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Applications of the theory of the metaplectic representation to quadratic Hamiltonians on the two-dimensional Euclidean space

Dedicated to Professor Nobuyuki Ikeda on his seventieth birthday

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Abstract. The spectra of the quadratic Hamiltonians on the two-dimensional Euclidean space are determined completely by using the theory of the metaplectic representation. In some cases, the corresponding heat kernels are studied in connection with the well-definedness of the Wiener integrations. A proof of the Lévy formula for the stochastic area and a relation between the real and complex Hermite polynomials are given in our framework.

1. Introduction.

Let $P(\xi, x)$ be a real homogeneous quadratic polynomial in $\xi \in \boldsymbol{R}^2$ and $x \in \boldsymbol{R}^2$, and $P(D, x)$ be the corresponding second order differential operator defined by the Weyl correspondence (cf. [**5**] (2.1)), where $D = \left(2\pi\sqrt{-1}\right)^{-1}\nabla$. It is known that $P(D, x)$ is essentially self-adjoint on the space $\mathscr{S}(\boldsymbol{R}^2)$ of the rapidly decreasing functions on $\boldsymbol{R}^2$. We call the corresponding self-adjoint operator a quadratic Hamiltonian and denote it by the same symbol. The purpose of this paper is to determine the spectrum of quadratic Hamiltonians by using the theory of the metaplectic representation (cf. [**5**], [**6**]) and Hörmander's classification of quadratic forms [**9**]. We will also discuss the heat kernels of the semigroups generated by quadratic Hamiltonians whose principal symbols are $|\xi|^2$.

The metaplectic representation μ is a unitary representation of the double cover of the symplectic group $Sp(n; \boldsymbol{R})$ and its differential $d\mu$ is a Lie algebra isomorphism between the symplectic Lie algebra $sp(n; \boldsymbol{R})$ and an algebra consisting of the quadratic Hamiltonians on $\boldsymbol{R}^n$. By this representation, symplectically equivalent quadratic forms correspond to unitary equivalent quadratic Hamiltonians. On the other hand, Hörmander [**9**] has classified all of the quadratic forms on $\boldsymbol{R}^{2n}$ into several types of simple forms. Therefore, by determining the spectra of the quadratic Hamiltonians

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corresponding to these simple forms, we can determine the spectra of all quadratic Hamiltonians. In this paper we will carry out this plan in the two-dimensional case.

As an example of quadratic Hamiltonians, we will study the following Schrödinger operator with a uniform magnetic field:

$$H = \frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_1} - bx_2\right)^2 + \frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_2} + bx_1\right)^2 + c_1 x_1^2 + c_2 x_2^2 \tag{1.1}$$

on $L^2(\boldsymbol{R}^2)$, where b, c_1, c_2 are real constants. When both of c_1 and c_2 are non-negative, the operators of this type naturally appear in the study of the semi-classical approximation for the Schrödinger operators (cf. [**7**], [**16**]). In this case the complete description of the spectral properties of the operator H has been given in [**15**] and it is systematically generalized in [**16**] for the operator of the form

$$\boldsymbol{H} = \frac{1}{2}\sum_{j=1}^{n}\left(\sqrt{-1}\frac{\partial}{\partial x_j} - (Bx)_j\right)^2 + \frac{1}{2}\langle x, Kx\rangle \tag{1.2}$$

on $\boldsymbol{R}^n$, where B is a real skew-symmetric $n \times n$ matrix, K is a real symmetric non-negative $n \times n$ matrix and $\langle \cdot, \cdot \rangle$ is the usual inner product in $\boldsymbol{R}^n$. This operator is unitary equivalent to a direct sum of the harmonic oscillators and the Laplacians and the spectral property is fairly simple. However, if we remove the assumption on the nonnegativity of the operators, the problem is complicated. We will describe this situation in the two-dimensional case and determine the spectrum of the operator H completely.

Secondly we will study the heat kernel $\exp(-tH)(x,y)$ of the semigroup $\exp(-tH)$ generated by the Schrödinger operator H when $c_1 = 0$ and $c_2 = -\ell^2$. By the above considerations, we know that there exists an element $S \in Sp(2; \boldsymbol{R})$ such that

$$\mu(S) H \mu(S)^{-1} = H_0, \tag{1.3}$$

where H_0 is a quadratic Hamiltonian of a simpler form. For example, when $|\ell| \neq \sqrt{2}|b|$, H_0 is a direct sum of the harmonic oscillator and a multiplication operator (see Section 5 below). Then the calculation related to the heat semigroup for H_0 can be accomplished by the Mehler formula. Moreover the operator $\mu(S)$, which is originally defined as an isomorphism on $\mathscr{S}(\boldsymbol{R}^2)$, extends to an isomorphism on the space $\mathscr{S}'(\boldsymbol{R}^2)$ of Schwartz tempered distributions (cf. [**5**] Proposition (4.27)). In fact $\mu(S)$ consists of simple integral transform (see Theorem 2.2 below). Therefore we can consider an expression

$$e^{-tH}(x,y) = \{\mu(S) e^{-tH_0} \mu(S)^{-1} \delta_y\}(x). \tag{1.4}$$

In general, this holds only in the sense of distributions. However we can show that this holds as smooth functions in $(t, x, y) \in (0, t_0) \times \boldsymbol{R}^2 \times \boldsymbol{R}^2$ for some $t_0 > 0$ and that t_0 is infinity if $|b|$ is large enough.

This result on the heat kernel is also interesting from a probabilistic point of view. In fact the heat kernel of $\exp(-tH)$ is expressed by means of a Wiener integral (see (5.2) below) and the range where this Wiener integral is absolutely convergent is smaller. Therefore we may regard this Wiener integral as an example of convergent Wiener integrals which are not absolutely convergent. Ikeda, Kusuoka and Manabe [**10**], [**11**] has also pointed it out and they have identified t_0 with the conjugate point in their Lagrangian mechanics (see also [**4**]). The important points in our method are that our representation (1.4) consists of simple Gaussian integrations and that it is given in terms of the corresponding usual Hamiltonian mechanics.

As another application of the equality (1.4), we will give the explicit representation of the heat kernel of $\exp(-tH)$ when $c_1 = c_2 = 0$. This brings us a proof of the Lévy formula for the stochastic area by using the equivalence of the circular motion and the one-dimensional harmonic oscillator in $\boldsymbol{R}^2$. Moreover, by studying the eigenfunctions, we will show that the complex Hermite polynomials are obtained through an integral transform from the usual Hermite polynomials.

The organization of this paper is as follows. In Section 2 we will review the theory of the metaplectic representations following Folland [**5**] and prepare some formulae which will be used in the subsequent sections. In Section 3 we will determine the spectra of the quadratic Hamiltonians on $\boldsymbol{R}^2$. In Section 4 we apply the results in Section 3 to the Schrödinger operator with a uniform magnetic field. In Section 5 we will study the heat kernel $\exp(-tH)(x, y)$ when $c_1 = 0$ and $c_2 = -\ell^2$. In Section 6 we will give a proof of the explicit representation of the heat kernel $\exp(-tH)(x, y)$ when $c_1 = c_2 = 0$ and will show the correspondence between the usual and the complex Hermite polynomials.

2. The metaplectic representation.

In this section we review the theory of the metaplectic representation and give some formulae which will be used in the following sections. We follow the convention in Folland [**5**]. For example, we denote by $\hat{f}$ or $\mathscr{F}f$ the Fourier transform of a function f on $\boldsymbol{R}^2$ defined by

$$\hat{f}(\xi) = \int_{\boldsymbol{R}^2} \exp(-2\pi\sqrt{-1}\langle \xi, x\rangle) f(x)\, dx,$$

where $\langle \cdot , \cdot \rangle$ is the usual inner product in $\boldsymbol{R}^2$. We will denote by $\langle \cdot , \cdot \rangle$ the usual inner product in any dimensional Euclidean space since there is no fear of confusion.

Let $Sp(2;\boldsymbol{R})$ be the symplectic group, that is, the group of 4×4 real matrices which preserves the symplectic form

$$[w_1, w_2] = \langle w_1, \mathscr{I} w_2 \rangle, \quad \mathscr{I} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the 2×2 identity matrix. That is, $\mathscr{A} \in Sp(2;\boldsymbol{R})$ if and only if $\mathscr{A}^* \mathscr{I} \mathscr{A} = \mathscr{I}$ holds, where $\mathscr{A}^*$ is the transpose of $\mathscr{A}$. We will often write 4×4 matrix in block form by using 2×2 matrices. Then $\mathscr{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2;\boldsymbol{R})$ if and only if $A^*C = C^*A$, $B^*D = D^*B$ and $A^*D - C^*B = I$.

