J. Math. Soc. Japan
Vol. 47, No. 3, 1995

On the structure of the moduli space of harmonic eigenmaps

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(Received Feb. 8, 1993)
(Revised Nov. 1, 1993)

1. Introduction and preliminaries.

It is well-known that a map $F: M\to S^N$ of a Riemannian manifold M into the Euclidean N-sphere $S^N\subset \boldsymbol{R}^{N+1}$ is *harmonic* [3] iff the induced vector-valued function $F: M\to \boldsymbol{R}^{N+1}$ satisfies the equation

$$\Delta^M F=\mu F, \tag{1}$$

where Δ^M is the Laplacian on M and the scalar μ is uniquely determined by F, in fact, μ is nothing but the energy density $e(F)=\operatorname{trace}\|F_*\|^2$ of F. (We work in the C^∞-category, i.e., we assume that all manifolds, maps, bundles etc. are of class C^∞.)

Applying an $(n+1)\times(N+1)$-matrix $A\in M(n+1, N+1)$ to both sides of (1) we infer that $f=A\cdot F$ defines a harmonic map of M into S^n of energy density $e(f)=\mu(=e(F))$ provided that A maps the image of F into S^n. In this case, we say that f is *derived* from F and write $f\leftarrow F$. Define a symmetric endomorphism $\langle f\rangle$ of $\boldsymbol{R}^{N+1}$ by $\langle f\rangle=A^\top A-I_{N+1}$. The condition $|f|^2=1$ is equivalent to that $\langle f\rangle$ is perpendicular to $\operatorname{proj}[F(x)]$ for all $x\in M$ with respect to the usual inner product $\langle C, C'\rangle=\operatorname{trace}(C'^\top\cdot C)$, $C, C'\in S^2(\boldsymbol{R}^{N+1})$. Clearly $\langle f\rangle$ depends only on the equivalence class of f, where two maps $f', f'': M\to S^n$ (derived from F) are said to be *equivalent* if $f''=U\cdot f'$ for some $U\in O(n+1)$. Restricting ourselves from here on to full maps (i.e., assuming that the image always spans the range) we obtain that, given a full harmonic map $F: M\to S^N$, the equivalence classes of full harmonic maps $f: M\to S^n$ that are derived from F can be parametrized (via $f\to\langle f\rangle$) by the convex body

$$\mathcal{L}_F=\{C\in\mathcal{E}_F \mid C+I_{N+1}\geqq 0\} \tag{2}$$

('$\geqq$' stands for positive semidefinite), where

$$\mathcal{E}_F=(\operatorname{span}\{\operatorname{proj}[F(x)] \mid x\in M\})^\perp\subset S^2(\boldsymbol{R}^{N+1}). \tag{3}$$

(Our main reference for the facts recalled here and below is [**11**].) It follows from (2) that $\partial\mathcal{L}_F=\mathcal{L}_F\setminus\operatorname{int}\mathcal{L}_F$ parametrizes those full harmonic maps $f: M\to S^n$ derived from F for which $n<N$.

The relation $\leftharpoonup$ is transitive. Hence, if $f\leftharpoonup F$, the moduli space $\mathcal{L}_f$ can be thought of as being contained in $\mathcal{L}_F$. More precisely, setting $f=A\cdot F$ as above, the injective affine map

$$\iota: S^2(\boldsymbol{R}^{n+1}) \longrightarrow S^2(\boldsymbol{R}^{N+1})$$

defined by

$$\iota(C)=A^{\top}(C+I_{n+1})A-I_{N+1}\,, \qquad C\in S^2(\boldsymbol{R}^{n+1})$$

satisfies

$$\iota(\mathcal{L}_f)=\iota(\mathcal{E}_f)\cap\mathcal{L}_F\,. \tag{4}$$

We call $I_f=\iota(\operatorname{int}\mathcal{L}_f)\subset\mathcal{L}_F$ the (open) *cell* associated to $f(\leftharpoonup F)$. By (4), ι establishes an affine isomorphism between $\mathcal{L}_f$ and the closure of I_f in $\mathcal{L}_F$. Clearly, I_f is convex and open in $\iota(\mathcal{E}_f)$. Moreover, $\langle f\rangle\in I_f$ and by passing to the boundary of I_f the range dimension decreases. $\mathcal{I}_F=\{I_f\mid f\leftharpoonup F\}$ is a decomposition of $\mathcal{L}_F$ into disjoint convex sets. We call $\mathcal{I}_F$ the *natural stratification* of $\mathcal{L}_F$.

In the most important cases $F: M\to S^N$ posesses symmetries, i.e., it is equivariant with respect to a homomorphism $\rho_F: G\to O(N+1)$, where G is a closed subgroup of the group of isometries of M. This means that, for $a\in G$, we have

$$F\cdot a=\rho_F(a)\cdot F\,. \tag{5}$$

The homomorphism ρ_F, which is uniquely determined by fullness of F, induces an orthogonal G-module structure on $\boldsymbol{R}^{N+1}$ and hence an orthogonal G-module structure on $S^2(\boldsymbol{R}^{N+1})$. Using (5) in (2)-(3), we obtain that $\mathcal{E}_F\subset S^2(\boldsymbol{R}^{N+1})$ is a G-submodule. In fact, the moduli space $\mathcal{L}_F\subset\mathcal{E}_F$ is G-invariant. More precisely, for $f\leftharpoonup F$ and $a\in G$, we have

$$a\cdot\langle f\rangle=\langle f\cdot a^{-1}\rangle\,. \tag{6}$$

The action of G on $\mathcal{L}_F$ preserves the natural stratification $\mathcal{I}_F$, i.e., as expected from (6), we have

$$a\cdot I_f=I_{f\cdot a^{-1}}\,.$$

In particular, $G\cdot I_f$ is a smooth submanifold of $\mathcal{E}_F$, so that $G\cdot\mathcal{I}_F=\{G\cdot I_f\mid f\leftharpoonup F\}$ is a G-invariant smooth stratification of $\mathcal{L}_F$.

REMARK. A somewhat more involved argument shows [**11**] that if $\langle f\rangle$ is the center of mass of I_f then $G\cdot\langle f\rangle$ and I_f intersect (weakly) transversally at $\langle f\rangle$ so that

$$\dim(G\cdot I_f)=\dim I_f+\dim G-\dim G_f\,,$$

where $G_f \subset G$, the isotropy subgroup of G at $\langle f \rangle$, is nothing but the symmetry group of f, i.e., we have

$$G_f = \{a \in G \mid \text{there exists } A \in O(n+1) \text{ such that } f \cdot a = A \cdot f\}.$$

Our specific interest motivating the general framework lies in the standard moduli space $\mathcal{L}_\lambda \subset \mathcal{E}_\lambda$ associated to the standard λ-eigenmap $f_\lambda : M \to S^{n(\lambda)}$, where $M = G/K$ is a compact Riemannian homogeneous space and f_λ is defined by having its components comprise an orthonormal basis $\{f_\lambda^i\}_{i=0}^{n(\lambda)}$ in the eigenspace V_λ of Δ^M corresponding to $\lambda \in \mathrm{Spec}(M)$. Here, V_λ is endowed with the normalized L^2-scalar product

$$\langle \phi, \phi' \rangle = \frac{n(\lambda)+1}{\mathrm{vol}(M)} \int_M \phi \phi' \cdot \nu_M, \qquad \phi, \phi' \in V_\lambda,$$

where ν_M stands for the Riemannian volume element. By the very definition of f_λ, the standard moduli space $\mathcal{L}_\lambda$ parametrizes the (equivalence classes of) full *λ-eigenmaps* $f : M \to S^n$, $n \leqq n(\lambda)$, i.e., whose components belong to V_λ. (Equivalently, $f : M \to S^n$ is said to be a λ-eigenmap if the induced vector-valued function $f : M \to \boldsymbol{R}^{n+1}$ satisfies

$$\Delta^M f = \lambda f$$

or, which is the same, $f : M \to S^n$ is a harmonic map of energy density $e(f) = \lambda$.) Clearly, $f_\lambda : M \to S^{n(\lambda)}$ is equivariant with respect to the homomorphism $\rho_\lambda : G \to O(n(\lambda)+1)$ that defines the orthogonal G-module structure on $V_\lambda (\cong \boldsymbol{R}^{n(\lambda)+1})$. For $M = G/K$ isotropy irreducible, f_λ is actually a homothetic (standard) minimal immersion.

