x \mapsto \mathscr{A}(x, \xi)$ is measurable on Ω for every $\xi \in \boldsymbol{R}^N$ and $\xi \mapsto \mathscr{A}(x, \xi)$ is continuous for a.e. $x \in \Omega$;

(A.2) $\mathscr{A}(x, \xi) \cdot \xi \geq \alpha_1 w(x)|\xi|^p$ for all $\xi \in \boldsymbol{R}^N$ and a.e. $x \in \Omega$ with a constant $\alpha_1 > 0$;

(A.3) $|\mathscr{A}(x, \xi)| \leq \alpha_2 w(x)|\xi|^{p-1}$ for all $\xi \in \boldsymbol{R}^N$ and a.e. $x \in \Omega$ with a constant $\alpha_2 > 0$;

(A.4) $(\mathscr{A}(x, \xi_1) - \mathscr{A}(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0$ whenever $\xi_1, \xi_2 \in \boldsymbol{R}^N$, $\xi_1 \neq \xi_2$, for a.e. $x \in \Omega$;

(B.1) $x \mapsto \mathscr{B}(x, t)$ is measurable on Ω for every $t \in \boldsymbol{R}$ and $t \mapsto \mathscr{B}(x, t)$ is continuous for a.e. $x \in \Omega$;

(B.2) For any open set $D \Subset \Omega$, there is a constant $\alpha_3(D) \geq 0$ such that $|\mathscr{B}(x, t)| \leq \alpha_3(D) w(x)(|t|^{p-1} + 1)$ for all $t \in \boldsymbol{R}$ and a.e. $x \in D$;

(B.3) $t \mapsto \mathscr{B}(x, t)$ is nondecreasing on $\boldsymbol{R}$ for a.e. $x \in \Omega$.

We remark that if $\mathscr{A}$ and $\mathscr{B}$ satisfy the above conditions, then $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{B}}$ which are defined by

$$\tilde{\mathscr{A}}(x, \xi) = -\mathscr{A}(x, -\xi) \quad \text{and} \quad \tilde{\mathscr{B}}(x, t) = -\mathscr{B}(x, -t)$$

also satisfy these conditions with the same constants α_1, α_2 and $\alpha_3(D)$.

For the nonnegative measure $\mu : d\mu(x) = w(x)\,dx$ and an open subset D of Ω, we consider the weighted Sobolev spaces $H^{1,p}(D;\mu)$, $H_0^{1,p}(D;\mu)$ and $H_{\mathrm{loc}}^{1,p}(D;\mu)$ (see [**HKM**] for details).

$u \in H_{\mathrm{loc}}^{1,p}(D;\mu)$ is said to be a (weak) *solution* of (E) in D if

$$\int_D \mathscr{A}(x, \nabla u) \cdot \nabla \varphi \, dx + \int_D \mathscr{B}(x, u) \varphi \, dx = 0$$

for all $\varphi \in C_0^\infty(D)$. $u \in H^{1,p}_{\mathrm{loc}}(D;\mu)$ is said to be a *supersolution* (resp. *subsolution*) of (E) in D if

$$\int_D \mathscr{A}(x, \nabla u) \cdot \nabla \varphi \, dx + \int_D \mathscr{B}(x, u) \varphi \, dx \geq 0 \quad (\text{resp. } \leq 0)$$

for all nonnegative $\varphi \in C_0^\infty(D)$.

The following proposition can be proved in the same way as [**HKM**; Lemma 3.18] by using (B.3) as well as (A.4); see [**O**; Lemma 3.6] for details.

Proposition 1.1 (Comparison Principle I). *Let D be an open set such that $D \Subset \Omega$ and let $u \in H^{1,p}(D;\mu)$ be a supersolution and $v \in H^{1,p}(D;\mu)$ a subsolution of* (E) *in D. If* $\min(u - v, 0) \in H_0^{1,p}(D;\mu)$, *then $u \geq v$ a.e. in D.*

Corollary 1.1. *Let D be any open set in Ω. If u is a supersolution and v is a subsolution of* (E) *in D and if $u + \varepsilon \geq v$ a.e. outside a compact set in D for any $\varepsilon > 0$, then $u \geq v$ a.e. in D.*

Proof. For $\varepsilon > 0$, let $u + \varepsilon \geq v$ a.e. in $D \setminus K_\varepsilon$ with a compact set K_ε and let G be an open set such that $K_\varepsilon \subset G \Subset D$. By (B.3), $u + \varepsilon$ is a supersolution of (E). Since $\min(u + \varepsilon - v, 0) \in H_0^{1,p}(G;\mu)$, it follows from Proposition 1.1 that $u + \varepsilon \geq v$ a.e. in G. Therefore $u + \varepsilon \geq v$ a.e. in D. Since ε is arbitrary, $u \geq v$ a.e. in D.

Proposition 1.2. *Let D be any open set in Ω and let $\{u_n\}$ be a nondecreasing sequence of supersolutions of* (E) *in D. If there is $g \in H^{1,p}_{\mathrm{loc}}(D;\mu)$ such that $u_n \leq g$ a.e. for all n, then $u = \lim_{n\to\infty} u_n$ is a supersolution of* (E) *in D.*

Proof. First we show that $\{\int_G |\nabla u_n|^p \, d\mu\}$ is bounded for any $G \Subset D$. Choose $\eta \in C_0^\infty(D)$ such that $0 \leq \eta \leq 1$ in D and $\eta = 1$ on G. Since each u_n is a supersolution of (E) in D,

$$\int_D \mathscr{A}(x, \nabla u_n) \cdot \nabla[(g - u_n)\eta^p] \, dx + \int_D \mathscr{B}(x, u_n)(g - u_n)\eta^p \, dx \geq 0.$$

It follows that

$$\begin{aligned} \int_D [\mathscr{A}(x, \nabla u_n) \cdot \nabla u_n] \eta^p \, dx \leq & \int_D [\mathscr{A}(x, \nabla u_n) \cdot \nabla g] \eta^p \, dx \\ & + p \int_D [\mathscr{A}(x, \nabla u_n) \cdot \nabla \eta] \eta^{p-1} (g - u_n) \, dx \\ & + \int_D \mathscr{B}(x, u_n)(g - u_n)\eta^p \, dx. \end{aligned}$$

Hence, by (A.2), (A.3), (B.2) and (B.3), we have

$$\alpha_1 \int_D |\nabla u_n|^p \eta^p \, d\mu \le \alpha_2 \int_D |\nabla u_n|^{p-1} [|\nabla g| \eta^p + p|\nabla \eta|(g - u_1)\eta^{p-1}] \, d\mu$$

$$+ \alpha_3(D') \int_{D'} (1 + |g|^{p-1})(g - u_1) \, d\mu$$

$$\le \alpha_2 \left(\int_D |\nabla u_n|^p \eta^p \, d\mu \right)^{(p-1)/p} \left(\int_D [|\nabla g| \eta + p|\nabla \eta|(g - u_1)]^p \, d\mu \right)^{1/p}$$

$$+ \alpha_3(D') \int_{D'} (1 + |g|^{p-1})(g - u_1) \, d\mu,$$

where D' is an open set such that $\mathrm{Spt}\,\eta \subset D' \Subset D$. Noting that g, $u_1 \in H^{1,p}(D';\mu)$, we deduce that $\{\int_D |\nabla u_n|^p \eta^p \, d\mu\}$, and hence $\{\int_G |\nabla u_n|^p \, d\mu\}$ is bounded.

Since $u_1 \le u_n \le g$, $\{\int_G |u_n|^p \, d\mu\}$ is also bounded. Thus, by [**HKM**; Theorem 1.32], $u \in H^{1,p}(G;\mu)$ and $\nabla u_n \to \nabla u$ weakly in $L^p(G;\mu)$.

The rest of the proof can be carried out by suitably modifying the proof of [**HKM**; Theorem 3.75]; we may use Lebegue's convergence theorem to treat the terms involving $\mathscr{B}(x, u_n)$.