Let $sp(2;\boldsymbol{R})$ be the symplectic Lie algebra. $\mathscr{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in sp(2;\boldsymbol{R})$ if and only if $\mathscr{I}\mathscr{A} + \mathscr{A}^*\mathscr{I} = 0$ or, in block form, $D = -A^*$, $B = B^*$ and $C = C^*$. $\mathscr{A} \in sp(2;\boldsymbol{R})$ corresponds to the quadratic polynomial $P_{\mathscr{A}}$ on $\boldsymbol{R}^2$ by

$$P_{\mathscr{A}}(w) = -\frac{1}{2}\langle w, \mathscr{A}\mathscr{I} w \rangle$$

or

$$P_{\mathscr{A}}(\xi, x) = \frac{1}{2}\langle \xi, B\xi \rangle - \langle \xi, Ax \rangle - \frac{1}{2}\langle x, Cx \rangle, \quad w = \begin{pmatrix} \xi \\ x \end{pmatrix}, \quad \xi, x \in \boldsymbol{R}^2.$$

We denote by $\mathscr{Q}$ the totality of the real homogeneous quadratic polynomials on $\boldsymbol{R}^4$. If we regard $\mathscr{Q}$ as a Lie algebra by considering the Poisson bracket, the above correspondence gives rise a Lie algebra isomorphism from $sp(2;\boldsymbol{R})$ to $\mathscr{Q}$. It should be noted that, if we consider $P_{\mathscr{A}}(\xi, x)$ as a classical Hamiltonian and consider the classical mechanics, the corresponding Hamilton equation is

$$\frac{d}{dt}\begin{pmatrix} \xi \\ x \end{pmatrix} = \begin{pmatrix} A^* & C \\ B & -A \end{pmatrix}\begin{pmatrix} \xi \\ x \end{pmatrix}$$

and $\mathscr{A}$ itself does not appear. If we consider the correspondence between $sp(2;\boldsymbol{R})$ and $\mathscr{Q}$ given by

$$\tilde{P}_{\mathscr{A}}(w) = \frac{1}{2}\langle w, \mathscr{I}\mathscr{A} w \rangle, \tag{2.1}$$

$\mathscr{A}$ itself determines the Hamilton equation. Although we can construct the theory of metaplectic representation based on (2.1), we will follow the convention in [**5**] in order to apply directly the results therein.

The metaplectic representation μ of $Sp(2;\boldsymbol{R})$ is defined as follows. Let $\boldsymbol{H}_2$ be the real 5-dimensional Heisenberg group, which is $\boldsymbol{R}^5$ endowed with the group law

$$(p,q,t)(p',q',t') = (p+p', q+q', t+t' + (\langle p,q'\rangle - \langle q,p'\rangle)/2)$$

for $p=(p_1,p_2)$, $q=(q_1,q_2)\in\boldsymbol{R}^2$ and $t\in\boldsymbol{R}$. For $(p,q,t)\in\boldsymbol{H}_2$, we let $\rho(p,q,t)$ be the unitary operator on $L^2(\boldsymbol{R}^2)$ defined by

$$\rho(p,q,t)f(x) = \exp\bigl(2\pi\sqrt{-1}t + 2\pi\sqrt{-1}\langle q,x\rangle + \pi\sqrt{-1}\langle p,q\rangle\bigr)f(x+p), \quad f\in L^2(\boldsymbol{R}^2).$$

Then ρ is an irreducible unitary representation of $\boldsymbol{H}_2$ on the Hilbert space $L^2(\boldsymbol{R}^2)$, which is called the Schrödinger representation. The Stone-von Neumann theorem says that any irreducible unitary representation ν of $\boldsymbol{H}_2$ on a Hilbert space such that $\nu(0,0,t) = \exp\bigl(2\pi\sqrt{-1}t\bigr)$ is unitary equivalent to ρ.

Now let $\mathscr{A}\in Sp(2;\boldsymbol{R})$. Then $\mathscr{A}$ defines an automorphism $T_{\mathscr{A}}$ of $\boldsymbol{H}_2$ by

$$T_{\mathscr{A}}(p,q,t) = (p',q',t), \quad \begin{pmatrix} p' \\ q' \end{pmatrix} = \mathscr{A}\begin{pmatrix} p \\ q \end{pmatrix},$$

and $\rho\circ T_{\mathscr{A}}$ is another irreducible unitary representation of $\boldsymbol{H}_2$ satisfying $(\rho\circ T_{\mathscr{A}})(0,0,t) = \exp\bigl(2\pi\sqrt{-1}t\bigr)$. Therefore, by the Stone-von Neumann theorem, there exists a unitary operator $\mu(\mathscr{A})$ on $L^2(\boldsymbol{R}^2)$ which satisfies

$$\rho\circ T_{\mathscr{A}} = \mu(\mathscr{A})\circ\rho\circ\mu(\mathscr{A})^{-1}.$$

In [**5**], it has been shown that $\mu(\mathscr{A})$ is determined up to factors ± 1 and μ is a double-valued unitary representation of $Sp(2;\boldsymbol{R})$. Therefore μ is a homomorphism from $Sp(2;\boldsymbol{R})$ into the group of unitary operators on $L^2(\boldsymbol{R}^2)$ modulo $\{\pm I\}$ and is called the metaplectic representation. We will sometimes neglect the sign in the following. In some special cases we will discuss later, $\mu(\mathscr{A})$ can be written down explicitly as an integral operator and the problem of the sign appears as an ambiguity in that of a square root.

The metaplectic representation μ gives us the representation $d\mu$ of $sp(2;\boldsymbol{R})$ by

$$(d\mu)(\mathscr{A}) = \frac{d}{dt}\mu(e^{t\mathscr{A}})|_{t=0}, \quad \mathscr{A}\in sp(2;\boldsymbol{R}),$$

where the sign of $\mu(e^{t\mathscr{A}})$ is determined so that $\mu(e^{t\mathscr{A}})$ is continuous in t and is equal to I at $t=0$. Then the following is known:

THEOREM 2.1 (Theorem (4.45) in [**5**]). *For any f in the space $\mathscr{S}(\boldsymbol{R}^2)$ of the rapidly decreasing functions on $\boldsymbol{R}^2$, it holds that*

$$d\mu(\mathscr{A})f = 2\pi\sqrt{-1}P_{\mathscr{A}}(D, x)f,$$

where $D = \left(2\pi\sqrt{-1}\right)^{-1}\nabla$.

We proceed to the explicit expression of $\mu(\mathscr{A})$ in some special case. $Sp(2;\boldsymbol{R})$ is generated by the elements of the form $\mathscr{A}^{(1)}(A) = \begin{pmatrix} I & 0 \\ A & I \end{pmatrix}$, $\mathscr{A}^{(2)}(B) = \begin{pmatrix} B & 0 \\ 0 & B^{*-1} \end{pmatrix}$ and $\mathscr{I}$, where A is symmetric and $B \in GL(2;\boldsymbol{R})$. The metaplectic representation takes the following simple forms for these elements:

$$\mu(\mathscr{A}^{(1)}(A))f(x) = \exp\left(-\pi\sqrt{-1}\langle x, Ax\rangle\right)f(x),$$

$$\mu(\mathscr{A}^{(2)}(B))f(x) = (\det B)^{-1/2}f(B^{-1}x),$$

$$\mu(\mathscr{I}) = \sqrt{-1}\mathscr{F}^{-1}.$$

By combining these simple formulae, we can show the following:

THEOREM 2.2 (Theorems (4.51), (4.53) in [**5**]). *Let* $\mathscr{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2;\boldsymbol{R})$.
(i) *If* $\det A \neq 0$, *it holds that*

$$\mu(\mathscr{A})f(x) = (\det A)^{-1/2}\int_{\boldsymbol{R}^2} \exp\left(2\pi\sqrt{-1}S(\xi, x)\right)\hat{f}(\xi)\,d\xi$$

$$S(x, \xi) = -\frac{1}{2}\langle x, CA^{-1}x\rangle + \langle \xi, A^{-1}x\rangle + \frac{1}{2}\langle \xi, A^{-1}B\xi\rangle.$$

(ii) *If* $\det B \neq 0$, *it holds that*

$$\mu(\mathscr{A})f(x) = \sqrt{-1}(\det B)^{-1/2}\int_{\boldsymbol{R}^2} \exp(2\pi\sqrt{-1}S'(x, y))f(y)\,dy,$$

$$S'(x, y) = -\frac{1}{2}\langle x, DB^{-1}x\rangle + \langle y, B^{-1}x\rangle - \frac{1}{2}\langle y, B^{-1}Ay\rangle.$$

3. Spectra of quadratic Hamiltonians.

In this section we determine the spectra of the quadratic Hamiltonians on $\boldsymbol{R}^2$. The main result is Theorem 3.5 below.

Any quadratic Hamiltonian is expressed in the form $P_{\mathscr{A}}(D, x)$ by some $\mathscr{A} \in sp(2;\boldsymbol{R})$.

