J. Math. Soc. Japan
Vol. 48, No. 4, 1996

On morphisms into contractible surfaces of Kodaira logarithmic dimension 1

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(Received Dec. 27, 1993)
(Revised Apr. 7, 1995)

1. Introduction.

The Abhyankar-Moh-Suzuki theorem [**AM**], [**Su**] and the Lin-Zaidenberg theorem [**LZ**] give a complete classification of polynomial injection of the complex line $\boldsymbol{C}$ into the complex plane $\boldsymbol{C}^2$. There is also classification of morphisms of $\boldsymbol{C}$ into $\boldsymbol{C}^2$ for which the only singular point of the image is a node [**N**]. Since smooth contractible complex algebraic surfaces have a lot in common with the plane, it is interesting to consider similar questions for them. There are some results in this direction which we remind now. It is known that every smooth contractible complex algebraic surface has Kodaira logarithmic dimension either 1 or 2 [**GM**], [**F**] unless this surface is isomorphic to $\boldsymbol{C}^2$. Zaidenberg proved that there is no polynomial injection of $\boldsymbol{C}$ into a smooth contractible surface of Kodaira logarithmic dimension 2 [**Z**]. But it is not so for contractible surfaces with Kodaira logarithmic dimension 1. Every of these surfaces contains a curve isomorphic to $\boldsymbol{C}$ [**Z**]. Miyanishi, Sugie and Tsunoda [**MS**], [**MT**] reproved the result of Zaidenberg in the case Kodaira logarithmic dimension 2, and in the case of Kodaira logarithmic dimension 1 they showed that there exists only one contractible curve which, of course, coincides with the curve we mentioned above. (They do not make the assumption about the smoothness of this curve).

In this paper we study contractible surfaces of Kodaira logarithmic dimension $\bar{k}=1$ only. We obtain a powerful generalization of the result of Zaidenberg, Miyanishi and Sugie. Namely we described all morphisms (not necessarily injections) from $\boldsymbol{C}$ into such surfaces. Our classification is unexpectedly simple due to the result of Petrie and tom Dieck who represent some of contractible surfaces with $\bar{k}=1$ as hypersurface in $\boldsymbol{C}^3$ [**PtD**]. Moreover, using the same approach we obtain a complete classification of morphisms from a once-punctured Riemann surface into a contractible surface W with $\bar{k}(W)=1$ (section 3). From this result follows that the Abhyankar-Singh property [**AS**] holds for W, i.e., if f is a regular function on W whose zero fiber is isomorphic to a once-punctured Riemann surface then every fiber of f has one puncture only.

We cannot obtain a similar result for morphisms from C^* into a smooth contractible surface with $\bar{k}=1$, but we study the case when these morphisms depend holomorphically on a parameter. This enables us to classify all morphisms from contractible surfaces with $\bar{k}=1$ into contractible surfaces with $\bar{k}=1$ (Theorems 6.2, 6.4). As a result we can strengthen the theorem from [**PtD**] which says that there is no nontrivial automorphism of a smooth contractible surface W with $\bar{k}(W)=1$. Actually, there is no nontrivial nondegenerate morphism from W into itself.

2. Preliminaries.

From now on n and m will be always coprime natural numbers satisfying $1<m<n$. Let $h_{n,m}$ be the polynomial on $\boldsymbol{C}^2$ with coordinates (x, y) defined by $h_{n,m}(x, y)=(x+1)^n-(y+1)^m$. Put $f_{n,m}(x, y, z)=h_{n,m}(xz, yz)/z$. Consider the hypersurface $V(n, m)=\{(x, y, z)\in\boldsymbol{C}^3 | f_{n,m}(x, y, z)=1\}$. Then $V(n, m)$ is a contractible surface of Kodaira logarithmic dimension 1 [**PtD**]. It contains a line $L_{n,m}$ which coincides with the zeros of the function z on $V(n, m)$. We shall need the following fact which may be extracted from ([**PtD**], pp. 150–151).

LEMMA 2.1. *The surface $V(n, m)$ is isomorphic to the complement of the proper transform of the curve $\{(u, v)\in\boldsymbol{C}^2 | u^n-v^m=0\}$ in the blow-up of $\boldsymbol{C}^2$ at the point $p=(1, 1)$. For every contractible surface W of Kodaira logarithmic dimension 1 there exists a unique pair (n, m) such that*

(1) either W is isomorphic to $V(n, m)$ or

(2) W can be obtained by the following procedure. Let $\rho=\rho_j\circ\cdots\circ\rho_1 : \overline{W}\to V(n, m)$ be a blow-up of $V(n, m)$ at a point $q\in L_{n,m}$ and infinitely near points and such that the center of the blow-up ρ_i lies on the exception divisor of ρ_{i-1} for $i\geqq 2$. Then W coincides with the complement of the proper transform in $\overline{W}$ of the curve $(\rho_{j-1}\circ\cdots\circ\rho_1)^{-1}(L_{n,m})$ under the blow-up ρ_j. □

DEFINITION 2.1.1. Let W be the same is in the previous lemma. Then we say that the contractible surface W is of type (n, m).

Denote by ρ_W the restriction of ρ to W in case (2) and the identical mapping in case (1). Put $L_W=\rho_W^{-1}(L_{n,m})$. Then L_W is a line and the restriction of ρ_W to $W-L_W$ is an isomorphism between $W-L_W$ and $V(n, m)-L_{n,m}$. Put $u=1+xz$ and $v=1+yz$. The following fact is clear.

LEMMA 2.2. *The image of $V(n, m)$ under the mapping $\tau_{n,m}:=(u, v)$ is $(\boldsymbol{C}^2-\Gamma_{n,m})\cup p$ where $\Gamma_{n,m}$ is the curve $\{(u, v)\in\boldsymbol{C}^2 | u^n-v^m=0\}$ and the point $p=(1, 1)\in\Gamma_{n,m}$. Moreover, $\tau_{n,m}(L_{n,m})=p$ and $\tau_{n,m}(V(n, m)-L_{n,m})=\boldsymbol{C}^2-\Gamma_{n,m}$.* □

Put $\tau_W = \tau_{n,m} \circ \rho_W$. Then Lemmas 2.1 and 2.2 imply

COROLLARY 2.3. *$\tau_W(W) = (\boldsymbol{C}^2 - \Gamma_{n,m}) \cup p$, $\tau_W(L_W) = p$, and the restriction of τ_W to $W - L_W$ is an isomorphism between $W - L_W$ and $\boldsymbol{C}^2 - \Gamma_{n,m}$.* □

We would like to emphasize one important property of functions z, u, v on $V(n, m): z = u^n - v^m$. This implies that the function $z_W := z \circ \rho_W$ coincides with $(u^n - v^m) \circ \tau_W$.

