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Transience conditions for self-similar additive processes

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Abstract. The class of self-similar additive processes is an important subclass of stochastically continuous processes with independent increments which are not assumed to be time-homogeneous. It is shown that this process is transient if it is proper and the dimension is three or more. Furthermore sufficient conditions for transience are given in one- or two-dimensional cases.

1. Introduction and results.

In this paper an additive process means a stochastically continuous process with independent increments, which is not assumed to be time-homogeneous. In general it is an unexplored field to investigate transience and recurrence for time-inhomogeneous Markov processes. They form too large a class to be analyzed. So we want to find a large enough subclass containing important additive processes. One of such subclasses is the class of self-similar additive processes. In fact it contains all strictly stable processes. The class was first studied by Sato [**8**], [**9**]. He made its characterization by the correspondence with the class of selfdecomposable distributions. After Sato's paper, the class was studied by Watanabe [**12**] and Sato and Yamamuro [**10**]. There are a lot of studies on self-similar processes (for example [**4**], [**5**], and [**11**]). But almost all of them deal with the case with not independent but stationary increments, so that no preceding results are useful.

Criteria for transience and recurrence of Lévy processes are well-known, where Lévy process means a time-homogeneous additive process. Lévy processes are transient if the dimension of the space is three or more. If the dimension is one or two, the Brownian motion is recurrent, and there are both recurrent case and transient case in general Lévy processes. Transience and recurrence are characterized by an integrability condition near the origin of the characteristic function of the distribution at time 1. This fact is based on the expression of transience and recurrence by finiteness and infiniteness of the expected occupation times on compact sets.

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In [**10**] we showed that a self-similar additive process is either transient or recurrent, and, furthermore, we made comparison in transience and recurrence of the Lévy process and the self-similar additive process associated with a common selfdecomposable distribution. In this paper, as the sequel to the problem, we consider the following problems for self-similar additive processes:

(1) Are all self-similar additive processes transient if the dimension is three or more?

(2) Can we decide transience or recurrence, in terms of the characteristic functions, if the dimension is one or two?

The problem (1) is answered in the affirmative. Concerning the problem (2), we give some sufficient conditions for transience in dimension one or two in Section 2 and some sufficient conditions for recurrence in dimension one in Section 5. We note that, unlike Lévy processes, the expected occupation times on compact sets cannot determine transience and recurrence (see [**10**]).

The definition of a self-similar additive process is as follows.

Definition. *A stochastic process $\{X_t : t \geq 0\}$ on $\boldsymbol{R}^d$, which is defined on a probability space $(\Omega, \mathscr{F}, P)$, is called a self-similar additive process, or a process of class L, with exponent $H > 0$ if it satisfies the following conditions*:

(i) *$\{X_{ct}\}$ and $\{c^H X_t\}$ have the same finite-dimensional distributions for every $c > 0$,*

(ii) *$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent for any n and any choice of $0 \leq t_0 < t_1 < t_2 < \cdots < t_n$,*

(iii) *almost surely X_t is right-continuous in $t \geq 0$ and has left limits in $t > 0$.*

Let $\{X_t\}$ be a self-similar additive process on $\boldsymbol{R}^d$ with exponent H. Then, by its definition, $\{X_t\}$ is stochastically continuous and $X_0 = 0$ almost surely. The distribution at any fixed time is self-decomposable (see [**8**], [**9**]). Here, the distribution μ is said to be self-decomposable if for any $c \in (0,1)$, there exists an infinitely divisible distribution ρ_c such that $\hat{\mu}(z) = \hat{\mu}(cz)\hat{\rho}_c(z)$, $z \in \boldsymbol{R}^d$, where $\hat{\mu}$ and $\hat{\rho}_c$ are characteristic functions of μ and ρ_c, respectively. It is known that the characteristic function of its distribution at time t is given by the following: Let $\langle x, y \rangle$ be the usual inner product and $|x| = \sqrt{\langle x, x \rangle}$. Let $S = \{x \in \boldsymbol{R}^d : |x| = 1\}$ and $D = \{x \in \boldsymbol{R}^d : |x| \leq 1\}$. Then

$$\begin{aligned} Ee^{i\langle z, X_t\rangle} &= Ee^{i\langle t^H z, X_1\rangle} \\ &= \exp\left[-2^{-1}t^{2H}\langle z, Az\rangle + \int_{\boldsymbol{R}^d}(e^{it^H\langle z,x\rangle} - 1 - it^H\langle z,x\rangle 1_D(x))\rho(dx) + it^H\langle \gamma, z\rangle\right] \end{aligned}$$

for $z \in \boldsymbol{R}^d$, where

$$\rho(EB) = \int_B \sigma(d\xi) \int_E k_\xi(u) u^{-1}\, du$$

for $E \in \mathscr{B}((0,\infty))$, $B \in \mathscr{B}(S)$. Here $\gamma \in \boldsymbol{R}^d$, A (called Gaussian variance matrix) is a symmetric and nonnegative-definite matrix, and ρ (called Lévy measure) is a measure on $\boldsymbol{R}^d$ such that $\rho(\{0\}) = 0$, and σ is a probability measure on S, and $k_\xi(u)$ is nonnegative, nonincreasing right continuous in u and Borel measurable in ξ, and

$$\int_S \sigma(d\xi) \int_0^\infty k_\xi(u) u (1+u^2)^{-1}\, du < \infty.$$

We define transience and recurrence as follows.

Definition. *The process $\{X_t\}$ is called transient if*

$$P\Big(\lim_{t\to\infty} |X_t| = \infty\Big) = 1.$$

The process $\{X_t\}$ is called recurrent if

$$P\Big(\liminf_{t\to\infty} |X_t - X_s| = 0\Big) = 1 \quad \textit{for every } s \geq 0.$$

In order to formulate our main theorems, we prepare some terminology. A measure on $\boldsymbol{R}^d$ is said to be full if it is not concentrated on any $(d-1)$-dimensional hyperplane. The process $\{X_t\}$ is said to be proper if the distribution of X_t is full for each $t > 0$. For any random variable Z we denote by P_Z the distribution of Z, and by $\hat{P}_Z(z)$ the characteristic function of P_Z. The following are our main results.

Theorem 1.1. *Suppose that $\{X_t\}$ is a proper self-similar additive process and that*

$$\int_S \sigma(d\xi) k_\xi(0+) < \infty.$$

Let X_1 have a distribution such that

$$\hat{P}_{X_1}(z) = \exp\bigg[\int_S \sigma(d\xi) \int_0^\infty (e^{i\langle z, u\xi\rangle} - 1) \frac{k_\xi(u)}{u}\, du + i\langle z, \gamma_0\rangle\bigg].$$

If $d = 2$, or if $d = 1$ and $\gamma_0 = 0$, then $\{X_t\}$ is transient.

Theorem 1.2. *Suppose that $\{X_t\}$ is a proper self-similar additive process. If $d \geq 3$, then the process $\{X_t\}$ is transient.*

Theorem 1.3. *Let $d = 1$. Suppose that $\rho((0,\infty)) = 0$. A self-similar additive process $\{X_t\}$ is recurrent if it satisfies one of the following conditions*:

(i) $A \neq 0$,

(ii) $A = 0$ *and* $\int_{(-1,0)} |x|\rho(dx) = \infty$,

(iii) $A = 0$, $\int_{(-1,0)} |x|\rho(dx) < \infty$, $\rho((-\infty,0)) > 0$, *and* $-\int_{(-1,0)} x\rho(dx) + \gamma > 0$.

The dual sufficient conditions for recurrence are obtained in case $\rho((-\infty,0)) = 0$. It is the case where $\{-X_t\}$ satisfies one of the conditions in Theorem 1.3.