Hörmander [**9**] has classified symplectically all of the elements in $sp(2;\boldsymbol{R})$ into several types of simple forms in the following manner:

Proposition 3.1 (cf. [**9**]). *Any* $\mathscr{A} \in sp(2;\boldsymbol{R})$ *corresponds to one of the following*:

(i) *If the Jordan decomposition of* $\mathscr{A}$ *includes a* 1×1 *block for a real or purely imaginary eigenvalue or a* 2×2 *block for* 0 *eigenvalue, then there exists an* $S \in Sp(2;\boldsymbol{R})$ *such that* $S^{-1}\mathscr{A}S = A \oplus A'$ *for some* $A, A' \in sp(1;\boldsymbol{R})$. *If* A *has a non-zero real eigenvalue* α, *then there exists a* $T \in Sp(1;\boldsymbol{R})$ *such that*

$$T^{-1}AT = A_1 \equiv \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix}.$$

If A *has a pure imaginary non-zero eigenvalue* $\sqrt{-1}\alpha$, *then there exists a* $T \in Sp(1;\boldsymbol{R})$ *such that* $T^{-1}AT = \varepsilon A_2$, *where*

$$A_2 = \begin{pmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{pmatrix},$$

$\varepsilon = 1$ *if the quadratic form* $P_A(\xi_1, x_1)$, $\xi_1, x_1 \in \boldsymbol{R}$, *is positive definite and* $\varepsilon = -1$ *if it is negative definite. If* A *has a zero eigenvalue, then* $A = 0$ *or there exists a* $T \in Sp(1;\boldsymbol{R})$ *such that* $T^{-1}AT = \varepsilon A_3$, *where*

$$A_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

$\varepsilon = 1$ *if the quadratic form* $P_A(\xi_1, x_1)$ *is non-negative definite and* $\varepsilon = -1$ *if that is non-positive definite. The same statements hold for* A'.

(ii) *If one of* $\pm\alpha \pm \sqrt{-1}\beta$ *is an eigenvalue of* $\mathscr{A}$ *for some* $\alpha, \beta > 0$, *then there exists an* $S \in Sp(2;\boldsymbol{R})$ *such that*

$$S^{-1}\mathscr{A}S = \mathscr{A}_4 \equiv \begin{pmatrix} A_4 & 0 \\ 0 & D_4 \end{pmatrix}, \quad A_4 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad D_4 = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix}.$$

(iii) *If the Jordan decomposition of* $\mathscr{A}$ *includes a* 2×2 *block for an eigenvalue* α *or* $-\alpha$ *for some* $\alpha > 0$, *then there exists an* $S \in Sp(2;\boldsymbol{R})$ *such that*

$$S^{-1}\mathscr{A}S = \mathscr{A}_5 \equiv \begin{pmatrix} A_5 & 0 \\ C_5 & -A_5 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}, \quad C_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(iv) *When the Jordan decomposition of* $\mathscr{A}$ *includes a* 2×2 *block for an eigenvalue* $\sqrt{-1}\alpha$ *or* $-\sqrt{-1}\alpha$ *for some* $\alpha > 0$, *we let* $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \boldsymbol{C}^4$ *be solutions of* $(\mathscr{A} + \sqrt{-1}\alpha)\boldsymbol{w}_1 = 0$, $(\mathscr{A} + \sqrt{-1}\alpha)\boldsymbol{w}_2 = \boldsymbol{w}_1$, *and* $\overline{\boldsymbol{w}_2}$ *be the vector whose elements are given by the complex* *conjugate of the corresponding ones of* $\boldsymbol{w}_2$. *Then there exists an* $S \in Sp(2;\boldsymbol{R})$ *such that* $S^{-1}\mathscr{A}S = \varepsilon\mathscr{A}_6$, *where*

$$\mathscr{A}_6 = \begin{pmatrix} A_6 & I \\ 0 & A_6 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix},$$

$\varepsilon = 1$ *if* $\langle \boldsymbol{w}_1, \mathscr{I}\overline{\boldsymbol{w}_2}\rangle > 0$ *and* $\varepsilon = -1$ *if* $\langle \boldsymbol{w}_1, \mathscr{I}\overline{\boldsymbol{w}_2}\rangle < 0$.

(v) *If* $\mathscr{A}^4 = 0$ *and* $\mathscr{A}^3 \neq 0$, *then there exists an* $S \in Sp(2;\boldsymbol{R})$ *such that* $S^{-1}\mathscr{A}S = \varepsilon\mathscr{A}_7$, *where*

$$\mathscr{A}_7 = \begin{pmatrix} 0 & B_7 \\ C_7 & 0 \end{pmatrix}, \quad B_7 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_7 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $\varepsilon = 1$ *or* -1 *corresponding to whether the signature of the quadratic form* $P_{\mathscr{A}}(\xi, x)$ *is* $(2,1)$ *or* $(1,2)$.

Although the matrices A_j, $j = 1,2,3$ and $\mathscr{A}_k$, $k = 4,5,6,7$, are different from the corresponding ones in Hörmander [**9**], we can easily see the equivalence. The corresponding operators are given by the following:

LEMMA 3.2. *Let* A_j, $j = 1,2,3$ *and* $\mathscr{A}_k$, $k = 4,5,6,7$, *be the matrices which appeared in Proposition* 3.1. *Then the corresponding quadratic operators are given by*

$$P_{A_1}(D_1, x_1) = \frac{1}{(2\pi)^2}\left(-\frac{1}{2}\frac{d^2}{dx_1^2} - 2\pi^2\alpha^2 x_1^2\right),$$

$$P_{A_2}(D_1, x_1) = \frac{1}{(2\pi)^2}\left(-\frac{1}{2}\frac{d^2}{dx_1^2} + 2\pi^2\alpha^2 x_1^2\right),$$

$$P_{A_3}(D_1, x_1) = \frac{1}{2}x_1^2$$

on $L^2(\boldsymbol{R})$, *where* $D_1 = \left(2\pi\sqrt{-1}\right)^{-1} d/dx_1$, *and*

$$P_{\mathscr{A}_4}(D, x) = \frac{-\alpha}{2\pi\sqrt{-1}}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 1\right) + \frac{\beta}{2\pi\sqrt{-1}}\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right),$$

$$P_{\mathscr{A}_5}(D, x) = \frac{\sqrt{-1}\alpha}{2\pi}\left(x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\right) + \frac{1}{2}(x_1^2 - x_2^2),$$

$$P_{\mathscr{A}_6}(D, x) = \frac{1}{(2\pi)^2}\left\{\frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_1} - 2\pi\alpha x_2\right)^2 + \frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_2} + 2\pi\alpha x_1\right)^2 - 2\pi^2\alpha^2|x|^2\right\},$$

$$P_{\mathscr{A}_7}(D, x) = \frac{1}{(2\pi)^2}\left(-\frac{1}{2}\frac{\partial^2}{\partial x_1^2} - 4\pi^2 x_1 x_2\right)$$

on $L^2(\boldsymbol{R}^2)$.

Now we obtain the spectrum of $P_{\mathscr{A}}(D,x)$ by determining the spectra of $P_{A_j}(D_1,x_1)$ for $j=1,2,3$ and those of $P_{\mathscr{A}_j}(D,x)$ for $j=4,5,6,7$.

PROPOSITION 3.3. (i) *The spectrum of the operator* $P_{A_2}(D_1,x_1)$ *consists of the simple eigenvalues given by* $(n+1/2)\alpha/(2\pi), n\in \boldsymbol{Z}_+$.

(ii) *The spectrum of the operator* $P_{A_3}(D_1,x_1)$ *is* $[0,\infty)$ *and is absolutely continuous.*

(iii) *The spectrum of* $P_{A_1}(D_1,x_1)$ *and those of* $P_{\mathscr{A}_j}(D,x)$ *for* $j=4,5,6,7$ *are* $\boldsymbol{R}$ *and are absolutely continuous.*

PROOF. $P_{A_2}(D_1,x_1)$ is a harmonic oscillator and (i) is well known. (ii) is obvious. The proof for $j=1$ and 7 is easy.

For $j=4$ and 6, we use the fact that $(2\pi)^2P_{\mathscr{A}_j}(D,x)$ is unitary equivalent to the Schrödinger operator with the magnetic field defined by

$$H=\frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_1}-2\pi\lambda x_2\right)^2+\frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_2}+2\pi\lambda x_1\right)^2-2\pi^2\kappa^2|x|^2, \tag{3.1}$$

where λ and κ are positive constants such that $\lambda\le\kappa$. In fact, for $j=6$, we have $(2\pi)^2P_{\mathscr{A}_6}(D,x)=H$ with $\lambda=\kappa=\alpha$. For $j=4$, we note that $H=(2\pi)^2P_{\mathscr{A}_4'}(D,x)$, where

$$\mathscr{A}_4'=\begin{pmatrix} A_4' & I\\ C_4' & A_4'\end{pmatrix},\quad A_4'=\begin{pmatrix}0 & -\lambda\\ \lambda & 0\end{pmatrix},\quad C_4'=\begin{pmatrix}-\lambda^2+\kappa^2 & 0\\ 0 & -\lambda^2+\kappa^2\end{pmatrix}.$$

The eigenvalues of $\mathscr{A}_4'$ are $\pm\alpha(\kappa,\lambda)\pm\sqrt{-1}\beta(\kappa,\lambda)$, where

$$\alpha(\kappa,\lambda)=\sqrt{\kappa^2/2-\lambda^2+\sqrt{(\lambda^2-\kappa^2/2)^2+\lambda^2(\lambda^2-\kappa^2)}},$$

$$\beta(\kappa,\lambda)=\sqrt{\lambda^2-\kappa^2/2+\sqrt{(\lambda^2-\kappa^2/2)^2+\lambda^2(\lambda^2-\kappa^2)}}.$$

By setting $\lambda=\beta$ and $\kappa=\sqrt{\alpha^2+\beta^2}$, we have $\alpha(\kappa,\lambda)=\alpha$ and $\beta(\kappa,\lambda)=\beta$. Then $(2\pi)^2P_{\mathscr{A}_4}(D,x)$ and H are unitary equivalent by Theorem 3.1 (ii).

