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Modified analytic trivialization for weighted homogeneous function-germs

Dedicated to Professor Tzee-Char Kuo's 60th birthday

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Abstract. We show that a real analytic family of real analytic functions admits a modified analytic trivialization (MAT) via an appropriate toric modification, if their weighted initial forms define isolated singularities near zero.

We consider the classification problem of real singularities defined by real analytic function-germs. If one classifies them by the differentiable equivalence, the modulus may appear, as Whitney's example (1.1) shows, thus we consider this equivalence relation too strong for the classification purpose. The equivalence relation by homeomorphisms does not cause modulus, but we do not expect a workable theory for this equivalence relation. So a fundamental problem in real singularities is to find a "nice and natural" equivalence relation for real analytic function-germs.

T.-C. Kuo considered homeomorphisms becoming bianalytic isomorphisms after some blowing up, and he showed some basic trivialization theorems in [**11**], [**12**], and a finite classification theorem in [**10**], [**13**] for the equivalence relation induced by such homeomorphisms. Of course, such homeomorphisms define an equivalence relation which is weaker than bianalytic isomorphisms and stronger than topological equivalence. In this paper, we discuss the construction of a trivialization with respect to this equivalence relation. In particular we want to know more families admitting such trivialization under some reasonable assumptions.

Let $x=(x_1,\dots,x_n)$ be a system of coordinates of $\boldsymbol{R}^n$ at the origin 0, and let $I=[a_1,b_1]\times\cdots\times[a_m,b_m]$ be a cuboid in $\boldsymbol{R}^m$. Let $F(x,t):(\boldsymbol{R}^n,0)\times I\to(\boldsymbol{R},0)$ be a real analytic family of real analytic function-germs. We set $f_t(x)=F(x;t)$. We denote the Taylor expansion of $f_t(x)$ at the origin by $\sum_\nu c_\nu(t)x^\nu$. Let $\omega=(\omega_1,\dots,\omega_n)$ be an

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n-tuple of positive integers. Setting $H_j(x;t) = \sum_\nu c_\nu(t)x^\nu$ where the sum is taken over ν with $\langle \omega, \nu \rangle = \omega_1\nu_1 + \cdots + \omega_n\nu_n = j$, we can write

$$F(x;t) = H_d(x;t) + H_{d+1}(x;t) + \cdots, \quad H_d(x;t) \not\equiv 0.$$

We call H_d the *weighted initial form* of f_t about the weight ω. If $\omega = (1,\dots,1)$, this H_d is the usual initial form of f_t. We now quote the first trivialization theorem obtained by T.-C. Kuo in [**11**].

Theorem (0.1). *Let $f_t : (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}, 0)$ $(t \in I)$ be a real analytic family of real analytic function-germs. If the initial forms of f_t $(t \in I)$ define an isolated singularity at the origin, then f_t admits a* MAT *via the blowing up of $\boldsymbol{R}^n$ at the origin, that is, f_t is trivialized by a family of homeomorphisms coming from a family of bianalytic isomorphisms of the space obtained by the blowing up of $\boldsymbol{R}^n$ at the origin.*

In this paper, we show the weighted version of this theorem. Namely, we have the following:

Theorem (0.2). *Let $f_t : (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}, 0)$ $(t \in I)$ be a real analytic family of real analytic function-germs. If the weighted initial forms of f_t $(t \in I)$ define an isolated singularity at the origin, then f_t admits a* MAT *via some toric modification of $\boldsymbol{R}^n$, that is, f_t is trivialized by a family of homeomorphisms coming from a family of bianalytic isomorphisms of the space obtained by some toric modification of $\boldsymbol{R}^n$.*

For the definitions of the notions involved in the above theorems we send the reader to §1 and §4.

Speaking about topological trivialization only, many authors showed such trivialization theorems. For example, J. Damon [**4**] showed a topological trivialization theorem for $\mathscr{A}$-finite weighted homogeneous map-germs. But his discussion does not apply to our situation, because we cannot use the partition of unity.

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§1. Modified analytic trivialization.

The motivation to introduce the modified analytic trivialization is explained by the following example.

EXAMPLE (1.1) (H. Whitney [**16**]). Let $W_t : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}, 0)$ $(t \in I)$ be a homogeneous polynomial-germ defined by $W_t(x, y) = xy(x-y)(x-ty)$ for $I = (1, \infty)$. Then, W_t $(t \in I)$ is a topologically trivial family that is not C^1-trivial. Here, we are interested in the following observation due to T.-C. Kuo in [**11**]. Let $\pi : X \to \boldsymbol{R}^2$ be the blowing up at the origin. Then the family $W_t \circ \pi : (X, \pi^{-1}(0)) \to (\boldsymbol{R}, 0)$ is real analytically trivial along I.

Since we may find a modulus situation like (1.1) after some blowing up, we should consider such a trivialization using at least compositions of blowing-ups or more widely proper analytic modifications. Motivated by such observations, we are led to the definition of modified analytic trivialization.

DEFINITION (1.2). Let $\pi : X \to \boldsymbol{R}^n$ be a proper analytic modification. We say that f_t $(t \in I)$ admits a modified analytic trivialization (MAT, for short) via π along I if there exists a t-level preserving real analytic isomorphism between two neighborhoods of $\pi^{-1}(0) \times I$ which induces a t-level preserving homeomorphism h between two neighborhoods of $0 \times I$ so that $F \circ h(x; t)$ is independent of t.