Note that there is a $\boldsymbol{C}^*$-action $G_{n,m}$ on $\boldsymbol{C}^2 - \Gamma_{n,m}$ given by $(u, v) \to (\lambda^m u, \lambda^n v)$ where $\lambda \in \boldsymbol{C}^*$. The pullback of $G_{n,m}$ to $W - L_W$ will be denoted by G_W. This action G_W cannot be extended to L_W since p is not invariant under $G_{n,m}$. On the other hand the mapping $W - L_W \to (W - L_W)/G_W \cong \boldsymbol{C}$ can be extended to a morphism $\Phi_W : W \to \boldsymbol{CP}^1$ where $\Phi_W^{-1}(\infty) = L_W$. We shall fix all the notation of this section for the rest of the paper.

3. Morphisms from once-punctured Riemann surfaces.

First we shall discuss some simple facts about plane curves.

LEMMA 3.1. *There is no closed plane affine algebraic curve R which meets $\Gamma_{n,m}$ at $p = (1, 1)$ only.*

PROOF. Assume the contrary. Let R be the zero fiber of a polynomial $P(u, v)$, and let $g : \boldsymbol{C} \to \Gamma$ be the normalization of $\Gamma_{n,m}$. One may suppose that the coordinate form of g is $g(t) = (t^m, t^n)$. Since $p = g(1)$ and $R \cap \Gamma_{n,m} = p$, the polynomial $P \circ g$ has the only zero at 1, i.e., $P \circ g = (t-1)^l$ up to a constant factor.

On the other hand $P \circ g$ is an element of the algebra generated by t^n and t^m. Hence the derivative of $P \circ g$ at $t = 0$ is zero, but it is not so for the function $(t-1)^l$. This contradiction implies the desired conclusion. □

Recall that a compact Riemann surface without a point is called a once-punctured Riemann surface.

LEMMA 3.2. *Let S be an algebraic curve which is homeomorphic to a once-punctured Riemann surface, and let $f : S \to W$ be a morphism. Then the function z_W is constant on $f(S)$.*

PROOF. Put $h = \tau_W \circ f$. Since z_W coincides with the function $z \circ \tau_W$ (recall $z = u^n - v^m$) it suffices to show that z is constant on $R = h(S)$. Note that R must be closed since S is once-punctured. If $z|_R \neq \text{const}$ then R meets $\Gamma_{n,m}$. Since $R \subset \tau_{n,m}(V_{n,m}) = (\boldsymbol{C}^2 - \Gamma_{n,m}) \cup p$, we have $R \cap \Gamma_{n,m} = p$. This contradicts Lemma 3.1 and we are done. □

REMARK. Using notation of the previous lemma consider two cases: $z_W|_{f(S)} = 0$ and $z_W|_{f(S)} = \text{const} \neq 0$. In the first case, clearly, $f(S) \subset L_W$. In the second case $f(S)$ is contained in a nonzero fiber of z_W which is isomorphic to the curve $F_{n,m}=\{(u, v) \mid u^n-v^m=1\}$. Note that $F_{n,m}$ is once-punctured and has positive genus.

COROLLARY 3.3. *For every nonconstant morphism $f: \boldsymbol{C} \rightarrow W$ the image of $\boldsymbol{C}$ is contained in L_W. In particular, L_W is the only contractible curve in W.*

PROOF. Suppose that $z_W|_{f(C)} \neq 0$. Then f generates a mapping from $\boldsymbol{C}$ into the surface $F_{n,m}$ from the remark above. Since the genus of $F_{n,m}$ is positive this mapping must be constant. Hence $f(\boldsymbol{C})$ is contained in the zero fiber of $z \circ \rho_W$ which is L_W. □

4. Some properties of morphisms from C^*.

LEMMA 4.1. *Let $f: \boldsymbol{C}^* \rightarrow W-L_W$ be a morphism. Then $f(\boldsymbol{C}^*)$ is contained in an orbit of the action G_W.*

PROOF. Since G_W is generated by $G_{n,m}$ it is enough to prove that for every morphism $g: \boldsymbol{C}^* \rightarrow \boldsymbol{C}^2-\Gamma_{n,m}$ the image is contained in the orbit of $G_{n,m}$. The function $z=u^n-v^m$ makes $\boldsymbol{C}^2-\Gamma_{n,m}$ a fibration over $\boldsymbol{C}^*$ whose fibers are isomorphic to $F_{n,m}$ where $F_{n,m}$ is isomorphic to the curve $\{(u, v) \in \boldsymbol{C}^2 \mid u^n-v^m=1\}$. This fibration is not a direct product and its monodromy has order nm. Consider $Y=\{(u, v, \zeta) \in \boldsymbol{C}^3 \mid (u, v) \notin \Gamma_{n,m}, \zeta^{nm}=u^n-v^m\}$ and the natural projection $\pi: Y \rightarrow \boldsymbol{C}^2-\Gamma_{n,m}$. The function ζ makes Y a fibration over $\boldsymbol{C}^*$ with fiber $F_{n,m}$, and Y can be already identified with the direct product $\boldsymbol{C}^* \times F_{n,m}$. The natural $\boldsymbol{C}^*$-action on this direct product is the pullback of $G_{n,m}$. Its orbits are $\boldsymbol{C}^* \times q$, where $q \in F_{n,m}$ and $\pi(\boldsymbol{C}^* \times q)$ is an orbit of $G_{n,m}$.

Since π is a covering there exists a mapping $h: \boldsymbol{C}^* \rightarrow Y$ such that $\pi \circ h(t)=g(t^{nm})$. Since Y is the direct product h generates two morphisms $h_1: \boldsymbol{C}^* \rightarrow \boldsymbol{C}^*$ and $h_2: \boldsymbol{C}^* \rightarrow F_{n,m}$. But the genus of $F_{n,m}$ is positive, and thus h_2 is constant. Hence $h(\boldsymbol{C}^*) \subset \boldsymbol{C}^* \times q$ for some $q \in F_{n,m}$ which implies the desired conclusion. □

Let $\psi: \boldsymbol{C}^* \rightarrow W$ be a morphism. Put $\varphi=\tau_W \circ \psi$. There are two possibilities: either $\varphi(\boldsymbol{C}^*)$ is a closed curve or it is not. In the first case we can apply Lemma 3.1. In the second case one may suppose that φ may be extended to $\boldsymbol{C}$. If $\varphi(0) \notin \Gamma_{n,m}$ or $\varphi(0)=p$ we can again apply Lemma 3.1 to $\varphi(\boldsymbol{C})$. Thus the only difficult case is when $\varphi(0) \in \Gamma_{n,m}-p$.