When we consider the case $d = 1$, transience and recurrence of many self-similar additive processes are left undecided by Theorems 1.1 and 1.3. But, our results show distinct difference from the case of Lévy processes on $\boldsymbol{R}^1$. For example, if μ is a stable distribution of index $\alpha \in (1,2)$ with non-zero mean and one-sided Lévy measure or if μ is a stable distribution of index 1 with one-sided Lévy measure, then the self-similar additive process with distribution μ at time 1 is recurrent but the Lévy process with distribution μ at time 1 is transient. Let $\{B_t\}$ be a one-dimensional Brownian motion and let $\gamma \neq 0$. It is shown in [**10**] that $\{B_t + t^{1/2}\gamma\}$, which is a self-similar additive process with exponent 2^{-1}, is recurrent. Theorem 1.3 is a generalization of this fact. We know, on the other hand, that the Lévy process $\{B_t + t\gamma\}$ is transient.

2. Transience conditions.

In this section we shall obtain some conditions important to our discussion. Namely we shall show that if a certain integral with respect to the distribution P_{X_1}, which is irrelevant to the expected occupation time directly, is finite, then the process $\{X_t\}$ is transient. At first we prepare a lemma for the proof. Let r be a positive real number. Let $p_r(s,x;t,\Gamma)$ be the transition function of $\{X_{t^r}\}$ for $0 \leq s \leq t$, $x \in \boldsymbol{R}^d$, and $\Gamma \in \mathscr{B}(\boldsymbol{R}^d)$. Now we define a time-homogeneous transition function $\bar{p}_r(h,y,B)$ by

$$\bar{p}_r(h,y,B) = p_r(t,x;t+h,\Gamma)$$

for $h \geq 0$, $y = (t,x) \in [0,\infty) \times \boldsymbol{R}^d$, and $B \in \mathscr{B}([0,\infty) \times \boldsymbol{R}^d)$, where $\Gamma = \{z \in \boldsymbol{R}^d : (t+h,z) \in B\}$ (see [**3**] p. 87). We denote by $\{Y_h^r\}$ the time-homogeneous Markov process with this transition probability $\bar{p}_r(h,y,B)$. Let $\bar{P}_h^r$ be the transition operator of $\{Y_h^r\}$.

Lemma 2.1. *The process* $\{Y_h^r\}$ *is a Hunt process.*

Proof. Denote by C_0 the real Banach space of continuous functions on $[0,\infty) \times \boldsymbol{R}^d$ vanishing at infinity with the norm of uniform convergence. For any $f \in C_0$ and $y = (t,x) \in [0,\infty) \times \boldsymbol{R}^d$, we have

$$\bar{P}_h^r f(y) = \int_\Omega P(d\omega) f(t+h, X_{(t+h)^r}(\omega) - X_{t^r}(\omega) + x).$$

For each t, almost surely the limit of X_s as $s \downarrow t$ is equal to the limit as $s \uparrow t$, so we have $\bar{P}_h^r f \in C_0$. By virtue of Theorem 9.4 in [**2**] p. 46, the process $\{Y_h^r\}$ is a Hunt process. □

Consider the following three conditions on P_{X_1}.

(1) $\displaystyle\int_{\boldsymbol{R}^d} \frac{1}{|x|^2} P_{X_1}(dx) < \infty.$

(2) $\displaystyle\int_{\boldsymbol{R}^d} \frac{1}{|x|} P_{X_1}(dx) < \infty.$

(3) $\displaystyle\int_{\boldsymbol{R}^d} \frac{1}{|x|^\alpha} P_{X_1}(dx) < \infty$ for some α with $0 < \alpha < 1$.

THEOREM 2.2. *If* (1) *holds, then* $\{X_t\}$ *is transient.*

THEOREM 2.3. *Let* X_1 *have a distribution such that*

$$\hat{P}_{X_1}(z) = \exp\left[\int_S \sigma(d\xi) \int_0^\infty (e^{i\langle z, u\xi\rangle} - 1) \frac{k_\xi(u)}{u}\, du + i\langle \gamma_0, z\rangle\right],$$

where $\int_S \sigma(d\xi) k_\xi(0+) < \infty$. *If* (2) *holds, then the process* $\{X_t\}$ *is transient. If* $\gamma_0 = 0$ *and* (3) *holds, then the process* $\{X_t\}$ *is transient.*

PROOF OF THEOREMS 2.2 AND 2.3. From now on without loss of generality we assume that $H = 1$, because the exponent H of $\{X_t\}$ can be changed to any number by the nonrandom time change $\{X_{t^r}\}$ and the transience property is invariant under the nonrandom time change.

For $a > 0$ let

$$f_a(x) = \prod_{j=1}^d ((a - |x_j|) \vee 0).$$

For any positive integer n we denote by $f_a^n(x)$ the n times convolution of $f_a(x)$ with itself. Then the Fourier transform of f_a^n is as follows:

$$\hat{f}_a^n(z) = \int_{\boldsymbol{R}^d} e^{i\langle z, x\rangle} f_a^n(x)\, dx = \prod_{j=1}^d \left(\frac{\sin 2^{-1} a z_j}{2^{-1} z_j}\right)^{2n}.$$

First step. Let r be a positive integer multiple of 2^{-1} and suppose that $\int_{\boldsymbol{R}^d} 1/|x|^{1/r} P_{X_1}(dx) < \infty$. Let K be an arbitrary compact set in $\boldsymbol{R}^d$. We first suppose prove that, if we choose a sufficiently small, then

$$\text{(2.1)} \qquad \inf_{\substack{t\geq 0\\ w\in K}} E\left[\int_0^\infty \hat{f}_a^{4r}(X_{(t+s)^r} - X_{t^r} + w)\,ds\right] > 0.$$

Then we can show transience of $\{X_t\}$ as follows. Let $T_K = \inf\{s > 0 : Y_s^r \in [0,\infty)\times K\}$. Let $g_a(t,x) = \hat{f}_a^{4r}(x)$ for each $(t,x)\in[0,\infty)\times \boldsymbol{R}^d$. The process $\{Y_t^r\}$ is expressed by a system of probability measures $\{\tilde{P}_r^y : y\in[0,\infty)\times\boldsymbol{R}^d\}$ on the space of paths on $[0,\infty)\times\boldsymbol{R}^d$. The expectation with respect to $\tilde{P}_r^y$ is denoted by $\tilde{E}_r^y$. Now from Lemma 2.1 $\{Y_t^r\}$ is a Hunt process. So, using the strong Markov property, we have

$$\begin{aligned}\text{(2.2)} \quad \tilde{E}_r^y\left[\int_0^\infty g_a(Y_s^r)\,ds\right] &\geq \tilde{E}_r^y\left[\tilde{E}_r^{Y_{T_K}^r}\left[\int_0^\infty g_a(Y_s^r)\,ds\right]; T_K<\infty\right]\\ &\geq \tilde{P}_r^y(T_K<\infty)\inf_{w\in K, t\geq 0}\tilde{E}_r^{(t,w)}\left[\int_0^\infty g_a(Y_s^r)\,ds\right]\\ &= \tilde{P}_r^y(T_K<\infty)\inf_{w\in K, t\geq 0} E\left[\int_0^\infty \hat{f}_a^{4r}(X_{(t+s)^r} - X_{t^r}+w)\,ds\right].\end{aligned}$$