Even more specifically, we will be interested in the case when $M = S^m = SO(m+1)/SO(m)$, $m \geqq 2$, with $\lambda = \lambda_k = k(k+m-1) \in \mathrm{Spec}(S^m)$, $k \geqq 2$. Then $V_{\lambda_k} = \mathcal{H}_{S^m}^k$ is the linear space of spherical harmonics of order k on S^m and $f_{\lambda_k} : S^m \to S^{n(\lambda_k)}$ is a standard minimal immersion. (For $m=2$, $f_{\lambda_k} : S^2 \to S^{2k}$ is nothing but the classical Veronese map.) The decomposition of $S^2(\mathcal{H}_{S^m}^k)$ into irreducible $SO(m+1)$-modules has been given by Wallach [**15**] from which $\mathcal{E}_{\lambda_k}$ can be determined [**11**]. In particular, we have

$$\dim \mathcal{L}_{\lambda_k} = \frac{1}{2}(n(\lambda_k)+1)(n(\lambda_k)+2) - \sum_{j=0}^{k} (n(\lambda_{2j})+1), \tag{7}$$

where

$$n(\lambda_k)+1 = \dim \mathcal{H}_{S^m}^k = (m+2k-1) \frac{(m+k-2)!}{k!(m-1)!}. \tag{8}$$

In [**11**] we showed that, for each $m \geqq 3$, the cardinality of the stratification $SO(m+1) \cdot \mathcal{I}_{\lambda_k}$ is $\aleph_1$ provided that

$$(n(\lambda_k)+1)(n(\lambda_k)+2) > 2\sum_{j=0}^{k}(n(\lambda_{2j})+1)+(n(\lambda_k)+1)(m(m+1)+2). \tag{9}$$

Note that, by (8), for fixed $m \geqq 3$, there exists $k(m)$ such that (9) is satisfied for all $k \geqq k(m)$.

In view of this fact it is natural to ask whether there exists a (cruder) stratification of $\mathcal{L}_{\lambda_k}$ with finitely many strata. Clearly, this is closely related to the study of smoothness of $\partial\mathcal{L}_{\lambda_k}$.

REMARK. In the work of DoCarmo and Wallach [2] the moduli space parametrizing the (equivalence classes of) full homothetic minimal immersions $f: S^m \to S^n$ (with homothety constant λ_k/m) is the intersection of $\mathcal{L}_{\lambda_k}$ with an appropriate linear subspace of $\mathcal{E}_{\lambda_k}$. The question of determining the 'polyhedral structure' of the boundary of the moduli space has been raised there.

The main objective of this paper is to construct and study a finite stratification on $\mathcal{L}_{\lambda_k}$ with almost everywhere smooth strata. On doing this, we will discover a fascinating relation between the tangent space at the smooth points of the strata and the linear space of divergencefree Jacobi fields along the λ_k-eigenmaps the points represent. As a byproduct, we obtain a sharp lower bound for the nullity of such maps. For λ_k-eigenmaps corresponding to smooth points we also obtain a positive solution to the fundamental question whether any divergencefree Jacobi field along a λ_k-eigenmap arises from a variation through λ_k-eigenmaps.

Since the construction of the stratification is completely general, we, in Section 2, prefer the general framework introduced above. In Section 3 we show that the strata of the stratification are real (semi-)algebraic and obtain a sufficient condition for a point of $\partial\mathcal{L}_F$ to be smooth. This is then specialized and made very explicit in Section 4 to the standard moduli space $\mathcal{L}_{\lambda_k}$. Finally, we devote Section 5 to applications to bi-eigenmaps and orthogonal multiplications.

REMARK. One of the main results of [11] is the computation of the dimension of the moduli space $\mathcal{L}_{\lambda_p}$ corresponding to the standard minimal immersion $f_{\lambda_p}: \boldsymbol{C}P^m \to S^{n(\lambda_p)}$, $\lambda_p \in \mathrm{Spec}\,(\boldsymbol{C}P^m)$. Though technically more involved, similar results to those in Section 3 and 4 can be derived for this case in an analogous way. Same is true for the corresponding moduli spaces parametrizing minimal immersions. We feel however that the spherical case reveals all the subtlety involved and thereby confine ourselves only to that.

ACKNOWLEDGEMENT. The author wishes to thank the referee for pointing out various errors and improvements of the original manuscript.

2. The rank stratification of the moduli space.

Let $F: M \to S^N$ be a full harmonic map and consider the moduli space $\mathcal{L}_F$ associated to F via (2)-(3). For $1 \leqq n \leqq N$, we define

$$\mathcal{L}^{n+1} = \{X \in S^2(\boldsymbol{R}^{N+1}) \mid X + I_{N+1} \geqq 0,\ \operatorname{rank}(X + I_{N+1}) = n+1\} \tag{10}$$

and

$$\mathcal{L}_F^{n+1} = \mathcal{L}^{n+1} \cap \mathcal{E}_F \subset \mathcal{L}_F\,. \tag{11}$$

By the construction of $\mathcal{L}_F$, the subset $\mathcal{L}_F^{n+1} \subset \mathcal{L}_F$ corresponds to those full harmonic maps $f \leftharpoonup F$ whose range is S^n.

Given a full harmonic map $f: M \to S^n$ we denote by $K(f)$ the linear space of divergencefree Jacobi fields along f [**13**]. Using the natural shift $\check{}: T(\boldsymbol{R}^{n+1}) \to \boldsymbol{R}^{n+1}$, it follows that $v \in K(f)$ iff the vector-valued function $\check{v}: M \to \boldsymbol{R}^{n+1}$ satisfies

$$\Delta^M \check{v} = \mu \check{v}\,. \tag{12}$$

Actually, there is a linear isomorphism between $K(f)$ and the linear space of vector-valued functions $\check{v}: M \to \boldsymbol{R}^{n+1}$ satisfying (12) and $\langle f, \check{v} \rangle = 0$. Whenever convenient, we will identify $K(f)$ with this linear space. Clearly, $so(n+1) \cdot f \subset K(f)$ is a linear subspace.

We now assume that $f \leftharpoonup F$. A vector field v along f is said to be *derived* from F, written as $v \leftharpoonup F$, if there exists $B \in M(n+1, N+1)$ such that $\check{v} = B \cdot F$. We set

$$K_F(f) = \{v \in K(f) \mid v \leftharpoonup F\}.$$

Notice that, since $f \leftharpoonup F$, we have $so(n+1) \cdot f \subset K_F(f)$ as a linear subspace.

REMARK. Given a smooth variation $f_t: M \to S^n$, $|t| < \delta$, of $f = f_0$ through harmonic maps, $v = \partial f_f / \partial t|_{t=0}$ is a Jacobi field along f. Moreover, v is divergencefree if $e(f_t) = e(f)$, $|t| < \delta$. This is because we have

$$\left.\frac{\partial e(f_t)}{\partial t}\right|_{t=0} = \operatorname{trace} \left.\frac{\partial}{\partial t} \|(f_t)_*\|^2\right|_{t=0} = 2 \operatorname{trace} \langle f_*, \nabla v \rangle = 2 \operatorname{div}_f v\,.$$

Since in the construction of $\mathcal{L}_F$ we factored out the action of the orthogonal group on the range, we may intuitively think of $K_F(f)/so(n+1) \cdot f$ as being the 'tangent space' of $\mathcal{L}_F^{n+1}$ at $\langle f \rangle$. The key question is of course whether $\mathcal{L}_F^{n+1}$ is smooth near $\langle f \rangle$.