In [**7**], [**8**], [**12**], [**15**], there are several theorems that establish MAT.

§2. Review of the homogeneous case.

Because of its importance, we, very briefly, review the proof of Theorem (0.1). Let $\pi : X \to \boldsymbol{R}^n$ denote the blowing up at the origin. We first assume that $m = 1$ and set $t = t_1$. We consider the vector field V defined by

$$V = -\sum_{i=1}^{n} \frac{Q_i}{P} \frac{\partial}{\partial x_i} + \frac{\partial}{\partial t}, \quad \text{where } P = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i}\right)^2, \text{ and } Q_i = \frac{\partial F}{\partial t} \frac{\partial F}{\partial x_i}.$$

We see V is tangent to each level of F, whenever V is defined. Remember that X is the analytic submanifold of $\boldsymbol{R}^n \times P^{n-1}(\boldsymbol{R})$ defined by $x_i \xi_j = \xi_i x_j$ $(1 \le i < j \le n)$ where $P^{n-1}(\boldsymbol{R})$ is the $(n-1)$-dimensional real projective space with homogeneous coordinate $\xi = [\xi_1 : \cdots : \xi_n]$. On the subset of X defined by $\xi_s \neq 0$, we have a coordinate system $y = (y_1, \ldots, y_n)$ defined by $y_s = x_s$, $y_i = \xi_i / \xi_s$, $i \neq s$, and π is expressed as $x_s = y_s$, $x_i = y_i y_s$, $i \neq s$. Then for $\xi_s \neq 0$,

$$\frac{\partial}{\partial x_s} = \frac{\partial}{\partial y_s} - \sum_{j \neq s} \frac{y_j}{y_s} \frac{\partial}{\partial y_j}; \quad \frac{\partial}{\partial x_j} = \frac{1}{y_s} \frac{\partial}{\partial y_j}, \quad j \neq s.$$

Since $\partial F / \partial t$, $\partial F / \partial x_i$ are order d and $d-1$ respectively, they are thus divisible by y_s^d and

y_s^{d-1} respectively. We set $P = y_s^{2d-2}\tilde{P}$ and $Q_i = y_s^{2d-1}\tilde{Q}_i$. Then $\tilde{P}$ and $\tilde{Q}_i$ are analytic. Since we can prove that $\tilde{P}$ is strictly positive near $\pi^{-1}(0) \times I$, $d(\pi^{-1} \times \mathrm{id})V$ is analytically extendible there. The trajectory of this vector field yields the desired analytic isomorphism.

We next see the case $m > 1$. Let V_j be the vector field defined by

$$V_j = -\sum_{i=1}^{n} \frac{Q_i}{P}\frac{\partial}{\partial x_i} + \frac{\partial}{\partial t_j}, \quad \text{where } P = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2, \text{ and } Q_i = \frac{\partial F}{\partial t_j}\frac{\partial F}{\partial x_i}.$$

Then the same proof as above shows that it admits an analytic lift by $\pi \times \mathrm{id}$. We denote its trajectory by H_j. If we define

$$H(y;t) = H_m(t_m - a_m; H_{m-1}(\cdots; H_1(t_1 - a_1; y, a)))$$

for $t = (t_1, \ldots, t_m) \in I$, and $a = (a_1, \ldots, a_m)$, we obtain the desired analytic isomorphism.

§3. A vector field for the weighted homogeneous case.

The purpose of the paper is to present a "weighted version" of the proof above. By the discussion above, it is enough to show the case $m = 1$. Thus, throughout the rest of this paper, we assume that $m = 1$ and set $t = t_1$. Unfortunately the direct generalization is not possible, the vector field V above does not have an analytic lift by the blowing up at the origin or by some other modification (for example, a toric modification that will be defined in §4) under the assumption in (0.2). The reason is that somehow the above V is not "weighted homogeneous" in some sense. In the general setup, we know no method to construct an appropriate vector field which admits an analytic lift via some modification. Several authors tried this problem, see [**11**], [**12**], [**15**]. The first idea of the proof is to give an explicit expression of the weighted version of the vector field V.

Let $F(x;t) = \sum_\nu c_\nu(t)x^\nu$, $t \in I$, be an analytic family of real analytic function-germs. We assume that the initial forms $H_d(x;t)$, $t \in I$ define isolated singularities at the origin of $\boldsymbol{R}^n$. Then $\omega_i < d$ for each $i = 1, \ldots, n$. In fact, suppose that $\omega_{i_0} \geq d$ for some i_0. Then $H_d(x;t)$ can contain only monomials in x_i's with $i \neq i_0$ or x_{i_0}. Since $H_d(x;t)$ defines a singularity at 0, $H_d(x;t)$ cannot contain the monomial x_{i_0} and thus is a polynomial in the variables x_i's with $i \neq i_0$. This contradicts that $H_d(x;t)$ defines an isolated singularity.