A continuous solution of (E) in D will be called $(\mathscr{A}, \mathscr{B})$-*harmonic* in D. Note that if h is $(\mathscr{A}, \mathscr{B})$-harmonic in D, then $-h$ is $(\tilde{\mathscr{A}}, \tilde{\mathscr{B}})$-harmonic in D.

The following three theorems can be proved by suitably modifying the arguments in [**S**] (see [**O**] for details).

Theorem 1.1. *Any solution of* (E) *in* D *is equal a.e. to an* $(\mathscr{A}, \mathscr{B})$*-harmonic function.*

Theorem 1.2 (Harnack Inequality). *Given an open set* $D \Subset \Omega$ *and* $R_0 > 0$, *there exists a constant* $c = c(p, c_\mu, \alpha_1, \alpha_2, \alpha_3(D), R_0) > 0$ *such that whenever* $0 < R \le R_0$ *and* $B(x, 3R) \subset D$

$$\sup_{B(x,R)} h \le c \left(\inf_{B(x,R)} h + R \right)$$

for any nonnegative $(\mathscr{A}, \mathscr{B})$*-harmonic function* h *in* $B(x, 3R)$.

Here, c_μ denotes a constant depending only on those constants which appear in the conditions for w to be p-admissible.

Theorem 1.3. *A locally uniformly bounded family of* $(\mathscr{A}, \mathscr{B})$*-harmonic functions in* D *is equi-continuous at each point of* D.

We say that an open set D in Ω is $(\mathscr{A},\mathscr{B})$-*regular*, if $D \Subset \Omega$ and for any $\theta \in H^{1,p}_{\mathrm{loc}}(\Omega;\mu)$ which is continuous at each point of ∂D, there exists a unique $h \in C(\bar{D}) \cap H^{1,p}(D;\mu)$ such that $h = \theta$ on ∂D and h is $(\mathscr{A},\mathscr{B})$-harmonic in D.

Theorem 1.4. *Let D be a bounded open set such that $\bar{D} \subset \Omega$. If $\boldsymbol{R}^N \setminus D$ is (p,μ)-thick in the sense of* [**HKM**] *at each point of ∂D, then D is $(\mathscr{A},\mathscr{B})$-regular.*

Outline of the proof. Given $\theta \in H^{1,p}_{\mathrm{loc}}(\Omega;\mu)$ which is continuous at each point of ∂D, by using the theory of monotone operators (cf. [**KS**]), we can show as in Appendix I of [**HKM**] the existence of $h \in H^{1,p}(D;\mu)$ such that $h - \theta \in H^{1,p}_0(D;\mu)$ and h is a solution of (E) in D (see [**O**] for details). By Theorem 1.1 above, we can take h to be continuous in D. We can prove that h assumes the values θ on ∂D by modifying the arguments in [**GZ**] (again, see [**O**] for details). The uniqueness follows from the comparison principle (Corollary 1.1 above).

From [**HKM**; Corollary 6.32], we obtain the following

Corollary 1.2. *For any compact set K and an open set D such that $K \subset D \subset \Omega$, there exists an $(\mathscr{A},\mathscr{B})$-regular open set G such that $K \subset G \subset D$.*

The following theorem can be proved in the same way as in the proof of Proposition 1.2; we may take $h_n\eta^p$ instead of $(g - u_n)\eta^p$:

Theorem 1.5. *If $\{h_n\}$ is a locally uniformly convergent sequence of $(\mathscr{A},\mathscr{B})$-harmonic functions in D, then $h := \lim_{n\to\infty} h_n$ is $(\mathscr{A},\mathscr{B})$-harmonic in D.*

Corollary 1.3. *If D is an $(\mathscr{A},\mathscr{B})$-regular open set in Ω, then for any $\psi \in C(\partial D)$, there exists a unique $h \in C(\bar{D})$ such that $h = \psi$ on ∂D and h is $(\mathscr{A},\mathscr{B})$-harmonic in D.*

Proof. Choose $\psi_n \in C^\infty(\Omega)$ such that $\psi_n \to \psi$ uniformly on ∂D. Put $\varepsilon_n = \sup_{\partial D}|\psi - \psi_n|$. Since D is an $(\mathscr{A}, \mathscr{B})$-regular open set, there exists $h_n \in C(\bar{D})$ such that $h_n = \psi_n$ on ∂D and h_n is $(\mathscr{A}, \mathscr{B})$-harmonic in D. Thus $h_n \le h_m + \varepsilon_n + \varepsilon_m$ on ∂D. Since $h_m + \varepsilon_n + \varepsilon_m$ is a supersolution of (E) in D, by Corollary 1.1, we have $h_n \le h_m + \varepsilon_n + \varepsilon_m$ in D. Similarly $h_m \le h_n + \varepsilon_n + \varepsilon_m$ in D. Therefore $\{h_n\}$ converges uniformly on $\bar{D}$. Hence the existence follows from Theorem 1.5. The uniqueness follows from Corollary 1.1.

Theorem 1.6 (Harnack principle). *If $\{h_n\}$ is a nondecreasing or nonincreasing sequence of $(\mathscr{A},\mathscr{B})$-harmonic functions in a domain D and if $\{h_n(x_0)\}$ is bounded for some $x_0 \in D$, then $h := \lim_{n\to\infty} h_n$ is $(\mathscr{A},\mathscr{B})$-harmonic in D.*

Proof. Suppose first that $\{h_n\}$ is nondecreasing. Let G be any domain such that $x_0 \in G \Subset D$ and let $\alpha = \inf_G h_1$. Let $\mathscr{B}_1(x,t) = \mathscr{B}(x, t+\alpha)$. Then, $\mathscr{B}_1$ satisfies conditions (B1)–(B3) (with possibly different $\alpha_3(D)$). Each $h_n - \alpha$ is nonnegative $(\mathscr{A}, \mathscr{B}_1)$-harmonic in G.

Let x_1 be any point in G such that $\{h_n(x_1)\}$ is bounded and let $B(x_1, 3R) \Subset G$. By the Harnack inequality applied to $(\mathscr{A}, \mathscr{B}_1)$, we have

$$h_n(x) - \alpha \le \sup_{B(x_1,R)} (h_n - \alpha) \le c\left(\inf_{B(x_1,R)} (h_n - \alpha) + R\right) \le c(h_n(x_1) - \alpha + R)$$

for each $x \in B(x_1, R)$. Therefore $\{h_n\}$ is uniformly bounded on $B(x_1, R)$ whenever $B(x_1, 3R) \Subset G$. From this, we infer that $\{h_n\}$ is locally uniformly bounded in G. Thus $\{h_n\}$ is equi-continuous by Theorem 1.3. Hence it follows from Ascoli-Arzela's theorem that $\{h_n\}$ has a locally uniformly convergent subsequence, and hence h is $(\mathscr{A}, \mathscr{B})$-harmonic in G by Theorem 1.5. Since G is arbitrary, it is $(\mathscr{A}, \mathscr{B})$-harmonic in D.

If $\{h_n\}$ is nonincreasing, then we apply the above result to $\{-h_n\}$, which is a nondecreasing sequence of $(\tilde{\mathscr{A}}, \tilde{\mathscr{B}})$-harmonic functions in D.

§2. $(\mathscr{A}, \mathscr{B})$-superharmonic functions.

Let D be an open subset of Ω. A function $u : D \to \boldsymbol{R} \cup \{\infty\}$ is said to be $(\mathscr{A}, \mathscr{B})$-*superharmonic* in D if it is lower semicontinuous, finite on a dense set in D and, for each open set $G \Subset D$ and for $h \in C(\overline{G})$ which is $(\mathscr{A}, \mathscr{B})$-harmonic in G, $u \ge h$ on ∂G implies $u \ge h$ in G. $(\mathscr{A}, \mathscr{B})$-*subharmonic* functions are similarly defined. v is $(\mathscr{A}, \mathscr{B})$-subharmonic in D if and only if $-v$ is $(\tilde{\mathscr{A}}, \tilde{\mathscr{B}})$-superharmonic in D.