From (2.2) we have, for any $\alpha>0$,

$$\begin{aligned}\text{(2.3)} \qquad & P(\inf\{u>0 : X_{(u+\alpha)^r}\in K\}<\infty)\\ &= \tilde{P}_r^{(0,0)}(\inf\{u>0 : Y_{u+\alpha}^r\in[0,\infty)\times K\}<\infty)\\ &= \int_{[0,\infty)\times\boldsymbol{R}^d}\bar{p}_r(\alpha,(0,0),dy)\tilde{P}_r^y(T_K<\infty)\\ &\leq \frac{1}{c}\int_{[0,\infty)\times\boldsymbol{R}^d}\bar{p}_r(\alpha,(0,0),dy)\tilde{E}_r^y\left[\int_0^\infty g_a(Y_s^r)\,ds\right]\\ &= \frac{1}{c}\tilde{E}_r^{(0,0)}\left[\int_\alpha^\infty g_a(Y_s^r)\,ds\right],\end{aligned}$$

where c is a positive constant independent of α. Since r is a positive integer multiple of 2^{-1}, the change of variable $u=|x_k|s^r$ gives the following:

$$\int_0^\infty \hat{f}_a^{4r}(s^r x)\,ds \leq \frac{a^{8r(d-1)}}{r|x_k|^{1/r}}\int_0^\infty u^{(1/r)-1}\left|\frac{\sin(aux_k/2|x_k|)}{ux_k/2|x_k|}\right|^{8r} du \leq \text{const.}\times\frac{1}{|x_k|^{1/r}}$$

for $1\leq k\leq d$. Hence we have

$$\int_0^\infty \hat{f}_a^{4r}(s^r x)\,ds \leq \text{const.}\times\frac{1}{\sum_{k=1}^d |x_k|^{1/r}} \leq \text{const.}\times\frac{1}{|x|^{1/r}}.$$

Then we have

$$\tilde{E}_r^{(0,0)}\left[\int_\alpha^\infty g_a(Y_s^r)\,ds\right] = \int_{\boldsymbol{R}^d} P_{X_1}(dx)\int_\alpha^\infty \hat{f}_a^{4r}(s^r x)\,ds \tag{2.4}$$

$$\leq \text{const.} \times \int_{\boldsymbol{R}^d} \frac{1}{|x|^{1/r}} P_{X_1}(dx) < \infty.$$

It follows from (2.4) that the last member of (2.3) converges to 0 as $\alpha \to \infty$. This shows that the process $\{X_t\}$ is transient. Then it is enough to show (2.1) for proving transience of the process. Now we have, for $\varepsilon > 0$,

$$\inf_{t\geq 0,\, w\in K} E\left[\int_0^\infty \hat{f}_a^{4r}(X_{(t+s)^r} - X_{t^r} + w)\,ds\right] \tag{2.5}$$

$$\geq \inf_{t\geq 0,\, w\in K}\int_0^\varepsilon ds \int_{\boldsymbol{R}^d} f_a^{4r}(z)\frac{\hat{P}_{X_{(t+s)^r}}(z)}{\hat{P}_{X_{t^r}}(z)} e^{i\langle w,z\rangle}\,dz$$

$$= \inf_{t\geq 0,\, w\in K}\int_0^\varepsilon ds \int_{|z|\leq 4ra\sqrt{d}} f_a^{4r}(z)\cos F_t(s,z)\exp G_t(s,z)\,dz = Q, \quad (say).$$

Here $F_t(s,z)$ and $G_t(s,z)$ are, respectively, the imaginary and the real part of $\Psi_t(s,z)$ defined by

$$\Psi_t(s,z) = i\langle z,w\rangle - 2^{-1}((t+s)^{2r} - t^{2r})\langle Az,z\rangle + i((t+s)^r - t^r)\langle z,\gamma\rangle$$

$$+ \int_S \sigma(d\xi)\left\{\int_0^\infty\left(e^{i\langle z,u\xi\rangle} - 1 - i\langle z,u\xi\rangle 1_D\left(\frac{u\xi}{(t+s)^r}\right)\right)\frac{k_\xi(u/(t+s)^r)}{u}\,du\right.$$

$$\left. - \int_0^\infty\left(e^{i\langle z,u\xi\rangle} - 1 - i\langle z,u\xi\rangle 1_D\left(\frac{u\xi}{t^r}\right)\right)\frac{k_\xi(u/t^r)}{u}\,du\right\}.$$

Second step. We shall prove Theorem 2.2 by showing $Q > 0$ in (2.5). Note that the assumption of the first step holds, namely, $\int_{\boldsymbol{R}^d} 1/|x|^{1/r} P_{X_1}(dx) < \infty$ with $r = 2^{-1}$. Now we estimate the last term of $\Psi_t(s,z)$. We will repeatedly use nonincrease of $k_\xi(u)$ in u. We have

$$\left|\int_S \sigma(d\xi)\int_0^{\sqrt{t}}(e^{i\langle z,u\xi\rangle} - 1 - i\langle z,u\xi\rangle)\frac{k_\xi(u/\sqrt{t+s}) - k_\xi(u/\sqrt{t})}{u}\,du\right| \tag{2.6}$$

$$\leq \int_S \sigma(d\xi)\int_0^{\sqrt{t}}\frac{|z|^2}{2}u\left(k_\xi\left(\frac{u}{\sqrt{t+s}}\right) - k_\xi\left(\frac{u}{\sqrt{t}}\right)\right)du$$

$$\leq \frac{|z|^2}{2}\int_S \sigma(d\xi)\left(\int_0^{\sqrt{t/(t+s)}}(t+s)uk_\xi(u)\,du - \int_0^1 tuk_\xi(u)\,du\right)$$

$$\leq \frac{s|z|^2}{2}\int_S \sigma(d\xi)\int_0^1 uk_\xi(u)\,du.$$

Let $M \geq \sqrt{t+s}$. And we have

$$
\begin{aligned}
(2.7)\qquad & \left| \int_S \sigma(d\xi) \left\{ \int_{\sqrt{t}}^{M} \left(e^{i\langle z, u\xi\rangle} - 1 - i\langle z, u\xi\rangle 1_D\left(\frac{u\xi}{\sqrt{s+t}}\right)\right) \frac{k_\xi(u/\sqrt{t+s})}{u}\, du \right.\right.\\
&\qquad \left.\left. - \int_{\sqrt{t}}^{M} (e^{i\langle z, u\xi\rangle} - 1) \frac{k_\xi(u/\sqrt{t})}{u}\, du \right\} \right| \\
&\leq \int_S \sigma(d\xi) \left\{ \int_{\sqrt{t}}^{M} |e^{i\langle z, u\xi\rangle} - 1 - i\langle z, u\xi\rangle| \frac{k_\xi(u/\sqrt{t+s}) - k_\xi(u/\sqrt{t})}{u}\, du \right.\\
&\qquad \left. + \int_{\sqrt{t+s}}^{M} |\langle z, u\xi\rangle| \frac{k_\xi(u/\sqrt{t+s})}{u}\, du + \int_{\sqrt{t}}^{M} |\langle z, u\xi\rangle| \frac{k_\xi(u/\sqrt{t})}{u}\, du \right\} \\
&\leq \int_S \sigma(d\xi) \left\{ \int_{\sqrt{t}}^{M} \frac{|z|^2}{2} u k_\xi\left(\frac{u}{\sqrt{t+s}}\right) du \right.\\
&\qquad \left. + |z|(M - \sqrt{t+s}) k_\xi(1) + |z|(M - \sqrt{t}) k_\xi(1) \right\} = I, \quad (say)
\end{aligned}
$$

and

$$
\begin{aligned}
(2.8)\qquad & \left| \int_S \sigma(d\xi) \int_M^\infty (e^{i\langle z, u\xi\rangle} - 1) \frac{k_\xi(u/(t+s)^r) - k_\xi(u/t^r)}{u}\, du \right| \\
&\leq 2 \int_S \sigma(d\xi) \left\{ \int_{M/(t+s)^r}^{\infty} \frac{k_\xi(u)}{u}\, du - \int_{M/t^r}^{\infty} \frac{k_\xi(u)}{u}\, du \right\} \\
&= 2 \int_S \sigma(d\xi) \int_{M/(t+s)^r}^{M/t^r} \frac{k_\xi(u)}{u}\, du = J, \quad (say).
\end{aligned}
$$