THEOREM 1. *$\mathcal{L}^{n+1} \subset S^2(\boldsymbol{R}^{N+1})$ is a real analytic submanifold of dimension $(n+1)(N+1-n/2)$. Moreover, if $f: M \to S^n$ is a full harmonic map with $f \leftharpoonup F$ then*

$$T_{\langle f \rangle}(\mathcal{L}^{n+1}) \cap \mathcal{E}_F \cong K_F(f)/so(n+1) \cdot f\,. \tag{13}$$

In particular, we have

$$\dim K_F(f) \geqq (n+1)(N+1) - \operatorname{codim} \mathcal{E}_F . \tag{14}$$

PROOF. Let $X_0 \in \mathcal{L}^{n+1}$ and choose $U_0 \in O(N+1)$ such that $U_0 X_0 U_0^\top$ is diagonal. We write

$$X_0 + I_{N+1} = U_0^\top \begin{bmatrix} D_0 & 0 \\ 0 & 0 \end{bmatrix} U_0 ,$$

where $D_0 \in M(n+1, n+1)$ is a diagonal matrix with elements $d_0 \geqq \cdots \geqq d_n > 0$ along the main diagonal. Choose $\varepsilon > 0$ such that, for $D \in S^2(\boldsymbol{R}^{n+1})$, the condition $\|D - D_0\| < \varepsilon$ implies that D is positive definite, in particular, D^{-1} exists and is also positive definite. Finally, Let

$$\mathcal{N} = \{X \in S^2(\boldsymbol{R}^{N+1}) \mid X + I_{N+1} = U_0^\top \begin{bmatrix} D & E \\ E^\top & \tilde{D} \end{bmatrix} U_0,\ \|D - D_0\| < \varepsilon\} . \tag{15}$$

Clearly, $\mathcal{N}$ is an open neighborhood of X_0 in $S^2(\boldsymbol{R}^{N+1})$. Using the identity

$$\begin{bmatrix} D & E \\ E^\top & \tilde{D} \end{bmatrix} = \begin{bmatrix} D & 0 \\ E^\top & \tilde{D} - E^\top D^{-1} E \end{bmatrix} \begin{bmatrix} I_{n+1} & D^{-1}E \\ 0 & I_{N-n} \end{bmatrix}$$

we obtain that, for $X \in \mathcal{N}$, with $X + I_{N+1}$ decomposed as in (15), $\operatorname{rank}(X + I_{N+1}) = n+1$ iff $\tilde{D} = E^\top D^{-1} E$. Since, in this case, $X + I_{N+1} \geqq 0$ is automatically satisfied, we have $X \in \mathcal{L}^{n+1}$. In other words, we have

$$\mathcal{N} \cap \mathcal{L}^{n+1} = \{X \in S^2(\boldsymbol{R}^{N+1}) \mid X + I_{N+1} = U_0^\top \begin{bmatrix} D & E \\ E^\top & E^\top D^{-1} E \end{bmatrix} U_0,\ \|D - D_0\| < \varepsilon\} . \tag{16}$$

We now define

$$\Phi : \mathcal{N} \cap \mathcal{L}^{n+1} \longrightarrow \boldsymbol{R}^{(n+1)(N+1-n/2)}$$

by

$$\Phi(X) = [D \mid E] ,$$

where $X + I_{N+1}$ has the decomposition as in (15). Clearly, Φ is an analytic diffeomorphism and defines a local chart of $\mathcal{L}^{n+1}$ around X_0.

To prove the second statement, let $f = A_0 \cdot F$, where $A_0 \in M(n+1, N+1)$ (necessarily of maximal rank). By the singular values decomposition of rectangular matrices [**6**], we have

$$A_0 = V_0^\top [B_0 \mid 0] U_0 \tag{17}$$

for some $V_0 \in O(n+1)$ and $U_0 \in O(N+1)$, where $B_0 \in M(n+1, n+1)$ is a diagonal matrix with elements $b_0 \geqq \cdots \geqq b_n > 0$ along the main diagonal. By (17), the matrix $\langle f \rangle$ that represents $f : M \to S^n$ in $\mathcal{L}_F$ decomposes as

$$\langle f\rangle = A_0^{\mathrm{T}} A_0 - I_{N+1} = U_0^{\mathrm{T}} \begin{bmatrix} B_0^2 & 0 \\ 0 & 0 \end{bmatrix} U_0 - I_{N+1} \tag{18}$$

so that, setting $X_0 = \langle f\rangle$ and $D_0 = B_0^2$, the construction of the local chart around $\langle f\rangle$ applies.

Now let $Y \in T_{\langle f\rangle}(\mathcal{L}^{n+1})$ and choose a smooth curve $X: (-\delta, \delta) \to \mathcal{H} \cap \mathcal{L}^{n+1}$ such that $X(0) = \langle f\rangle$ and $\dot{X}(0) = Y$. By (16), we can write

$$X(t) + I_{N+1} = U_0^{\mathrm{T}} \begin{bmatrix} D(t) & E(t) \\ E(t)^{\mathrm{T}} & E(t)^{\mathrm{T}} D(t)^{-1} E(t) \end{bmatrix} U_0 \cdot \quad |t| < \delta,$$

with $D(0) = D_0$ and $E(0) = 0$. Differentiating at $t = 0$, we obtain the decomposition of the tangent vector Y as

$$Y = U_0^{\mathrm{T}} \begin{bmatrix} D_0' & E_0' \\ E_0'^{\mathrm{T}} & 0 \end{bmatrix} U_0, \tag{19}$$

where $D_0' = \dot{D}(0) \in S^2(\boldsymbol{R}^{n+1})$ and $E_0' = \dot{E}(0) \in M(n+1, N-n)$. Now, for $Y \in T_{\langle f\rangle}(\mathcal{L}^{n+1})$ of the form (19), we define the vector-valued function

$$\Psi(Y): M \longrightarrow \boldsymbol{R}^{n+1}$$

by

$$\Psi(Y) = V_0^{\mathrm{T}} B_0^{-1} [D_0' | 2E_0'] U_0 \cdot F. \tag{20}$$

We claim that $Y \in \mathcal{E}_F$ iff $\langle \Psi(Y), f\rangle = 0$. Indeed, using (19)-(20), we have

$$\begin{aligned} \langle \Psi(Y), f\rangle &= \langle B_0^{-1}[D_0' | 2E_0'] U_0 F, [B_0 | 0] U_0 F\rangle \\ &= \langle [D_0' | 2E_0'] U_0 F, [I_{n+1} | 0] U_0 F\rangle \\ &= \langle \begin{bmatrix} D_0' & 2E_0' \\ 0 & 0 \end{bmatrix} U_0 F, U_0 F\rangle \\ &= \langle \begin{bmatrix} D_0' & E_0' \\ E_0'^{\mathrm{T}} & 0 \end{bmatrix} U_0 F, U_0 F\rangle \\ &= \langle Y \cdot F, F\rangle \end{aligned}$$

and the claim follows. We obtain that, for $Y \in T_{\langle f\rangle}(\mathcal{L}^{n+1}) \cap \mathcal{E}_F$, (20) defines a vector field along f that is automatically in $K_F(f)$. Equivalently, the restriction $\Psi | \mathcal{E}_F$ gives rise to a linear map

$$\Psi | \mathcal{E}_F : T_{\langle f\rangle}(\mathcal{L}^{n+1}) \cap \mathcal{E}_F \longrightarrow K_F(f)$$

which, by (19)-(20), is injective.

We finally claim that the image of $\Psi | \mathcal{E}_F$ is transversal to $so(n+1) \cdot f$. In fact, using (17), we compute

$$\begin{aligned} so(n+1)\cdot f &= so(n+1)A_0F \\ &= so(n+1)V_0^{\top}[B_0|0]U_0F \\ &= V_0^{\top}so(n+1)[B_0|0]U_0F \\ &= V_0^{\top}B_0^{-1}[so(n+1)|0]U_0F . \end{aligned}$$

On the other hand, by (19)-(20), the image of Ψ (on $T_{\langle f\rangle}(\mathcal{L}^{n+1})$) is

$$V_0^{\top}B_0^{-1}[S^2(\boldsymbol{R}^{n+1})|M(n+1, N-n)]$$

which is transversal to

$$V_0^{\top}B_0^{\top}[so(n+1)|0]U_0$$

in $M(n+1, N+1)$. Since $K_F(f)=\mathcal{E}_f\oplus so(N+1)\cdot f$ ([**11**, p. 16]) we get (13).