LEMMA 4.2. *Let $g: \boldsymbol{C} \rightarrow \Gamma_{n,m}$ be the normalization of $\Gamma_{n,m}$ given by $t \rightarrow (t^m, t^n)$. Let $\varphi: \boldsymbol{C} \rightarrow \boldsymbol{C}^2$ be a morphism such that $\varphi(\boldsymbol{C})$ meets $\Gamma_{n,m}$ at p and at another point p_0 only. Suppose that the curve $\varphi(\boldsymbol{C})$ is the zero fiber of an irreducible polynomial $P(u, v)$. Then*

(1) *$P \circ g(t)=t^k(t-1)^l$ when $m>2$. In particular, p_0 is the origin.*

(2) *$P\circ g(t)$ is either $t^k(t-1)^l$ or $(t^2-1)^k$ when $m=2$ and $n>3$. In particular, p_0 is either the origin or $(1, -1)$;*

(3) *$P\circ g(t)$ is one of the polynomials $t^k(t-1)^l$, $(t^2-1)^k$, $(t-1)^k(t+l/k)^l$ when $(n, m)=(3, 2)$. In particular p_0 belongs to a discrete subset of $\Gamma_{3,2}$.*

PROOF. Suppose $g(-a)=p_0$. Since $P\circ g$ is zero at 1 and $-a$ only, then $P\circ g(t)=(t-1)^k(t+a)^l$ up to a constant factor. On the other hand $P\circ g$ belongs to the algebra generated by t^m and t^n, i.e., its derivative at 0 is 0. Suppose that $a\neq 0$. Then $[(t-1)^k(t+a)^l]'=0$ for $t=0$ only when $a=l/k$ which implies case (3). In case (1) the second derivative of $P\circ g$ at $t=0$ is also 0. The direct computation shows that the second derivative of $(t-1)^k(t+l/k)^l$ is nonzero for $t=0$. This implies case (1). In case (2) similar consideration of the third derivative implies the desired conclusion. □

5. Families of morphisms from C^*.

It is difficult to describe all morphisms from C^* into a contractible surface of Kodaira logarithmic dimension 1. But if we deal with a family of morphisms from C^* we can extract some information which is the purpose of this section.

LEMMA 5.1. *Let $\Delta_\varepsilon=\{\zeta\in C \mid |\zeta|<\varepsilon\}$ for positive ε, $\Delta=\Delta_1$, $\Delta^*=\Delta-\{0\}$, let $\pi:\Delta\times C\to\Delta$ be the natural projection, and let $\varphi:\Delta^*\times C\to C^2$ be a regular mapping with the following properties:*

- *$\varphi_\zeta(0)=p_0\in\Gamma_{n,m}$ for every $\zeta\in\Delta^*$ where φ_ζ is the restriction of φ to the line $\pi^{-1}(\zeta)$ and $p_0\neq p=(1, 1)$,*
- *for every $\zeta\in\Delta^*$ the curve $\varphi_\zeta(C^*)$ meets $\Gamma_{n,m}$ at the point p only.*

Denote by R the curve $\varphi^{-1}(p)$. Suppose that $\bar{R}\cap\pi^{-1}(0)$ is not empty where $\bar{R}$ is the closure of R in $\Delta\times C$. Then $\bar{R}\cap\pi^{-1}(\Delta_\varepsilon)$ is relatively compact in $\Delta\times C$ when $\varepsilon>0$ is sufficiently small.

PROOF. Assume the contrary. Let $(\zeta, t)\in R\cap\pi^{-1}(\Delta_\varepsilon^*)$. Then we may treat t as a multi-valued function of ζ. Reducing ε and replacing ζ by $\zeta^{1/l}$ for some natural l, if necessary, one may suppose that this multi-valued function is actually the union of k single-valued functions $t_1(\zeta), \cdots, t_k(\zeta)$. The absence of relative compactness just means that $t_i(\zeta)$ has a pole at 0 for some i (say $i=1$). Let $\varphi_\zeta=(f_\zeta, g_\zeta)$ where f_ζ, g_ζ are polynomials. Consider the regular mapping $\tilde{\varphi}:\Delta^*\times C\to C^2$ such that its restriction to the line $\pi^{-1}(\zeta)$ is given by $\tilde{\varphi}_\zeta(t)=(\tilde{f}_\zeta(t), \tilde{g}_\zeta(t))$ where $\tilde{f}_\zeta(t)=f_\zeta(t+t_1(\zeta))$ and $\tilde{g}_\zeta(t)=g_\zeta(t+t_1(\zeta))$. In particular, $\tilde{\varphi}_\zeta(0)=p$. Consider the three possibilities:

(1) $\tilde{\varphi}_0=\lim_{\zeta\to 0}\tilde{\varphi}_\zeta$ exists and it is different from a constant,
(2) this limit does not exist,
(3) $\tilde{\varphi}_0$ exists and it is constant.

(1). Put $S=\tilde{\varphi}_0(\boldsymbol{C})$. Assume that $S\neq\Gamma_{n,m}$ and that $\tilde{\varphi}_0(t^0)=p_1\in\Gamma_{n,m}$ for some $t^0\in\boldsymbol{C}$ where $p_1\neq p$. By the Hurwitz theorem, there exists $t^1(\zeta)$ such that $\tilde{\varphi}_\zeta(t^1(\zeta))=p(\zeta)\in\Gamma_{n,m}-p$ when $|\zeta|$ is sufficiently small. Moreover, $t^1(\zeta)\to t^0$ as $\zeta\to 0$. On the other hand $\tilde{\varphi}_\zeta(t)\in\Gamma_{n,m}-p$ only when $t=-t_1(\zeta)$, i.e., $t^1(\zeta)=-t_1(\zeta)$. It cannot be so, since $|t_1(\zeta)|\to\infty$ as $\zeta\to 0$. Hence $S\cap\Gamma_{n,m}=p$. But this contradicts Lemma 3.1.

Now let $S=\Gamma_{n,m}$. Then for some $t^0\in\boldsymbol{C}$ the point $\tilde{\varphi}_0(t^0)$ is the origin $o\in\boldsymbol{C}^2$. Since o is a singular point of $\Gamma_{n,m}$ then, by [Proposition 2.2, Z], for small $|\zeta|$ there exists $t^1(\zeta)$ such that $t^1(\zeta)\to t^0$ as $\zeta\to 0$ and $\tilde{\varphi}_\zeta(t^1(\zeta))\in\Gamma_{n,m}-p$. The same argument as above shows that $t^1(\zeta)=-t_1(\zeta)$, but it cannot be so since $|t_1(\zeta)|\to\infty$ as $\zeta\to 0$.