Here we consider two cases. If $t > 1$, then, choosing $M = \sqrt{t+s}$, we have

$$
\begin{aligned}
I &\leq \int_S \sigma(d\xi) \left\{ \frac{|z|^2}{2} k_\xi\left(\sqrt{\frac{t}{t+s}}\right) \int_{\sqrt{t}}^{\sqrt{t+s}} u du + |z|\sqrt{s} k_\xi(1) \right\} \\
&\leq \int_S \sigma(d\xi) \left\{ \frac{s|z|^2}{4} k_\xi\left(\frac{1}{\sqrt{1+s}}\right) + |z|\sqrt{s} k_\xi(1) \right\}
\end{aligned}
$$

and

$$
J \leq 2 \int_S \sigma(d\xi) \int_1^{\sqrt{1+s}} \frac{k_\xi(u)}{u}\, du.
$$

Let $0 < \varepsilon < 1$. If $t \leq 1$, then, choosing $M = \varepsilon^{-1}\sqrt{1+s}$, we have

$$I \leq \int_S \sigma(d\xi) \left\{ \frac{|z|^2}{2} \int_0^{\varepsilon^{-1}\sqrt{1+s}} u k_\xi \left(\frac{u}{\sqrt{1+s}} \right) du + 2|z|\varepsilon^{-1}\sqrt{1+s} k_\xi(1) \right\}$$

and

$$J \leq 2 \int_S \sigma(d\xi) \int_{1/\varepsilon}^{\infty} \frac{k_\xi(u)}{u} \, du.$$

Hence the three inequalities (2.6), (2.7), and (2.8) give the following estimate: for small enough a and ε,

$$|\Psi_t(s,z)| \leq \pi/4 \quad \text{for } 0 < s < \varepsilon, |z| \leq 4ra\sqrt{d}, \text{ and } w \in K,$$

and hence $\cos F_t(s,z) \geq 1/\sqrt{2}$ and $\exp G_t(s,z) \geq e^{-\pi/4}$. Therefore

$$Q \geq \frac{\varepsilon e^{-\pi/4}}{\sqrt{2}} \int_{|z| \leq 4ra\sqrt{d}} f_a^{4r}(z) \, dz > 0,$$

completing the proof of Theorem 2.2.

Third step. Next we shall prove Theorem 2.3. Let $r = m$, where m is a positive integer. From the assumption of the theorem, $\Psi_t(s,z)$ is represented by

$$\begin{aligned} \Psi_t(s,z) &= i\langle z, w\rangle + i((s+t)^m - t^m)\langle \gamma_0, z\rangle \\ &\quad + \int_S \sigma(d\xi) \int_0^\infty (e^{i\langle z, u\xi\rangle} - 1) \frac{k_\xi(u/(s+t)^m) - k_\xi(u/t^m)}{u} \, du. \end{aligned}$$

We have

$$\left| \int_S \sigma(d\xi) \int_0^{\varepsilon^{-1}} (e^{i\langle z, u\xi\rangle} - 1) \frac{k_\xi(u/(s+t)^m) - k_\xi(u/t^m)}{u} \, du \right| \leq \varepsilon^{-1}|z| \int_S \sigma(d\xi) k_\xi(0+),$$

and, let $M = \varepsilon^{-1}$, then J in (2.8) has the following estimate: for $t > 1$,

$$J \leq 2m \log(1+s) \int_S \sigma(d\xi) k_\xi(0+),$$

and, for $t \leq 1$,

$$J \leq 2 \int_S \sigma(d\xi) \int_{1/\varepsilon(1+s)^m}^{\infty} \frac{k_\xi(u)}{u} \, du.$$

It suffices to show (2.1) that $Q > 0$ in (2.5), if $\gamma_0 = 0$ and (3) holds in the case that m is an arbitrary positive integer and if (2) holds in the case that $m = 1$. In the same way as in the second step, we can show it in the respective cases. This completes the proof. $\square$

3. Lemmas.

In this section, in order to prove Theorems 1.1 and 1.2, we prepare some lemmas. We denote by H_A the range of the linear transformation defined by the matrix A. We denote by Π_A the orthogonal projector from $\boldsymbol{R}^d$ to H_A. In general denote by $\mathrm{Supp}(\mu)$ the support of a measure μ. Let G be the smallest linear subspace containing H_A and $\mathrm{Supp}(\rho)$. Let $S(G) = \{x \in G : |x| = 1\}$. For $\theta \in S(G)$ and $\varepsilon > 0$, let $\beta(\theta, \varepsilon) = \int_{|\langle \xi, \theta \rangle| \geq \varepsilon} \sigma(d\xi) k_\xi(0+)$ if $|\Pi_A \theta| < \varepsilon$, and let $\beta(\theta, \varepsilon) = \infty$ if $|\Pi_A \theta| \geq \varepsilon$. Let

$$\beta = \sup_{\varepsilon > 0} \inf_{\theta \in S(G)} \beta(\theta, \varepsilon).$$

This quantity β depends only on the matrix A and the Lévy measure ρ. But, making G explicit, we call β the concentration order of the pair $(A, \rho)_G$. Sato [**6**] defined a similar quantity β, but our β is different from Sato's β, since Sato defined β with $\inf_{\theta \in S(\boldsymbol{R}^d)}$ in place of $\inf_{\theta \in S(G)}$. We have the following lemma.

Lemma 3.1. *The process $\{X_t\}$ is proper if and only if $G = \boldsymbol{R}^d$.*

Proof. The process $\{X_t\}$ is proper if and only if P_{X_1} is full. By a general theory of infinitely divisible distributions, P_{X_1} is full if and only if $G = \boldsymbol{R}^d$ (see [**7**]). □

We need more lemmas. Here recall that a distribution P_{X_1} is self-decomposable.

Keeping Lemma 3.1 in mind, we introduce the following terminology. Let V be a linear subspace of $\boldsymbol{R}^d$. We say that a self-similar additive process $\{X_t\}$ is V-proper if $P(X_t \in V) = 1$ and $G = V$, where G is the linear subspace defined at the beginning of this section. For any measure μ and any Borel set B, denote by $\mu|_B$ the restriction of μ to B.

Lemma 3.2. *Let V be a linear subspace of $\boldsymbol{R}^d$. Denote by T the orthogonal projector from $\boldsymbol{R}^d$ to V. Then we have the following.*

(a) *The process $\{TX_t\}$ is a self-similar additive process with exponent H, and its characteristic function is as follows:*

$$Ee^{i\langle z, TX_t \rangle} = Ee^{i\langle t^H z, TX_1 \rangle} = \exp\Bigg[-2^{-1} t^{2H} \langle TATz, z \rangle + \int_{\boldsymbol{R}^d} (e^{it^H \langle z, x \rangle} - 1 - it^H \langle z, x \rangle 1_D(x)) \rho T^{-1}|_{V \setminus \{0\}}(dx) + it^H \langle z, T\gamma \rangle \Bigg].$$

Here $\rho T^{-1}(B) = \rho(T^{-1}(B))$ for any Borel set B.