By Corollary 1.1, we see that a continuous supersolution (resp. subsolution) of (E) in D is $(\mathscr{A}, \mathscr{B})$-superharmonic (resp. $(\mathscr{A}, \mathscr{B})$-subharmonic).

The following proposition is clear.

Proposition 2.1. *If u, v are $(\mathscr{A}, \mathscr{B})$-superharmonic in D, then so is* $\min(u, v)$.

Theorem 2.1 (Comparison Principle II). *Let u be $(\mathscr{A}, \mathscr{B})$-superharmonic in D and v be $(\mathscr{A}, \mathscr{B})$-subharmonic in D. If*

$$\liminf_{x \to \xi} \{u(x) - v(x)\} \ge 0$$

for all $\xi \in \partial^a D$, then $u \ge v$ in D, where $\partial^a D$ is the boundary of D in the one point compactification of $\boldsymbol{R}^N$.

Proof. Fix $\xi \in \partial^a D$ and $\varepsilon > 0$. There is a compact set $K \subset D$ such that $u + \varepsilon \geq v$ in $D \backslash K$. Choose an $(\mathscr{A}, \mathscr{B})$-regular open set G such that $K \subset G \Subset D$. We can find $\psi \in C(\partial G)$ such that $u + 2\varepsilon \geq \psi \geq v$ on ∂G. Then there exist $(\mathscr{A}, \mathscr{B})$-harmonic functions $h_1 \in C(\overline{G})$ and $h_2 \in C(\overline{G})$ such that $h_1 = \psi - 2\varepsilon$ on ∂G and $h_2 = \psi$ on ∂G. Since $h_1 + 2\varepsilon$ is a supersolution and h_2 is a subsolution,

$$h_1 + 2\varepsilon \geq h_2 \quad \text{in } G$$

by Corollary 1.1. On the other hand $u \geq h_1$ on ∂G implies $u \geq h_1$ in G and $v \leq h_2$ on ∂G implies $v \leq h_2$ in G. Consequently $u + 2\varepsilon \geq v$ in G. Hence $u + 2\varepsilon \geq v$ in D. Since ε is arbitrary, the theorem follows.

Corollary 2.1. *If u is $(\mathscr{A}, \mathscr{B})$-superharmonic* (resp. *$(\mathscr{A}, \mathscr{B})$-subharmonic*) *in D and $\alpha > 0$ is a constant, then $u + \alpha$* (resp. $u - \alpha$) *is $(\mathscr{A}, \mathscr{B})$-superharmonic* (resp. *$(\mathscr{A}, \mathscr{B})$-subharmonic*) *in D.*

Proof. Let $G \Subset D$ be an open set and let $h \in C(\overline{G})$ be $(\mathscr{A}, \mathscr{B})$-harmonic in G such that $u + \alpha \geq h$ on ∂G. Then $h - \alpha$ is a continuous subsolution of (E), so that it is $(\mathscr{A}, \mathscr{B})$-subharmonic in G. Further,

$$\liminf_{x \to \xi} \{u(x) - (h(x) - \alpha)\} \geq 0$$

for all $\xi \in \partial G$. Therefore, by Theorem 2.1, $u + \alpha \geq h$ in G. Hence $u + \alpha$ is $(\mathscr{A}, \mathscr{B})$-superharmonic in D. The proof for $u - \alpha$ is similar.

We can easily prove the following proposition.

Proposition 2.2. *Let $\{u_n\}$ be a sequence of $(\mathscr{A}, \mathscr{B})$-superharmonic functions in D.*

(1) *If $\{u_n\}$ converges locally uniformly in D, then $u := \lim_{n \to \infty} u_n$ is $(\mathscr{A}, \mathscr{B})$-superharmonic in D.*

(2) *If $\{u_n\}$ is nondecreasing and $u := \lim_{n \to \infty} u_n$ is finite on a dense set in D, then u is $(\mathscr{A}, \mathscr{B})$-superharmonic in D.*

The following proposition can be shown in the same manner as [**HKM**, Lemma 7.14].

Proposition 2.3. *Let D be an open set in Ω and let G be an $(\mathscr{A}, \mathscr{B})$-regular open set such that $\overline{G} \subset D$. For an $(\mathscr{A}, \mathscr{B})$-superharmonic function u on D, we define*

$$u_G = \sup\{h \in C(\overline{G}) \mid h \leq u \text{ on } \partial G \text{ and } h \text{ is } (\mathscr{A}, \mathscr{B})\text{-harmonic in } G\}.$$

Then

$$P(u, G) := \begin{cases} u & \text{in } D \setminus G \\ u_G & \text{in } G \end{cases}$$

is $(\mathscr{A}, \mathscr{B})$*-superharmonic in* D, $(\mathscr{A}, \mathscr{B})$*-harmonic in* G *and* $P(u, G) \le u$ *in* D.

Remark 2.1. If $u \in H^{1,p}_{\mathrm{loc}}(\Omega; \mu) \cap C(\overline{G})$, then u_G coincides with the $(\mathscr{A}, \mathscr{B})$-harmonic function h in G such that $h - u|_G \in H^{1,p}_0(G; \mu)$.

§3. Dirichlet problems with respect to an ideal boundary.

Let Ω^* be a compactification of Ω and let $\partial^* \Omega = \Omega^* \setminus \Omega$. Given a bounded function ψ on $\partial^* \Omega$, let

$$\mathscr{U}_\psi = \left\{ u : \begin{array}{l} (\mathscr{A}, \mathscr{B})\text{-superharmonic in } \Omega \text{ and} \\ \liminf_{x \to \xi} u(x) \ge \psi(\xi) \text{ for all } \xi \in \partial^* \Omega \end{array} \right\}$$

and

$$\mathscr{L}_\psi = \left\{ v : \begin{array}{l} (\mathscr{A}, \mathscr{B})\text{-subharmonic in } \Omega \text{ and} \\ \limsup_{x \to \xi} v(x) \le \psi(\xi) \text{ for all } \xi \in \partial^* \Omega \end{array} \right\}.$$

Theorem 3.1. *If both* $\mathscr{U}_\psi$ *and* $\mathscr{L}_\psi$ *are nonempty, then*

$$\overline{H}(\psi; \Omega^*) := \inf \mathscr{U}_\psi \quad \text{and} \quad \underline{H}(\psi; \Omega^*) := \sup \mathscr{L}_\psi$$

are $(\mathscr{A}, \mathscr{B})$*-harmonic in* Ω *and* $\underline{H}(\psi; \Omega^*) \le \overline{H}(\psi; \Omega^*)$.

Proof. By the comparison principle Theorem 2.1, we see that $\underline{H}(\psi; \Omega^*) \le \overline{H}(\psi; \Omega^*)$. Since we have Propositions 2.1 and 2.3 as well as Theorem 1.6, we can carry out the Perron's method to obtain the $(\mathscr{A}, \mathscr{B})$-harmonicity of $\underline{H}(\psi; \Omega^*)$ and $\overline{H}(\psi; \Omega^*)$ (cf. [**HKM**; Theorem 9.2]).

We say that ψ is $(\mathscr{A}, \mathscr{B})$-*resolutive* if both $\mathscr{U}_\psi$ and $\mathscr{L}_\psi$ are nonempty and $\underline{H}(\psi; \Omega^*) = \overline{H}(\psi; \Omega^*)$. Ω^* is called an $(\mathscr{A}, \mathscr{B})$-*resolutive* compactification, if all $\psi \in C(\partial^* \Omega)$ are $(\mathscr{A}, \mathscr{B})$-resolutive.

