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Fatou sets in complex dynamics on projective spaces

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Introduction.

The theory of complex dynamical systems defined by holomorphic maps on complex projective spaces, which generalizes the iteration theory of one variable rational functions, has been studied by several authors [**FS1**], [**FS2**], [**HP**], [**U3**]. Concerning Julia sets and Fatou sets, analogies to the one variable case are pursued to some extent. There are also many problems which we encounter first in higher dimensional case.

In this paper, we prove two fundamental results on Fatou sets for complex dynamical systems of degree greater than 1 on complex projective spaces: Fatou sets are pseudoconvex, hence Stein (Theorem 2.3); Fatou sets are Carathéodory hyperbolic, hence Kobayashi hyperbolic (Theorems 2.5 and 2.6). With the latter theorem, we can derive some results analogous to the one dimensional case. It is proved that the immediate basin of an attractive periodic point contains critical points. The same result is proved for a parabolic periodic point in two dimensional case under an additional condition.

To prove the above fundamental theorems we employ the method originated by Hubbard and Papadopol [**HP**]. Namely we consider, for a holomorphic map f on $\boldsymbol{P}^n$, the corresponding homogeneous polynomial map F on $\boldsymbol{C}^{n+1}$ and the Green function h with respect to F. It is shown in Theorem 2.2 that a point $p\in\boldsymbol{P}^n$ is in the Fatou set if and only if the Green function is pluriharmonic in a neighborhood of the fiber $\pi^{-1}(p)$ of the projection $\pi:\boldsymbol{C}^{n+1}-\{O\}\to\boldsymbol{P}^n$. The "only if" part is proved in [**HP**, Proposition 5.4] and the "if" part provides the answer to a problem posed in [**HP**, below Prop. 5.4].

The outline of the present paper is as follows: In section 1, properties of homogeneous maps and Green functions are described. Using these, we prove the two main theorems on Fatou set in section 2. As applications of the hyperbolicity, results on the critical points in the Fatou sets are proved in section 3. Although the results on Green functions are due to [**HP**], we include the proofs for the sake of completeness.

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1. Homogeneous maps.

Let $\boldsymbol{C}^{n+1}$ denote the space of $n+1$ tuples $x=(x_0, \cdots, x_n)$ of complex numbers. A holomorphic map $F: \boldsymbol{C}^{n+1} \to \boldsymbol{C}^{n+1}$ is said to be a homogeneous map of degree d if F is defined by an $n+1$ tuple $F(x)=(f_0(x), \cdots, f_n(x))$ of homogeneous polynomials of degree d. A homogeneous map F is said to be degenerate if $F(x)=O$ for some $x \neq O$, where O denotes the origin $(0, \cdots, 0)$. Otherwise F is said to be non-degenerate.

We denote a point in the complex projective space $\boldsymbol{P}^n$ by homogeneous coordinates $p=[x_0 : \cdots : x_n]$. The projection $\pi : \boldsymbol{C}^{n+1} - \{O\} \to \boldsymbol{P}^n$ given by $\pi(x_0, \cdots, x_n) = [x_0 : \cdots : x_n]$ defines a holomorphic $\boldsymbol{C}^*$-bundle over $\boldsymbol{P}^n$. See [**FS2**] for the proof of the following proposition.

PROPOSITION 1.1. *For a non-degenerate homogeneous map $F: \boldsymbol{C}^{n+1} \to \boldsymbol{C}^{n+1}$ there exists a unique holomorphic map $f : \boldsymbol{P}^n \to \boldsymbol{P}^n$ such that $\pi \circ F = f \circ \pi$. Conversely, for a holomorphic map $f : \boldsymbol{P}^n \to \boldsymbol{P}^n$ there exists a non-degenerate homogeneous map $F: \boldsymbol{C}^{n+1} \to \boldsymbol{C}^{n+1}$ such that $\pi \circ F = f \circ \pi$. Such a map F is unique up to a non-zero constant.*

In this section, we study (non-degenerate) homogeneous maps F to know properties of the corresponding holomorphic maps f.