Finally, the estimate in (14) is an easy consequence of (13). In fact, we have

$$\begin{aligned} \dim (K_F(f)/so(n+1)\cdot f) &= \dim (T_{\langle f\rangle}(\mathcal{L}^{n+1})\cap\mathcal{E}_F) \\ &\geqq \dim \mathcal{L}^{n+1}+\dim \mathcal{E}_F-\dim (S^2(\boldsymbol{R}^{N+1})) \\ &= (n+1)(N+1-n/2)-\operatorname{codim} \mathcal{E}_F . \quad \square \end{aligned}$$

REMARK. Let $f: M\to S^n$ be a full λ-eigenmap, where $M=G/K$ is a compact Riemannian homogeneous space and $\lambda\in\operatorname{Spec}(M)$. Then

$$so(n+1)\cdot f \neq K(f)$$

unless f is equivariant with respect to a homomorphism $\rho_f: G\to O(n+1)$ with *irreducible* G-module structure on $\boldsymbol{R}^{n+1}$. This can be seen as follows. We think of the elements of the Lie algebra $\mathcal{G}$ of G as being infinitesimal isometries of M. Then we have $f_*(\mathcal{G})\subset K(f)$. (In fact, for $X\in\mathcal{G}$, the vector field $f_*(X)$ along f is clearly a Jacobi field. It is divergencefree iff $Xe(f)=0$ which is the case.) Now assuming $so(n+1)\cdot f=K(f)$, we obtain $f_*(\mathcal{G})\subset so(n+1)\cdot f$ from which equivariance of f follows. For the irreducibility, if

$$\boldsymbol{R}^{n+1} = V_1\oplus V_2$$

is a nontrivial orthogonal G-invariant decomposition then $f=(f^1, f^2)$ with $f^j: M\to V_j$, $j=1, 2$. By equivariance, $\|f^1\|=\cos t$ and $\|f^2\|=\sin t$ for some $0<t<\pi/2$. The vector-valued function $\check{v}=(-\tan t\, f^1, f^2): M\to\boldsymbol{R}^{n+1}$ gives rise to an element of $K(f)$ which obviously does not belong to $so(n+1)\cdot f$. Much stronger results can be proved when $M=G/K$ is naturally reductive. Then $so(n+1)\cdot f=K(f)$ implies that

- $\dim \operatorname{Fix}(K, \boldsymbol{R}^{n+1})=1$, 2 or 4 and $\boldsymbol{R}^{n+1}$ is a real, complex or quaternionic G-module, accordingly.

- If $M=M_1\times M_2$ then f factors through one of the projections $\pi_j: M\to M_j$, $j=1, 2$.
- If $M=G/K$ is isotropy irreducible then $f: M\to S^n$ is a homothetic minimal immersion (with trivial moduli space). In particular, in the spherical case, f is standard and $m=2$.

(For further results cf. [**13**][**14**].)

REMARK. For $n=N$, equality holds in (14). In fact, for $f=F$, we have

$$K_F(F)=so(N+1)\cdot F\oplus\mathcal{E}_F\subset S^2(\boldsymbol{R}^{N+1}).$$

Moreover, if $f: M\to S^N$ is a full harmonic map with $f\leftharpoondown F$ then $f=A\cdot F$ with A invertible and $\check{v}\mapsto A^T\cdot\check{v}$ gives rise to an isomorphism between $K_F(f)$ and $K_F(F)$. We will also see in Section 4 that the lower bound in (14) is the best possible in a number of nontrivial examples.

We now return to the general situation and consider $\mathcal{L}_F^{n+1}$ in (10)-(11). Since $\mathcal{L}^{n+1}$ is real analytic Weierstrass preparation [7] applies yielding that any point $X_0\in\mathcal{L}_F^{n+1}$ has a neighborhood $\mathcal{N}_0$ in $\mathcal{E}_F$ such that $\mathcal{N}_0\cap\mathcal{L}_F^{n+1}$ is a finite union of real analytic (in fact, affine algebraic) submanifolds of $\mathcal{E}_F$. If $\mathcal{N}_0$ can be chosen such that $\mathcal{N}_0\cap\mathcal{L}_F^{n+1}$ is a single submanifold then X_0 is said to be a *smooth point*. Otherwise, X_0 is a *singular point*. The set $\mathcal{O}_F^{n+1}$ of smooth points is clearly open and dense in $\mathcal{L}_F^{n+1}$, in fact, the singular set $\mathcal{S}_F^{n+1}=\mathcal{L}_F^{n+1}\backslash\mathcal{O}_F^{n+1}$ is at least of codimension 1. We set $\mathcal{O}_F=\bigcup_{n=1}^N\mathcal{O}_F^{n+1}$ and $\mathcal{S}_F=\bigcup_{n=1}^N\mathcal{S}_F^{n+1}$. Clearly, $\mathcal{O}_F$ contains the interior of $\mathcal{L}_F$ and $\mathcal{S}_F$ is at least of codimension 1 in $\partial\mathcal{L}_F$.

We call

$$\mathcal{T}_F=\{\mathcal{L}_F^{n+1}\,|\,1\leqq n\leqq N\}$$

the *rank stratification* of $\mathcal{L}_F$. By the above, each stratum of $\mathcal{T}_F$ is real analytic almost everywhere. Clearly, $\mathcal{T}_F$ is also G-invariant. By the rank condition, we have

$$\partial\mathcal{L}_F^{n+1}=\mathrm{cl}\,\mathcal{L}_F^{n+1}\backslash\mathcal{L}_F^{n+1}\subset\bigcup_{n'<n}\mathcal{L}_F^{n'+1}.$$

Moreover, for a full harmonic map $f: M\to S^n$, $f\leftharpoondown F$, we have $I_f\subset\mathcal{L}_F^{n+1}$ and $\partial I_f\cap\mathcal{L}_F^{n+1}=\emptyset$. We express this by saying that $G\cdot\mathcal{I}_F$ is a substratification of $\mathcal{T}_F$.

REMARK. In the spherical case, for $m\geqq 3$, we have (with obvious notation)

$$SO(m+1)\cdot\mathcal{I}_{\lambda_k}\neq\mathcal{T}_{\delta_k}$$

provided that (9) holds. This is because the L.H.S. has cardinality $\aleph_1$ as mentioned in Section 1.

REMARK. Let $f: M\to S^n$ be a full harmonic map with $f\leftharpoondown F$. If

$$\dim K_F(f)\leqq(n+1)(N+1)-\mathrm{codim}\,\mathcal{E}_F$$

then $\langle f\rangle$ is a smooth point (and equality holds). (This is clear since $\mathcal{L}^{n+1}$ and $\mathcal{E}_F$ intersect transversally at $\langle f\rangle$.)

It is a general fact that on the boundary of a compact convex body (in a finite dimensional linear space) the set of smooth points forms an open and dense subset [1]. In our case, let $N_0<N$ be the largest range dimension n on $\partial\mathcal{L}_F$ for which $\mathcal{L}_F^{n+1}\neq\emptyset$. Clearly, $\mathcal{L}_F^{N_0+1}$ and hence $\mathcal{O}_F^{N_0+1}$ are open in $\partial\mathcal{L}_F$. If $N_0=N-1$, we can say more:

PROPOSITION. *Assume that there exists a full harmonic map $f: M\to S^{N-1}$ with $f\leftharpoonup F$. Then $\mathcal{L}_F^N\subset\partial\mathcal{L}_F$ is real analytic everywhere.*

PROOF. The condition guarantees that $\mathcal{L}_F^N$ is nonempty. For $X_0\in\mathcal{L}_F^N$, we have

$$T_{X_0}(\mathcal{L}^N)\oplus\boldsymbol{R}\cdot X_0=S^2(\boldsymbol{R}^{N+1}).$$

This follows by comparing (18) and (19). Hence $\mathcal{L}^N$ and $\mathcal{E}_F$ intersect transversally. $\square$

REMARK. As we will see in Section 4, the assumption of the Proposition is satisfied in the spherical case for $m\geqq 5$.

Continuing as above, let $N_0>\cdots>N_l$ be those range dimensions n for which the interior $\operatorname{int}\mathcal{L}_F^{n+1}$ in $\partial\mathcal{L}_F$ is nonempty. (An example in Section 4 shows that $\mathcal{L}_F^{n+1}$ is not necessarily dense in $\partial\mathcal{L}_F$ so that $\mathcal{L}_F^{N_1+1}$ may exist.) As noted above, for $j=0,\cdots,l$, $\mathcal{O}_F^{N_j+1}$ is open and dense in $\mathcal{L}_F^{N_j+1}$ and so is $\operatorname{int}\mathcal{O}_F^{N_j+1}$ in $\operatorname{int}\mathcal{L}_F^{N_j+1}$. Hence

$$\bigcup_{j=0}^{l}\operatorname{int}\mathcal{O}_F^{N_j+1}\subset\partial\mathcal{L}_F$$

is an open and (by the Baire Category theorem) dense subset of smooth points in $\partial\mathcal{L}_F$.