We here consider the vector field V defined by

$$V = -\sum_{i=1}^{n} \frac{Q_i}{P}\frac{\partial}{\partial x_i} + \frac{\partial}{\partial t}, \quad \text{where } P = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^{2k_i s}, \quad Q_i = \frac{\partial F}{\partial t}\left(\frac{\partial F}{\partial x_i}\right)^{2k_i s - 1},$$

$k_i = k/(d - \omega_i)$, $k = \prod_{i=1}^{n}(d - \omega_i)$ and s is a positive integer. It is not hard to see this vector field V is tangent to each level of F, whenever V is defined. We also remark the weighted initial form of P, about this weight, is $\sum_{i=1}^{n}(\partial H_d/\partial x_i)^{2k_is}$.

§4. Toric modification.

Let us recall some basic definitions and properties of toric modifications and fans. See [**5**], [**6**], [**14**], for a detailed discussion. Let M be the lattice $\boldsymbol{Z}^n$ and N its dual lattice $\mathrm{Hom}_{\boldsymbol{Z}}(M,\boldsymbol{Z})$. We write an element of M as a row vector and an element of N as a column vector. Let e_i denote the i-th unit row vector $(0,\dots,0,1,0,\dots,0)$ and e^i the i-th unit column vector ${}^t(0,\dots,0,1,0,\dots,0)$. We set $M_{\boldsymbol{R}} = M\otimes_{\boldsymbol{Z}}\boldsymbol{R}$, and $N_{\boldsymbol{R}} = N\otimes_{\boldsymbol{Z}}\boldsymbol{R}$, and denote $\langle\,,\rangle : N_{\boldsymbol{R}}\times M_{\boldsymbol{R}}\to\boldsymbol{R}$ the canonical bilinear map defined by $\langle e^i, e_j\rangle = \delta_{ij}$ where $\delta_{ij}=1$, if $i=j$; 0, otherwise. Let $v_1,\dots,v_s$ be vectors in N (resp. M). We set

$$\sigma = \{r_1v_1+\cdots+r_sv_s \in N_{\boldsymbol{R}}\ (\text{resp. } M_{\boldsymbol{R}}) : r_i\geq 0,\ i=1,\dots,s\}$$

and call it the *cone generated by* $v_1,\dots,v_s$. We simply call such a set a *cone* in N (resp. M). For a cone σ in N, set $\sigma^\vee = \{u\in M_{\boldsymbol{R}} : \langle v,u\rangle\geq 0,\ \forall v\in\sigma\}$. It is easy to see that $\sigma^\vee$ is a cone in M. Let $\boldsymbol{R}[M\cap\sigma^\vee]$ denote the algebra generated by the additive semigroup $M\cap\sigma^\vee$ over the real number field $\boldsymbol{R}$, and U_σ the set of $\boldsymbol{R}$-valued points of $\mathrm{Spec}(\boldsymbol{R}[M\cap\sigma^\vee])$. If a cone σ_1 in N is a face of another cone σ_2 in N, then the natural inclusion $\sigma_2^\vee\subset\sigma_1^\vee$ induces an open embedding $U_{\sigma_1}\subset U_{\sigma_2}$, since $\boldsymbol{R}[M\cap\sigma_1^\vee]$ and $\boldsymbol{R}[M\cap\sigma_2^\vee]$ have a same quotient field and hence have a same generic point. Therefore, if Σ is a *fan*, that is, a finite collection of cones in N that form a complex, we can glue U_σ's ($\sigma\in\Sigma$) together and we obtain a (real) *toric variety* denoted by X_Σ. We say a cone σ in N is *regular* if σ is generated by a part of a basis of N. We say a fan is *regular* if σ is regular for each $\sigma\in\Sigma$. If Σ is a regular fan, then X_Σ is nonsingular.

Example (4.1). Let σ be the positive orthant in $N_{\boldsymbol{R}}$. Then $\sigma^\vee$ is the positive orthant in $M_{\boldsymbol{R}}$ and $\boldsymbol{R}[M\cap\sigma^\vee]$ is the polynomial ring $\boldsymbol{R}[x_1,\dots,x_n]$. The set of faces of σ is a regular fan and the corresponding toric variety is $\boldsymbol{R}^n$.

Let Σ be a fan that forms a subdivision of the positive orthant, that is, $|\Sigma| := \bigcup_{\sigma\in\Sigma}\sigma$ is the positive orthant in $N_{\boldsymbol{R}}$. Then the natural embedding of the positive orthant in $\sigma^\vee$ ($\sigma\in\Sigma$) induces an embedding $\boldsymbol{R}[x_1,\dots,x_n]\subset\boldsymbol{R}[M\cap\sigma^\vee]$. This defines a proper modification $\pi=\pi_\Sigma : X_\Sigma\to\boldsymbol{R}^n$. We call this a *toric modification* of $\boldsymbol{R}^n$.

Assume that Σ is a regular fan that is a subdivision of the positive orthant. Then, for each n-cone σ in Σ, U_σ is isomorphic to $\boldsymbol{R}^n$ with some coordinate system $(y_1,\dots,y_n)$, and the map $\pi|_{U_\sigma} : U_\sigma\to\boldsymbol{R}^n$ can be expressed by $x_i=\prod_{j=1}^{n} y_j^{a_i^j}$ for some integral n by n matrix (a_i^j) whose determinant is ± 1. By definition, $a^j = {}^t(a_1^j,\dots,a_n^j)$ $(j=1,\dots,n)$ generate the cone σ.