We define the Euclidean norm of $x=(x_0, \cdots, x_n) \in \boldsymbol{C}^{n+1}$ by

$$\|x\| = (|x_0|^2 + \cdots + |x_n|^2)^{1/2}.$$

LEMMA 1.2. *Let $F: \boldsymbol{C}^{n+1} \to \boldsymbol{C}^{n+1}$ be a homogeneous map of degree d.*

(1) *There exists a constant $M>0$ such that $\|F(x)\| \leqq M\|x\|^d$ for all $x \in \boldsymbol{C}^{n+1}$. If F is non-degenerate, then there exists a constant $m>0$ such that $\|F(x)\| \geqq m\|x\|^d$ for all $x \in \boldsymbol{C}^{n+1}$.*

(2) *If $d \geqq 2$, then there exists a constant $r>0$ such that*

$$\|F(x)\| < \frac{1}{2}\|x\| \qquad \textit{whenever } \|x\| < r.$$

If F is non-degenerate and $d \geqq 2$, then there exists a constant $R>0$ such that

$$\|F(x)\| > 2\|x\| \qquad \textit{whenever } \|x\| > R.$$

PROOF. (1) Let $M := \sup_{\|x\|=1} \|F(x)\|$. Since F is homogeneous of degree d, it follows that

$$\|F(x)\| = \|x\|^d \|F(x/\|x\|)\| \leqq M\|x\|^d.$$

If F is non-degenerate, then $m := \inf_{\|x\|=1} \|F(x)\| > 0$. Hence

$$\|F(x)\| = \|x\|^d \|F(x/\|x\|)\| \geqq m\|x\|^d.$$

(2) We choose r so that $0<r\leqq(2M)^{-1/(d-1)}$. Then

$$\|F(x)\| \leqq M\|x\|^d < Mr^{d-1}\|x\| \leqq (1/2)\|x\| \qquad \text{for } \|x\|<r.$$

When F is non-degenerate, we choose R so that $R\geqq(2m)^{-1/(d-1)}$. Then

$$\|F(x)\| \geqq m\|x\|^d > mR^{d-1}\|x\| \geqq 2\|x\| \qquad \text{for } \|x\|>R. \qquad \square$$

DEFINITION. For a homogeneous map F of degree $d\geqq 2$, we define the basin of attraction $\mathcal{A}$ of the origin O by

$$\mathcal{A} = \{x \in \boldsymbol{C}^{n+1} \mid F^j(x) \to O \text{ as } j\to\infty\}.$$

PROPOSITION 1.3. (1) *$\mathcal{A}$ is non-empty and pseudoconvex.*

(2) *$\mathcal{A}$ is a complete circular domain, i.e., if $x\in\mathcal{A}$ and $c\in\boldsymbol{C}$ with $|c|\leqq 1$, then $cx\in\mathcal{A}$.*

(3) *$\mathcal{A}$ is bounded if and only if F is non-degenerate.*

PROOF. We put $\mathcal{B}_r=\{x\in\boldsymbol{C}^{n+1} \mid \|x\|<r\}$. Then $\mathcal{A}$ is the union of the increasing sequence of pseudoconvex open sets $F^{-j}(\mathcal{B}_r)$, $j=1, 2, \cdots$. Hence $\mathcal{A}$ is pseudoconvex. It follows immediately from the homogeneity of F that $\mathcal{A}$ is a complete circular domain. If F is non-degenerate, then $\mathcal{A}$ is contained in the ball $\{x\in\boldsymbol{C}^{n+1} \mid \|x\|\leqq R\}$, hence bounded. If F is degenerate, then there exists an $x\neq O$ such that $F(x)=O$, and $F(cx)=O$ for all $c\in\boldsymbol{C}$; hence $\mathcal{A}$ is unbounded. $\square$

THEOREM 1.4. *There exists a unique function $h: \boldsymbol{C}^{n+1}\to\boldsymbol{R}\cup\{-\infty\}$ with the following properties*:

(i) *$\alpha(x)=h(x)-\log\|x\|$ is homogeneous of degree 0, i.e., $\alpha(cx)=\alpha(x)$ for $x\in\boldsymbol{C}^{n+1}-\{O\}$, $c\in\boldsymbol{C}^*$.*

(ii) *$\mathcal{A}=\{x\in\boldsymbol{C}^{n+1} \mid h(x)<0\}$*

PROOF. We set

$$\rho(x) = \sup\{a>0 \mid ax \in \mathcal{A}\}$$

and

$$h(x) = -\log\rho(x).$$

It can be easily verified that h satisfies the required properties. $\square$

DEFINITION. h is said to be the Green function for the homogeneous map F.