According to the remark before Theorem 1, for a smooth variation $f_t: M\to S^n$, $|t|<\delta$, consisting of full harmonic maps derived from F, the vector field $v=\partial f_t/\partial t|_{t=0}$ along f_0 belongs to $K_F(f_0)$. For the converse, we have:

THEOREM 2. *Let $f: M\to S^n$ be a full harmonic map with $f\leftharpoonup F$ and assume that $\langle f\rangle$ is a smooth point in $\mathcal{L}_F$. Then, for any $v\in K_F(f)$, there exists a smooth variation $f_t: M\to S^n$, $|t|<\delta$, of full harmonic maps derived from F such that $f_0=f$ and $\partial f_t/\partial t|_{t=0}=v$.*

PROOF. We retain the notation in the proof of Theorem 1. We work modulo $so(n+1)\cdot f$ and assume that $v\in K_F(f)$ is actually contained in the image of $\Psi|\mathcal{E}_F$, i.e., $\Psi(Y)=\check{v}$ for some $Y\in T_{\langle f\rangle}(\mathcal{L}_F^{n+1})$. Since $\mathcal{L}_F^{n+1}$ is smooth near $\langle f\rangle$, we can choose a smooth curve $X:(-\delta,\delta)\to\mathcal{L}_F^{n+1}$ with $X(0)=\langle f\rangle$ and

$\dot{X}(0)=Y$.

We first claim that $(X(t)+I_{N+1})^{1/2}:(-\delta,\delta)\to S^2(\boldsymbol{R}^{N+1})$ is smooth. Indeed, for $|t|<\delta$, the rank of $X(t)+I_{N+1}$ is constant $(=n+1)$ and hence $K(t)=\ker(X(t)+I_{N+1})$ is a smooth curve in the Grassmannian $Gr_{N-n}(\boldsymbol{R}^{N+1})$. Now, $X(t)+I_{N+1}$ restricted to the orthogonal complement $L(t)=K(t)^{\perp}$, which is a smooth curve in $Gr_{n+1}(\boldsymbol{R}^{N+1})$, is positive definite so that $(X(t)+I_{N+1})^{1/2}>0$ is also smooth on L. Extending to zero on $K(t)$ the claim follows.

Let $U:(-\delta,\delta)\to O(N+1)$ be a smooth curve such that, for $|t|<\delta$, we have $U(t)\cdot L(t)=\boldsymbol{R}^{n+1}\subset\boldsymbol{R}^{N+1}$ with $U(0)=U_0$, where U_0 is given in (18). Using (18), we define the full harmonic map $f_t: M\to S^n$ by

$$f_t=V_0^{\top}[I_{n+1}|0]U(t)(X(2t)+I_{N+1})^{1/2}F, \qquad |t|<\delta. \tag{21}$$

Since $f_t \leftarrow F$, it remains to show that $\partial f_t/\partial t|_{t=0}=\check{v}$ $(\mathrm{mod}\,(so(n+1)\cdot f))$. Differentiating (21) at $t=0$, we obtain

$$\begin{aligned}\left.\frac{\partial f_t}{\partial t}\right|_{t=0} &= V_0^{\top}[I_{n+1}|0]\dot{U}(0)(\langle f\rangle+I_{N+1})^{1/2}F\\ &\quad+V_0^{\top}[I_{n+1}|0]U_0\left.\frac{d}{dt}(X(2t)+I_{N+1})^{1/2}\right|_{t=0}F.\end{aligned}$$

The first term on the R.H.S. rewrites as

$$V_0^{\top}[I_{n+1}|0]U_0'U_0^{\top}\begin{bmatrix}B_0 & 0\\ 0 & 0\end{bmatrix}U_0F=V_0^{\top}[I_{n+1}|0]U_0'U_0^{\top}\begin{bmatrix}I_{n+1}\\ 0\end{bmatrix}V_0f,$$

where $U_0'=\dot{U}(0)$. This term belongs to $so(n+1)\cdot f$, since $U_0'U_0^{\top}$ is skew symmetric (which can be seen by differentiating $U(t)U(t)^{\top}=I_{N+1}$, $|t|<\delta$).

For the second term on the R.H.S. we first write

$$\left.\frac{d}{dt}(X(2t)+I_{N+1})^{1/2}\right|_{t=0}=U_0^{\top}\begin{bmatrix}P & Q\\ Q^{\top} & \tilde{P}\end{bmatrix}U_0.$$

By (18) and (19), we have

$$\begin{aligned}2Y &= 2U_0^{\top}\begin{bmatrix}D_0' & E_0'\\ E_0'^{\top} & 0\end{bmatrix}U_0\\ &= U_0^{\top}\begin{bmatrix}P & Q\\ Q^{\top} & \tilde{P}\end{bmatrix}\begin{bmatrix}B_0 & 0\\ 0 & 0\end{bmatrix}U_0\\ &\quad+U_0^{\top}\begin{bmatrix}B_0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}P & Q\\ Q^{\top} & \tilde{P}\end{bmatrix}U_0\end{aligned}$$

so that

$$D_0'=(1/2)(PB_0+B_0P)$$

and

$$E_0'=(1/2)B_0Q$$

follow. Hence the second term rewrites as

$$
\begin{aligned}
V_0^{\top}[I_{n+1}|0]\begin{bmatrix} P & Q \\ Q^{\top} & \tilde{P} \end{bmatrix} U_0 F &= V_0^{\top} B_0^{-1}[B_0 P | B_0 Q] U_0 F \\
&= V_0^{\top} B_0^{-1}[(1/2)(B_0 P + P B_0) | B_0 Q] U_0 F + V_0^{\top} B_0^{-1}[(1/2)(B_0 P - P B_0) | 0] U_0 F \\
&= V_0^{\top} B_0^{-1}[D_0' | 2E_0'] U_0 F + V_0^{\top} B_0^{-1}[(1/2)(B_0 P - P B_0) | 0] U_0 F .
\end{aligned}
$$

Here, by (20), the first term on the R.H.S. is $\check{v}$ and the second belongs to $so(n+1)\cdot f$. $\square$

REMARK. R.T. Smith constructed a 1-parameter family of nonharmonic diffeomorphisms f_t, $t \in \boldsymbol{R}$, of $\boldsymbol{R}^3$ such that $\partial f_t / \partial t|_{t=0}$ is a Jacobi field along f_0 [**10**]. Theorem 2 shows that in our case the positive solution to this 'inverse problem' depends on the nonexistence of nonsmooth points on the boundary of the moduli space.

3. A criterion for smoothness.

To get more precise information about the structure of the stratum $\mathcal{L}_F^{n+1}$ near $\langle f \rangle$ we now return to the general setup given at the beginning of the proof of Theorem 1. To simplify the discussion of the forthcoming examples we, however, require only that

$$X_0 + I_{N+1} = U_0^{\top} \begin{bmatrix} D_0 & E_0^{\top} \\ E_0 & \tilde{D}_0 \end{bmatrix} U_0$$

holds for some *permutation matrix* $U_0 \in O(N+1)$ with $D_0 \in S^2(\boldsymbol{R}^{n+1})$ positive definite. Choose $\varepsilon > 0$ such that, for $D \in S^2(\boldsymbol{R}^{n+1})$, the condition $\|D - D_0\| < \varepsilon$ implies that D is positive definite. Finally, we define the open neighborhood $\mathfrak{N}$ of X_0 in $S^2(\boldsymbol{R}^{N+1})$ by (15). For the next lemma we also introduce the following notation: Given any matrix $Z \in M(N+1, N+1)$, the matrix obtained from Z by *deleting* the $i_1, \cdots, i_p$ rows and $j_1, \cdots, j_q$ columns of Z will be denoted by

$$Z_{j_1, \cdots, j_q}^{i_1, \cdots, i_p} \in M(N-p+1, N-q+1). \tag{22}$$

If $p = q$ we denote its (minor) determinant by

$$|Z_{j_1, \cdots, j_q}^{i_1, \cdots, i_p}|.$$

(Note that (22) is usually called the complementary minor. Nevertheless, for our purposes it is more preferable to keep track of the entries that are deleted from Z.)