Let Δ be a convex polyhedron in $M_{\boldsymbol{R}}$, which coincides with the positive orthant outside some compact set. We set $\sigma_F = \bigcup_{r\geq 0} r\cdot(\Delta - m)$, where m is a point in the relative interior of F, for a face F of Δ. Then the system $\Sigma_\Delta = \{\sigma_F : F \text{ a face of } \Delta\}$ forms a fan that is a subdivision of the first orthant, and we obtain a proper modification $X_{\Sigma_\Delta} \to \boldsymbol{R}^n$, and denote it by π_Δ. We remark that π_Δ is an isomorphism except over the origin, that is, $\pi|_{X-\pi_\Delta^{-1}(0)} : X - \pi_\Delta^{-1}(0) \to \boldsymbol{R}^n - 0$ is an isomorphism.

§5. Analytic lift of the vector field.

The purpose of this section is to show that the vector field V defined in §3 has an analytic lift by some toric modification under the supposition in (0.2). Set

$$\Delta_0 = \{v = (v_1, \dots, v_n) \in M_{\boldsymbol{R}} : \langle \omega, v\rangle \geq d, v_i \geq 0, i = 1, \dots, n\}.$$

Suppose Σ is a regular fan that is a subdivision of Σ_{Δ_0}. For $a = {}^t(a_1, \dots, a_n) \in N$, we set

$$\ell(a) = \min\{\langle a, v\rangle : c_v \neq 0\},$$

$$\ell'(a) = \min\left\{\langle a, v\rangle : \frac{dc_v}{dt} \neq 0\right\},$$

$$\ell^{(i)}(a) = \min\{\langle a, v\rangle : c_v \neq 0, v_i \geq 1\}.$$

Obviously $\ell, \ell', \ell^{(i)}$ are convex, and $\ell'(a) \geq \ell(a)$ and $\ell^{(i)}(a) \geq \ell(a)$.

Let σ is a regular cone in Σ, and $a^1, \dots, a^n$ be a system of generators for it. We consider the pullback of V on $U_\sigma \times I$ with the system of coordinates $y = (y_1, \dots, y_n)$ and t. For shortness, we write $a_i = (a_i^1, \dots, a_i^n)$, $\ell = (\ell(a^1), \dots, \ell(a^n))$, $\ell' = (\ell'(a^1), \dots, \ell'(a^n))$, and $\ell^{(i)} = (\ell^{(i)}(a^1), \dots, \ell^{(i)}(a^n))$. Then, we can write

$$\frac{\partial F}{\partial x_i} = y^{\ell^{(i)} - a_i}\tilde{F}_i, \quad \frac{\partial F}{\partial t} = y^{\ell'}\tilde{F}_t$$

for some analytic functions $\tilde{F}_i$, $\tilde{F}_t$ in (y, t) near $\pi^{-1}(0) \times I$. Thus,

$$P = \sum_{i=1}^n \left(\frac{\partial F}{\partial x_i}\right)^{2k_i s} = \sum_{i=1}^n y^{2k_i s(\ell^{(i)} - a_i)}\tilde{F}_i^{2k_i s}.$$

Since the weighted initial form of f_t defines an isolated singularity at the origin, the coefficient of x_i^{2ks/ω_i} in the Taylor expansion of P at 0 is not zero. Here we set

$$m(a) = \min\left\{\frac{a_1}{\omega_1}, \dots, \frac{a_n}{\omega_n}\right\}, \quad R(a) = \left\{i : \frac{a_i}{\omega_i} = m(a)\right\},$$

and $m = (m(a^1), \dots, m(a^n))$. Then we obtain $P = y^{2mks}\tilde{P}$ for some analytic function $\tilde{P}$ in (y,t) near $\pi^{-1}(0) \times I$. We also have

$$m(a^j) = \frac{1}{k}\min_i\{k_i(\ell^{(i)}(a^j) - a_i^j)\} = \min_i\left\{\frac{\ell^{(i)}(a^j) - a_i^j}{d - \omega_i}\right\},$$

because the weighted initial form of P is $\sum_{i=1}^n (\partial H_d/\partial x_i)^{2k_i s}$ which has non-zero coefficient at the term x_i^{2ks/ω_i}.

Lemma (5.1). *$\tilde{P}$ is everywhere positive near $\pi^{-1}(0) \times I$.*

Now we set

$$\gamma^{(i)}(a) = \{v \in M : \langle a, v\rangle = \ell^{(i)}(a), c_v \neq 0, v_i \geq 1\}, \quad \text{and}$$

$$\gamma^{[i]}(a) = \{v \in M : \langle a, v\rangle = a_i + m(a)(d - \omega_i), c_v \neq 0, v_i \geq 1\}.$$

Lemma (5.2). *Let Λ be a subset of $\{1, \dots, n\}$, and a^j $(j \in \Lambda)$ vectors with $\bigcap_{j\in\Lambda} R(a^j) \neq \emptyset$. If we set $a = \sum_{j\in\Lambda} c_j a^j$ for positive numbers c_j $(j \in \Lambda)$, then we have $\gamma^{[i]}(a) = \bigcap_{j\in\Lambda}\gamma^{[i]}(a^j)$.*

Proof. The inclusion $\supset$ is obvious. We see the converse. For any $v \in \gamma^{[i]}(a)$,

$$\langle a, v\rangle - \sum_{j\in\Lambda} c_j a_i^j = \sum_{j\in\Lambda} c_j m(a)(d - \omega_i).$$