(2). Suppose that $f_\zeta(t)=\sum_{i=0}^{s} a_i(\zeta)t^i$ and $g_\zeta(t)=\sum_{i=0}^{r} b_i(\zeta)t^i$. Since φ is regular, the functions $a_i(\zeta)$ and $b_i(\zeta)$ have no essential singularities at 0. Let $\tilde{f}_\zeta(t)=\sum_{i=0}^{s}\tilde{a}_i(\zeta)t^i$ and $\tilde{g}_\zeta(t)=\sum_{i=0}^{r}\tilde{b}_i(\zeta)t^i$. Recall that $t_1(\zeta)$ has a pole at 0. Hence, by construction, functions $\tilde{a}_i$ and $\tilde{b}_i$ have at most poles at 0. In fact one of these functions must have a pole since $\lim_{\zeta\to 0}\tilde{\varphi}_\zeta$ does not exist. Note also that $\tilde{a}_0(\zeta)=\tilde{b}_0(\zeta)=1$ since $\tilde{\varphi}_\zeta(0)=p$. Let k_i be 0 when $\tilde{a}_i$ has a removable singularity at 0, and let it be the order of the pole of $\tilde{a}_i$ at zero otherwise. Similarly, l_i is 0 when $\tilde{b}_i$ has a removable singularity at 0, and it is the order of the pole of $\tilde{b}_i$ at 0 otherwise. Put $\alpha=\max\{k_i/i,\ l_j/j\,|\,i=1,\cdots,s;\ j=1,\cdots,r\}$. Then α is positive rational number of form k/l (where k and l are integers) and, replacing ζ by $\zeta^{1/l}$, one may suppose that α is integer. Consider $'t=t\zeta^{-\alpha}$. Then $(\zeta,\ 't)$ is a new coordinate system on $\Delta^*\times\boldsymbol{C}$ which provides the embedding of this manifold into another sample of $\Delta\times\boldsymbol{C}$. In this new coordinate system denote by $'\tilde{\varphi}_\zeta('t)$ the restriction of $\tilde{\varphi}$ to the fiber $\pi^{-1}(\zeta)$. Every holomorphic function $h(\zeta,t)=\sum c_i(\zeta)t^i$ on $\Delta^*\times\boldsymbol{C}$ may be rewritten in the new coordinate system as $\sum c_i(\zeta)\zeta^{i\alpha}('t)^i$. Applying this observation to the coordinate functions of the mapping $'\tilde{\varphi}_\zeta$ one can see that this mapping has limit when $\zeta\to 0$ and, moreover, $\lim_{\zeta\to 0}{}'\tilde{\varphi}_\zeta$ is not constant. Note also that the function $'t_1(\zeta)=t_1(\zeta)\zeta^{-\alpha}$ still has a pole at 0. Therefore, we have reduced case (2) to case (1).

(3). Since $\tilde{\varphi}_0$ is constant $\tilde{a}_i(0)$ and $\tilde{b}_i(0)$ are zero for $i>0$. Consider $''t=t\zeta^\beta$ where β is a positive rational number. Consider the restriction of φ to $\pi^{-1}(\zeta)$ as a function of $''t$ which is denoted by $''\tilde{\varphi}_\zeta(''t)$. Following the scheme of (2) one may suppose that β is integer and that $''\tilde{\varphi}_0=\lim_{\zeta\to 0}{}''\tilde{\varphi}_\zeta$ exists and is different from a constant mapping. When we change t to $''t$ the function $t_1(\zeta)$ must be replaced by $''t_1(\zeta)=t_1(\zeta)\zeta^\beta$. If $''t_1(\zeta)$ has a pole at 0 then the same argument as in (1) implies that this case cannot hold. Assume that $''t_1$ has a removable singularity at 0. Then $''\tilde{\varphi}_0(-''t_1(0))=p_0$, by construction. On the other hand some point $(0,t_0)$ in the old coordinate system (ζ,t) belongs to $\bar{R}$, i.e., in every neighborhood of this point there are points from R. When we switch from

(ζ, t) to $(\zeta, {}''t)$ these points will appear in every neighborhood of $(0, -{}''t_1(0))$ since the limits of $(t_0-t_1(\zeta))\zeta^{\beta}$ and $-t_1(\zeta)\zeta^{\beta}$ are the same when $\zeta\to 0$. Hence ${}''\bar{\varphi}_0(-{}''t_1(0))=p$. Contradiction. □

LEMMA 5.2. *Let S be a smooth algebraic curve, let X be a smooth algebraic surface admitting a morphism $\pi: X\to S$ whose fibers are isomorphic to $\boldsymbol{C}$. Suppose that $h: S\to X$ is a section (i.e., $\pi\circ h=id$) and $\varphi: X\to \boldsymbol{C}^2$ is a morphism such that*

- $\varphi(h(S))=p_0\in\Gamma_{n,m}$ where $p_0\neq p$,

- the intersection of $\varphi(X-h(S))$ and $\Gamma_{n,m}$ consists of the point p only.

Then $\varphi(X)$ is a curve.

PROOF. Put $R=\varphi^{-1}(p)$. Removing some points from S and the corresponding fibers from X, if necessary, one may suppose that $\pi|_R: R\to S$ is a finite morphism. The curve R consists of components $R_1, R_2, \cdots, R_k$. Let $S_1\to R_1$ be a normalization of R_1 and let $\nu: S_1\to S$ be the composition of this normalization and π. Consider $X_1=X\otimes_S S_1$. Then we have a section $h_1: S_1\to X_1$ and a natural morphism $\varphi_1: X_1\to\boldsymbol{C}^2$ so that this pair (h_1, φ_1) has the same properties as the pair (h, φ). It suffices to prove that $\varphi_1(X_1)$ is a curve. Replace S, X, h, φ by S_1, X_1, h_1, φ_1. The advantage of this procedure is that one may suppose that $\pi|_{R_1}: R_1\to S$ is an isomorphism, i.e., we have the second section $h^1: S\to X$ so that $h^1=(\pi|_{R_1})^{-1}$. Hence we can consider X as the direct product $\boldsymbol{C}\times S$ where $0\times S=h(S)$ and $1\times S=h^1(S)$. Note that for $i\geqq 2$ each R_i may be treated the graph of a multi-valued holomorphic function ψ_i on S. Let $\bar{S}$ be a smooth compactification of S, then $\bar{S}-S$ consists of a finite number of points $s_1, \cdots, s_l$. Consider a neighborhood U of s_j such that $U-s_j$ is isomorphic to a once-punctured disc Δ^*. Applying Lemma 5.1 to the restriction φ to $\pi^{-1}(U-s_j)$ one can see ψ_i may be extended to s_j, by the Riemann theorem about removing singularities. Hence we obtain a multi-valued holomorphic function on $\bar{S}$ which must be constant due to the maximum principle. This implies that $R_i=t_i\times S$ for some $t_i\in\boldsymbol{C}$. Consider a generic point $s\in S$ and a local coordinate ζ on S in a neighborhood of s. Put φ_ζ equal to the restriction of φ to $\pi^{-1}(\zeta)$. Then $\varphi_\zeta=(f_\zeta, g_\zeta)$, where $f_\zeta, g_\zeta\in\boldsymbol{C}[t]$. Since $\varphi_\zeta(\boldsymbol{C})$ meet $\Gamma_{n,m}$ when $t=0, 1, t_2, \cdots, t_k$ only, $f_\zeta^n(t)-g_\zeta^m(t)=\lambda(\zeta)q(t)$ where $q\in\boldsymbol{C}[t]$ and has roots at $0, 1, t_2, \cdots, t_k$ only, and $\lambda(\zeta)$ is a holomorphic function. Since s is generic one may suppose that $\lambda(\zeta)\neq 0$. Put $\tilde{f}_\zeta=f_\zeta/\lambda^{1/n}$ and $\tilde{g}_\zeta=g_\zeta/\lambda^{1/m}$, then $\tilde{f}_\zeta^n(t)-\tilde{g}_\zeta^m(t)=q(t)$. We shall show that $\tilde{f}_\zeta$, $\tilde{g}_\zeta$ do not depend on ζ. Let $\hat{f}_\zeta$, $\hat{g}_\zeta$ be the derivatives of $\tilde{f}_\zeta$, $\tilde{g}_\zeta$ with respect to ζ. Since s is generic, $\deg\hat{f}_\zeta=\deg\tilde{f}_\zeta=$const and $\deg\tilde{g}_\zeta=\deg\hat{g}_\zeta=$const. Note that $n\hat{f}_\zeta\tilde{f}_\zeta^{n-1}-m\hat{g}_\zeta\tilde{g}_\zeta^{m-1}=0$. Suppose that $\tilde{f}_\zeta$ and $\tilde{g}_\zeta$ are relatively prime. Assume that $\hat{f}_\zeta$, $\hat{g}_\zeta$ are not identically zero. Then $\deg\hat{f}_\zeta\geqq(m-1)\deg\tilde{g}_\zeta$ and $\deg\hat{g}_\zeta\geqq(n-1)\deg\tilde{f}_\zeta$ which is impossible. Thus $\tilde{f}_\zeta$, $\tilde{g}_\zeta$ does not depend on ζ in this case.