Proposition 3.1. *If* $\{\psi_n\}$ *is a uniformly convergent sequence of* $(\mathscr{A}, \mathscr{B})$*-resolutive functions on* $\partial^* \Omega$, *then* $\psi := \lim_{n \to \infty} \psi_n$ *is* $(\mathscr{A}, \mathscr{B})$*-resolutive.*

Proof. Let $\varepsilon_n = \sup_{\xi \in \partial^* \Omega} |\psi_n(\xi) - \psi(\xi)|$. By assumption, $\varepsilon_n \to 0$. Since $\psi_n + \varepsilon_n \ge \psi$, $u + \varepsilon_n \in \mathscr{U}_\psi$ for any $u \in \mathscr{U}_{\psi_n}$. It follows that $\overline{H}(\psi_n; \Omega^*) + \varepsilon_n \ge \overline{H}(\psi; \Omega^*)$. Similarly,

$\underline{H}(\psi_n;\Omega^*) - \varepsilon_n \le \underline{H}(\psi;\Omega^*)$. Since ψ_n is $(\mathscr{A},\mathscr{B})$-resolutive, it follows that $\overline{H}(\psi;\Omega^*) - \underline{H}(\psi;\Omega^*) \le 2\varepsilon_n$ for all n. Hence, $\overline{H}(\psi;\Omega^*) = \underline{H}(\psi;\Omega^*)$.

We consider the following condition:

(C_0) There exist a lower bounded $(\mathscr{A},\mathscr{B})$-superharmonic function on Ω and an upper bounded $(\mathscr{A},\mathscr{B})$-subharmonic function on Ω.

LEMMA 3.1. *If condition* (C_0) *is satisfied, then both* $\mathscr{U}_\psi$ *and* $\mathscr{L}_\psi$ *are nonempty for any bounded function* ψ *on* $\partial^*\Omega$.

PROOF. Let v_1 be an $(\mathscr{A},\mathscr{B})$-superharmonic function on Ω such that $v_1 \ge c_1$ and v_2 be an $(\mathscr{A},\mathscr{B})$-subharmonic function on Ω such that $v_2 \le c_2$. Given a bounded ψ on $\partial^*\Omega$, $v_1 + c_1' \in \mathscr{U}_\psi$ whenever $c_1' \ge 0$ and $c_1 + c_1' \ge \sup\psi$ as well as $v_2 - c_2' \in \mathscr{L}_\psi$ whenever $c_2' \ge 0$ and $c_2 - c_2' \le \inf\psi$.

REMARK 3.1. If $\mathscr{B}$ satisfies the following condition (B.4), then condition (C_0) is satisfied:

(B.4) There exist t_+ and t_- such that $\mathscr{B}(x,t_+) \ge 0$ and $\mathscr{B}(x,t_-) \le 0$ a.e. in Ω.

In fact, the constant function t_+ is $(\mathscr{A},\mathscr{B})$-superharmonic and t_- is $(\mathscr{A},\mathscr{B})$-subharmonic on Ω.

Now we consider the following spaces:

$$\mathscr{D}^p(\Omega;\mu) := \{f \in H^{1,p}_{\mathrm{loc}}(\Omega;\mu) \mid |\nabla f| \in L^p(\Omega;\mu),\ f \text{ is bounded continuous}\},$$

$$\mathscr{D}^p_0(\Omega;\mu) := \left\{ f \in \mathscr{D}^p(\Omega;\mu) \,\middle|\, \begin{array}{l} {}^\exists\varphi_n \in C_0^\infty(\Omega) \text{ s.t. } \varphi_n \to f \text{ a.e., } \{\varphi_n\} \text{ is} \\ \text{uniformly bounded, } \nabla\varphi_n \to \nabla f \text{ in } L^p(\Omega;\mu) \end{array} \right\}.$$

We say that Ω is (p,μ)-*hyperbolic* if $1 \notin \mathscr{D}^p_0(\Omega;\mu)$.

EXAMPLES. (1) Any bounded domain is (p,μ)-hyperbolic. This fact follows from the Poincaré inequality ([**HKM**; 1.4]).

(2) For $\delta > -N$, $\boldsymbol{R}^N$ is $(p,|x|^\delta\,dx)$-hyperbolic if and only if $p < N+\delta$.

To show that $\boldsymbol{R}^N$ is not $(p,|x|^\delta\,dx)$-hyperbolic if $p \ge N+\delta$, it is enough to consider the functions

$$\varphi_n(x) = \begin{cases} 1, & |x| \le 1 \\ 1 - \dfrac{\log|x|}{\log n}, & 1 < |x| < n\ , \quad n = 2,3,\dots \\ 0, & |x| \ge n. \end{cases}$$

We see that $\int_{\boldsymbol{R}^N} |\nabla \varphi_n(x)|^p |x|^\delta \, dx \to 0$ $(n \to \infty)$ if $p \ge N + \delta$, while $\{\varphi_n\}$ is uniformly bounded and $\varphi_n \to 1$, so that $1 \in \mathscr{D}_0^p(\boldsymbol{R}^N; \mu)$.

On the other hand, if $p < N + \delta$, then we can show that

$$\int_{1 \le |x| \le 2} |\varphi(x)|^p \, dx \le C(N, p, \delta) \int_{|x| \ge 1} |\nabla \varphi(x)|^p |x|^\delta \, dx$$

for any $\varphi \in C_0^\infty(\boldsymbol{R}^N)$. From this it follows that $\boldsymbol{R}^N$ is $(p, |x|^\delta \, dx)$-hyperbolic if $p < N + \delta$.

Now we state our main theorem.

Theorem 3.2. *Suppose Ω is (p, μ)-hyperbolic and the following conditions* (C_1) *and* (B.5) *are satisfied*:

(C_1) *There exist a bounded supersolution of* (E) *in Ω and a bounded subsolution of* (E) *in Ω.*

(B.5) $\int_\Omega |\mathscr{B}(x, t)| \, dx < \infty$ *for any $t \in \boldsymbol{R}$.*

If $Q \subset \mathscr{D}^p(\Omega; \mu)$, then the Q-compactification Ω_Q^* of Ω (see [**CC**]) *is an $(\mathscr{A}, \mathscr{B})$-resolutive compactification.*

Remark 3.2. Condition (C_1) implies condition (C_0) (cf. Corollary 4.1 in the next section). As in Remark 3.1, condition (B.4) implies (C_1). The following example shows that (C_1) is satisfied even when (B.4) does not hold.

Example. In the case $\mathscr{A}(x, \xi) = |\xi|^{p-2}\xi$, if Ω is a bounded domain and $\mathscr{B}(x, 0)$ is bounded on Ω, then (C_1) is satisfied.

Proof. Let $q = p/(p-1)$. Suppose $\Omega \subset \{|x| < R\}$ and $|\mathscr{B}(x, 0)| \le M$ on Ω. Put

$$a = \frac{1}{q} \left(\frac{M}{N} \right)^{1/(p-1)} \quad \text{and} \quad v(x) = a(R^q - |x|^q).$$

Since $v \ge 0$ on Ω,

$$\mathscr{B}(x, v(x)) \ge \mathscr{B}(x, 0) \ge -M = -N(aq)^{p-1}.$$

Now, $\nabla v(x) = -aq|x|^{q-2}x$, so that $v \in H^{1,p}(\Omega; dx)$ and

$$\mathscr{A}(x, \nabla v(x)) = -(aq)^{p-1} x.$$

Thus,

$$-\mathrm{div}\,\mathscr{A}(x,\nabla v(x))+\mathscr{B}(x,v(x))=N(aq)^{p-1}+\mathscr{B}(x,v(x))\geq 0.$$

Therefore, v is a bounded supersolution of (E) in Ω. Similarly, we see that $-v$ is a bounded subsolution of (E).

For the proof of Theorem 3.2, we need discussions on obstacle problems which will be given in the next section. The theorem will be proved in §5.

Corollary 3.1. *If $D\Subset\Omega$, then the closure $\bar{D}$ is an $(\mathscr{A},\mathscr{B})$-resolutive compactification of D.*

Proof. Choose an $(\mathscr{A},\mathscr{B})$-regular open set G containing $\bar{D}$ and let u_0 be the $(\mathscr{A},\mathscr{B})$-harmonic function in G assuming values 0 on ∂G. Then condition (C_1) is satisfied with the function $u_0|_D$. Condition (B.5) for D follows from (B.2). If we take $Q=C^\infty(\bar{D})$, then the Q-compactification coincides with $\bar{D}$. Since $C^\infty(\bar{D})\subset\mathscr{D}^p(\Omega;\mu)$, this corollary is a consequence of Theorem 3.2.