Since $\langle a^j, v\rangle - a_i^j \geq m(a^j)(d - \omega_i)$ for v with $c_v \neq 0$, we have $v \in \gamma^{[i]}(a^j)$ for $j \in \Lambda$. $\square$

Proof of (5.1). Let (y,t) be a point in $\pi^{-1}(0)\times I$. We assume that $y = (y_1, \dots, y_n)$ satisfies $y_j = 0$, $j \in \Lambda$; $y_j \neq 0$, $j \notin \Lambda$, for some subset Λ of $\{1, \dots, n\}$. It is enough to see $\tilde{P}$ is positive at (y,t). By definition of $\tilde{P}$, we have

$$\tilde{P} = \sum_{i=1}^n \prod_{j=1}^n y_j^{2ks\{((\ell^{(i)(a^j)} - a_i^j)/(d-\omega_i)) - m(a^j)\}}\tilde{F}_i^{2k_i s}.$$

Since $y_j = 0$ for $j \in \Lambda$, our assertion is equivalent to

$$\left\{y : \sum_{v\in A} v_i c_v \prod_{j\notin\Lambda} y_j^{\langle a^j, v\rangle - \ell^{(i)}(a^j)} = 0 \text{ for } i \text{ with } \frac{\ell^{(i)}(a^j) - a_i^j}{d - \omega_i} = m(a^j),\ j \in \Lambda\right\}$$

is a subset of $\{\prod_{j\notin\Lambda} y_j = 0\}$, where $A = \bigcap_{j\in\Lambda}\gamma^{(i)}(a^j)$. Here, we set $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_n)$ where $\tilde{y}_j = y_j$, if $j \notin \Lambda$; 1 if $j \in \Lambda$. Then this is equivalent to

$$\left\{\tilde{y} : \sum_{v \in A} v_i c_v \prod_{j=1}^{n} \tilde{y}_j^{\langle a^j, v\rangle - \ell^{(i)}(a^j)} = 0 \text{ for } i \text{ with } \frac{\ell^{(i)}(a^j) - a_i^j}{d - \omega_i} = m(a^j), j \in \Lambda\right\}$$

is a subset of $\{\prod_{j=1}^{n} \tilde{y}_j = 0\}$. By setting $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$ where $\tilde{x}_i = \prod_{j=1}^{n} \tilde{y}_j^{a_i^j} = \prod_{j \notin \Lambda} y_j^{a_i^j}$, this comes from

$$\left\{\tilde{x} : \sum_{v \in A} v_i c_v \tilde{x}^v = 0 \text{ for } i \text{ with } \frac{\ell^{(i)}(a^j) - a_i^j}{d - \omega_i} = m(a^j),\ j \in \Lambda\right\}$$

is a subset of $\{\prod_{i=1}^{n} \tilde{x}_i = 0\}$. Now we set $B = \bigcap_{j \in \Lambda} \gamma^{[i]}(a^j)$. Since $\gamma^{[i]}(a^j) \neq \emptyset$ if and only if $\ell^{(i)}(a^j) = m(a^j)(d - \omega_i) + a_i^j$, our assertion is equivalent to

$$\left\{x : \frac{\partial}{\partial x_i}\left(\sum_{v \in B} c_v x^v\right) = 0 : i = 1, \dots, n\right\} \subset \{x_1 \cdots x_n = 0\}. \tag{5.3}$$

By (5.2), if we set $a = \sum_{j \in \Lambda} c_j a^j$ for $c_j > 0$, we have $B = \gamma^{[i]}(a)$.

We set $a = \bar{c}_0 \omega + \sum_{j=1}^{n} \bar{c}_j e^j$ $(\bar{c}_j \geq 0,\ \bar{c}_0 > 0,\ \bar{c}_1 \cdots \bar{c}_n = 0)$ and $E(a) = \{j : \bar{c}_j > 0\}$. Then we have

$$\begin{aligned}\gamma^{[i]}(a) &= \gamma^{[i]}(\omega) \cap \bigcap_{j \in E(a)} \gamma^{[i]}(e^j) \ \text{(by (5.2))}\\ &= \gamma^{[i]}(\omega) \cap \{v : v_j = 0 \text{ for } j \in E(a) - \{i\}, v_i = 1 \text{ if } i \in E(a)\}.\end{aligned}$$

Thus (5.3) is equivalent to

$$\{\operatorname{grad} H_d(x; t) = 0\} \cap \{x_j = 0, j \in C\} \subset \left\{\prod_{j \notin C} x_j = 0\right\},$$

where $C = \bigcup_{j \in \Lambda} E(a^j)$. This always holds, since H_d defines an isolated singularity at the origin. This completes the proof of (5.1). □

We next write down Q_i as a function in variables y and t. By definition,

$$Q_i = \frac{\partial F}{\partial t}\left(\frac{\partial F}{\partial x_i}\right)^{2k_i s - 1} = y^{\ell' + (2k_i s - 1)(\ell^{(i)} - a_i)} \tilde{F}_t \tilde{F}_i^{2k_i s - 1}.$$

Therefore the exponent of y in Q_i/P is $\ell' + (2k_i s - 1)(\ell^{(i)} - a_i) - 2mks$, which is equal to

$$(\ell' - \ell) + (\ell - \ell^{(i)}) + 2sk\left\{\frac{\ell^{(i)} - a_i}{d - \omega_i} - m\right\} + a_i. \tag{5.4}$$

We remark that each term of a component of (5.4) is not negative except the second. We say that $a = {}^t(a_1, \dots, a_n) \in N$ is *i-good* if $\ell^{(i)}(a) > a_i + m(a)(d - \omega_i)$ or $\ell(a) = \ell^{(i)}(a)$.