(b) *If the process $\{TX_t\}$ is transient, then $\{X_t\}$ is transient.*

(c) *If $\{X_t\}$ is proper, then $\{TX_t\}$ is V-proper.*

PROOF. We immediately see the first assertion of (a) and (b) from the definitions of a self-similar additive process and transience, respectively. Since T equals the transposed matrix of T, by virtue of Proposition 2.4.11 in [**7**] p. 60 the second assertion of (a) holds. The proof of (c) is as follows. Let $V^{\perp}$ be the orthogonal complement of V. If $P(TX_1 \in H) = 1$ for some $(\dim V - 1)$-dimensional hyperplane H in V, then $P(X_1 \in H + V^{\perp}) = 1$, contrary to that $\{X_t\}$ is proper. Hence $\{TX_t\}$ is V-proper. $\square$

LEMMA 3.3 (Sato [**6**]). *Suppose that P_{X_1} is not a delta measure. Let β be the concentration order of $(A, \rho)_G$. Then $\beta > 0$ and, for any β' satisfying $0 < \beta' < \beta$, there is a constant M such that*

$$|\hat{P}_{X_1}(z)| \le M|z|^{-\beta'} \quad \text{for } z \neq 0 \text{ and } z \in G.$$

We use the following terminology. Let V be a linear subspace of $\boldsymbol{R}^d$ and let $S(V) = \{x \in V : |x| = 1\}$. Suppose that a measure Λ on V satisfies $\Lambda(\{0\}) = 0$ and has a decomposition

$$\Lambda(EB) = \int_B \lambda(d\xi) \int_E \frac{\tau_\xi(u)}{u}\, du$$

for $E \in \mathscr{B}((0, \infty))$ and $B \in \mathscr{B}(S(V))$. We say that the decomposition satisfies the condition (A) if λ is a probability measure on $S(V)$, and $\tau_\xi(u)$ is nonnegative, non-increasing right continuous in u and Borel measurable in ξ, and

$$\int_{S(V)} \lambda(d\xi) \int_0^\infty \tau_\xi(u) u (1 + u^2)^{-1}\, du < \infty.$$

LEMMA 3.4. *Let V be a linear subspace of $\boldsymbol{R}^d$. Denote by T the orthogonal projector from $\boldsymbol{R}^d$ to V. Then the Lévy measure $\rho T^{-1}|_{V \setminus \{0\}}$ of $\{TX_t\}$ is decomposed into*

$$\rho T^{-1}|_{V \setminus \{0\}}(EB) = \int_B \tilde{\sigma}(d\xi) \int_E \frac{\tilde{k}_\xi(u)}{u}\, du$$

for $B \in \mathscr{B}(S(V))$ and $E \in \mathscr{B}((0, \infty))$, where this decomposition satisfies the condition (A). *Furthermore, we have*

$$\int_{S(V)} \tilde{\sigma}(d\xi) \tilde{k}_\xi(0+) = \int_{S \setminus V^{\perp}} \sigma(d\xi) k_\xi(0+).$$

Here $V^{\perp}$ is the orthogonal complement of V.

PROOF. The first assertion is a consequence of the fact that $\{TX_t\}$ is also a self-similar additive process. We have, for $a > 0$,

$$\int_a^\infty \frac{du}{u}\int_{S(V)} \tilde{\sigma}(d\xi)\tilde{k}_\xi(u) = \int_0^\infty \frac{du}{u}\int_{uT\xi\in(a,\infty)S(V)} \sigma(d\xi)k_\xi(u)$$

$$= \int_{((T\xi)/|T\xi|)\in S(V),\, T\xi\neq 0} \sigma(d\xi)\int_{a/|T\xi|}^\infty \frac{du}{u}k_\xi(u)$$

$$= \int_a^\infty \frac{du}{u}\int_{T\xi\neq 0} \sigma(d\xi)k_\xi\left(\frac{u}{|T\xi|}\right).$$

Hence, from the right continuity of $\tilde{k}_\xi(u)$ and $k_\xi(u)$ in u, we conclude the lemma. □

Lemma 3.5. *Let $d=3$. Suppose that $\{X_t\}$ is proper. Assume that the characteristic function of X_1 is decomposed as*

$$\hat{P}_{X_1}(z) = \hat{\mu}_1(z)\hat{\mu}_2(z),$$

where μ_1 and μ_2 are self-decomposable distributions. For $k=1,2$ let A_k and ρ_k be the Gaussian variance matrix and the Lévy measure of μ_k, respectively. Let V_k be the smallest linear subspace containing H_{A_k} and $\mathrm{Supp}(\rho_k)$. *Suppose that V_1 is two- or three-dimensional. If the concentration order of $(A_1,\rho_1)_{V_1}$ is infinite, then we have*

$$\int_{|z|>1} \frac{|\hat{P}_{X_1}(z)|}{|z|}\, dz < \infty.$$

Proof. At first suppose that V_1 is three-dimensional. Then from Lemma 3.3 we have, for any α with $\alpha > 2$,

$$\int_{|z|>1} \frac{|\hat{P}_{X_1}(z)|}{|z|}\, dz \le \text{const.} \times \int_{|z|>1} \frac{1}{|z|^{1+\alpha}}\, dz < \infty.$$

Next suppose that V_1 is two-dimensional. Denote by T_1 and T_2 the orthogonal projectors from $\boldsymbol{R}^3$ to V_1 and V_2, respectively. Choose a vector e such that $e\in V_2$, $|e|=1$, and $e\notin V_1$. Such a vector e exists because $\{X_t\}$ is proper. From Lemma 3.3 we have, for any $\alpha>0$ and some $0<\beta'<1$,

$$\int_{|z|>1} \frac{|\hat{P}_{X_1}(z)|}{|z|}\, dz = \int_{|z|>1} \frac{|\hat{\mu}_1(z)|\,|\hat{\mu}_2(z)|}{|z|}\, dz$$

$$\le \text{const.} \times \int_{|z|>1} \frac{dz}{|z|}\cdot\frac{1}{|T_1 z|^\alpha}\cdot\frac{1}{|\langle z,e\rangle|^{\beta'}} = J, \quad (say),$$

since $|T_2 z| \ge |\langle z,e\rangle|$. Let the vectors e_1,e_2,e_3 be the orthonormal basis of $\boldsymbol{R}^3$ such that V_1 is generated by the vectors e_1,e_2. Let $e=c_1e_1+c_2e_2+c_3e_3$, where $c_1,c_2,c_3\in\boldsymbol{R}$ and

$c_3 \neq 0$. Write the integral using the coordinates in the basis e_1, e_2, e_3. Then we have

$$J = \text{const.} \times \int_{|z|>1} \frac{dz}{|z|} \cdot \frac{1}{(z_1^2 + z_2^2)^{\alpha/2}} \cdot \frac{1}{|c_1 z_1 + c_2 z_2 + c_3 z_3|^{\beta'}}.$$

Here we introduce a new norm $\|\cdot\|$. Let $\|z\| = (z_1^2 + z_2^2)^{1/2} + |c_1 z_1 + c_2 z_2 + c_3 z_3|$ for any $z \in \boldsymbol{R}^3$. Note that there are positive constants a and b such that $a\|z\| \le |z| \le b\|z\|$. Hence we have

$$\begin{aligned}
J &\le \text{const.} \times \int_{\|z\|>1/b} \frac{dz}{\|z\|} \cdot \frac{1}{(z_1^2 + z_2^2)^{\alpha/2}} \cdot \frac{1}{|c_1 z_1 + c_2 z_2 + c_3 z_3|^{\beta'}} \\
&= \text{const.} \times \int_{\boldsymbol{R}^2} \frac{dz_1 dz_2}{(z_1^2 + z_2^2)^{\alpha/2}} \int_{(z_1^2+z_2^2)^{1/2}+|s|>1/b} \frac{ds}{((z_1^2 + z_2^2)^{1/2} + |s|)|s|^{\beta'}} \\
&= \text{const.} \times \int_0^\infty \frac{ds}{s^{\beta'}} \int_{\substack{r+s>1/b \\ r>0}} \frac{r^{1-\alpha}}{r+s}\, dr \\
&= \text{const.} \times \left\{ \int_0^{1/b} \frac{ds}{s^{\alpha+\beta'-1}} \int_{\substack{s(u+1)>1/b \\ u>0}} \frac{u^{1-\alpha}}{1+u}\, du + \int_{1/b}^\infty \frac{ds}{s^{\alpha+\beta'-1}} \int_{\substack{s(u+1)>1/b \\ u>0}} \frac{u^{1-\alpha}}{1+u}\, du \right\}.
\end{aligned}$$