Corollary 3.2. *Suppose $w(x)\equiv 1$, Ω is p-hyperbolic (i.e., (p,dx)-hyperbolic) and conditions (C_1) and (B.5) are satisfied. Then the p-Kuramochi compactification (see [**T**]) of Ω is $(\mathscr{A},\mathscr{B})$-resolutive.*

§4. Obstacle problems.

Let D be an open set such that $D\Subset\Omega$, f be a $[-\infty,\infty]$-valued function on D and $\theta\in H^{1,p}(D;\mu)$. As in [**HKM**], let

$$\mathscr{K}_{f,\theta}(D)=\{v\in H^{1,p}(D;\mu)\,|\,v\geq f \text{ a.e. in } D,\ v-\theta\in H_0^{1,p}(D;\mu)\}.$$

We shall say that $u\in H^{1,p}(D;\mu)$ is a solution to the obstacle problem $\mathrm{OBP}(\mathscr{A},\mathscr{B};f,\theta;D)$, if $u\in\mathscr{K}_{f,\theta}(D)$ and

$$\int_D\mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx+\int_D\mathscr{B}(x,u)\varphi\,dx\geq 0$$

for all $\varphi\in H_0^{1,p}(D;\mu)$ such that $u+\varphi\geq f$ a.e. in D (i.e., $u+\varphi\in\mathscr{K}_{f,\theta}(D)$).

A solution to $\mathrm{OBP}(\mathscr{A},\mathscr{B};f,\theta;D)$ is a supersolution of (E) in D.

If u is a solution to $\mathrm{OBP}(\mathscr{A},\mathscr{B};f,\theta;D)$, then $u|_{D'}$ is a solution to $\mathrm{OBP}(\mathscr{A},\mathscr{B};f,u;D')$ for any open set $D'\subset D$.

Lemma 4.1. *Suppose u is a solution to $\mathrm{OBP}(\mathscr{A},\mathscr{B};f,\theta;D)$. If $v\in H^{1,p}(D;\mu)$ is a supersolution of (E) in D and $\min(u,v)\in\mathscr{K}_{f,\theta}(D)$, then $v\geq u$ a.e. in D.*

This lemma can be proved by suitably modifying the proof of [**HKM**; Lemma 3.22] (cf. [**O**; Lemma 3.7]).

Theorem 4.1. *If $\mathcal{K}_{f,\theta}(D) \neq \emptyset$, then the obstacle problem* $\mathrm{OBP}(\mathcal{A},\mathcal{B}; f,\theta; D)$ *admits a unique solution. Furthermore, if f is locally bounded above, then the solution has a representative u which satisfies*

$$u(x) = \lim_{\rho\to+0} \frac{1}{\mu(B(x,\rho))}\int_{B(x,\rho)} u\,d\mu = \liminf_{y\to x} u(x) \tag{4.1}$$

for all $x \in D$.

The existence of a solution in Theorem 4.1 can be shown by using the theory of monotone operators as in Appendix I of [**HKM**] and the uniqueness follows from Lemma 4.1 (see [**O**] for details). We give an outline of the proof of the last half of Theorem 4.1 as well as that of the next theorem in the Appendix. In case $w = 1$, these results have been shown in a more general setting, e.g. in [**MZ**; pp. 1439–1441]. A straightforward extension of the arguments in [**MZ**] to the weighted case seems to be invalid.

Theorem 4.2. *If $\mathcal{K}_{f,\theta}(D) \neq \emptyset$ and f is continuous, then the solution u of* $\mathrm{OBP}(\mathcal{A},\mathcal{B}; f,\theta; D)$ *which satisfies* (4.1) *is continuous in D. Furthermore, it is $(\mathcal{A},\mathcal{B})$-harmonic in the open set $\{x \in D \mid u(x) > f(x)\}$.*

As a consequence of Theorem 4.1, we obtain the following (cf. [**HKM**; Theorem 7.16 and Corollary 7.18]):

Corollary 4.1. *Any supersolution of* (E) *has an $(\mathcal{A},\mathcal{B})$-superharmonic representative.*

Proof. Let u be a supersolution of (E) in an open set $G \subset \Omega$ and let $\hat{u}(x) = \operatorname{ess\,lim\,inf}_{y\to x} u(y)$ for all $x \in G$. If we can show that $\hat{u} = u$ a.e. on G, then we see that $\hat{u}$ is $(\mathcal{A},\mathcal{B})$-superharmonic in G as in the proof of [**HKM**; Theorem 7.16].

Let $G_0 \Subset G$ be an arbitrary $(\mathcal{A},\mathcal{B})$-regular open set. Then there is a bounded $(\mathcal{A},\mathcal{B})$-harmonic function h_0 in G_0. For any $n \in \boldsymbol{N}$, $h_0 + n$ is a supersolution of (E). Thus we see by Lemma 4.1 that $u_n := \min(u, h_0 + n)$ is the solution of $\mathrm{OBP}(\mathcal{A},\mathcal{B}; u_n, u_n; D)$ for any $D \Subset G_0$ (cf. the proof of [**HKM**; Theorem 3.23]). Hence, u_n has a representative $\tilde{u}_n$ satisfying (4.1) for all $x \in G_0$ by Theorem 4.1. Then, $\tilde{u} := \lim_{n\to\infty} \tilde{u}_n$ is equal to u a.e. and is lower semicontinuous in G_0. It then follows that $\tilde{u} = \hat{u}$ a.e. in G_0 (see the last half of the proof of [**HKM**; Theorem 3.63]), so that $\hat{u} = u$ a.e. on G_0. Since G_0 is arbitrary, $\hat{u} = u$ a.e. on G.

§5. Proof of Theorem 3.2.

In order to prove Theorem 3.2, we prepare two lemmas.

Lemma 5.1. *Let $\{u_n\}$ be a uniformly bounded sequence of functions in $H_0^{1,p}(\Omega;\mu)$ such that $\{\int_\Omega |\nabla u_n|^p\,d\mu\}$ is bounded and $u_n \to u$ a.e. in Ω. If u is continuous, then $u \in \mathscr{D}_0^p(\Omega;\mu)$.*

Proof. By [**HKM**; Lemma 1.33], we see that $u \in H_{\mathrm{loc}}^{1,p}(\Omega;\mu)$, $\int_\Omega |\nabla u|^p\,d\mu < \infty$ and $\nabla u_n \to \nabla u$ weakly in $L^p(\Omega;\mu)$. Obviously, u is bounded continuous in Ω. Hence, $u \in \mathscr{D}^p(\Omega;\mu)$.

Now, let $|u_n| \le M$ for all n and choose $\eta_n \in C_0^\infty(\Omega)$ such that $|\eta_n| \le M$ and

$$\int_\Omega |\nabla\eta_n - \nabla u_n|^p\,d\mu + \int_\Omega |\eta_n - u_n|^p\,d\mu < \frac{1}{2^n}, \quad n = 1, 2, \dots.$$

Then, $\nabla\eta_n \to \nabla u$ weakly in $L^p(\Omega;\mu)$. Thus, using Mazur's lemma, we can find a sequence $\{\varphi_k\}$ such that each φ_k is a convex combination of functions in $\{\eta_n\}_{n\ge k}$ and $\nabla\varphi_k \to \nabla u$ strongly in $L^p(\Omega;\mu)$. Then $\varphi_k \in C_0^\infty(\Omega)$ and $|\varphi_k| \le M$ for each k. Since $u_n \to u$ in $L^p(D;\mu)$ for any $D \Subset \Omega$ by Lebegue's convergence theorem, $\eta_n \to u$ in $L^p(D;\mu)$, and hence $\varphi_k \to u$ in $L^p(D;\mu)$ for any $D \Subset \Omega$. Hence, taking a subsequence, we may assume that $\varphi_k \to u$ a.e. in Ω. Therefore, $u \in \mathscr{D}_0^p(\Omega;\mu)$.