LEMMA (5.5). *If $a = {}^t(a_1, \dots, a_n)$ is i-good and s is large enough, then we have*

$$(\ell'(a) - \ell(a)) + (\ell(a) - \ell^{(i)}(a)) + 2sk\left\{\frac{\ell^{(i)}(a) - a_i}{d - \omega_i} - m(a)\right\} \geq 0.$$

PROOF. Obvious. □

LEMMA (5.6). *If the map $\pi|_{U_\sigma} : U_\sigma \to \boldsymbol{R}^n$ is expressed by*

$$x_i = y_i y_{k+1}^{a_i^{k+1}} \cdots y_n^{a_i^n}, \quad i = 1, \dots, k; \quad x_i = y_{k+1}^{a_i^{k+1}} \cdots y_n^{a_i^n}, \quad i = k+1, \dots, n,$$

then $y_{k+1}^{a_i^{k+1}} \cdots y_n^{a_i^n} \dfrac{\partial}{\partial x_i}$ has an analytic lift on U_σ.

PROOF. By elementary calculation, we have $\sum_{i=1}^n a_i^j x_i(\partial/\partial x_i) = y_j(\partial/\partial y_j)$. Thus, we obtain

$$x_i \frac{\partial}{\partial x_i} = y_i \frac{\partial}{\partial y_i}, \quad i = 1, \dots, k; \quad x_i \frac{\partial}{\partial x_i} = \sum_{j=1}^n b_i^j y_j \frac{\partial}{\partial y_j}, \quad i = k+1, \dots, n,$$

where $(b_i^j)_{i,j=k+1,\dots,n}$ is the inverse matrix of $(a_i^j)_{i,j=k+1,\dots,n}$. This implies the lemma. □

We say that a cone is *good* if each its 1-dimensional face is generated by a vector that is either an *i*-good vector for each i, or an e^i. We say a fan is *good* if each cone in it is good. Therefore, if there is a good subdivision Σ of Σ_{Δ_0} and s is sufficiently large, V has an analytic lift by the toric modification π corresponding to Σ, because of (5.1), (5.4), (5.5) and (5.6). By the proof of (5.6), this analytic lift is tangent to each irreducible component of $\pi^{-1}(0) \times I$. If the toric modification π is an isomorphism except over the origin, the trajectories of this vector field define a t-level preserving homeomorphism of $\boldsymbol{R}^n \times I$ near $0 \times I$, and this completes the proof. In the next section, we will show the existence of such a good subdivision.

§6. Good subdivision.

In this section, we show the following:

PROPOSITION (6.1). *There is a good subdivision of Σ_{Δ_0} so that the corresponding toric modification is an isomorphism except over the origin.*

LEMMA (6.2). *Let k be a positive integer with $k < n$, $a^{k+1}, \dots, a^n$ i-good vectors and suppose that $\bigcap_{j=1}^k R(e^j) \cap \bigcap_{j=k+1}^n R(a^j) \neq \emptyset$. Let $a = \sum_{j=1}^k c_j e^j + \sum_{j=k+1}^n c_j a^j$ be an integral vector with $0 \leq c_i < 1$. Then a is i-good.*

PROOF. In general, we have $\ell^{(i)}(a) \geq \sum_{j=1}^k c_j \ell^{(i)}(e^j) + \sum_{j=k+1}^n c_j \ell^{(i)}(a^j)$. If $\ell^{(i)}(a) > (d - \omega_i) m(a) + a_i$, (6.2) is obvious. We assume $\ell^{(i)}(a) = (d - \omega_i) m(a) + a_i$. Then it follows that

$$\ell^{(i)}(a) = \sum_{j=1}^k c_j \ell^{(i)}(e^j) + \sum_{j=k+1}^n c_j \ell^{(i)}(a^j),$$

and $\ell^{(i)}(e^j) = \delta_{ij}$, $j = 1, \dots, k$; $\ell^{(i)}(a^j) = \ell(a^j) = (d - \omega_i) m(a^j) + a_i^j$, $j = k+1, \dots, n$. So

$$\ell^{(i)}(a) = \sum_{j=1}^k c_j \ell^{(i)}(e^j) + \sum_{j=k+1}^n c_j \ell^{(i)}(a^j) = c_i + \sum_{j=k+1}^n c_j \ell^{(i)}(a^j).$$

Thus we obtain

$$\begin{aligned} \ell(a) &\geq \sum_{j=1}^k c_j \ell(e^j) + \sum_{j=k+1}^n \ell(a^j) \\ &= \sum_{j=1}^k c_j \ell(e^j) + \sum_{j=k+1}^n \ell^{(i)}(a^j) \\ &= \sum_{j=1}^k c_j \ell(e^j) + \ell^{(i)}(a) - c_i. \end{aligned}$$

Since the weighted initial form of f_t defines an isolated singularity at the origin, $\ell(e^j)$ can be 0 or 1, and only one can be 1. We thus obtain $0 \leq \ell^{(i)}(a) - \ell(a) < 1$, which implies $\ell^{(i)}(a) = \ell(a)$. □