Choose α with $\alpha + \beta' > 2$ and $1 < \alpha < 2$. Then

$$\int_{1/b}^\infty \frac{ds}{s^{\alpha+\beta'-1}} \int_{\substack{s(1+u)>1/b \\ u>0}} \frac{u^{1-\alpha}}{1+u}\, du \le \int_{1/b}^\infty \frac{ds}{s^{\alpha+\beta'-1}} \left(\int_0^1 u^{1-\alpha}\, du + \int_1^\infty \frac{du}{u^\alpha} \right) < \infty.$$

Furthermore, we have

$$\begin{aligned}
\int_0^{1/b} \frac{ds}{s^{\alpha+\beta'-1}} \int_{\substack{s(1+u)>1/b \\ u>0}} \frac{u^{1-\alpha}}{1+u}\, du &\le \int_0^{1/b} \frac{ds}{s^{\alpha+\beta'-1}} \int_{u>(1/bs)-1} \frac{du}{u^\alpha} \\
&= \frac{b^{\alpha-1}}{\alpha-1} \int_0^{1/b} \frac{ds}{s^{\beta'}(1-bs)^{\alpha-1}} < \infty.
\end{aligned}$$

Hence we have $J < \infty$. This completes the proof of the lemma. □

LEMMA 3.6. *Let B be a subset of $\boldsymbol{R}^d$ such that if $x \in B$, then $ax \in B$ for any $a > 0$. Let W be a linear subspace of $\boldsymbol{R}^d$, and let U be the smallest linear subspace that contains* $\mathrm{Supp}(\rho|_{B \cap W})$. *Denote by α the concentration order of $(0, \rho|_{B \cap W})_U$. If $\alpha < \infty$ and*

$$\int_{B \cap W \cap S} \sigma(d\xi) k_\xi(0+) = \infty,$$

then there is a point $\zeta_0 \in S \cap U$ such that

$$\int_{(B\cap W\cap S)\setminus S_0}\sigma(d\xi)k_\xi(0+)<\infty, \tag{3.1}$$

$$\int_{B\cap W\cap S_0}\sigma(d\xi)k_\xi(0+)=\infty, \tag{3.2}$$

where $S_0=\{\xi\in S\cap W:\langle\xi,\zeta_0\rangle=0\}$.

PROOF. From the definition of the concentration order, there is a sequence ζ_n^ε which converges to some $\zeta_0^\varepsilon\in S\cap U$ and which satisfies that

$$\alpha=\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\int_{\substack{|\langle\xi,\zeta_n^\varepsilon\rangle|\geq\varepsilon\\ \xi\in B\cap W\cap S}}\sigma(d\xi)k_\xi(0+).$$

By Fatou's lemma, we have

$$\alpha\geq\liminf_{\varepsilon\downarrow 0}\int_{\substack{|\langle\xi,\zeta_0^\varepsilon\rangle|\geq 2\varepsilon\\ \xi\in B\cap W\cap S}}\sigma(d\xi)k_\xi(0+)=\lim_{\varepsilon_j\downarrow 0}\int_{\substack{|\langle\xi,\zeta_0^{\varepsilon_j}\rangle|\geq 2\varepsilon_j\\ \xi\in B\cap W\cap S}}\sigma(d\xi)k_\xi(0+).$$

Choosing a sequence $\varepsilon_j\downarrow 0$ such that $\zeta_0=\lim_{j\to\infty}\zeta_0^{\varepsilon_j}$ exists, we have, again by Fatou's lemma,

$$\alpha\geq\int_{\substack{|\langle\xi,\zeta_0\rangle|>0\\ \xi\in B\cap W\cap S}}\sigma(d\xi)k_\xi(0+).$$

The lemma has been proved. □

LEMMA 3.7. *Set*

$$I_\alpha=\int_{\{z\in\boldsymbol{R}^d:|z|>1\}}\frac{|\hat{P}_{X_1}(z)|}{|z|^{d-\alpha}}\,dz \tag{3.3}$$

for $0<\alpha<d$. *Then* $\{X_t\}$ *is transient if it satisfies one of the following*:

(a) $d\geq 3$ *and* $I_2<\infty$.

(b) $d\geq 2$, $I_1<\infty$, *and* $\{X_t\}$ *satisfies the assumption in Theorem* 2.3.

(c) $d\geq 1$, $I_\alpha<\infty$ *with some* $0<\alpha<1$, $\{X_t\}$ *satisfies the assumption in Theorem* 2.3, *and* $\gamma_0=0$.

PROOF. Let $u(z)=\Pi_{j=1}^d((1-|z_j|)\vee 0)$, then its Fourier transform is given by $v(x)=\Pi_{j=1}^d((\sin x_j/2)/(x_j/2))^2$. Then we have, for $r>0$,

$$\begin{aligned}P_{X_1}(|x|\leq r)&\leq c\int_{\boldsymbol{R}^d}v(r^{-1}x)P_{X_1}(dx)=c\int_{\boldsymbol{R}^d}u(z)\hat{P}_{X_1}(r^{-1}z)\,dz\\ &=cr^d\int_{\boldsymbol{R}^d}\hat{P}_{X_1}(z)u(rz)\,dz\leq cr^d\int_{|z|\leq(\sqrt{d}/r)}|\hat{P}_{X_1}(z)|\,dz,\end{aligned} \tag{3.4}$$

where c is a positive constant not depending on r. By using (3.4), we have

$$\int_{\boldsymbol{R}^d} \frac{1}{|x|^{\alpha}} P_{X_1}(dx) = \alpha \int_0^{\infty} P_{X_1}(|x| < r) \frac{dr}{r^{1+\alpha}}$$

$$\le \text{const.} \times \int_0^{\sqrt{d}} \frac{dr}{r^{1+\alpha}} r^d \int_{|z| \le (\sqrt{d}/r)} |\hat{P}_{X_1}(z)|\, dz + \alpha \int_{\sqrt{d}}^{\infty} \frac{dr}{r^{1+\alpha}}.$$

Furthermore we have

$$\int_0^{\sqrt{d}} \frac{dr}{r^{1+\alpha}} r^d \int_{|z| \le (\sqrt{d}/r)} |\hat{P}_{X_1}(z)|\, dz = \frac{d^{(d-\alpha)/2}}{d-\alpha} \int_{|z|>1} \frac{|\hat{P}_{X_1}(z)|}{|z|^{d-\alpha}}\, dz$$

$$+ \int_{|z| \le 1} |\hat{P}_{X_1}(z)|\, dz \int_0^{\sqrt{d}} dr r^{d-1-\alpha}$$

$$\le \frac{d^{(d-\alpha)/2}}{d-\alpha} \int_{|z|>1} \frac{|\hat{P}_{X_1}(z)|}{|z|^{d-\alpha}}\, dz + \frac{d^{(d-\alpha)/2}}{d-\alpha} \int_{|z| \le \sqrt{d}} dz.$$

Hence, by virtue of Theorems 2.2 and 2.3, the process $\{X_t\}$ is transient. □

4. Proofs of Theorems 1.1 and 1.2.

At first, by using lemmas in Section 3, we shall prove our first main theorem.