The spectral property of the operator H defined by (3.1) has been studied in Arai-Yamada [**1**] by using the invariance under the rotation in $\boldsymbol{R}^2$. They have shown that the spectrum of H is continuous and equal to $\boldsymbol{R}$. Moreover, letting A be the generator of the dilation given by

$$A=\sum_{j=1}^{2}\frac{1}{2\sqrt{-1}}\left(x_j\frac{\partial}{\partial x_j}+\frac{\partial}{\partial x_j}x_j\right),$$

we get

$$[H, \sqrt{-1}A] = (-\Delta + c^2|x|^2)/2,$$

where Δ is the Laplacian on $\boldsymbol{R}^2$. Therefore, by the Mourre estimate, we can show that H has no singular continuous spectrum (cf. [**3**]).

The proof for $j = 5$ is given separately in the next proposition. □

PROPOSITION 3.4. *Let H_0 be the self-adjoint operator on $L^2(\boldsymbol{R}^2)$ defined by*

$$H_0 = \frac{\sqrt{-1}c}{2\pi}\left(x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\right) + x_1^2 - x_2^2,$$

where $0 \neq c \in \boldsymbol{R}$. Then H_0 is unitary equivalent to the direct sum

$$L \oplus L \oplus (-L) \oplus (-L),$$

where $L = r^2 + c\eta$ is the self-adjoint operator on $L^2((0,\infty), rdr) \otimes L^2(\boldsymbol{R}, d\eta)$.

PROOF. Let D_j, $1 \leqq j \leqq 4$, be the domains in $\boldsymbol{R}^2$ given by

$$D_1 = \{(x_1,x_2) \in \boldsymbol{R}^2 : x_1 > |x_2|\}, \quad D_2 = \{(x_1,x_2) \in \boldsymbol{R}^2 : x_2 > |x_1|\},$$

$$D_3 = \{(x_1,x_2) \in \boldsymbol{R}^2 : x_1 < -|x_2|\}, \quad D_4 = \{(x_1,x_2) \in \boldsymbol{R}^2 : x_2 < -|x_1|\},$$

and let H_j be the Friedrichs extension of the symmetric operator $H_0 \restriction C_0^\infty(\overline{D_j})$, respectively. Then H_0 is regarded as the direct sum $\bigoplus_{j=1}^4 H_j$ (cf. [**18**] Proposition 3 in Section XIII.15).

On D_1, we introduce the coordinate $(r,\theta), r > 0$, $\theta \in \boldsymbol{R}$, defined by

$$r = \sqrt{x_1^2 - x_2^2} \quad \text{and} \quad \theta = \tanh^{-1}\frac{x_2}{x_1}.$$

This is the coordinate such that $x_1 = r\cosh\theta$ and $x_2 = r\sinh\theta$. Then we get

$$H_1 = r^2 + \frac{\sqrt{-1}c}{2\pi}\frac{\partial}{\partial\theta}$$

on the space $L^2((0,\infty), rdr) \otimes L^2(\boldsymbol{R}, d\theta)$. We can show that this operator is unitary equivalent to L by considering the inverse Fourier transform in the variable θ. Note that L is essentially self-adjoint on the space $\mathscr{S}([0,\infty) \times \boldsymbol{R})$ of the rapidly decreasing functions. We can show that H_3 is unitary equivalent to L in the same way.

On D_2 and D_4, we introduce the coordinate (r,θ), $r > 0$, $\theta \in \boldsymbol{R}$, defined by

$$r = \sqrt{x_2^2 - x_1^2} \quad \text{and} \quad \theta = \tanh^{-1}\frac{x_1}{x_2},$$

and use the Fourier transform in the variable θ. Then we can show that H_2 and H_4 are unitary equivalent to $-L$. $\square$

The complete description of the spectrum of $P_{\mathscr{A}}(D,x)$ is the following:

THEOREM 3.5. *For any* $0 \neq \mathscr{A} \in sp(2;\boldsymbol{R})$, *the spectrum of the operator* $P_{\mathscr{A}}(D,x)$ *is determined as follows*:

(i) *If the Jordan decomposition of* $\mathscr{A}$ *consists of four* 1×1 *blocks of the eigenvalues* $\pm\sqrt{-1}\alpha_j$, $j=1,2$, *for some* $\alpha_j>0$, *then the spectrum of the operator* $P_{\mathscr{A}}(D,x)$ *consists of the eigenvalues given by* $\{\varepsilon_1(n_1+1/2)\alpha_1+\varepsilon_2(n_2+1/2)\alpha_2\}/(2\pi)$, $n_1,n_2\in\boldsymbol{Z}_+$, *including the multiplicity, where* $\varepsilon_j=1$ *if* $\langle \boldsymbol{a}_j,\mathscr{I}\boldsymbol{b}_j\rangle>0$ *and* $\varepsilon_j=-1$ *if* $\langle \boldsymbol{a}_j,\mathscr{I}\boldsymbol{b}_j\rangle<0$ *for the eigenvectors* $\boldsymbol{a}_j+\sqrt{-1}\boldsymbol{b}_j$ *of* $\mathscr{A}$ *corresponding to the eigenvalues* $\sqrt{-1}\alpha_j$, $j=1,2$, *respectively. In particular*, $\varepsilon_1=\varepsilon_2=1$ *if the quadratic form* $P_{\mathscr{A}}(\xi,x)$ *is positive definite and* $\varepsilon_1=\varepsilon_2=-1$ *if it is negative definite. When* $\varepsilon_1\cdot\varepsilon_2=-1$, *this spectral set is a dense subset in* $\boldsymbol{R}$ *if* $\alpha_1/\alpha_2\notin\boldsymbol{Q}$ *and is a discrete subset if* $\alpha_1/\alpha_2\in\boldsymbol{Q}$.

(ii) *If the Jordan decomposition of* $\mathscr{A}$ *consists of two* 1×1 *blocks of the eigenvalues* $\pm\sqrt{-1}\alpha$ *for some* $\alpha>0$ *and two* 1×1 *blocks of the eigenvalue* 0, *then the spectrum of* $P_{\mathscr{A}}(D,x)$ *consists only of the eigenvalues with infinite multiplicity and is given by* $\varepsilon(n+1/2)\alpha/(2\pi)$, $n\in\boldsymbol{Z}_+$, *where* $\varepsilon=1$ *if* $P_{\mathscr{A}}(\xi,x)$ *is non-negative definite and* $\varepsilon=-1$ *if it is non-positive definite.*

(iii) *If the Jordan decomposition of* $\mathscr{A}$ *consists of two* 1×1 *blocks of the eigenvalues* $\pm\sqrt{-1}\alpha$ *for some* $\alpha>0$ *and one* 2×2 *block of the eigenvalue* 0 *and* $P_{\mathscr{A}}(\xi,x)$ *is non-negative* (*resp. non-positive*) *definite, then the spectrum of* $P_{\mathscr{A}}(D,x)$ *is* $[\alpha/(4\pi),\infty)$ (*resp.* $(-\infty,-\alpha/(4\pi)]$) *and is absolutely continuous.*

(iv) *If* $\mathscr{A}^2=0$, $\mathscr{A}\neq 0$ *and* $P_{\mathscr{A}}(\xi,x)$ *is non-negative* (*resp. non-positive*) *definite, then the spectrum of* $P_{\mathscr{A}}(D,x)$ *is* $[0,\infty)$ (*resp.* $(-\infty,0]$) *and is absolutely continuous.*

(v) *For the other nonzero* $\mathscr{A}$, *the spectrum of* $P_{\mathscr{A}}(D,x)$ *is* $\boldsymbol{R}$ *and is absolutely continuous.*

REMARK. In (i), if $\alpha_1=\alpha_2$, the eigenvectors $\boldsymbol{a}_j+\sqrt{-1}\boldsymbol{b}_j$ should be taken so that $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ are linearly independent.