LEMMA. *Let* $X \in \mathfrak{N}$. *Then* $\operatorname{rank}(X + I_{N+1}) = n+1$ *iff*

$$|(U_0(X+I_{N+1})U_0^{\top})^{n+2,\cdots,\widehat{n+i+1},\cdots,N+1}_{n+2,\cdots,\widehat{n+j+1},\cdots,N+1}|=0,\qquad i,\,j=1,\cdots,N-n\,. \tag{23}$$

(*Here* ^ *means that the corresponding term is absent.*)

PROOF. Recall that the upper left block $D\in S^2(\boldsymbol{R}^{n+1})$ of $U_0(X+I_{N+1})U_0^{\top}$ in (15) is positive definite, in particular, its rows are linearly independent (in $\boldsymbol{R}^{n+1}$). Given $i=1,\cdots,N-n$, consider the $(n+i+1)$th row of $U_0(X+I_{N+1})U_0^{\top}$. Subtracting appropriate multiples of the first $n+1$ rows of $U_0(X+I_{N+1})U_0^{\top}$ we can achieve the first $n+1$ entries of this row to become zero. By (23) which we assume now, the $(n+j+1)$th entry of this resulting row must also become zero for $j=1,\cdots,N-n$. Since i and j were arbitrary, $\operatorname{rank}(X+I_{N+1})=n+1$ follows. The converse is obvious. □

By the Lemma, the polynomial equations (23) define $\mathcal{L}^{n+1}$ in the neighborhood $\mathcal{N}$ of X_0 in $S^2(\boldsymbol{R}^{N+1})$. Note that, by symmetry, we can assume that $1\leqq i\leqq j\leqq N-n$, so that the number of distinct equations in (23) is $(N-n)(N-n+1)/2$.

If $f:M\to S^n$ is a full harmonic map with $f\leftarrow F$ then, setting $X_0=\langle f\rangle$, to describe $\mathcal{L}_F^{n+1}=\mathcal{L}^{n+1}\cap\mathcal{E}_F$, we have to add $\operatorname{codim}\mathcal{E}_F$ homogeneous linear equations to (23).

We now determine the gradient of the polynomial in the L.H.S. of (23). For notational simplicity, we omit the conjugation by the permutation matrix U_0. If $X_0=\langle f\rangle$ then the effect of conjugation by U_0 can always be achieved by permuting the components of F. (In any case, it is clear that we could have done without U_0 on the first place by giving up the position of D_0 in the upper left corner and sacrificing the notational simplicity.) First let $\{e_{kl}\}_{1\leq k\leq l\leq N+1}\subset S^2(\boldsymbol{R}^{N+1})$ be the standard orthonormal basis given by

$$e_{kl}=\begin{cases}(f_{kl}+f_{lk})/\sqrt{2}\,, & \text{if } k<l\\ f_{kl}\,, & \text{if } k=l\,,\end{cases}$$

where $f_{kl}\in M(N+1,N+1)$ is the matrix with 1 at the kl position and zeros elsewhere. Differentiating the L.H.S. of (23) in the direction of e_{kl}, for $k<l$, we obtain

$$\begin{aligned}e_{kl}|(X+I_{N+1})^{n+2,\cdots,\widehat{n+i+1},\cdots,N+1}_{n+2,\cdots,\widehat{n+j+1},\cdots,N+1}|\,\Big|_{X_0}\\ =\frac{(-1)^{k(i)+l(j)}}{\sqrt{2}}|(X_0+I_{N+1})^{n+2,\cdots,\widehat{n+i+1},\cdots,N+1,k}_{n+2,\cdots,\widehat{n+j+1},\cdots,N+1,l}|\\ +\frac{(-1)^{l(i)+k(j)}}{\sqrt{2}}|(X_0+I_{N+1})^{n+2,\cdots,\widehat{n+i+1},\cdots,N+1,l}_{n+2,\cdots,\widehat{n+j+1},\cdots,N+1,k}|,\end{aligned}$$

where $|(X_0+I_{N+1})^{\cdots r\cdots r\cdots}_{\cdots}|=|(X_0+I_{N+1})^{\cdots}_{\cdots s\cdots s\cdots}|=0$ and $k(i)=(-1)^{k-\nu(k,i)}$ with $\nu(k,i)$ being the number of indices preceding k in the set

$$n+2, \cdots, n+\widehat{i}+1, \cdots, N+1 .$$

In a similar vein, for $k=l$, we have

$$e_{kk}|(X+I_{N+1})^{n+2,\cdots,\widehat{n+i+1},\cdots,N+1}_{n+2,\cdots,\widehat{n+j+1},\cdots,N+1}|\Big|_{X_0} = |(X_0+I_{N+1})^{n+2,\cdots,\widehat{n+i+1},\cdots,N+1,k}_{n+2,\cdots,\widehat{n+j+1},\cdots,N+1,k}|$$

(Note that $k(i)+k(j)$ is even provided that the minor determinant on the R.H.S. is nonzero.) Using these derivatives as coefficients of the gradient, we finally obtain

$$\begin{aligned}&\operatorname{grad}|(X+I_{N+1})^{n+2,\cdots,\widehat{n+i+1},\cdots,N+1}_{n+2,\cdots,\widehat{n+j+1},\cdots,N+1}|\Big|_{X_0}\\ &\quad = \mathcal{S}\{((-1)^{k(i)+l(j)}|(X_0+I_{N+1})^{n+2,\cdots,\widehat{n+i+1},\cdots,N+1,k}_{n+2,\cdots,\widehat{n+j+1},\cdots,N+1,l}|)_{kl}\}, \end{aligned} \tag{24}$$

where $\mathcal{S}(Z)=(Z+Z^{\top})/2$ stands for symmetrization. (Note that, for $i=j$, $\mathcal{S}$ can be omitted.)

THEOREM 3. *The set of matrices*

$$\{\operatorname{grad}|(X+I_{N+1})^{n+2,\cdots,\widehat{n+i+1},\cdots,N+1}_{n+2,\cdots,\widehat{n+j+1},\cdots,N+1}|\Big|_{X_0}\}_{1\le i\le j\le N-n} \tag{25}$$

is linearly independent in $S^2(\boldsymbol{R}^{N+1})$. Moreover, for $X_0=\langle f\rangle$ with $f\leftarrow F$, the projection of

$$\operatorname{grad}|(X+I_{N+1})^{n+3,\cdots,N+1}_{n+3,\cdots,N+1}|\Big|_{X_0} \tag{26}$$

to $\mathcal{E}_F$ is nonzero. If (25) *projected to $\mathcal{E}_F$ is linearly independent then $\langle f\rangle$ is a smooth point and, around $\langle f\rangle$, we have*

$$\dim \mathcal{L}_F^{n+1} = (n+1)(N+1-n/2)-\operatorname{codim}\mathcal{E}_F . \tag{27}$$

PROOF. The $(n+i+1, n+j+1)$th entry of (24) is

$$\frac{1}{2-\delta_{ij}}|(X_0+I_{N+1})^{n+2,\cdots,N+1}_{n+2,\cdots,N+1}| = \frac{1}{2-\delta_{ij}}|D_0|>0$$

while the $(n+i'+1, n+j'+1)$th entry is zero for $i\neq i'$ or $j\neq j'$. This proves the first statement. To prove the second, we compute

$$\begin{aligned}&\langle X_0, \operatorname{grad}|(X+I_{N+1})^{n+3,\cdots,N+1}_{n+3,\cdots,N+1}|\,|_{X_0}\rangle\\ &\quad = \sum_{k,l=1}^{N+1} x^0_{kl}(-1)^{k(1)+l(1)}|(X_0+I_{N+1})^{n+3,\cdots,N+1,k}_{n+3,\cdots,N+1,l}|\\ &\quad = \sum_{k,l=1}^{n+2}(x^0_{kl}+\delta_{kl})(-1)^{k+l}|(X_0+I_{N+1})^{k,n+3,\cdots,N+1}_{l,n+3,\cdots,N+1}|\\ &\qquad -\sum_{k=1}^{n+2}|(X_0+I_{N+1})^{k,n+3,\cdots,N+1}_{k,n+3,\cdots,N+1}|\end{aligned}$$

$$= (n+2)|(X_0+I_{N+1})^{n+3,\cdots,N+1}_{n+3,\cdots,N+1}|$$

$$-\sum_{k=1}^{n+2}|(X_0+I_{N+1})^{k,n+3,\cdots,N+1}_{k,n+3,\cdots,N+1}|.$$

Now, by the rank condition and $X_0+I_{N+1}\geqq 0$, we have $|(X_0+I_{N+1})^{n+3,\cdots,N+1}_{n+3,\cdots,N+1}|=0$ and the principal minor determinants $|(X_0+I_{N+1})^{k,n+3,\cdots,N+1}_{k,n+3,\cdots,N+1}|$ are all nonnegative with at least one positive. Thus (26) projects to $X_0\in\mathcal{E}_F$ nontrivially and the claim follows. Finally, the last statement is a reformulation of the implicit function theorem. □

REMARK. Notice that, for $n=N-1$, the set (25) consists of the single matrix $\mathrm{grad}|X+I_{N+1}||_{X_0}=\mathrm{adj}(X_0+I_{N+1})$ so that Theorem 3 reduces to the proposition in Section 2. In the next section we give an example showing that the projection of (25) to $\mathcal{E}_F$ may well be a linearly dependent set.