Proof of Theorem 1.1. It suffices to prove transience of $\{X_t\}$ in the case that $d = 1$ and $\gamma_0 = 0$. For, in the case that $d = 2$, choose a rotation matrix R such that the first coordinate of $R\gamma_0$ is 0 and let T be the orthogonal projector from $\boldsymbol{R}^2$ to the first coordinate. Then, from Lemma 3.4 $\{TRX_t\}$ satisfies the assumption of the theorem for $d = 1$, and transience of $\{TRX_t\}$ implies that of $\{X_t\}$ by Lemma 3.2(b).

Now suppose $d = 1$ and $\gamma_0 = 0$. Let β be the concentration order of $(A, \rho)_{\boldsymbol{R}^1}$, where $A = 0$. We know that $\beta > 0$ (see Lemma 3.3). Let $0 < \alpha < \min\{\beta, 1\}$. By virtue of Lemma 3.3, for any β' satisfying $\alpha < \beta' < \beta$, we have

$$\int_{|z|>1} \frac{|\hat{P}_{X_1}(z)|}{|z|^{1-\alpha}}\, dz \le \text{const.} \times \int_{|z|>1} \frac{1}{|z|^{1-\alpha+\beta'}}\, dz < \infty.$$

Hence, from Lemma 3.7(c), we obtain that $\{X_t\}$ is transient. □

To prove Theorem 1.2 we need the following lemma.

Lemma 4.1. *Let $d = 3$. Let V be a two-dimensional linear subspace of $\boldsymbol{R}^3$, and let T be the orthogonal projector from $\boldsymbol{R}^3$ to V. Suppose that the Gaussian variance matrix of*

TX_1 vanishes, namely, $TAT = 0$. Let $V^{\perp}$ be the orthogonal complement of V. If we have

$$J = \int_{S \setminus V^{\perp}} \sigma(d\xi) k_{\xi}(0+) < \infty, \tag{4.1}$$

then $\{X_t\}$ is transient.

Proof. As $\{TX_t\}$ satisfies the assumption of Theorem 1.1 by (4.1) combined with Lemma 3.4, $\{TX_t\}$ is transient. Hence, by Lemma 3.2(b), $\{X_t\}$ is transient. $\square$

We shall prove our second main theorem by using Lemmas 3.7 and 4.1.

Proof of Theorem 1.2. By Lemma 3.2 (b) and (c), it suffices to prove transience of $\{X_t\}$ in the case that $d = 3$. From Lemma 3.7 and Lemma 4.1, it suffices to show that I_{α} in (3.3) is finite or that J in (4.1) is finite under the assumption of Lemma 4.1. Let l be the dimension of H_A, where H_A is the linear subspace defined at the beginning of Section 3. If $l = 3$, then $I_2 < \infty$ from Lemma 3.3, because the concentration order of $(A, \rho)_{\boldsymbol{R}^3}$ is infinite. From now on suppose that $l \leq 2$.

Let B be a subset of $\boldsymbol{R}^3$ such that if $x \in B$, then $ax \in B$ for any $a > 0$. Now we have

$$\hat{P}_{X_1}(z) = \hat{\mu}_1(z) \hat{\mu}_2(z), \tag{4.2}$$

where

$$\hat{\mu}_1(z) = \exp\left[-2^{-1}\langle Az, z\rangle + \int_B (e^{i\langle z, x\rangle} - 1 - i\langle z, x\rangle 1_D(x))\rho(dx) + i\langle z, \gamma\rangle\right],$$

$$\hat{\mu}_2(z) = \exp\left[\int_{B^c} (e^{i\langle z, x\rangle} - 1 - i\langle z, x\rangle 1_D(x))\rho(dx)\right].$$

Here $B^c = \boldsymbol{R}^3 \setminus B$.

Suppose that $l = 2$ and $B = H_A$ above. Then the concentration order of $(A, \rho|_{H_A})_{H_A}$ is infinite. So from Lemma 3.5 we have $I_2 < \infty$. From now on we shall consider the case $l = 1$ and the case $A = 0$ together. Here, if $l = 1$, let $H = H_A$, and if $A = 0$, let H be a one-dimensional linear subspace such that

$$\int_{S \cap H} \sigma(d\xi) k_{\xi}(0+) < \infty. \tag{4.3}$$

From now on, in the case $A = 0$, suppose that such a linear subspace H exists, because if it does not exist, then the concentration order β of $(0, \rho)_{\boldsymbol{R}^3}$ is infinite from the definition, and $I_2 < \infty$ by Lemma 3.5.

Denote by U the smallest linear subspace containing $\mathrm{Supp}(\rho|_{H^c})$. Notice that U is two- or three-dimensional. Now we divide this case into two cases.

Case 1. Suppose that $\int_{H^c \cap S} \sigma(d\xi) k_\xi(0+) < \infty$. Denote by $H^\perp$ the orthogonal complement of H, and by T the orthogonal projector from $\boldsymbol{R}^3$ to $H^\perp$. Then we have $TAT = 0$ and $J < \infty$.

Case 2. Suppose that $\int_{H^c \cap S} \sigma(d\xi) k_\xi(0+) = \infty$. Let β_2 be the concentration order of $(0, \rho|_{H^c})_U$. Here we divide this case into two cases: $\beta_2 = \infty$ and $\beta_2 < \infty$.

Case 2-1. Suppose that $\beta_2 = \infty$. Let $B = H$ in (4.2). Apply Lemma 3.5 to P_{X_1}. Since U is two- or three-dimensional, we have $I_2 < \infty$.

Case 2-2. Suppose that $\beta_2 < \infty$. By the assumption $\int_{H^c \cap S} \sigma(d\xi) k_\xi(0+) = \infty$, then from Lemma 3.6 there is a point $\zeta_0 \in S \cap U$ such that

$$\int_{(H^c \cap S) \setminus S_0} \sigma(d\xi) k_\xi(0+) < \infty, \tag{4.4}$$

$$\int_{H^c \cap S_0} \sigma(d\xi) k_\xi(0+) = \infty, \tag{4.5}$$

where $S_0 = \{\xi \in S : \langle \xi, \zeta_0 \rangle = 0\}$. Let $W_0 = \{x \in \boldsymbol{R}^3 : \langle x, \zeta_0 \rangle = 0\}$. Let β_3 be the concentration order of $(0, \rho|_{W_0 \cap H^c})_{W_0'}$, where W_0' is the smallest linear subspace containing $\mathrm{Supp}(\rho|_{W_0 \cap H^c})$. Furthermore we divide this case into two cases: $\beta_3 < \infty$ and $\beta_3 = \infty$.

Case 2-2-1. Suppose that $\beta_3 < \infty$. From (4.5), using Lemma 3.6 again, we can decompose S_0, namely, there is a point $\zeta_1 \in W_0' \cap S$ such that

$$\int_{(H^c \cap S_0) \setminus S_1} \sigma(d\xi) k_\xi(0+) < \infty, \tag{4.6}$$

$$\int_{H^c \cap S_1} \sigma(d\xi) k_\xi(0+) = \infty, \tag{4.7}$$

where $S_1 = \{\xi \in S_0 : \langle \xi, \zeta_1 \rangle = 0\}$. Hence combining this with (4.4), we have

$$\int_{(H^c \cap S) \setminus S_1} \sigma(d\xi) k_\xi(0+) < \infty, \tag{4.8}$$

$$\int_{H^c \cap S_1} \sigma(d\xi) k_\xi(0+) = \infty. \tag{4.9}$$

Here we divide this case into two cases.