Lemma 5.2. *Let $f \in \mathscr{D}^p(\Omega;\mu)$ and suppose there is a bounded supersolution g of* (E) *in Ω such that $g \ge f$ in Ω and suppose*

$$\int_\Omega \mathscr{B}(x,f)^-\,dx < \infty. \tag{5.1}$$

Then there exists an $(\mathscr{A},\mathscr{B})$-superharmonic function u in Ω such that $u \ge f$ in Ω and $u - f \in \mathscr{D}_0^p(\Omega;\mu)$.

Proof. Let $\{D_n\}$ be an exhaustion of Ω by domains $D_n \Subset \Omega$ and let u_n be the solution to the obstacle problem $\mathrm{OBP}(\mathscr{A},\mathscr{B};f,f;D_n)$. Then, by Theorem 4.2, we may assume that u_n is continuous in D_n and by Lemma 4.1, $u_n \le g$ a.e. in D_n.

Now, let $n < m$. Let $v = u_m|_{D_n}$. Since $v \ge f$ on D_n,

$$0 \le \min(v, u_n) - f \le u_n - f \quad \text{in } D_n.$$

Hence, $\min(v,u_n) - f \in H_0^{1,p}(D_n;\mu)$, so that $\min(v,u_n) \in \mathscr{K}_{f,f}(D_n)$. Since v is a supersolution of (E) in D_n, Lemma 4.1 implies that $v \ge u_n$, namely, $u_m \ge u_n$ in D_n. Let $u_0 = \lim_{n\to\infty} u_n$. Then, $u_0 \ge f$ on Ω. Since $u_n \le g$ a.e. in D_n, u_0 is a supersolution of (E) in Ω by Proposition 1.2.

Next, let D be any open set such that $D \Subset \Omega$ and let u_D be the continuous solution to the obstacle problem $\mathrm{OBP}(\mathscr{A},\mathscr{B}; f, u_0; D)$ (cf. Theorems 4.1 and 4.2). Since u_0 is a supersolution of (E) in D and $\min(u_0, u_D) \in \mathscr{K}_{f,u_0}(D)$, we have $u_0 \geq u_D$ a.e. on D by Lemma 4.1. On the other hand, if $D_n \supset D$, then u_n, being the solution of $\mathrm{OBP}(\mathscr{A},\mathscr{B}; f, f; D_n)$, is also the solution of $\mathrm{OBP}(\mathscr{A},\mathscr{B}; f, u_n; D)$. Since u_D is a supersolution of (E) in D and $\min(u_n, u_D) \in \mathscr{K}_{f,u_n}(D)$ (note that $0 \leq u_n - \min(u_n, u_D) \leq \max(u_0 - u_D, 0) \in H_0^{1,p}(D)$), $u_D \geq u_n$ in D by Lemma 4.1 again. Letting $n \to \infty$, we have $u_D \geq u_0$, and hence $u_0 = u_D$ a.e. on D. Thus, there is a continuous function u in Ω such that $u = u_0$ a.e. on Ω. Then $u \geq f$ and, being a continuous supersolution of (E), u is $(\mathscr{A},\mathscr{B})$-superharmonic in Ω.

Since u_n is the solution to the obstacle problem $\mathrm{OBP}(\mathscr{A},\mathscr{B}; f, f; D_n)$,

$$\int_{D_n} \mathscr{A}(x, \nabla u_n) \cdot \nabla(f - u_n)\, dx + \int_{D_n} \mathscr{B}(x, u_n)(f - u_n)\, dx \geq 0,$$

so that

$$\alpha_1 \int_{D_n} |\nabla u_n|^p\, d\mu \leq \alpha_2 \int_{D_n} |\nabla u_n|^{p-1} |\nabla f|\, d\mu + \int_{D_n} \mathscr{B}(x, u_n)(f - u_n)\, dx. \tag{5.2}$$

Now, since $f \leq u_n \leq g$ a.e.,

$$\int_{D_n} \mathscr{B}(x, u_n)(f - u_n)\, dx \leq \int_{D_n} \mathscr{B}(x, f)(f - u_n)\, dx \leq \int_{\Omega} \mathscr{B}(x, f)^-(g - f)\, dx < \infty$$

by assumption (5.1).

Therefore, from (5.2), we deduce that $\{\int_{D_n} |\nabla u_n|^p\, d\mu\}$ is bounded. Extend each u_n by f on $\Omega \setminus D_n$ and denote the extended function again by u_n. Then $u_n - f \in H_0^{1,p}(\Omega;\mu)$, $\{u_n - f\}$ is uniformly bounded on Ω, $u_n - f \to u - f$ a.e. in Ω and $\{\int_\Omega |\nabla u_n - \nabla f|^p\, d\mu\}$ is bounded. Hence, by Lemma 5.1, $u - f \in \mathscr{D}_0^p(\Omega;\mu)$.

Proof of Theorem 3.2. We may assume that Q is a linear subspace of $\mathscr{D}^p(\Omega;\mu)$ containing constant functions and closed under max and min operations. Let v_1 (resp. v_2) be a bounded supersolution (resp. subsolution) of (E) in Ω. We may assume that v_1 is $(\mathscr{A},\mathscr{B})$-superharmonic and v_2 is $(\mathscr{A},\mathscr{B})$-subharmonic in Ω (Corollary 4.1). For simplicity, let $\Gamma = \Omega_Q^* \setminus \Omega$. Let $f \in Q$ and let ψ be the continuous extension of f to Γ. Since f is bounded, there is a constant $c \geq 0$ such that $g := v_1 + c \geq f$ on Ω and by condition (B.5), (5.1) is satisfied. Hence, by the above lemma, there is an $(\mathscr{A},\mathscr{B})$-superharmonic function u in Ω such that $u \geq f$ and $u - f \in \mathscr{D}_0^p(\Omega;\mu)$. Choose an exhaustion $\{D_n\}$ of Ω by $(\mathscr{A},\mathscr{B})$-regular open sets and put $u_n = P(u, D_n)$ in the notation in Proposition 2.3. Then, $u_n \in \mathscr{U}_\psi$, so that $u_n \geq \overline{H}(\psi; \Omega_Q^*)$ for each n. On the other hand, by the comparison principle, $u \geq u_n \geq u_{n+1}$. Thus, by Theorem 1.6, $\bar{u} := \lim_{n\to\infty} u_n$ is $(\mathscr{A},\mathscr{B})$-harmonic in Ω and $u \geq \bar{u} \geq \overline{H}(\psi;\Omega_Q^*)$.

Since u_n is $(\mathscr{A},\mathscr{B})$-harmonic in D_n, $u_n - u \in H_0^{1,p}(D_n;\mu)$ and $u_n = u$ on $\Omega\backslash D_n$, we have

$$\int_\Omega \mathscr{A}(x,\nabla u_n)\cdot(\nabla u_n - \nabla u)\,dx + \int_\Omega \mathscr{B}(x,u_n)\,(u_n - u)\,dx = 0. \tag{5.3}$$

Now, $u \in \mathscr{D}^p(\Omega;\mu)$, so that $\int_\Omega |\nabla u|^p\,d\mu < \infty$. If $c' \geq 0$ is so chosen that $v_2 - c' \leq f$ in Ω, then by the comparison principle, $v_2 - c' \leq u_n \leq u$, so that $\{u - u_n\}$ is uniformly bounded. Hence, by condition (B.5), $\{\int_\Omega \mathscr{B}(x,u_n)(u-u_n)\,dx\}$ is bounded. It then follows from (5.3) that $\{\int_\Omega |\nabla u_n|^p\,d\mu\}$ is bounded, and hence $\{\int_\Omega |\nabla(u-u_n)|^p\,d\mu\}$ is bounded. Since $u - u_n \in H_0^{1,p}(\Omega;\mu)$ and $u - \bar{u}$ is continuous in Ω, $u - \bar{u} \in \mathscr{D}_0^p(\Omega;\mu)$ by Lemma 5.1, and hence $\bar{u} - f \in \mathscr{D}_0^p(\Omega;\mu)$.