4. Schrödinger operators with magnetic fields.

In this section, we apply the results in the last section to the Schrödinger operator with a uniform magnetic field defined by

$$H=\frac{1}{2}\left(\sqrt{-1}\,\frac{\partial}{\partial x_1}-b\pi x_2\right)^2+\frac{1}{2}\left(\sqrt{-1}\,\frac{\partial}{\partial x_2}+b\pi x_1\right)^2+2\pi^2(c_1x_1^2+c_2x_2^2), \tag{4.1}$$

where $0 \neq b \in \boldsymbol{R}$, $c_1, c_2 \in \boldsymbol{R}$. We will see that this class of operators correspond to all types of the fundamental operators discussed in the last section. This operator is expressed as $H = (2\pi)^2 P_{\mathscr{A}}(D, x)$, where $\mathscr{A}$ is an element in $sp(2; \boldsymbol{R})$ given by

$$(4.2) \quad \mathscr{A} = \begin{pmatrix} A & I \\ C & A \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -b/2 \\ b/2 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -b^2/4 - c_1 & 0 \\ 0 & -b^2/4 - c_2 \end{pmatrix}.$$

4.1. The case where $c_1, c_2 \geq 0$. We set $c_1 = k^2$, $c_2 = \ell^2$ with $k, \ell \geq 0$. Then the signature of the quadratic form $P_{\mathscr{A}}(\xi, x)$ is $(4,0)$ if $k, \ell > 0$, $(3,0)$ if $k > 0$ and $\ell = 0$, $(2,0)$ if $k = \ell = 0$. The roots of the characteristic equation for $\mathscr{A}$ are given by the following:

$$\begin{aligned} &\pm\sqrt{-1}\alpha_+, \ \pm\sqrt{-1}\alpha_-, && \text{if } k, \ell > 0, \\ &\pm\sqrt{-1}\sqrt{b^2 + k^2}, \ 0 \ (\text{with multiplicity } 2), && \text{if } k \geq 0 \text{ and } \ell = 0, \end{aligned}$$

where

$$(4.3) \qquad \alpha_\pm = \sqrt{b^2 + k^2 + \ell^2 \pm \sqrt{(b^2 + k^2 + \ell^2)^2 - 4k^2\ell^2}} \Big/ 2.$$

Therefore the matrix $\mathscr{A}$ is transformed as follows:

Proposition 4.1. (i) *If $k, \ell > 0$, then there exists an $S_1 \in Sp(2; \boldsymbol{R})$ such that $S_1^{-1} \mathscr{A} S_1 = \mathscr{A}_1$, where*

$$\mathscr{A}_1 = \begin{pmatrix} 0 & I \\ C_1 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -\alpha_+^2 & 0 \\ 0 & -\alpha_-^2 \end{pmatrix}$$

and $\alpha_\pm$ are the constants given by (4.3).

(ii) *If $k > 0$ and $\ell = 0$, then there exists an $S_2 \in Sp(2; \boldsymbol{R})$ such that $S_2^{-1} \mathscr{A} S_2 = \mathscr{A}_2$, where*

$$\mathscr{A}_2 = \begin{pmatrix} 0 & B_2 \\ C_2 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -(b^2 + k^2) & 0 \\ 0 & -1 \end{pmatrix}.$$

(iii) *If $k = \ell = 0$, then there exists an $S_3 \in Sp(2; \boldsymbol{R})$ such that $S_3^{-1} \mathscr{A} S_3 = \mathscr{A}_3$, where*

$$\mathscr{A}_3 = \begin{pmatrix} 0 & B_3 \\ C_3 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad C_3 = \begin{pmatrix} -b^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

This proposition has been proved in [**16**]. By Proposition 4.1, we obtain the following:

THEOREM 4.2. (i) *If $k, \ell > 0$, then H is unitary equivalent to the operator H_1 defined by*

$$H_1 = \left(-\frac{1}{2}\frac{\partial^2}{\partial x_1^2} + 2\pi^2\alpha_+^2 x_1^2\right) + \left(-\frac{1}{2}\frac{\partial^2}{\partial x_2^2} + 2\pi^2\alpha_-^2 x_2^2\right)$$

and the spectrum consists of the eigenvalues given by $2\pi\{(n_1 + 1/2)\alpha_+ + (n_2 + 1/2)\alpha_-\}$, n_1, $n_2 \in \boldsymbol{Z}_+$, including the multiplicity, where $\alpha_\pm$ are the constants given by (4.3).

(ii) *If $k > 0$ and $\ell = 0$, then H is unitary equivalent to the operator H_2 defined by*

$$H_2 = \left(-\frac{1}{2}\frac{\partial^2}{\partial x_1^2} + 2\pi^2(b^2 + k^2)x_1^2\right) + 2\pi^2 x_2^2$$

and the spectrum of is $\left[\pi\sqrt{b^2 + k^2}, \infty\right)$ and is absolutely continuous.

(iii) *If $k = \ell = 0$, then H is unitary equivalent to the operator H_3 defined by*

$$H_3 = -\frac{1}{2}\frac{\partial^2}{\partial x_1^2} + 2\pi^2 b^2 x_1^2 \tag{4.4}$$

on $L^2(\boldsymbol{R}^2)$ and the spectrum consists only of the eigenvalues $2\pi(n + 1/2)|b|$, $n \in \boldsymbol{Z}_+$, with infinite multiplicities.

(iii) of this proposition has been shown by Avron-Herbst-Simon [**2**] by a different method. This result is given in [**15**] by a more direct method and is extended to general dimensional case in [**16**].

4.2 THE CASE WHERE $c_1 \geq 0$, $c_2 < 0$. We set $c_1 = k^2$, $c_2 = -\ell^2$ with $k \geq 0$, $\ell > 0$. Then the signature of the quadratic form $P_{\mathscr{A}}(\xi, x)$ is $(2, 1)$ if $k = 0$, $(3, 1)$ if $k > 0$. The roots of the characteristic equation for $\mathscr{A}$ are given by the following:

$$\begin{array}{ll} \pm\sqrt{-1}\sqrt{b^2 - \ell^2},\ 0 \ (\text{with multiplicity } 2), & \text{if } k = 0 \text{ and } |b| > \ell, \\ 0 \ (\text{with multiplicity } 2),\ \pm\sqrt{\ell^2 - b^2}, & \text{if } k = 0 \text{ and } |b| < \ell, \\ 0 \ (\text{with multiplicity } 4), & \text{if } k = 0 \text{ and } |b| = \ell, \\ \pm\sqrt{-1}\alpha,\ \pm\beta, & \text{if } k \neq 0, \end{array}$$

where

$$\alpha = \sqrt{\left\{(b^2 + k^2 - \ell^2) + \sqrt{(b^2 + k^2 - \ell^2)^2 + 4k^2\ell^2}\right\}\Big/2} \tag{4.5}$$

and

$$\beta = \sqrt{\left\{-(b^2 + k^2 - \ell^2) + \sqrt{(b^2 + k^2 - \ell^2)^2 + 4k^2\ell^2}\right\} \Big/ 2}. \tag{4.6}$$

Therefore the matrix $\mathscr{A}$ is transformed as follows:

Proposition 4.3. (i) *If $k = 0$ and $|b| > \ell$, then there exists an $S_1 \in Sp(2; \boldsymbol{R})$ such that $S_1^{-1} \mathscr{A} S_1 = \mathscr{A}_1$, where*

$$\mathscr{A}_1 = \begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -(b^2 - \ell^2) & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) *If $k = 0$ and $|b| < \ell$, then there exists an $S_2 \in Sp(2; \boldsymbol{R})$ such that $S_2^{-1} \mathscr{A} S_2 = \mathscr{A}_2$, where*

$$\mathscr{A}_2 = \begin{pmatrix} 0 & B_2 \\ C_2 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} \ell^2 - b^2 & 0 \\ 0 & -1 \end{pmatrix}.$$

(iii) *If $k = 0$ and $|b| = \ell$, then there exists an $S_3 \in Sp(2; \boldsymbol{R})$ such that $S_3^{-1} \mathscr{A} S_3 = \mathscr{A}_3$, where*

$$\mathscr{A}_3 = \begin{pmatrix} 0 & B_3 \\ C_3 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad C_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(iv) *If $k \neq 0$, then there exists an $S_4 \in Sp(2; \boldsymbol{R})$ such that $S_4^{-1} \mathscr{A} S_4 = \mathscr{A}_4$, where*

$$\mathscr{A}_4 = \begin{pmatrix} 0 & I \\ C_4 & 0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} -\alpha^2 & 0 \\ 0 & \beta^2 \end{pmatrix}$$

and α, β are the constants given by (4.5) *and* (4.6).

By Proposition 4.3, we obtain the following:

Theorem 4.4. (i) *If $k = 0$ and $|b| > \ell$, then H is unitary equivalent to the operator H_1 defined by*

$$H_1 = \left(-\frac{1}{2}\frac{\partial^2}{\partial x_1^2} + 2\pi^2(b^2 - \ell^2)x_1^2\right) - 2\pi^2 x_2^2.$$

(ii) *If $k = 0$ and $|b| < \ell$, then H is unitary equivalent to the operator H_2 defined by*

$$H_2 = \left(-\frac{1}{2}\frac{\partial^2}{\partial x_1^2} - 2\pi^2(\ell^2 - b^2)x_1^2\right) + 2\pi^2 x_2^2.$$

(iii) *If* $k = 0$ *and* $|b| = \ell$, *then* H *is unitary equivalent to the operator* H_3 *defined by*

$$H_3 = -\frac{1}{2}\frac{\partial^2}{\partial x_1^2} - 4\pi^2 x_1 x_2.$$

(iv) *If* $k \neq 0$, *then* H *is unitary equivalent to the operator* H_4 *defined by*

$$H_4 = \left(-\frac{1}{2}\frac{\partial^2}{\partial x_1^2} + 2\pi^2\alpha^2 x_1^2\right) + \left(-\frac{1}{2}\frac{\partial^2}{\partial x_2^2} - 2\pi^2\beta^2 x_2^2\right),$$

where α *and* β *are the constants given by* (4.5) *and* (4.6).