When $\tilde{f}_\zeta$ and $\tilde{g}_\zeta$ are not relatively prime, f_ζ and g_ζ have common zeros and $\varphi_\zeta(\boldsymbol{C})$ contains the origin $(0, 0)\in\Gamma$. Hence p_0 is the origin. Since $p_0\notin\varphi_\zeta(\boldsymbol{C}^*)$ we see the common zero of $\tilde{f}_\zeta$ and $\tilde{g}_\zeta$ is 0 only. This implies that $\tilde{f}_\zeta(t)=\bar{f}_\zeta(t)t^\alpha$ and $\tilde{g}_\zeta(t)=\bar{g}_\zeta(t)t^\beta$ where $\bar{f}_\zeta$, $\bar{g}_\zeta$, t are pairwise relatively prime.

Since s is generic, $\hat{f}_\zeta(t)=\check{f}_\zeta(t)t^\alpha$ and $\hat{g}_\zeta(t)=\check{g}_\zeta(t)t^\beta$. Hence $n\check{f}_\zeta(t)\bar{f}_\zeta^{n-1}(t)t^{n\alpha}-m\check{g}_\zeta(t)\bar{g}_\zeta^{m-1}(t)t^{m\beta}=0$. Put $r=n\alpha-m\beta$. Without loss of generality suppose that $r\geqq 0$. Since $\deg\check{f}_\zeta=\deg\bar{f}_\zeta$ and $\deg\check{g}_\zeta=\deg\bar{g}_\zeta$ when $\hat{f}_\zeta$, $\hat{g}_\zeta\neq 0$ we have $\deg\bar{g}_\zeta\geqq(n-1)\deg\bar{f}+r$ and $\deg\bar{f}\geqq(m-1)\deg\bar{g}_\zeta$. Hence $\deg\bar{g}_\zeta\geqq(n-1)(m-1)\deg\bar{g}_\zeta+r$ and $\deg\bar{f}_\zeta\geqq(n-1)(m-1)\deg\bar{f}_\zeta+(m-1)r$, which is impossible. Thus $\hat{f}_\zeta$, $\hat{g}_\zeta=0$ again and $\tilde{f}_\zeta$, $\tilde{g}_\zeta$ do not depend on ζ. Put $\tilde{f}=\tilde{f}_\zeta$, $\tilde{g}=\tilde{g}_\zeta$. Then $f_\zeta=\lambda(\zeta)^{1/n}\tilde{f}$ and $g_\zeta=\lambda(\zeta)^{1/m}\tilde{g}$. This implies that $\lambda\equiv$const (otherwise $\varphi_\zeta(1)$ cannot be $p=(1, 1)$ for each ζ). Therefore, f_ζ and g_ζ do not depend on ζ and we are done.

LEMMA 5.3. *Let Y be an algebraic surface, let S be a smooth algebraic curve, and let $\nu: Y\to S$ be a morphism such that its generic fibers are $\boldsymbol{C}^*$. Suppose that $\psi: Y\to W$ is a morphism into contractible surface W with $\bar{k}(W)=1$. Then either ψ is degenerate or for every $s\in S$ the set $\psi(\nu^{-1}(s))$ is contained in a fiber of a function Φ_W on W (see preliminaries for notation).*

PROOF. Suppose that $\psi(\nu^{-1}(s))$ is contained in a fiber of Φ_W for a generic $s\in\boldsymbol{C}$. Then we have a morphism $\chi=\Phi_W\circ\psi\circ\nu^{-1}: S-\{s_1, \cdots, s_l\}\to\boldsymbol{CP}^1$ where the fibers $\nu^{-1}(s_1), \cdots, \nu^{-1}(s_l)$ are singular. Let $\bar{S}$ be a smooth completion of S. Then one can extend χ to a morphism $\bar{\chi}: \bar{S}\to\boldsymbol{CP}^1$ due to the Riemann theorem about deleting singularities. Hence $\psi(\nu^{-1}(s_i))$ is contained in a fiber $\Phi_W^{-1}(\bar{\chi}(s_i))$ of Φ_W. This argument enables us to consider generic fibers of ν only. That is why from now on we suppose that every fiber of ν is generic and therefore, is isomorphic to $\boldsymbol{C}^*$. Moreover, we suppose that if $\psi(\nu^{-1}(s))$ is contained in a fiber of Φ_W, this fiber is different from $L_W=\Phi_W^{-1}(\infty)$, i.e., $\psi(\nu^{-1}(s))$ is contained in an orbit of G_W.

Let $\bar{Y}$ be a completion of Y for which the divisor $\bar{Y}-Y$ is of normal crossing type. By Hironaka's theorem, we may suppose that there is an extension $\bar{\nu}: \bar{Y}\to\bar{S}$ of ν. Due to the remark about generic fibers one may also suppose that $\bar{\nu}^{-1}(s)$ is isomorphic to $\boldsymbol{CP}^1$ for every $s\in S$, and $\bar{\nu}^{-1}(S)-Y$ consists of either two disjoint smooth curves R_0 and R_∞ or a smooth curve S_1 such that $\bar{\nu}|_{S_1}$ is a 2-sheeted covering. Replacing S and Y by S_1 and $Y\otimes_S S_1$ respectively, we can reduce the second possibility to the first one. Clearly, the restrictions $\bar{\nu}|_{R_0}$ and $\bar{\nu}|_{R_\infty}$ give isomorphisms between R_0, R_∞, and S. For a generic s put $\bar{T}=\bar{\nu}^{-1}(s)$, $T=\nu^{-1}(s)$, $t_0=R_0\cap\bar{T}$, and $t_\infty=R_\infty\cap\bar{T}$. Consider $\varphi=\tau_W\circ\psi: Y\to\boldsymbol{C}^2$. Suppose that φ is not constant.