Similarly, applying the above arguments to $(\tilde{\mathscr{A}},\tilde{\mathscr{B}})$ and $-f$, we obtain a bounded $(\mathscr{A},\mathscr{B})$-harmonic function $\underline{u}$ in Ω such that $\underline{u} \leq \underline{H}(\psi;\Omega_Q^*)$ and $f - \underline{u} \in \mathscr{D}_0^p(\Omega;\mu)$.

Therefore, $\bar{u} - \underline{u} \in \mathscr{D}_0^p(\Omega;\mu)$, so that there is a uniformly bounded sequence $\{\varphi_n\}$ in $C_0^\infty(\Omega)$ such that $\varphi_n \to \bar{u} - \underline{u}$ a.e. in Ω and $\nabla\varphi_n \to \nabla\bar{u} - \nabla\underline{u}$ in $L^p(\Omega;\mu)$. By the $(\mathscr{A},\mathscr{B})$-harmonicity of $\bar{u}$ and $\underline{u}$,

$$\int_\Omega \mathscr{A}(x,\nabla\bar{u})\cdot\nabla\varphi_n\,dx + \int_\Omega \mathscr{B}(x,\bar{u})\varphi_n\,dx = 0,$$

$$\int_\Omega \mathscr{A}(x,\nabla\underline{u})\cdot\nabla\varphi_n\,dx + \int_\Omega \mathscr{B}(x,\underline{u})\varphi_n\,dx = 0.$$

Subtracting these two equations and letting $n \to \infty$, we have

$$\int_\Omega [\mathscr{A}(x,\nabla\bar{u}) - \mathscr{A}(x,\nabla\underline{u})]\cdot(\nabla\bar{u} - \nabla\underline{u})\,dx + \int_\Omega [\mathscr{B}(x,\bar{u}) - \mathscr{B}(x,\underline{u})](\bar{u} - \underline{u})\,dx = 0.$$

By (A.4) and (B.3), we deduce that $\nabla\bar{u} = \nabla\underline{u}$ a.e. in Ω, and hence $\bar{u} = \underline{u} + c$. By the assumption that Ω is (p,μ)-hyperbolic, we see that $c = 0$, namely, $\bar{u} = \underline{u}$, which implies that $\overline{H}(\psi;\Omega_Q^*) = \underline{H}(\psi;\Omega_Q^*)$, that is, ψ is $(\mathscr{A},\mathscr{B})$-resolutive.

Since the set of continuous extensions of functions in Q is dense in $C(\Gamma)$ with respect to the uniform convergence, we conclude that every $\psi \in C(\Gamma)$ is $(\mathscr{A},\mathscr{B})$-resolutive by Proposition 3.1. □

Appendix: Continuity of solutions to obstacle problems.

In this appendix, we give an outline of the proof of the last half of Theorem 4.1 and Theorem 4.2, namely the continuity of solutions to obstacle problems. For the most part, we follow the discussions in [**MZ**], in which the case $w(x) = 1$ is treated. However, we cannot completely follow [**MZ**] due to the fact that our weight w is not necessarily translation invariant.

Let $D \neq \emptyset$ be an open set in Ω and $C > 0$ be a constant. Let ε denote the symbol $+$ or $-$. If $v \in H^{1,p}(D;\mu)$ is nonnegative and satisfies

$$\int_D |\nabla(v-k)^\varepsilon|^p \eta^p \, d\mu \leq C \int_D \{(v-k)^\varepsilon\}^p \{\eta^p + |\nabla\eta|^p\} \, d\mu + Ck^p \int_{\{(v-k)^\varepsilon > 0\}} \eta^p \, d\mu$$

for all $k \geq 0$ and $\eta \in C_0^\infty(D)$ with $0 \leq \eta \leq 1$, then we write $v \in S_{p,\mu}^\varepsilon(D, C)$.

The following is the key lemma (cf. [**MZ**; Theorems 2.2, 2.3, 2.4]).

Lemma A.1. *Let $D \Subset \Omega$, $M_0 \geq 0$ and u be a solution to the obstacle problem* $\mathrm{OBP}(\mathscr{A}, \mathscr{B}; f, \theta; D)$. *Then there is a constant $C > 0$ depending only on p, c_μ, α_1, α_2 and $\alpha_3(D)$ such that whenever $B(x_0, \rho) \Subset D$ the following holds*:

(i) *for every constant M with $|M| \leq M_0$ and $f \leq M$ in $B(x_0, \rho)$,*

$$(u - M)^+ + 2(M_0 + 1)\rho^{1/(p-1)} \in S_{p,\mu}^+(B(x_0, \rho), C),$$

(ii) *for every constant M with $|M| \leq M_0$,*

$$(u - M)^- + 2(M_0 + 1)\rho^{1/(p-1)} \in S_{p,\mu}^+(B(x_0, \rho), C),$$

(iii) *for every constant M with $|M| \leq M_0$ and $u \geq M$ a.e. in $B(x_0, \rho)$,*

$$u - M + 2(M_0 + 1)\rho^{1/(p-1)} \in S_{p,\mu}^-(B(x_0, \rho), C).$$

Proof. We give a sketch of the proof of (i), the first half of which is analogous to the proof of [**MZ**; Theorem 2.2]. Namely put $v = (u - M)^+$ and let $g = (v - k)^+$ for $k \geq 0$. For $\eta \in C_0^\infty(B(x_0, \rho))$ with $0 \leq \eta \leq 1$, we take

$$\varphi(x) = -g(x)\eta(x)^p, \quad x \in B(x_0, \rho).$$

Then $u + \varphi \in \mathscr{K}_{f,\theta}(D)$, and hence

$$\int_{B(x_0,\rho)} \mathscr{A}(x, \nabla u) \cdot \nabla\varphi \, dx + \int_{B(x_0,\rho)} \mathscr{B}(x, u)\varphi \, dx \geq 0.$$

Let $B^+ = \{x \in B(x_0, \rho) : u(x) > M + k\}$. Then $\varphi = 0$ on $B(x_0, \rho) \backslash B^+$ and $\nabla g = \nabla u$ on B^+. Thus,

$$\int_{B^+} (\mathscr{A}(x, \nabla u) \cdot \nabla u)\eta^p \, dx \leq -p \int_{B^+} (\mathscr{A}(x, \nabla u) \cdot \nabla\eta) g\eta^{p-1} \, dx - \int_{B^+} \mathscr{B}(x, u) g\eta^p \, dx.$$

By (A.2), (A.3) and (B.2), it follows that

$$(*1)\qquad \alpha_1 \int_{B^+} |\nabla u|^p \eta^p \, d\mu \le p\alpha_2 \int_{B^+} |\nabla u|^{p-1} |\nabla \eta| g \eta^{p-1} \, d\mu + \alpha_3 \int_{B^+} (|u|^{p-1} + 1) g \eta^p \, d\mu,$$

where $\alpha_3 = \alpha_3(D)$. By Young's inequality, we have

$$(*2)\qquad p\alpha_2 \int_{B^+} |\nabla u|^{p-1} |\nabla \eta| g \eta^{p-1} \, d\mu \le \frac{\alpha_1}{3} \int_{B^+} |\nabla u|^p \eta^p \, d\mu + C_1 \int_{B^+} g^p |\nabla \eta|^p \, d\mu$$

with $C_1 = C_1(p, \alpha_1, \alpha_2)$. On the other hand, since $|u| \le g + M_0 + k$ on B^+, again by Young's inequality

$$(|u|^{p-1} + 1) g \le C_2(p) \left\{ g^p + k^p + (\varepsilon\rho)^{p'} \lambda_0 + \left(\frac{g}{\varepsilon\rho} \right)^p \right\}.$$