(v) *When* $c_1 \geqq 0$ *and* $c_2 < 0$, *then the spectrum of* H *is* $\boldsymbol{R}$ *and is absolutely continuous.*

4.3 The case where $c_1, c_2 < 0$. We set $c_1 = -k^2$, $c_2 = -\ell^2$ with $k, \ell > 0$. Then the signature of the quadratic form $P_{\mathscr{A}}(\xi, x)$ is $(2, 2)$. The roots of the characteristic equation for $\mathscr{A}$ are given by the following:

$$\begin{array}{ll} \pm\sqrt{-1}\alpha_+,\ \pm\sqrt{-1}\alpha_-, & \text{if } |b| > k+\ell, \\ \pm\beta_+,\ \pm\beta_-, & \text{if } |b| < |k-\ell|, \\ \pm\sqrt{-k\ell}\ \text{(with multiplicity 2)}, & \text{if } |b| = k+\ell, \\ \pm\sqrt{k\ell}\ \text{(with multiplicity 2)}, & \text{if } |b| = |k-\ell|, \\ \pm\gamma \pm \sqrt{-1}\delta, & \text{if } |k-\ell| < |b| < k+\ell, \end{array}$$

where

$$\alpha_\pm = \sqrt{\left\{b^2 - k^2 - \ell^2 \pm \sqrt{(b^2-k^2-\ell^2)^2 - 4k^2\ell^2}\right\}\Big/2}, \tag{4.7}$$

$$\beta_\pm = \sqrt{\left\{k^2 + \ell^2 - b^2 \pm \sqrt{(k^2+\ell^2-b^2)^2 - 4k^2\ell^2}\right\}\Big/2} \tag{4.8}$$

$$\gamma = \sqrt{(k+\ell)^2 - b^2}\Big/2 \tag{4.9}$$

and

$$\delta = \sqrt{b^2 - (k-\ell)^2}\Big/2. \tag{4.10}$$

When $|b| = k+\ell$ or $|k-\ell|$, the dimension of the eigenspace corresponding to each eigenvalue is 1. Moreover, when $|b| > k+\ell$, we have $\langle \boldsymbol{a}, \mathscr{I}\boldsymbol{b}\rangle > 0$ for an eigenvector $\boldsymbol{a} + \sqrt{-1}\boldsymbol{b}$ of $\mathscr{A}$ corresponding to the eigenvalue $\sqrt{-1}\alpha_+$. When $|b| = k+\ell$, we have $\langle \boldsymbol{w}_1, \mathscr{I}\overline{\boldsymbol{w}_2}\rangle > 0$ for $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \boldsymbol{C}^4$ such that $(\mathscr{A} + \sqrt{-k\ell})\boldsymbol{w}_1 = 0$, $(\mathscr{A} + \sqrt{-k\ell})\boldsymbol{w}_2 = \boldsymbol{w}_1$. Therefore the matrix $\mathscr{A}$ is transformed as follows:

Proposition 4.5. (i) *If $|b| > k+\ell$, then there exists an $S_1 \in Sp(2;\boldsymbol{R})$ such that $S_1^{-1}\mathscr{A}S_1 = \mathscr{A}_1$, where*

$$\mathscr{A}_1 = \begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -\alpha_+^2 & 0 \\ 0 & \alpha_-^2 \end{pmatrix}$$

and $\alpha_\pm$ are the constants given by (4.7).

(ii) *If $|b| < |k-\ell|$, then there exists an $S_2 \in Sp(2;\boldsymbol{R})$ such that $S_2^{-1}\mathscr{A}S_2 = \mathscr{A}_2$, where*

$$\mathscr{A}_2 = \begin{pmatrix} 0 & I \\ C_2 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} \beta_+^2 & 0 \\ 0 & \beta_-^2 \end{pmatrix}$$

and $\beta_\pm$ are the constants given by (4.8).

(iii) *If $|b| = k+\ell$, then there exists an $S_3 \in Sp(2;\boldsymbol{R})$ such that $S_3^{-1}\mathscr{A}S_3 = \mathscr{A}_3$, where*

$$\mathscr{A}_3 = \begin{pmatrix} A_3 & I \\ 0 & A_3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -\sqrt{k\ell} \\ \sqrt{k\ell} & 0 \end{pmatrix}.$$

(iv) *If $|b| = |k-\ell|$, then there exists an $S_4 \in Sp(2;\boldsymbol{R})$ such that $S_4^{-1}\mathscr{A}S_4 = \mathscr{A}_4$, where*

$$\mathscr{A}_4 = \begin{pmatrix} A_4 & 0 \\ C_4 & -A_4 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & \sqrt{k\ell} \\ \sqrt{k\ell} & 0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(v) *If $|k-\ell| < |b| < k+\ell$, then there exists an $S_5 \in Sp(2;\boldsymbol{R})$ such that $S_5^{-1}\mathscr{A}S_5 = \mathscr{A}_5$, where*

$$\mathscr{A}_5 = \begin{pmatrix} A_5 & 0 \\ 0 & D_5 \end{pmatrix}, \quad A_5 = \begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix}, \quad D_5 = \begin{pmatrix} -\gamma & \delta \\ -\delta & -\gamma \end{pmatrix}$$

and γ, δ are the constants given by (4.9), (4.10), *respectively.*

By Proposition 4.5, we obtain the following:

Proposition 4.6. (i) *If $|b| > k+\ell$, then H is unitary equivalent to the operator H_1 defined by*

$$H_1 = \left(-\frac{1}{2}\frac{\partial^2}{\partial x_1^2} + 2\pi^2\alpha_+^2 x_1^2\right) - \left(-\frac{1}{2}\frac{\partial^2}{\partial x_2^2} + 2\pi^2\alpha_-^2 x_2^2\right),$$

where $\alpha_\pm$ *are the constants given by* (4.7).

(ii) *If* $|b| < |k - \ell|$, *then* H *is unitary equivalent to the operator* H_2 *defined by*

$$H_2 = \left(-\frac{1}{2}\frac{\partial^2}{\partial x_1^2} - 2\pi^2\beta_+^2 x_1^2\right) + \left(-\frac{1}{2}\frac{\partial^2}{\partial x_2^2} - 2\pi^2\beta_-^2 x_2^2\right),$$

where $\beta_\pm$ *are the constants given by* (4.8).

(iii) *If* $|b| = k + \ell$, *then* H *is unitary equivalent to the operator* H_3 *defined by*

$$H_3 = \frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_1} - 2\sqrt{k\ell}\pi x_2\right)^2 + \frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_2} + 2\sqrt{k\ell}\pi x_1\right)^2 - 2\pi^2 k\ell|x|^2.$$

(iv) *If* $|b| = |k - \ell|$, *then* H *is unitary equivalent to the operator* H_4 *defined by*

$$H_4 = 2\pi\sqrt{-1}\sqrt{k\ell}\left(x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\right) + 2\pi^2(x_1^2 - x_2^2).$$

(v) *If* $|k - \ell| < |b| < k + \ell$, *then* H *is unitary equivalent to the operator* H_5 *defined by*

$$H_5 = \frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_1} - \beta\pi x_2\right)^2 + \frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_2} + \beta\pi x_1\right)^2 - 2\pi^2 k\ell|x|^2,$$

where $\beta = \sqrt{b^2 - (k-\ell)^2}$.

H_5 in Theorem 4.6 is the operator we used in the proof of Theorem 3.3. The spectrum of H is given as follows:

Theorem 4.7. (i) *If* $|b| > k + \ell$, *then the spectrum of the operator* H *consists of the eigenvalues given by* $2\pi\{(n_1 + 1/2)\alpha_+ - (n_2 + 1/2)\alpha_-\}$, $n_1, n_2 \in \boldsymbol{Z}_+$, *where* $\alpha_\pm$ *are the constants given by* (4.7). *This spectral set is a dense subset in* $\boldsymbol{R}$ *if* $\alpha_+/\alpha_- \notin \boldsymbol{Q}$ *and is a discrete subset if* $\alpha_+/\alpha_- \in \boldsymbol{Q}$.

(ii) *If* $|b| \le k + \ell$, *then the spectrum of* H *is* $\boldsymbol{R}$ *and is absolutely continuous.*

5. Heat kernels.

In this section we apply the metaplectic representation to the study on the heat kernel of the semigroup $\exp(-tH)$ generated by the operator H defined by

$$H = \frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_1} - b\pi x_2\right)^2 + \frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_2} + b\pi x_1\right)^2 - 2\pi^2\ell^2 x_2^2, \tag{5.1}$$

where $b \neq 0$, $\ell > 0$.