THEOREM 1.5. *Let F be a homogeneous map of degree $d\geqq 2$.*

(1) *If $h_0(x)$ is a function on $\boldsymbol{C}^{n+1}-\{O\}$ such that $h_0(x)-\log\|x\|$ is bounded, then $d^{-j}h_0(F^j(x))$ converges to the Green function $h(x)$.*

(2) *The Green function h is plurisubharmonic on $\boldsymbol{C}^{n+1}$ and satisfies the equation*

$$(*) \qquad h(F(x)) = d\cdot h(x).$$

(3) *If F is non-degenerate, then the convergence in* (1) *is uniform and the Green function h is continuous on $\boldsymbol{C}^{n+1}-\{O\}$.*

PROOF. First we consider the case where F is non-degenerate. Put

$$\gamma(x):=h_0(F(x))-d\cdot h_0(x).$$

Then γ is bounded in $\boldsymbol{C}^{n+1}-\{O\}$ by Lemma 1.2, (1), and

$$d^{-1}h_0(F(x))=h_0(x)+d^{-1}\gamma(x).$$

Replacing x by $F^{k-1}(x)$ and multiplying by $d^{-(k-1)}$, we have

$$d^{-k}h_0(F^k(x))=d^{-(k-1)}h_0(F^{k-1}(x))+d^{-k}\gamma(F^{k-1}(x)).$$

Adding up this for $k=1, \cdots, j$ we obtain

$$d^{-j}h_0(F^j(x))=h_0(x)+d^{-1}\gamma(x)+\cdots+d^{-j}\gamma(F^{j-1}(x)).$$

Since $\gamma(x)$ is bounded and $d\geqq 2$, the right-hand side converges uniformly as $j\to\infty$. We write

$$\tilde{h}(x)=\lim_{j\to\infty} d^{-j}h_0(F^j(x)).$$

This limit does not depend on the choice of h_0. Indeed, if h_1 is another such function then $h_1(x)-h_0(x)$ is bounded and hence $d^{-j}h_1(F^j(x))-d^{-j}h_0(F^j(x))$ converges to 0 as $j\to\infty$.

Consider the particular case $h_0(x)=\log\|x\|$. We know that $d^{-i}\log\|F^j(x)\|$ is plurisubharmonic, continuous on $\boldsymbol{C}^{n+1}-\{O\}$ and $d^{-j}\log\|F^j(x)\|-\log\|x\|$ is homogeneous of degree 0. Hence the limit $\tilde{h}(x)$ has the same properties. It is clear from the construction that $\tilde{h}$ satisfies the equation (*).

Now we show that $\tilde{h}$ coincides with the Green function h. It suffices to show that $x\in\mathcal{A}$ if and only if $\tilde{h}(x)<0$. But this is clear from the equation (*).

In the case F is degenerate, we start with the function $h_0=\log\|x\|-A$, where A is a sufficiently large positive number. Then $\gamma(x)$ is negative and $d^{-j}h_0(F^j(x))$ is a decreasing sequence of plurisubharmonic functions and converges to a plurisubharmonic function. The rest of the proof is the same as in the non-degenerate case. □

2. Fatou sets.

Let f be a holomorphic map on $\boldsymbol{P}^n$ of degree d. We will always assume that $d\geqq 2$. Let F be the corresponding non-degenerate homogeneous map on $\boldsymbol{C}^{n+1}$ and h the Green function for F.

DEFINITION. We define

$$\mathcal{H}:=\{x\in\boldsymbol{C}^{n+1}\mid h \text{ is pluriharmonic in a neighborhood of } x\}$$

and

$$\Omega := \pi(\mathcal{H}).$$

It is clear that $\pi^{-1}(\Omega)=\mathcal{H}$, i.e. the set $\mathcal{H}$ is a cone.

PROPOSITION 2.1. *A point $p_0 \in \boldsymbol{P}^n$ is contained in Ω if and only if there are a neighborhood V of p_0 and a holomorphic map $s: V \to \boldsymbol{C}^{n+1}$ such that $\pi \circ s = \mathrm{id}$. and that $s(V) \subset \partial \mathcal{A}$. Such a holomorphic map s is unique up to a constant factor with absolute value* 1.

PROOF. We choose an open ball V with center p_0 with respect to a local coordinate system. Then the restriction to V of the $\boldsymbol{C}^*$-bundle $\pi: \boldsymbol{C}^{n+1} \to \boldsymbol{P}^n$ is trivial. We identify $\pi^{-1}(V)$ with $V \times \boldsymbol{C}^*$ and denote the points in $\pi^{-1}(V)$ by (p, z). Then the function $h|\pi^{-1}(V)$ takes the form

$$h(p, z) = \log |z| + \eta(p).$$

Here $\eta(p)$ is a plurisubharmonic function on V and it is pluriharmonic if and only if $h(p, z)$ is pluriharmonic on $\pi^{-1}(V)$.