REMARK. The usual way of reformulating the positive semidefiniteness of $X+I_{N+1}$ is to require *all* principal minor determinants to be nonnegative. This, however, does not work here as the gradient of a principal minor determinant of order $\geqq n+3$ is zero even in $S^2(\boldsymbol{R}^{N+1})$.

4. Harmonic eigenmaps between spheres.

This section is devoted to examples that all belong to the spherical case so that we put $M=S^m=SO(m+1)/SO(m)$ and $\lambda_k=k(k+m-1)\in\mathrm{Spec}(S^m)$. We consider the standard moduli space $\mathcal{L}_{\lambda_k}$ associated to a standard minimal immersion $f_{\lambda_k}: S^m\to S^{m(\lambda_k)}$. (To simplify the notation, we write $\mathcal{L}_{\lambda_k}=\mathcal{L}_{f_{\lambda_k}}$, $\mathcal{E}_{\lambda_k}=\mathcal{E}_{f_{\lambda_k}}$ etc.) As noted in Section 1, $\mathcal{L}_{\lambda_k}$ parametrizes the equivalence classes of full λ_k-eigenmaps $f: S^m\to S^n$. Since the components of f_{λ_k} comprise an orthonormal basis in $\mathcal{H}^k_{S^m}$, a divergencefree Jacobi field along f is automatically derived from f_{λ_k} so that

$$K_{\lambda_k}(f)=K(f).$$

By (7), for $m\geqq 3$ and $k\geqq 2$, we have

$$\mathrm{codim}\,\mathcal{E}_{\lambda_k}=\sum_{j=0}^{k}(n(\lambda_{2j})+1)$$

and so (14) specializes to

$$\dim K(f)\geqq(n+1)(n(\lambda_k)+1)-\sum_{j=0}^{k}(n(\lambda_{2j})+1). \tag{28}$$

The lowest dimensional nontrivial standard moduli space occurs when $m=3$ and $k=2$. Since the complete description of this moduli space has been given in [**12**] we first apply the results of Sections 2 and 3 to this particular case.

Setting, $m=3$, first of all we note that full λ_2-eigenmaps $f: S^3 \to S^n$ exist iff $2 \leqq n \leqq 8$ and $n \neq 3$. (In particular, $\mathcal{L}^4_{\lambda_2} = \emptyset$.) Moreover, given $f: S^3 \to S^n$, we have

$$O(4) \cdot I_f = \mathcal{L}^{n+1}_{\lambda_2} \tag{29}$$

and hence

$$O(4) \cdot \mathcal{I}_{\lambda_2} = \mathcal{T}_{\lambda_2}.$$

(In contrast, as noted in Section 1, this is definitely false for k large.) Since the L.H.S. of (29) is smooth, we conclude that no singular points exist on $\mathcal{L}_{\lambda_2}$, i.e., $\mathcal{S}_{\lambda_2} = \emptyset$. In particular, Theorem 2 applies to any full λ_2-eigenmap $f: S^3 \to S^n$ (a fact that has been established by a case-by-case verification in [**12**]). Specific representatives of full λ_2-eigenmaps have been given in [**12**] in each admissible range dimension (e.g. the Hopf map for $n=2$) and using these a tedious computation leads to

$$\dim K(f)/so(n+1) \cdot f = \begin{cases} 2 & \text{if } n=2 \\ 5 & \text{if } n=4 \\ 4 & \text{if } n=5 \\ 7 & \text{if } n=6 \\ 9 & \text{if } n=7 \\ 10 & \text{if } n=8. \end{cases}$$

Since $\operatorname{codim} \mathcal{E}_{\lambda_2} = 35$ we see that the lower estimate in (28) becomes equality for $n \geqq 5$. Massive computation shows that, for $n \geqq 5$, the projection of (25) to $\mathcal{E}_{\lambda_2}$ is a linearly independent set so that Theorem 3 applies.

The fact that $\dim K(f)/so(n+1) \cdot f$ does not increase with n provides the following interesting:

EXAMPLE. Let $F: S^3 \to S^6$ be the full λ_2-eigenmap defined by

$$F(z, w) = \left(\frac{1}{\sqrt{2}} (|z|^2 - |w|^2),\ \frac{1}{\sqrt{2}} z^2,\ \sqrt{3} z \bar{w},\ \frac{1}{\sqrt{2}} w^2 \right),$$

where we used complex coordinates $z, w \in \boldsymbol{C}$ and $|z|^2 + |w|^2 = 1$ specifies $S^3 \subset \boldsymbol{C}^2$. Computation shows that $C \in \mathcal{E}_F \subset S^2(\boldsymbol{R}^7)$ iff $C = C(a, b, c)$, $a, b, c \in \boldsymbol{R}$, where

$$C(a, b, c) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & b & c \\ 0 & 0 & -a & 0 & 0 & c & -b \\ 0 & 0 & 0 & (a-b)/3 & -c/3 & 0 & 0 \\ 0 & 0 & 0 & -c/3 & (a+b)/3 & 0 & 0 \\ 0 & b & c & 0 & 0 & -a & 0 \\ 0 & c & -b & 0 & 0 & 0 & -a \end{pmatrix}.$$

Using (a, b, c) as coordinates on $\mathcal{E}_F \cong \boldsymbol{R}^3$ and evaluating $C(a, b, c) + I_7 \geqq 0$ we obtain that $\mathcal{L}_F \subset \mathcal{E}_F$ is a finite solid cone. The vertex is at $(1, 0, 0)$ and the base disk has radius 2, center at $(-1, 0, 0)$ and is perpendicular to the a-axis. The rank stratification has 2 open strata in $\partial\mathcal{L}_F$: $\mathcal{L}_F^6$ making up the base disk of the cone and $\mathcal{L}_F^5$ that corresponds to the nappe of the cone (that is further stratified by the natural stratification whose strata are the segments connecting the points of the base circle with the vertex). $\mathcal{L}_F^3$ is disconnected; it consists of the base circle and the vertex of the cone. We now pick $(0, 0, 1)$ with corresponding full λ_2-eigenmap $f: S^3 \to S^4$ and work out (24) for $X_0 = \langle f \rangle$. The set (23) reduces to

$$|(X+I_7)_7^7| = 0, \quad |(X+I_7)_6^7| = 0 \quad \text{and} \quad |(X+I_7)_6^6| = 0\,.$$

Using the orthonormal basis $\{e_{kl}\}_{1 \le k \le l \le 7}$ introduced in Section 3, by (24), we obtain

$$\operatorname{grad}|(X+I_7)_7^7|\,\Big|_{X_0} = \frac{8}{9}(e_{33} - \sqrt{2}e_{36} + e_{66}),$$

$$\operatorname{grad}|(X+I_7)_6^7|\,\Big|_{X_0} = \frac{4\sqrt{2}}{9}(e_{23} - e_{26} - e_{37} + e_{67})$$

and

$$\operatorname{grad}|(X+I_7)_6^6|\,\Big|_{X_0} = \frac{8}{9}(e_{22} - \sqrt{2}e_{27} + e_{77})\,.$$

Combining these with $C(a, b, c)$ above, we have

$$\langle \operatorname{grad}|(X+I_7)_7^7|\,\Big|_{X_0}, C(a, b, c)\rangle = \langle \operatorname{grad}|(X+I_7)_6^6|\,\Big|_{X_0}, C(a, b, c)\rangle = -\frac{16}{9}(a+c)$$

and

$$\langle \operatorname{grad}|(X+I_7)_6^7|\,\Big|_{X_0}, C(a, b, c)\rangle = 0\,.$$

Hence the condition of Theorem 3 is not satisfied.

Higher dimensional examples involve tedious computations. We mention here only the full λ_2-eigenmap

$$f_\otimes : S^5 \longrightarrow S^9$$

associated to the tensor product

$$\otimes : \boldsymbol{R}^3 \times \boldsymbol{R}^3 \longrightarrow \boldsymbol{R}^9$$

by the Hopf-Whitehead construction [**3**], i.e.,

$$f_\otimes(x, y) = (|x|^2 - |y|^2, 2x \otimes y), \quad x, y \in \boldsymbol{R}^3, \quad |x|^2 + |y|^2 = 1\,.$$

By (8), $\operatorname{codim} \mathcal{E}_{\lambda_2} = 126$ so that the lower bound in (28) is 74. On the other hand, massive computation shows that $\dim K(f_\otimes) = 81$.