The heat kernel has the following representation:

$$\begin{aligned} e^{-tH}(x,y) = E\Bigg[\exp\Bigg(& - b\pi\sqrt{-1}\int_0^t \{(x_1 + w_1(s))\,dw_2(s) - (x_2 + w_2(s))\,dw_1(s)\} \\ & + 2\pi^2\ell^2\int_0^t (x_2 + w_2(s))^2\,ds\Bigg)\Bigg| x + w(t) = y\Bigg]\frac{1}{2\pi t}\exp\left(-\frac{|x-y|^2}{2t}\right), \end{aligned} \tag{5.2}$$

where $E[\cdot|\cdot]$ is the conditional expectation with respect to the 2-dimensional Wiener process $w(s) = (w_1(s), w_2(s))$ satisfying $w(0) = 0$. For details, we refer to [**12**] and [**19**].

Note that the expectation of the right hand side converges absolutely for $0 < t < 1/(2\ell)$ (cf. [**13**] Section 5.13). On the other hand, in Section 4, we showed that H is unitary equivalent to the operators H_j, $j = 1, 2, 3$, in Theorem 4.4. The heat kernels $\exp(-tH_j)(x,y)$ of the corresponding semigroups are explicitly written in the following way:

$$e^{-tH_1}(x,y) = p_\alpha^+(t, x_1, y_1)\exp(2\pi^2 t x_2^2)\delta_{y_2}(x_2), \tag{5.3}$$

$$e^{-tH_2}(x,y) = p_\beta^-(t, x_1, y_1)\exp(-2\pi^2 t x_2^2)\delta_{y_2}(x_2), \tag{5.4}$$

and

$$e^{-tH_3}(x,y) = \frac{1}{\sqrt{2\pi t}}\exp\left(2\pi^2(x_1 + y_1)x_2 t + \frac{2\pi^4 x_2^2 t^3}{3} - \frac{(x_1 - y_1)^2}{2t}\right)\delta_{y_2}(x_2), \tag{5.5}$$

where $\alpha = \sqrt{b^2 - \ell^2}$, $\beta = \sqrt{\ell^2 - b^2}$, δ_y is the Dirac delta function concentrated at the point y with respect to the Lebesgue measure and $p_\gamma^\pm(t, x_1, y_1)$, $\gamma > 0$, are the heat kernels of the semigroups generated by the operators defined by $-d^2/2dx_1^2 \pm 2\pi^2\gamma^2 x_1^2$, respectively, which are given by

$$p_\gamma^+(t, x_1, y_1) = \sqrt{\frac{\gamma}{\sinh 2\pi\gamma t}}\exp\{-\gamma\pi\coth 2\pi\gamma t(x_1^2 - 2x_1 y_1 \operatorname{sech} 2\pi\gamma t + y_1^2)\}, \tag{5.6}$$

$$p_\gamma^-(t, x_1, y_1) = \sqrt{\frac{\gamma}{\sin 2\pi\gamma t}}\exp\{-\gamma\pi\cot 2\pi\gamma t(x_1^2 - 2x_1 y_1 \sec 2\pi\gamma t + y_1^2)\}. \tag{5.7}$$

(5.4) and (5.7) hold for $0 < t < 1/(2\beta)$ and the others hold for all $t > 0$. We have

$1/(2\beta) > 1/(2\ell)$. Therefore, if the relation

$$e^{-tH}(x,y) = \{\mu(S_j)e^{-tH_j}\mu(S_j)^{-1}\delta_y\}(x) \tag{5.8}$$

is established for some $S_j \in Sp(2;\boldsymbol{R})$ as smooth functions, the heat kernel $\exp(-tH)(x,y)$ is well defined also for $t \geqq 1/(2\ell)$.

In fact we show the following:

Theorem 5.1. (i) *If $|b| > \ell$, there exists an $S_1 \in Sp(2;\boldsymbol{R})$ such that the equation* (5.8) *with $j=1$ holds as a smooth function in $(t,x,y) \in (0,\infty)\times\boldsymbol{R}^2\times\boldsymbol{R}^2$.*

(ii) *If $|b| < \ell$, there exists an $S_2 \in Sp(2;\boldsymbol{R})$ such that the equation* (5.8) *with $j=2$ holds as a smooth function in $(t,x,y) \in (0,1/(2\beta))\times\boldsymbol{R}^2\times\boldsymbol{R}^2$ where $\beta = \sqrt{\ell^2-b^2}$.*

(iii) *If $|b| = \ell$, there exists an $S_3 \in Sp(2;\boldsymbol{R})$ such that the equation* (5.8) *with $j=3$ holds as a smooth function in $(t,x,y) \in \left(0,\sqrt{3}/(\pi|b|)\right)\times\boldsymbol{R}^2\times\boldsymbol{R}^2$.*

Remark 5.1. $t = 1/(2\beta)$ and $t = \sqrt{3}/(\pi|b|)$ in the theorem are the conjugate points of the corresponding classical mechanics in the sense of Ikeda, Kusuoka and Manabe [**10**], [**11**].

Proof. We denote vectors in $\boldsymbol{R}^4$ by $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ in the following arguments.

(i) Set $\alpha' = \sqrt{b^2-\ell^2}$. Then the matrix $S_1 = (\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3\boldsymbol{a}_4)$ satisfies $S_1^{-1}\mathscr{A}S_1 = \mathscr{A}_1$ if the vectors $\boldsymbol{a}_1 + \sqrt{-1}\alpha'\boldsymbol{a}_3$ and $\boldsymbol{a}_4$ are eigenvectors corresponding to the eigenvalues $\sqrt{-1}\alpha'$ and 0, respectively, and the vector $\boldsymbol{a}_2$ is a generalized eigenvector satisfying $\mathscr{A}\boldsymbol{a}_2 = \boldsymbol{a}_4$. We automatically obtain

$$\langle \gamma_1\boldsymbol{a}_1 + \gamma_3\boldsymbol{a}_3, \mathscr{I}(\gamma_2\boldsymbol{a}_2 + \gamma_4\boldsymbol{a}_4)\rangle = 0 \tag{5.9}$$

for any $\gamma_j \in \boldsymbol{C}$ (see [**8**], Lemma 21.5.2). Thus S_1 belongs to $Sp(2;\boldsymbol{R})$ if we can choose $\boldsymbol{a}_i$'s so that $\langle \boldsymbol{a}_1, \mathscr{I}\boldsymbol{a}_3\rangle = \langle \boldsymbol{a}_2, \mathscr{I}\boldsymbol{a}_4\rangle = 1$ holds. For example, if we set $\boldsymbol{a}_1 = (b/\alpha', 0, 0, \alpha' - b^2/(2\alpha'))^*$, $\boldsymbol{a}_2 = (0, b/(\alpha'\ell), b^2/(2\alpha'\ell) - \ell/\alpha', 0)^*$, $\boldsymbol{a}_3 = (0, -1/\alpha', b/(2\alpha'), 0)^*$ and $\boldsymbol{a}_4 = (-\ell/\alpha', 0, 0, (b\ell)/(2\alpha'))^*$, these conditions are satisfied. Moreover, by this choice, we can show the smoothness of the right hand side of (5.8).

(ii) Let $\boldsymbol{b}_1$ and $\boldsymbol{b}_3$ be eigenvectors corresponding to the eigenvalues $\beta' = \sqrt{\ell^2 - b^2}$ and $-\beta'$, respectively. We can show $\langle \boldsymbol{b}_1, \mathscr{I}\boldsymbol{b}_3\rangle \neq 0$ by straightforward calculations and, choosing $\boldsymbol{b}_1$ and $\boldsymbol{b}_3$ so that $\langle \boldsymbol{b}_1, \mathscr{I}\boldsymbol{b}_3\rangle = \beta'/2$, we set $\boldsymbol{a}_1 = \boldsymbol{b}_1 - \boldsymbol{b}_3$, $\boldsymbol{a}_3 = (\boldsymbol{b}_1 + \boldsymbol{b}_3)/\beta'$. Moreover let $\boldsymbol{a}_4$ be an eigenvector corresponding to the eigenvalue 0 and $\boldsymbol{a}_2$ be a generalized eigenvector satisfying $\mathscr{A}\boldsymbol{a}_2 = -\boldsymbol{a}_4$. Then the matrix $S_2 = (\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3\boldsymbol{a}_4)$ satisfies $S_2^{-1}\mathscr{A}S_2 = \mathscr{A}_2$ and $\boldsymbol{a}_i$'s satisfy (5.9). S_2 belongs to $Sp(2;\boldsymbol{R})$ if $\langle \boldsymbol{a}_2, \mathscr{I}\boldsymbol{a}_4\rangle = 1$. There are some choices and, for example, we have only to set $\boldsymbol{a}_2 = (0, -b/(\ell\beta'), \ell/\beta' - b^2/(2\ell\beta'), 0)^*$ and $\boldsymbol{a}_4 = (-\ell/\beta', 0, 0, \ell b/(2\beta'))^*$. To prove the smoothness of the right hand side of (5.8), we take $\boldsymbol{a}_1=(0,1,-b/2,0)^*$ and $\boldsymbol{a}_3=(-b/\beta^2,0,0,\ell^2/\beta^2-b^2/(2\beta^2))^*$ and replace $\boldsymbol{a}_2$ by $\boldsymbol{a}_2+(\beta/\ell)\boldsymbol{a}_4$ in order to apply Theorem 2.2 in a simple way.