If $\varphi(T)$ is a closed curve in $\boldsymbol{C}^2$ then $\psi(T)\subset W-L_W$, by Lemma 3.1, and, therefore, $\psi(T)$ is contained in an orbit of G_W, by Lemma 4.1. The argument about generic fibers in the beginning of this proof implies the statement of Lemma in this case.

Now let $\varphi(T)$ is not a closed curve. This means that φ can be extended to either $T\cup t_0\approx \boldsymbol{C}$ or $T\cup t_\infty\approx \boldsymbol{C}$ (say, to $T\cup t_0$). If $\varphi(t_0)\notin \Gamma_{n,m}$ or if $\varphi(t_0)=p$ then $\varphi(\boldsymbol{C})$ is a closed curve which meets $\Gamma_{n,m}$ at p only. This contradicts Lemma 3.1. Hence $\varphi(t_0)=p_0\in\Gamma_{n,m}-p$. If $\varphi(T)\not\ni p$, then $\phi(T)\subset W-L_W$. Again $\phi(T)$ is contained in an orbit of G_W, by Lemma 4.1, and the argument from the beginning of the proof works. Thus we have to consider the case when $p\in\varphi(T)$. Since $\varphi(\boldsymbol{C})$ contains now p and p_0, we see that p_0 belongs a discrete subset of $\Gamma_{n,m}$, by Lemma 4.2. Hence p_0 is independent of s (recall $T=\nu^{-1}(s)$). Put $X=\bar{\nu}^{-1}(S)-R_\infty$. Then we may extend φ to X (use the same letter φ for this extension), since we have supposed that all fibers of ν and $\bar{\nu}$ over S are generic. Put $h=(\nu|_{R_0})^{-1}$ and $\pi=\bar{\nu}|_X$. Note that $h: S\to X$ is a section such that $\varphi(h(S))=p_0$. Hence the data X, S, π, φ, h satisfies Lemma 5.2. Thus $\varphi(X)$ is a curve and ϕ is degenerate. □

We shall also need some information about non-generic fibers in a family of morphisms from $\boldsymbol{C}^*$ to W, which is given in the next two lemmas.

LEMMA 5.4. *Let Y be a Stein surface, let $\nu: Y\to\Delta$ be a surjective mapping every fiber of which is biholomorphic to $\boldsymbol{C}^*$, let $\phi: Y\to\boldsymbol{C}^*$ be a holomorphic mapping, and let ϕ_s be the restriction of ϕ to $\nu^{-1}(s)$ for $s\in\Delta$. Suppose that ϕ_s has no essential singularities at* 0 *and* ∞. *Then ϕ_0 is a constant mapping iff ϕ_s is constant for every $s\in\Delta$.*

PROOF. Without loss of generality one may suppose that all nonzero fibers of ν are generic. Choose a coordinate on the zero fiber so that 0 and ∞ correspond to the punctures. Since Y is Stein, this coordinate can be extended to a holomorphic function f on Y. Let $C_s'=\nu^{-1}(s)\cap f^{-1}(\Delta)$ and $c_s=\nu^{-1}(s)\cap f^{-1}(\partial\Delta)$. We may suppose that f has no critical points on c_0 and, hence, c_s is a circle for every s. There are two possibilities: C_s' is biholomorphic to either Δ or Δ^*. But the first possibility implies that $f^{-1}(\Delta)$ is not holomorphically convex which contradicts to the fact that Y is Stein. Therefore, C_s' and $C_s''=\nu^{-1}(s)-(C_s'\cup c_s)$ are biholomorphic to Δ^*. Assume that ϕ_0 is a constant mapping and ϕ_s is not constant for nonzero s. Let p_s' and p_s'' be the punctures of $\nu^{-1}(s)$ so that $C_s'\cup p_s'$ and $C_s''\cup p_s''$ are discs. Without loss of generality suppose that ϕ_s has zero at p_s' when $s\neq 0$. On the other hand the restriction of ϕ_s to c_s must be close to the nonzero constant ϕ_0. Hence ϕ_s does not take on the zero value due to the argument principle. This contradiction proves lemma. □

LEMMA 5.5. *Let Y be a smooth affine algebraic surface with a finite Picard group, let $\nu: Y\to S$ be a morphism from Y into a smooth algebraic curve S such* *that its generic fibers are isomorphic to* C^*. *Suppose that* $\varphi: Y \to W$ *is a morphism for which* $\varphi(\nu^{-1}(s))$ *is contained in a fiber of* Φ_W *for every* s. *Let* $\varphi(\nu^{-1}(s_0)) \subset L_W$ *for some* s_0. *Then either* φ *is degenerate or* $\nu^{-1}(s_0)$ *is a disjoint union of lines.*

PROOF. Suppose that $\varphi(Y) \not\subset L_W$. Let U be a neighborhood of s_0 such that U is biholomorphic to Δ, s_0 corresponds to the origin, every $s \in U - s_0$ is a generic value of ν, and $\varphi(\nu^{-1}(s))$ is contained in an orbit of G_W for $s \in U - s_0$. It is well-known that each component of $\nu^{-1}(s_0)$ must be either C^*, or C, or a couple of lines with one common point (e.g., see [**Z**]). Assume that $\nu^{-1}(s_0)$ is not a disjoint union of lines. If this fiber contains a C^*-component then remove all other components from $\nu^{-1}(U)$. If there is no C^*-component, choose a component which is a couple of lines with one common point and remove all other components and one of these lines from $\nu^{-1}(U)$. As a result of this procedure we obtain a complex manifold X such that X admits a holomorphic mapping on $U \cong \Delta$ whose fibers are isomorphic to C^*. By abusing notation, denote the projection $X \to U$ by ν again. Note that X is Stein. Indeed, since Y has a finite Picard group, for every divisor $D \subset Y$ there exists natural k so that kD coincides with the zeros of a regular function on Y. Thus $Y - D$ is affine and, therefore, X is Stein. Put $\tilde{\varphi} = \tau_W \circ \varphi|_X : X \to C^2 - (0, 0)$. The assumption of this lemma on fibers may be reformulated now: $\tilde{\varphi}(\nu^{-1}(s_0)) = p = (1, 1)$ and $\tilde{\varphi}(\nu^{-1}(s)) \subset \{(u, v) \in C^2 - (0, 0) \mid u^n - a_s v^m = 0\}$ for $s \in U - s_0$ where $a_s \in C - \{1\}$. Reducing U, one may suppose that $a_s \in \{t \in C \mid |t-1| < 1/2\}$. Note that $\{(u, v) \in C^2 - (0, 0) \mid |u^n/v^m - 1| < 1/2\}$ is isomorphic to $C^* \times \Delta$. Thus we have a mapping $\psi: X \to C^*$. Lemma 5.4 implies that $\psi|_{\nu^{-1}(s)}$ is constant. Hence $\tilde{\varphi}$ and, therefore, φ are degenerate. $\square$

6. Morphisms of contractible surfaces with $\bar{k}=1$.

First we consider the case of degenerate mappings.