for $\varepsilon > 0$, where $p' = p/(p-1)$ and $\lambda_0 = (M_0 + 1)^p$. By the Poincaré inequality

$$\begin{aligned} \frac{1}{\rho^p} \int_{B^+} (g\eta)^p \, d\mu = \frac{1}{\rho^p} \int_{B(x_0,\rho)} (g\eta)^p \, d\mu &\le C_3 \int_{B(x_0,\rho)} |\nabla (g\eta)|^p \, d\mu \\ &\le C_3 2^{p-1} \left(\int_{B^+} |\nabla u|^p \eta^p \, d\mu + \int_{B^+} g^p |\nabla \eta|^p \, d\mu \right) \end{aligned}$$

with $C_3 = C_3(c_\mu)$. Choose $\varepsilon = \varepsilon(c_\mu, p, \alpha_1, \alpha_3) > 0$ so that $\varepsilon^{-p} \alpha_3 C_2(p) C_3 2^{p-1} = \alpha_1/3$. Then

$$\begin{aligned} (*3)\qquad & \alpha_3 \int_{B^+} (|u|^{p-1} + 1) g \eta^p \, d\mu \\ & \le \alpha_3 C_2(p) \left(\int_{B^+} g^p \eta^p \, d\mu + k^p \int_{B^+} \eta^p \, d\mu \right) \\ & \quad + C_4 \rho^{p'} \lambda_0 \int_{B^+} \eta^p \, d\mu + \frac{\alpha_1}{3} \left(\int_{B^+} |\nabla u|^p \eta^p \, d\mu + \int_{B^+} g^p |\nabla \eta|^p \, d\mu \right) \end{aligned}$$

with $C_4 = C_4(c_\mu, p, \alpha_1, \alpha_3)$. Thus, from $(*1)$, $(*2)$ and $(*3)$ we obtain

$$\int_{B^+} |\nabla u|^p \eta^p \, d\mu \le C_5 \left(\int_{B^+} g^p (\eta^p + |\nabla \eta|^p) \, d\mu + (k^p + \rho^{p'} \lambda_0) \int_{B^+} \eta^p \, d\mu \right)$$

with $C_5 = C_5(p, c_\mu, \alpha_1, \alpha_2, \alpha_3)$. Then we can show that

$$v + 2\lambda_0^{1/p} \rho^{1/(p-1)} \in S^+_{p,\mu}(B(x_0, \rho), 2C_5)$$

in the same manner as [**MZ**; Lemma 2.7].

To prove assertion (ii) (resp. (iii)), we let $v = (u - M)^-$, $g = (v - k)^+$ (resp. $v = u - M$, $g = (v - k)^-$) and $\varphi = g\eta^p$. The rest of the proof proceeds just as above (cf. [**MZ**; Theorem 2.4 (resp. 2.5)]).

The next two lemmas can be proved in the same way as in [**MZ**] by using the Moser iteration technique and the John-Nirenberg lemma (cf. [**MZ**; Lemmas 2.8, 2.9, 2.10, 2.13 and Theorems 2.11, 2.12, 2.14]; we should suitably use conditions on the p-admissibility for μ).

Lemma A.2. *Let $C > 0$, $\gamma \in (0, p]$ and $\rho \in (0, 1]$. If $v \in S^+_{p,\mu}(B(x_0, \rho), C)$, then there is a constant $K > 0$ depending only on p, c_μ, C and γ such that*

$$\operatorname*{ess\,sup}_{B(x_0,\rho/2)} v \le K\left(\frac{1}{\mu(B(x_0,\rho))}\int_{B(x_0,\rho)} v^\gamma\, d\mu\right)^{1/\gamma}.$$

Lemma A.3. *Let $C > 0$ and $\rho \in (0, 1]$. If $v \in S^-_{p,\mu}(B(x_0, 2\rho), C)$ with $v > 0$ a.e. in $B(x_0, \rho)$, then there are constants $K > 0$ and $\gamma \in (0, 1)$ depending only on N, p, c_μ and C such that*

$$\operatorname*{ess\,inf}_{B(x_0,\rho/2)} v \ge K\left(\frac{1}{\mu(B(x_0,\rho))}\int_{B(x_0,\rho)} v^\gamma\, d\mu\right)^{1/\gamma}.$$

Lemma A.1 (i) (ii) and Lemma A.2 yield

Proposition A.1. *Let $D \Subset \Omega$, $M_0 \ge 0$, $\gamma \in (0, p]$, $x_0 \in D$, $\rho \in (0, 1]$ with $\bar{B}(x_0, \rho) \subset D$ and u be a solution to the obstacle problem* $\mathrm{OBP}(\mathscr{A}, \mathscr{B}; f, \theta; D)$. *Then there is a constant $C > 0$ depending only on p, $\alpha_1, \alpha_2, \alpha_3(D)$, c_μ, γ and M_0 such that*

(i) *for every constant M with $|M| \le M_0$ and $f \le M$ in $B(x_0, \rho)$,*

$$\operatorname*{ess\,sup}_{B(x_0,\rho/2)} (u - M)^+ \le C\left(\frac{1}{\mu(B(x_0,\rho))}\int_{B(x_0,\rho)} \{(u - M)^+\}^\gamma\, d\mu\right)^{1/\gamma} + C\rho^{1/(p-1)},$$

(ii) *for every constant M with $|M| \le M_0$,*

$$\operatorname*{ess\,sup}_{B(x_0,\rho/2)} (u - M)^- \le C\left(\frac{1}{\mu(B(x_0,\rho))}\int_{B(x_0,\rho)} \{(u - M)^-\}^\gamma\, d\mu\right)^{1/\gamma} + C\rho^{1/(p-1)}.$$

Lemma A.1 (iii) and Lemma A.3 yield

Proposition A.2. *Let $D \Subset \Omega$, $M_0 \ge 0$, $x_0 \in D$, $\rho \in (0, 1]$ with $\bar{B}(x_0, \rho) \subset D$ and u be a solution to the obstacle problem* $\mathrm{OBP}(\mathscr{A}, \mathscr{B}; f, \theta; D)$. *Then there are constants C, C'*

and $\gamma \in (0,1)$ *such that, for every constant* M *with* $|M| \le M_0$ *and* $u \ge M$ *a.e. in* $B(x_0,\rho)$,

$$\operatorname{ess}\inf_{B(x_0,\rho/2)}(u-M) \ge C\left(\frac{1}{\mu(B(x_0,\rho))}\int_{B(x_0,\rho)}(u-M)^{\gamma}\,d\mu\right)^{1/\gamma} - C'\rho^{1/(p-1)}.$$

Here C *and* γ *depend only on* N, p, $\alpha_1,\alpha_2,\alpha_3(D)$, c_μ *and* M_0, *and* C' *depends only on* p *and* M_0.

Now, suppose that f is locally bounded above and u is a solution to the obstacle problem $\mathrm{OBP}(\mathscr{A},\mathscr{B};f,\theta;D)$. Then from Proposition A.1 it follows that u is essentially locally bounded in D. Furthermore, by Proposition A.2 and Hölder's inequality, setting

$$\tilde{u}(x_0) = \lim_{\rho\to+0}\frac{1}{\mu(B(x_0,\rho))}\int_{B(x_0,\rho)} u\,d\mu$$

for each $x_0 \in D$, we can show that $\tilde{u}$ satisfies (4.1) (cf. the proof of [**MZ**; Lemma 3.4]). Thus we obtain the last half of Theorem 4.1.

To prove Theorem 4.2, suppose f is continuous. Then we can show that u satisfying (4.1) is continuous in D in the same way as the proof of [**MZ**; Theorem 3.6], using Proposition A.1 (i).

Finally, if $\varphi \in C_0^\infty(\{u > f\})$, then by the continuity of $u - f$ there exists $\lambda_0 > 0$ such that $u + \lambda\varphi \ge f$ in D for $|\lambda| < \lambda_0$. It then follows that u is $(\mathscr{A},\mathscr{B})$-harmonic in $\{u > f\}$, namely the last half of Theorem 4.2.

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