(iii) The matrix $S_3=(\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3\boldsymbol{a}_4)$ satisfies $S_3^{-1}\mathscr{A}S_3=\mathscr{A}_3$ if $\boldsymbol{a}_4$ is an eigenvector corresponding to the eigenvalue 0 and $\boldsymbol{a}_j$, $j=1,2,3$, are the generalized eigenvectors satisfying $\mathscr{A}\boldsymbol{a}_1=\boldsymbol{a}_4$, $\mathscr{A}\boldsymbol{a}_2=\boldsymbol{a}_3$ and $\mathscr{A}\boldsymbol{a}_3=\boldsymbol{a}_1$. Moreover S_3 belongs to $Sp(2;\boldsymbol{R})$ if $\langle \boldsymbol{a}_1,\mathscr{I}\boldsymbol{a}_3\rangle=\langle \boldsymbol{a}_2,\mathscr{I}\boldsymbol{a}_4\rangle=1$ and (5.9) hold. For example, if we set $\boldsymbol{a}_1=(1,1,-b/2,-b/2)^*$, $\boldsymbol{a}_2=(0,1/b^2,1/(2b),-1/b)^*$, $\boldsymbol{a}_3=(0,-1/b,1/2,1)^*$ and $\boldsymbol{a}_4=(-b,0,0,b^2/2)^*$, these conditions are satisfied. Moreover, by this choice, we can show the smoothness of the right hand side of (5.8). □

6. The Lévy formula and Hermite Polynomials.

In this section we consider the Schrödinger operator

$$H=\frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_1}-b\pi x_2\right)^2+\frac{1}{2}\left(\sqrt{-1}\frac{\partial}{\partial x_2}+b\pi x_1\right)^2$$

with a uniform magnetic field on $\boldsymbol{R}^2$. We assume $b>0$ for simplicity. We apply the theory of the metaplectic representation to show that the eigenfunctions and the heat kernel for H are obtained from those for the harmonic oscillator through some integral transform. This means that the complex Hermite polynomials and the Lévy formula are obtained from the usual, real Hermite polynomials and the Mehler formula through the integral transform.

It is well known that the spectrum of H is $\{2\pi b(m+1/2);\, m=0,1,\dots,\}$ consisting only of the eigenvalues with infinite multiplicities and the normalized eigenfunctions corresponding to the eigenvalue $2\pi b(m+1/2)$ are given by

$$\phi_{m,n}(z,\bar{z})=\sqrt{\frac{b}{m!n!}\left(\frac{2}{b\pi}\right)^{m+n}}H_{m,n}(\sqrt{2}z,\sqrt{2}\bar{z})\exp(-\pi b|z|^2/2),\quad n=0,1,\dots,$$

where we identify $\boldsymbol{R}^2$ with $\boldsymbol{C}$ in the usual manner and $\{H_{m,n}(z,\bar{z})\}_{m,n=0}^{\infty}$ are the complex Hermite polynomials defined by

$$H_{m,n}(z,\bar{z})=(-1)^{m+n}\exp(\pi b|z|^2/2)\frac{\partial^{m+n}}{\partial z^m\partial\bar{z}^n}\exp(-b\pi|z|^2/2).$$

It is also well known that the explicit form of the heat kernel of $\exp(-tH)$ is given by

$$e^{-tH}(x,y)=\frac{b}{2\sinh b\pi t}\exp\left(b\pi\sqrt{-1}(y_1x_2-x_1y_2)-\frac{1}{2}b\pi\coth b\pi t\cdot|y-x|^2\right). \tag{6.1}$$

This is equivalent to the following Lévy formula

$$E\left[\exp\left(2b\pi\sqrt{-1}S_t\right)|w(t)=y-x\right]=\frac{b\pi t}{\sinh b\pi t}\exp\left((1-b\pi t\coth b\pi t)\frac{|y-x|^2}{2t}\right), \tag{6.2}$$

where S_t is Lévy's stochastic area given by

$$S_t=\frac{1}{2}\int_0^t\{w_1(s)\circ dw_2(s)-w_2(s)\circ dw_1(s)\}.$$

On the other hand, as is discussed in Section 4, H is expressed as $(2\pi)^2P_{\mathscr{A}}(D,x)$, where $\mathscr{A}$ is an element in $sp(2;\boldsymbol{R})$ given by

$$\mathscr{A}=\begin{pmatrix}A & I\\ C & A\end{pmatrix},\quad A=\begin{pmatrix}0 & -b/2\\ b/2 & 0\end{pmatrix}\quad\text{and}\quad C=\begin{pmatrix}-b^2/4 & 0\\ 0 & -b^2/4\end{pmatrix}.$$

Let H_3 be the operator defined by (4.4). Then there exists an $S\in Sp(2;\boldsymbol{R})$ such that $H=\mu(S)H_3\mu(S)^{-1}$ by Proposition 4.1 and Theorem 4.2. It should be noted that the choice of S is not unique. We set

$$S=\begin{pmatrix}I & B\\ C & I/2\end{pmatrix},\quad B=\begin{pmatrix}0 & -1/b\\ -1/b & 0\end{pmatrix},\quad C=\begin{pmatrix}0 & b/2\\ b/2 & 0\end{pmatrix}. \tag{6.3}$$

Then it is easy to show that S is a desired matrix and we can compute $\mu(S)$ easily as will be seen in (6.4) below.

Letting $\{h_m(x_1)\}_{m=0}^{\infty}$ be the Hermite polynomials defined by

$$h_m(x_1)=(-1)^m\exp(b\pi x_1^2)\frac{d^m}{dx_1^m}\exp(-b\pi x_1^2),$$

we set

$$\varphi_0(x_1)=(2b)^{1/4}\exp(-b\pi x_1^2),\quad \varphi_m(x_1)=h_m\left(\sqrt{2}x_1\right)\varphi_0(x_1)\Big/\sqrt{m!(2b\pi)^m}\quad (m\geqq 1)$$

and

$$(\varphi_m\otimes\varphi_n)(x_1,x_2)=\varphi_m(x_1)\varphi_n(x_2).$$

$\{\varphi_m\otimes\varphi_n\}_{m,n=0}^{\infty}$ forms a complete orthonormal system in $L^2(\boldsymbol{R}^2)$ consisting of the normalized eigenfunctions for H_0.

Then we show the following:

Proposition 6.1. *Let S be the element in $Sp(2;\boldsymbol{R})$ given by* (6.3) *and μ be the metaplectic representation. Then the following hold*:

(i) *$\mu(S)$ is explicitly given by*

$$\mu(S)f(x) = \int_{\boldsymbol{R}^2} \exp\left(2\pi\sqrt{-1}\left(\langle \xi, x\rangle - \frac{bx_1x_2}{2} - \frac{\xi_1\xi_2}{b}\right)\right)\hat{f}(\xi)\,d\xi \tag{6.4}$$

for $f \in \mathscr{S}'(\boldsymbol{R}^2)$.

(ii) *It holds that*

$$\mu(S)(\varphi_m \otimes \varphi_n) = (-1)^m \sqrt{-1}^n \phi_{m,n}. \tag{6.5}$$

PROOF. (6.4) is an easy consequence of Theorem 2.2.

For the proof of (6.5), we consider the generating function:

$$\Psi_t(x_1) = \sum_{m=0}^{\infty} \frac{t^m}{m!} h_m(\sqrt{2}x_1)\varphi_0(x_1)$$

and

$$\Phi_{(\tau,\sigma)}(z,\bar{z}) = \sum_{m,n=0}^{\infty} \frac{\tau^m\bar{\sigma}^n}{m!n!} H_{m,n}(\sqrt{2}z, \sqrt{2}\bar{z}) \exp(-b\pi|z|^2/2)$$

for $t \in \boldsymbol{R}$, $\tau, \sigma \in \boldsymbol{C}$. Then we have

$$\Psi_t(x_1) = (2b)^{1/4} \exp(-b\pi t^2 + 2\sqrt{2}b\pi t x_1 - b\pi x_1^2)$$

and

$$\Phi_{(\tau,\sigma)}(z,\bar{z}) = \exp(-b\pi\tau\bar{\sigma}/2 + b\pi\bar{\sigma}z/\sqrt{2} + b\pi\tau\bar{z}/\sqrt{2} - b\pi z\bar{z}/2)$$

(cf. [**20**], [**21**]). Now we apply (6.4). Then, after some straightforward calculations, we get

$$\mu(S)(\Psi_t \otimes \Psi_s) = \sqrt{b}\Phi_{(-2t,\,-2\sqrt{-1}s)}$$

and (6.5). □

For the heat kernel, we use (6.4) and the Mehler formula

$$e^{-tH_0}(x,y) = p_b^+(t, x_1, y_1)\delta_{y_2}(x_2),$$

where $p_b^+(t, x_1, y_1)$ is the function given by (5.6). Then we can show the following:

PROPOSITION 6.2. *$(\mu(S)\exp(-tH_0)\mu(S)^{-1}\delta_y)(x)$ coincides with the right hand side of* (6.1).

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