LEMMA 6.1. *Let* $g: Y \to S$ *be a nonconstant morphism of a smooth algebraic variety* Y *onto an affine algebraic curve* S. *Suppose that* Y *has a finite first homology group and* $\nu: \tilde{S} \to S$ *is a normalization of* S. *Then* $\tilde{S}$ *is isomorphic to* C.

PROOF. Let $\bar{S}$ be a completion of S and $\bar{Y}$ be a smooth completion of Y. One may suppose that g may be extended to a regular morphism $\bar{g}: \bar{Y} \to \bar{S}$ due to the Hironaka theorem. Note that $\bar{g}^{-1}(S)$ has a finite first homology group, since $\bar{g}^{-1}(S) \supset Y$. Replacing Y by $\bar{g}^{-1}(S)$, one may suppose from the beginning that g is proper. There exists $\tilde{g}: Y \to \tilde{S}$ such that $g = \nu \circ \tilde{g}$ and $\tilde{g}$ is also proper [**Sh**]. By Stein factorization, there exist a finite morphism $\hat{\nu}: \hat{S} \to \tilde{S}$ and a proper morphism $\hat{g}: Y \to \hat{S}$ so that $\tilde{g} = \hat{\nu} \circ \hat{g}$ and the generic fibers of $\hat{g}$ are connected. Clearly, it suffices to prove that $\hat{S}$ is isomorphic to C. Suppose that $\gamma:[0, 1]\to\hat{S}$ is an arbitrary loop, i.e., $\gamma(0)=\gamma(1)=s$. One may think that s is a generic point and, in particular, $F=\hat{g}^{-1}(s)$ is connected. Since $\hat{g}$ is proper there exists a continuous mapping $\mu_1:[0, 1]\to Y$ for which $\gamma(t)=\hat{g}(\mu_1(t))$ for every $t\in[0, 1]$. Choose a path μ_2 in F joining the points $\mu_1(0)$ and $\mu_1(1)$. Then μ_1 and μ_2 generate a loop μ in Y. Consider the elements $[\gamma]\in H_1(\hat{S})$ and $[\mu]\in H_1(Y)$. Clearly, $\hat{g}_*([\mu])=[\gamma]$. Hence $[\gamma]$ has a finite order. But there is no nontrivial elements of finite order in the first homology group of a Riemann surface. Hence $H_1(\hat{S})=0$ and $\hat{S}$ is isomorphic to $\boldsymbol{C}$ which concludes the proof. □

THEOREM 6.2. *Let Y be a smooth algebraic variety whose first homology group is finite, let W be a smooth contractible surface with $\bar{k}(W)=1$, and let $\varphi: Y\to W$ be a morphism such that $\varphi(Y)$ is a curve. Then $\varphi(Y)\subset L_W$.*

PROOF. Put $S=\varphi(Y)$, and let $\nu:\tilde{S}\to S$ be a normalization of this curve S. By Lemma 6.1, $\tilde{S}$ is isomorphic to $\boldsymbol{C}$. By Corollary 3.3, $S=\nu(\tilde{S})\subset L_W$. □

In the nondegenerate case we shall need the following lemma.

LEMMA 6.3. *Let S_W be the curve $\{u^{n_1}-v^{m_1}=1\}$ where n_1 and m_1 are relatively prime and $n_1>m_1>0$, and let S_U be the curve $\{u^n-v^m=1\}$. Consider the action of a cyclic group $g_W\cong\boldsymbol{Z}_{n_1m_1}$ on the curve S_W given by $(u, v)\to(\varepsilon^{m_1}u, \varepsilon^{n_1}v)$ where ε is a primitive (n_1m_1)-root of unity, and the action of a cyclic group $g_U\cong\boldsymbol{Z}_{nm}$ on the curve S_U given by $(u, v)\to(\delta^m u, \delta^n v)$ where δ is an (nm)-root of unity. Suppose that h is a morphism from S_W to S_U such that the image of every orbit of g_W is contained in an orbit of g_U. Then h is given by one of the formulas: $h(u, v)=(u^k, v^l)$ or $h(u, v)=(v^l, u^k)$. When the first formula holds $n_1=kn$ and $m_1=lm$, and when the second formula holds $n_1=lm$ and $m_1=nk$ (this imposes some conditions on k and l.)*

PROOF. The assumption of Lemma implies that there exists a homomorphism $h_*: g_W\to g_U$ for which $h\circ\gamma=h_*(\gamma)\circ h$ for every $\gamma\in g_W$. Let $\ker h_*$ be generated by an element of g_W whose action on S_W is given by $(u, v)\to(\varepsilon^{m_1m_2n_2}u, \varepsilon^{n_1m_2n_2}v)$ where n_2 is a divisor of n_1 and m_2 is a divisor of m_1. Put $l=m_1/m_2$ and $k=n_1/n_2$. Consider $S_V=\{u^{n_2}-v^{m_2}=1\}$ and the action of the group $g_V\cong\boldsymbol{Z}_{n_2m_2}$ on S_V given by $(u, v)\to(\sigma^{m_2}u, \sigma^{n_2}v)$ where σ is an (n_2m_2)-root of unity. There is the natural mapping $h_1: S_W\to S_V$ given by $(u, v)\to(u^k, v^l)$ and a mapping $h_2: S_V\to S_U$ such that $h=h_2\circ h_1$ and the restriction of h_2 to every orbit of g_V is an embedding into an orbit of g_U. Hence $in_2m_2=nm$ for some natural i. Note that n_2, $m_2>1$, otherwise S_V is isomorphic to $\boldsymbol{C}$, but there is no nonconstant morphism from $\boldsymbol{C}$ into S_U whose genus is positive. The genus of S_V is $(n_2-1)(m_2-1)/2$ and the genus of S_U is $(n-1)(m-1)/2$. Since there is a nonconstant mapping from S_V to S_U, the Riemann-Hurwitz formula implies that $(n_2-1)(m_2-1)\geqq(n-1)(m-1)$, i.e., $n_2m_2-n_2-m_2\geqq nm-m-n$ and $n+m\geqq nm(i-1)/i$. Assume $i>1$, then the last inequality holds either when $m=3$, $n=4$, $i=2$, and $n_2m_2=6$, or when $m=i=2$. In the first case $n_2m_2-n_2-m_2<nm-n-m$. Contradiction.