Returning to the general situation, we wish to show that the assumption of the Proposition is satisfied for $m \geqq 5$.

THEOREM 4. *For $m \geqq 5$ there exist full λ_k-eigenmaps*

$$f : S^m \longrightarrow S^{n(\lambda_k)-1}.$$

PROOF. $\mathcal{H}^2_{S^3}$ is 9-dimensional and, using the variables $(x, y, u, v) \in \boldsymbol{R}^4$, is spanned by

$$\sqrt{\frac{2}{3}}\Big(\frac{1}{\sqrt{2}}(x^2+y^2-u^2-v^2),\ x^2-y^2,\ u^2-v^2,\ 2xy,\ 2xu,\ 2xv,\ 2yu,\ 2yv,\ 2uv\Big). \tag{30}$$

These quadratic spherical harmonics actually form an orthonormal basis (so that they are the components of a standard minimal immersion $f_{\lambda_2} : S^3 \to S^8$). We now choose a complex variable $z \in \boldsymbol{C}$ and multiply the spherical harmonics in (30) by z^{k-2}. We obtain a set of spherical harmonics of order k that are, by homogeneity, mutually orthogonal and have the same norm. Up to a common scaling factor, they can be considered to be part of an orthonormal basis in $\mathcal{H}^k_{S^m}$, for $m \geqq 5$. Extending this set to an orthonormal basis, we obtain the components of a standard minimal immersion $f_{\lambda_k} : S^m \to S^{n(\lambda_k)}$.

We now change the picture and consider the following 8 quadratic spherical harmonics

$$x^2-y^2,\ u^2-v^2,\ 2xy,\ \sqrt{2}xu,\ \sqrt{2}xv,\ \sqrt{2}yu,\ \sqrt{2}yv,\ 2uv. \tag{31}$$

These are in fact the components of a full λ_2-eigenmap of S^3 into S^7. We now multiply these by z^{k-2} and arrive in $\mathcal{H}^k_{S^m}$ as before. Then we scale and use the same elements as in the previous extension to get $f : S^m \to S^{n(\lambda_k)-1}$. (Note that the sum of squares of (30) and (31) are the same so that f is well-defined.) $\square$

REMARK. Let $f : S^3 \to S^5$ be a full λ_2-eigenmap. Then [8] $\mathcal{L}_f$ is a disk with $\mathcal{L}^3_f$ making up the boundary circle. This shows that Theorem 4 cannot be generalized to arbitrary moduli spaces.

REMARK. Using the explicit forms of the representatives of full λ_2-eigenmaps $f : S^3 \to S^n$, $5 \leqq n \leqq 10$, the same proof shows that, for $n \geqq 5$, there exist full λ_k-eigenmaps $f : S^m \to S^{n(\lambda_k)-c}$, where $0 \leqq c \leqq 4$.

5. Bi-eigenmaps and orthogonal multiplications.

A map $f : M_1 \times M_2 \to S^n$ of a product $M_1 \times M_2$ of Riemannian manifolds into the Euclidean n-sphere S^n is said to be a *bi-eigenmap* if f is harmonic map of constant energy density with respect to each variable separately, i.e., for each $x_j \in M_j$, $j=1, 2$, we have $e(f(\cdot, x_2)) = \lambda_1 \in \mathrm{Spec}\,(M_1)$ and $e(f(x_1, \cdot)) = \lambda_2 \in \mathrm{Spec}\,(M_2)$.

In this case f is said to have eigenvalues λ_1, λ_2. If $M_j=G_j/K_j$, $j=1, 2$, are compact Riemannian homogeneous spaces then f is a bi-eigenmap with eigenvalues λ_1 and λ_2 iff $f \leftharpoondown f_{\lambda_1}\otimes f_{\lambda_2}$, where

$$f_{\lambda_1}\otimes f_{\lambda_2}: M_1\times M_2 \longrightarrow S^{(n(\lambda_1)+1)(n(\lambda_2)+1)-1}$$

is defined by

$$(f_{\lambda_1}\otimes f_{\lambda_2})(x_1, x_2) = f_{\lambda_1}(x_1)\otimes f_{\lambda_2}(x_2), \quad x_j \in M_j,\ j=1, 2\,.$$

Hence the moduli space $\mathcal{L}_{f_{\lambda_1}\otimes f_{\lambda_2}}$ parametrizes the equivalence classes of full bi-eigenmaps with eigenvalues λ_1 and λ_2. We have [**11**]

$$\mathcal{E}_{f_{\lambda_1}\otimes f_{\lambda_2}} = so(V_{\lambda_1})\otimes so(V_{\lambda_2})+\mathcal{E}_{\lambda_1}\otimes S^2(V_{\lambda_2})+S^2(V_{\lambda_1})\otimes\mathcal{E}_{\lambda_2} \tag{32}$$

as $G_1\times G_2$-modules, where V_{λ_j} is the eigenspace of Δ^{M_j} associated to λ_j, $j=1, 2$. In particular, we have

$$\begin{aligned}\dim \mathcal{L}_{f_{\lambda_1}\otimes f_{\lambda_2}} &= (1/4)n(\lambda_1)(n(\lambda_1)+1)n(\lambda_2)(n(\lambda_2)+1)\\ &\quad +(1/2)(n(\lambda_1)+1)(n(\lambda_1)+2)\dim \mathcal{L}_{\lambda_2}\\ &\quad +(1/2)(n(\lambda_2)+1)(n(\lambda_2)+2)\dim \mathcal{L}_{\lambda_1}\,. \end{aligned}\tag{33}$$

It follows that all the previous constructions, e. g. the rank stratification applies to $\mathcal{L}_{f_{\lambda_1}\otimes f_{\lambda_2}}$.

Specializing, from here on, to the spherical case $M_j=S^{m_j}=SO(m_j+1)/SO(m_j)$, $j=1, 2$, the dimension formula (33) combined with (7)-(8) gives the *exact* dimension of the moduli space of bi-eigenmaps with eigenvalues λ_{k_1} and λ_{k_2}. The importance of bi-eigenmaps is twofold. First, for $k_1=k_2=1$, i. e., $f_{\lambda_j}=I_{m_j+1}$, $j=1, 2$, a bi-eigenmap is nothing but an *orthogonal multiplication*, i. e., a bilinear map

$$f: \boldsymbol{R}^{m_1+1}\times \boldsymbol{R}^{m_2+1} \longrightarrow \boldsymbol{R}^{n+1}$$

satisfying

$$|f(x_1, x_2)| = |x_1|\cdot|x_2|, \quad x_j\in \boldsymbol{R}^{m_j+1}, \qquad j=1, 2\,.$$

The moduli space $\mathcal{L}_{I_{m_1+1}\otimes I_{m_2+1}}\subset so(m_1+1)\otimes so(m_2+1)$ parametrizes the equivalence classes of full orthogonal multiplications. The stratum $\mathcal{L}^{m_2+1}_{I_{m_1+1}\otimes I_{m_2+1}}$ which is a real analytic manifold almost everywhere corresponds to those orthogonal multiplications that give rise to m_1 linearly independent vector fields on S^{m_2}. (For the relation to Clifford modules cf. [**5**].) By a result of Hurwitz, this stratum is nonempty iff $m_1\leqq 2^p+8q-1$ and $m_2=2^{p+4q}(2r+1)-1$, for integers p, q, r, such that $0\leqq p\leqq 3$, $0\leqq q$, r.

As a specific example, consider $\mathcal{L}_{I_2\otimes I_3}\subset so(2)\otimes so(3)\cong \boldsymbol{R}^3$. Little computation shows that this moduli space is a solid ball around the origin (of radius 2) and the boundary is a single smooth stratum corresponding to the range dimension 4. (For $m_1=m_2=2$, the rank stratification is still smooth and has 3 non-empty strata corresponding to the range dimensions 4, 7 and 8 [8].)

Second, Ratto [9] recently used bi-eigenmaps to manufacture harmonic maps between spheres by applying a homotopy method to the Hopf-Whitehead construction (assuming some damping conditions). In view of the fact that $\dim \mathcal{L}_{f_{\lambda_{k_1}} \otimes f_{\lambda_{k_2}}}$ in (33) is huge, it is expected that the ranges k_1, $k_2 \geqq 2$ will provide further harmonically represented homotopy classes of maps between spheres.

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