LEMMA (6.3). *Let k be a positive integer with $k < n$, $a^{k+1}, \dots, a^n$ vectors generating good cones and suppose that $\bigcap_{j=1}^k R(e^j) \cap \bigcap_{j=k+1}^n R(a^j) \neq \emptyset$. Then there is a good subdivision for the cone generated by $e^1, \dots, e^k$, $a^{k+1}, \dots, a^n$.*

PROOF. We may assume that this cone is not regular. Then the volume v of the parallelogram generated by e^s $(s = 1, \dots, k)$ and a^t $(t = k+1, \dots, n)$ is not 1. We also have an integral vector $a = \sum_{j=1}^k c_j e^j + \sum_{j=k+1}^n c_j a^j$ with $0 \leq c_j < 1$, $\sum_{j=k+1}^n c_j > 0$. According to (6.2), we have that a is i-good. Then a divides in n-cones, and the parallelogram generated by a, e^s $(s = 1, \dots, k)$ and a^t $(t = k+1, \dots, n)$ except e^j or a^j has volume $c_j v$, and less than v. Thus, we can show (6.3) by induction on v. □

By the proof of (6.3), it is easy to see (6.1). □

REMARK. If there is a regular subdivision given by the integral points in the parallelogram generated by $\omega, e_1, \dots, \widehat{e_j}, \dots, e_n$, this subdivision is good in our sense. In [**3**], C. Bouvier and G. Gonzalez-Sprinberg showed that such a subdivision always uniquely exists when $n = 3$, and the non-existence of such a subdivision for $n = 4$. This shows that sometimes it is a delicate problem to construct a subdivision with restrictions on its 1-cones.

§7. Examples.

When $n = 2$, the toric modification π is obtained by composition of blowing-ups at some points. In this case, our vector field is tangent to each exceptional set, and induces homeomorphisms of each stages of blow-ups. This answers the question asked by S. Koike to the first author. However, we cannot expect similar things when $n \geq 3$.

EXAMPLE (7.1) (J. Briançon–J. P. Speder [**2**]). Let $f_t : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}, 0)$ be the family defined by $f_t(x, y, z) = z^5 + ty^6z + xy^7 + x^{15}$. This is a weighted homogeneous polynomial with weight $(1, 2, 3)$ and weighted degree 15, and defines an isolated singularity at the origin, whenever $t \neq t_0 := -15^{1/7}(7/2)^{4/5}/3$. Thus this admits a MAT along I, where I is an interval not containing t_0. This fact was first remarked in [**8**].

In [**9**], S. Koike defined the notion of strong C^0-equivalence and showed that this family is not strong C^0-trivial. If this family admits a MAT which induces a family of homeomorphisms of the space M obtained by the blowing up at the origin, this induces strong C^0-equivalence. So no MAT of this family induces a homeomorphism of M.

EXAMPLE (7.2). Let $f_t : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}, 0)$ $(t \in I = (-\infty, \infty))$ be the family of function-germs defined by $f_t(x, y, z) = x^{15} + y^8x + z^5 + tzy^7$. This is a weighted homogeneous polynomial with weight $(4, 7, 12)$ and weighted degree 60 if $t = 0$. Since this family satisfies the assumption of (0.2), it admits a MAT along I.

For strong C^0-triviality of this example, it is possible to see similar things to the first example and we leave it to the reader.

EXAMPLE (7.3). Let $f : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}, 0)$ be the germ defined by $f(x, y, z) = z(x^2 + y^2 \pm z^2)$. Let f_t be the family defined by $f_t = f + t(x^4 + y^4)$. Then, by (0.1), f_t admit a MAT via the blow-up at the origin. Remark that f_1 is locally irreducible at 0, but f_0 is not.

Example (7.4). Let $f:(\boldsymbol{K}^4,0)\to(\boldsymbol{K},0)$ $(\boldsymbol{K}=\boldsymbol{R},\boldsymbol{C})$ be the germ defined by the weighted homogeneous polynomial $f(x,y,z,w)=x^{12}+(y^3+z^2)^2+y^4w+w^3$ with respect to weight $(1,2,3,4)$ and weighted degree 12. It is easy to see that this defines an isolated singularity at the origin, and not non-degenerate in the sense of [**1**] (or [**7**]). Let $f_t:(\boldsymbol{R}^4,0)\to(\boldsymbol{R},0)$ be the family defined by $f_t=f+ty^3z^2$. This admits a MAT for $|t|\ll 1$.

Example (7.5). Consider the polynomial $f=x^9+(x^3+y^2)^2w+xz^4+w^3$. This is a weighted homogeneous polynomial with respect to weight $(2,3,4,6)$ and degree 18, which defines a germ $(\boldsymbol{K}^4,0)\to(\boldsymbol{K},0)(\boldsymbol{K}=\boldsymbol{R},\boldsymbol{C})$ with an isolated singularity at the origin. This is not non-degenerate in the sense of [**1**], either. Then the family of germs defined by $f_t=f+tx^3y^2w$ of $(\boldsymbol{R}^4,0)$ to $(\boldsymbol{R},0)$ admits a MAT for $|t|\ll 1$.

§8. Epilogue.