Case 2-2-1-1. Suppose that $A = 0$. Denote by W_1 the linear subspace spanned by S_1. Let $(W_1)^\perp$ be the orthogonal complement of W_1, and let T be the orthogonal projector from $\boldsymbol{R}^3$ to $(W_1)^\perp$. From (4.3) and (4.8) we have

$$J = \int_{S \setminus S_1} \sigma(d\xi) k_\xi(0+) \le \int_{S \cap H} \sigma(d\xi) k_\xi(0+) + \int_{(S \cap H^c) \setminus S_1} \sigma(d\xi) k_\xi(0+) < \infty.$$

Case 2-2-1-2. Suppose that $l = 1$. From (4.9) the set $H^c \cap S_1$ is non-empty. Then, W_1 is one-dimensional and $H^c \cap S_1 = S_1$ consists of two points. Hence we have $H \cap W_1 = \{0\}$. Now let W be the linear subspace spanned by H and W_1. Denote by Π the orthogonal projector from W to H. Then we obtain that the concentration order of $(A, \rho|_W)_W$ is infinite. In fact, since S_1 consists of two points and $H \neq S_1$, we have

$$
\begin{aligned}
(4.10)\qquad \lim_{\varepsilon\to 0} \inf_{\substack{|\Pi_A\zeta|<\varepsilon \\ \zeta\in W\cap S}} \int_{\substack{|\langle\xi,\zeta\rangle|\ge\varepsilon \\ \xi\in W\cap S}} \sigma(d\xi)k_\xi(0+) &\ge \lim_{\varepsilon\to 0} \inf_{\substack{|\Pi_A\zeta|<\varepsilon \\ \zeta\in W\cap S}} \int_{\substack{|\langle\xi,\zeta\rangle|\ge\varepsilon \\ \xi\in S_1}} \sigma(d\xi)k_\xi(0+) \\
&= \int_{S_1} \sigma(d\xi)k_\xi(0+) = \infty
\end{aligned}
$$

by (4.9). Here Π_A is the orthogonal projector defined at the beginning of Section 3. Let $B = W$ in (4.2). Then from Lemma 3.5 we have $I_2 < \infty$.

Case 2-2-2. Suppose that $\beta_3 = \infty$. Notice that $W_0' \neq \{0\}$ by (4.5). At first suppose that W_0' is two-dimensional. In (4.2) choose $B^c = W_0 \cap H^c$. Then, by Lemma 3.5, we have $I_2 < \infty$. Next suppose that W_0' is one-dimensional. Here we divide this case into two cases: $l = 1$ and $A = 0$.

Case 2-2-2-1. Suppose that $l = 1$. Let W be the linear subspace spanned by H and W_0'. Then, W is two-dimensional. Similary to (4.10), considering (4.5) in place of (4.9), we conclude that the concentration order of $(A, \rho|_{W_0\cap H^c})_W$ is infinite. Hence from Lemma 3.5 we have $I_2 < \infty$.

Case 2-2-2-2. Suppose that $A = 0$. Let $(W_0')^\perp$ be the orthogonal complement of W_0', and let T be the orthogonal projector from $\boldsymbol{R}^3$ to $(W_0')^\perp$. Then we have, since $W_0 \cap H^c \supset W_0'$,

$$
\begin{aligned}
J &= \int_{S\setminus W_0'} \sigma(d\xi)k_\xi(0+) \\
&= \int_{S\setminus(S_0\cap H^c)} \sigma(d\xi)k_\xi(0+) + \int_{(S_0\cap H^c)\setminus W_0'} \sigma(d\xi)k_\xi(0+) \\
&= \int_{S\setminus(S_0\cap H^c)} \sigma(d\xi)k_\xi(0+) \\
&= \int_{(S\cap H^c)\setminus(S_0\cap H^c)} \sigma(d\xi)k_\xi(0+) + \int_{(S\cap H)\setminus(S_0\cap H^c)} \sigma(d\xi)k_\xi(0+) \\
&\le \int_{(S\setminus S_0)\cap H^c} \sigma(d\xi)k_\xi(0+) + \int_{S\cap H} \sigma(d\xi)k_\xi(0+) < \infty
\end{aligned}
$$

by (4.3) and (4.4). The proof is now complete. □

5. Proof of Theorem 1.3.

PROOF OF THEOREM 1.3. Without loss of generality we assume that the exponent $H = 1$ for $\{X_t\}$. Let $a > 1$. We have

$$
\begin{aligned}
(5.1)\qquad 1 &\geq \sum_{n=1}^{\infty} P(X_{a^n} < 0, X_{a^{n+k}} > 0 \text{ for all positive integer } k) \\
&\geq \sum_{n=1}^{\infty} P(X_{a^n} < 0 < X_{a^{n+1}}, X_{a^{n+k+1}} - X_{a^{n+1}} \geq 0 \text{ for all positive integer } k) \\
&= \sum_{n=1}^{\infty} P(X_{a^n} < 0 < X_{a^{n+1}}) P(X_{a^{n+k+1}} - X_{a^{n+1}} \geq 0 \text{ for all positive integer } k) \\
&= \sum_{n=1}^{\infty} P(X_1 < 0 < X_a) P(X_{a^k} - X_1 \geq 0 \text{ for all positive integer } k).
\end{aligned}
$$

In case (i) and (ii) both P_{X_1} and $P_{X_a - X_1}$ have supports $\boldsymbol{R}$ by Proposition 4.1 of Sato [**9**]. Let $\gamma_0 = \gamma - \int_{(-1,0)} x\rho(dx)$. In case (iii) we have $\mathrm{Supp}(P_{X_1}) = (-\infty, \gamma_0]$ and $(a-1)\gamma_0 \in \mathrm{Supp}(P_{X_a - X_1}) \subset (-\infty, (a-1)\gamma_0]$. These follow from the Lévy measures of X_1 and $X_a - X_1$ are, respectively, $\sigma(\{-1\})k_{-1}(-x)1_{(-\infty,0)}(x)/|x|\,dx$ and $\sigma(\{-1\})(k_{-1}(-x/a) - k_{-1}(-x))1_{(-\infty,0)}(x)/|x|\,dx$ on $(-\infty, 0)$. Since we have that

$$P(X_1 < 0 < X_a) \geq P(-\eta < X_1 < 0) P(X_a - X_1 > \eta)$$

for any $\eta > 0$, in all cases we have $P(X_1 < 0 < X_a) > 0$ by choosing η appropriately. From this and (5.1) we have

$$(5.2)\qquad P(X_{a^k} - X_1 \geq 0 \text{ for all positive integer } k) = 0.$$

In the same way we have

$$1 \geq \sum_{n=1}^{\infty} P(X_1 > 0 > X_a) P(X_{a^k} - X_1 \leq 0 \text{ for all positive integer } k).$$

Further,

$$P(X_1 > 0 > X_a) \geq P(\eta > X_1 > 0) P(X_a - X_1 < -\eta) > 0$$

for any $\eta > 0$, since $\mathrm{Supp}(P_{X_a - X_1})$ is unbounded below. Hence

$$(5.3)\qquad P(X_{a^k} - X_1 \leq 0 \text{ for all positive integer } k) = 0.$$

Since a can be arbitrarily large in (5.2) and (5.3), we have

$$P\left(\bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty}\{X_{a^k}-X_1>0\}\right)\geq P\left(\bigcap_{l=1}^{\infty}\bigcup_{k'=1}^{\infty}\{X_{a^{lk'}}-X_1>0\}\right)=1$$

$$P\left(\bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty}\{X_{a^k}-X_1<0\}\right)\geq P\left(\bigcap_{l=1}^{\infty}\bigcup_{k'=1}^{\infty}\{X_{a^{lk'}}-X_1<0\}\right)=1.$$

Hence the sample path of $X_t - X_1$ crosses the origin after arbitrarily large time. By using Lévy-Itô decomposition theorem, we see from $\rho((0,\infty)) = 0$ that the process $\{X_t\}$ has no upward jump. Hence $X_t - X_1$ visits the origin after arbitrarily large time. This shows that $\{X_t\}$ is not transient, so $\{X_t\}$ is recurrent. This completes the proof of Theorem 1.3. □

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