In the second case $n_2m_2-n_2-m_2=nm/2-n_2-m_2=n-m_2-n_2\geqq nm-m-n=n-2$. The last inequality obviously does not hold. Thus $i=1$ and $n_2m_2=nm$. Suppose that h_2 is s-sheeted. If $s>1$, then $(n_2, m_2)\neq(n, m)$ and $(n_2, m_2)\neq(m, n)$. Hence the number $nm=n_2m_2$ is a product of at least three prime numbers, i.e., $nm=n_2m_2\geqq 30$.

By the Riemann-Hurwitz formula, $(n_2-1)(m_2-1)>s[(n-1)(m-1)-2]$, but one can see that this inequality does not hold when $s>1$ and $nm\geqq 30$. Thus $s=1$ and h_2 is an isomorphism. Then $(n-1)(m-1)=(n_2-1)(m_2-1)$ since the S_U and S_V have the same genus. Thus $(n_2, m_2)=(n, m)$ or $(n_2, m_2)=(m, n)$. Suppose that $(n_2, m_2)=(n, m)$. Since h_2 maps the orbits of g_V into the orbits of g_U this mapping h_2 must be the identical mapping and $h(u, v)=(u^k, v^l)$. If $(n_2, m_2)=(m, n)$ we obtain similarly that $h(u, v)=(v^l, u^k)$. $\square$

THEOREM 6.4. *Let W, U be contractible smooth surfaces with $\bar{k}(W)=\bar{k}(U)=1$, and let $\varphi: W\to U$ be a nondegenerate morphism. Suppose that the type of U is (n, m) and the type of W is (n_1, m_1). Then*

(1) *$\tau_U\circ\varphi=f\circ\tau_W$ where $f: \boldsymbol{C}^2\to\boldsymbol{C}^2$ is given by one of the following formulas: (i) $(u, v)\to(u^k, v^l)$, or (ii) $(u, v)\to(v^l, u^k)$.*

(2) *The type (n_1, m_1) of W is either (kn, lm) if f is given by (i), or (lm, kn) if f is given by (ii) (note that since n_1 and m_1 are relatively prime and $n_1>m_1$, this property imposes some condition on the pair (k, l)).*

(3) *Moreover, for every W as above and every pairs (n, m) and (k, l) such that either $(n_1, m_1)=(nk, ml)$ or $(n_1, m_1)=(lm, kn)$ there exist a contractible surface U of type (n, m) and a morphism φ satisfying (1) and (2).*

PROOF. Let Φ_W, Φ_U, z_W, z_U have the same meaning as in preliminaries. Since φ is nondegenerate it sends the fibers of Φ_W into the fibers of Φ_U, by Lemma 5.3. Since the only fiber of Φ_W which is not isomorphic to $\boldsymbol{C}^*$, is $L_W=\Phi_W^{-1}(\infty)$, we have $\varphi^{-1}(L_U)\subset L_W$, by Lemma 5.5. Note that $\varphi(L_W)\subset L_U$, by Corollary 3.3. Since L_W is the zero fiber of z_W and L_U is the zero fiber of z_U, φ transforms the fibers of z_W into the fibers of z_U, by Nullstallenzats. The generic fiber of z_W is the curve $S_W=\{u^{n_1}-v^{m_1}=1\}$ and the generic fiber of z_U is the curve $S_U=\{u^n-v^m=1\}$. There is the action of a cyclic group $g_W\cong\boldsymbol{Z}_{n_1m_1}$ on the curve S_W given by $(u, v)\to(\varepsilon^{m_1}u, \varepsilon^{n_1}v)$ where ε is an (n_1m_1)-root of unity, and there is the action of a cyclic group $g_U\cong\boldsymbol{Z}_{nm}$ on the curve S_U given by $(u, v)\to(\delta^m u, \delta^n v)$ where δ is an (nm)-root of unity. Denote by h the morphism $S_W\to S_U$ generated by φ. Note that every orbit of g_W may be treated as the intersection of the generic fiber S_W of z_W with a fiber of Φ_W. Similarly, every orbit of g_U is the intersection of S_U with a fiber of Φ_U. Hence h transforms the orbits of g_W into the orbits of g_U and h has one of the forms prescribed, by Lemma 6.3. Consider the restriction of φ to $W-L_W$. Recall that $\varphi(W-L_W) \subset U-L_U$ and $\tau_W|_{W-L_W}: W-L_W \to (\boldsymbol{C}^2-\Gamma_{n_1,m_1})$ is an isomorphism such that the fibers of z_W are mapped into nonzero fibers of the function $u^{n_1}-v^{m_1}$, and the similar statement is true for τ_U, the fibers of z_U, and nonzero fibers of u^n-v^m. Thus φ generates the mapping $f: \boldsymbol{C}^2-\Gamma_{n_1,m_1} \to \boldsymbol{C}^2-\Gamma_{n,m}$ so that $\tau_U \circ \varphi = f \circ \tau_W$ and f maps each fiber of $u^{n_1}-v^{m_1}$ into a fiber of $z=u^n-v^m$. Suppose that $h(u, v) =(u^k, v^l)$ and consider h as a mapping of $\boldsymbol{C}^2$. Then $f=\phi \circ h$ where $\phi: \boldsymbol{C}^2-\Gamma_{n,m} \to \boldsymbol{C}^2-\Gamma_{n,m}$ is a mapping of form $\phi(u, v)=((c(u^n-v^m))^{im}u, (c(u^n-v^m))^{in}v)$ where i is integer and $c \in \boldsymbol{C}^*$. Therefore, $f(u, v)=((c(u^{n_1}-v^{m_1}))^{im}u^k, (c(u^{n_1}-v^{m_1}))^{in}v^l)$. Since f is generated by φ, one can see that f may be extended to p and $f(p)=p$. This implies that $i=0$, i.e., $f(u, v)=(u^k, v^l)$. This implies (1) and (2) in case (i).

Now we have to check the existence of such morphisms. Consider a small neighborhood B_W of p in $\boldsymbol{C}^2$. Then the restriction of $f(u, v)=(u^k, v^l)$ to this neighborhood is a biholomorphism between B_W and another neighborhood B_U of p. Let U be the surface obtained by gluing $\tau_W^{-1}(B_W)$ and $\boldsymbol{C}^2-\Gamma_{n,m}$ along the sets $\tau_W^{-1}(B_W)-L_W \approx B_W-\Gamma_{n_1,m_1}$ and $B_U-\Gamma_{n,m}$ by the mapping f. Then one can easily check that U is contractible algebraic surface with $\bar{k}(U)=1$, since it satisfies the construction of Petrie and tom Dieck, described in Lemma 2.1. The natural projection $W \to U$ produces the desired morphism. Theorem is proved in case (i).

One can obtain case (ii) in a similar way, by putting $h(u, v)=(v^l, u^k)$. □

COROLLARY 6.5. *Every nondegenerate morphism from W into W is the identical mapping.*

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