We set $\boldsymbol{K}=\boldsymbol{R}$ or $\boldsymbol{C}$. Let $f:(\boldsymbol{K}^n,0)\to(\boldsymbol{K},0)$ be an analytic function-germ. We say that a proper analytic modification $\pi:X\to\boldsymbol{K}^n$ gives a *resolution* of f if for each point $x\in X$ there is a system of coordinates at x so that $f\circ\pi$ is expressed by a monomial in this coordinates. Let $f_t:(\boldsymbol{K}^n,0)\to(\boldsymbol{K},0)$ $(t\in I)$ be an analytic family of analytic function-germs. We set $F(x;t)=f_t(x)$. We say a proper analytic modification $\pi:X\to\boldsymbol{K}^n$ gives a *simultaneous resolution* of $f_t(t\in I)$ if for each point $x\in X$ there is an analytic family of systems of coordinates at x depending on $t\in I$ so that $F\circ(\pi\times\mathrm{id})$ is expressed by a monomial in this coordinates.

Let $f:(\boldsymbol{R}^n,0)\to(\boldsymbol{R},0)$ be a real analytic function-germ. Suppose that the weighted initial form of f, with respect to the weight ω (and say of degree d), defines an isolated singularity at the origin. Setting $a=\sum_{j\neq s}c_je_j+c_0\omega$, $c_j\geq 0$, it is not difficult to see $\ell(a)\geq a_i$ for each $i=1,\dots,n$. Now we remark that there is a weighted homogeneous polynomial g with weight ω and weighted degree d and satisfying the following conditions:

(1) g defines an isolated singularity at the origin.

(2) g is non-degenerate in the sense of [**1**].

(3) there is an analytic family f_t $(t\in[0,1])$ so that $f_0=f$ and $f_1=g$, and that its weighted initial form $H_d(x;t)$ defines an isolated singularity at the origin for each $t\in[0,1]$.

Then, by (0.2), this family f_t admits a MAT via a toric modification corresponding to a good subdivision. We will see that a resolution of g is given by some toric modification corresponding to some good subdivision. We now assume the language of Newton polyhedron. See [**1**], or [**7**] for its definition. This is true when g is convenient, that is, the Newton polyhedron of g meets each coordinate axis. Then it is enough to see it in the case when g is not convenient. We set $g_t = g + \sum_{i=1}^{n} t x_i^m$ for sufficiently large m. Then g_t $(t \neq 0)$ is convenient and non-degenerate in the sense of [**1**]. Since each $n-1$ dimensional compact face of the Newton polyhedron of g_t $(t \neq 0)$ is not contained in any of the coordinate hyperplanes, the minimal integral vector a which is normal to this face satisfies that $\ell(a) = \ell^{(i)}(a)$, and then a is i-good vector for each $i = 1, \dots, n$. Then the construction in §6 works and we can obtain a good subdivision of the fan obtained as the dual of the Newton polyhedron of g_t $(t \neq 0)$ that was denoted by $\Gamma^*(g_t)$ in the notation of §3 in [**7**]. The corresponding toric modification gives a resolution of g_t $(t \neq 0)$. By (1.2), this g_t admits a MAT via this modification, and this modification gives a simultaneous resolution of g_t for $|t| \ll 1$, and thus gives a resolution of g. Moreover, using this discussion, this toric modification also gives a resolution of f.

Let $f_t : (\boldsymbol{C}^n, 0) \to (\boldsymbol{C}, 0)$ $(t \in I)$ be a real analytic family of holomorphic function-germs. Suppose that the weighted initial form H_d of f_t, about the weight ω, defines an isolated singularity at the origin. We set $F(x; t) = f_t(x)$. Considering the vector field V defined by

$$V = -\sum_{i=1}^{n} \frac{Q_i}{P} \frac{\partial}{\partial x_i} + \frac{\partial}{\partial t}, \quad \text{where } P = \sum_{i=1}^{n} \left| \frac{\partial F}{\partial x_i} \right|^{2k_i s}, \quad Q_i = \frac{\partial F}{\partial t} \left| \frac{\partial F}{\partial x_i} \right|^{2k_i s - 2} \overline{\frac{\partial F}{\partial x_i}},$$

we can construct a t-level preserving homeomorphism of $\boldsymbol{C}^n \times I$ near $0 \times I$ coming from a real analytic isomorphism $X \times I$ near $\pi^{-1}(0) \times I$ where $\pi : X \to \boldsymbol{C}^n$ is a (complex) toric modification constructed by the way similar to the above. Here, $\bar{F}$ denotes the complex conjugation of F.

Thus, we obtain the following facts, which do not seem to have appeared in literature.

(1) *Let* $f : (\boldsymbol{K}^n, 0) \to (\boldsymbol{K}, 0)$ *be an analytic function-germ. If the weighted initial form of* f *about the weight* ω *defines an isolated singularity at the origin, then a toric modification corresponding to some subdivision of* Σ_{Δ_0} *gives a resolution of* f.

(2) *Let* $f_t : (\boldsymbol{K}^n, 0) \to (\boldsymbol{K}, 0)$ $(t \in I)$ *be an analytic family of analytic function-germs. If the weighted initial form of* f_t $(t \in I)$ *about the weight* ω *defines an isolated singularity at the origin, then a toric modification corresponding to some subdivision of* Σ_{Δ_0} *gives a simultaneous resolution of* f_t $(t \in I)$.

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