Suppose that $s(p)=(p, \sigma(p))$ is a holomorphic map satisfying the condition of the theorem. Then $0=h(s(p))=\log|\sigma(p)|+\eta(p)$. Hence $\eta(p)=-\log|\sigma(p)|$ is pluriharmonic.

Conversely suppose that $\eta(p)$ is pluriharmonic. Then there is a pluriharmonic function η^* on V such that $\eta+i\eta^*$ is holomorphic. We define $\sigma(p)=\exp(-\eta(p) -i\eta^*(p))$ and $s(p)=(p, \sigma(p))$. Then $h(s(p))=\log|\sigma(p)|+\eta(p)=0$. This implies $s(V) \subset \partial \mathcal{A}$. Since η^* is unique up to an additive real constant, σ is unique up to a constant of absolute value 1. □

THEOREM 2.2. *We define the sets Ω', Ω'' as follows*:

$$\Omega' = \left\{ p \in \boldsymbol{P}^n \;\middle|\; \begin{array}{l} \text{there exists a neighborhood } V \text{ of } p \text{ such that} \\ f^j|V \ (j=1, 2, \cdots) \text{ is a normal family} \end{array} \right\},$$

$$\Omega'' = \left\{ p \in \boldsymbol{P}^n \;\middle|\; \begin{array}{l} \text{there exist a neighborhood } V \text{ of } p \text{ and a uniformly} \\ \text{convergent subsequence } f^{j_\nu}|V \ (\nu=1, 2, \cdots) \end{array} \right\}.$$

Then $\Omega=\Omega'=\Omega''$.

DEFINITION. The set $\Omega=\Omega'=\Omega''$ is said to be the Fatou set of f. Each connected component of Ω is said to be a Fatou component.

PROOF OF THEOREM 2.2. It is clear by definition that $\Omega' \subseteq \Omega''$. We will prove that $\Omega'' \subseteq \Omega$ and $\Omega \subseteq \Omega'$.

First we prove that $\Omega'' \subseteq \Omega$ (cf. [**HP**, Proposition 5.4]). Let $p \in \Omega''$ and choose a neighborhood V of p and a uniformly convergent subsequence $\{f^{j_\nu}|V\}$. Put $\varphi=\lim_{\nu\to\infty}(f^{j_\nu}|V)$. Choose a hyperplane H which does not contain $\varphi(p)$.

Taking a suitable homogeneous coordinates $[x_0:\cdots:x_n]$, we suppose that $H=\{x_0=0\}$. We choose an "ε neighborhood" $N_\varepsilon=\{|x_0|<\varepsilon\|x\|\}$ of H so that $\varphi(p)\notin N_\varepsilon$. By shrinking V, we suppose that $f^{j_\nu}(V)\cap N_\varepsilon=\emptyset$ for sufficiently large ν. We define

$$h_0(x):=\begin{cases}\log\|x\| & \text{for } x\in\pi^{-1}(N_\varepsilon),\\ \log(|x_0|/\varepsilon) & \text{for } x\in\pi^{-1}(\boldsymbol{P}^n-N_\varepsilon).\end{cases}$$

Then $h_0(x)-\log\|x\|$ is bounded since $0\leqq h_0(x)-\log\|x\|\leqq\log(1/\varepsilon)$. Therefore by Theorem 1.5, $d^{-j_\nu}h_0(F^{j_\nu}(x))$ converges uniformly to the Green function $h(x)$ as $\nu\to\infty$. If $x\in\pi^{-1}(V)$, we have $F^{j_\nu}(x)\in\pi^{-1}(\boldsymbol{P}^n-N_\varepsilon)$. Hence $h_0(F^{j_\nu}(x))$ is pluriharmonic on $\pi^{-1}(V)$. Therefore the limit h is pluriharmonic on $\pi^{-1}(V)$. Thus $p\in\Omega$.

Now we prove $\Omega\subseteqq\Omega'$. Suppose that $p\in\Omega$ and choose V and $s:V\to\boldsymbol{C}^{n+1}$ as in Proposition 2.1. Since $F^j(s(V))\subset\partial\mathcal{A}$, the sequence $\{F^j\circ s\}$ is uniformly bounded, hence is a normal family. Suppose that $\{F^{j_\nu}\circ s\}$ is a subsequence which is uniformly convergent on compact sets and let $\Phi:V\to\boldsymbol{C}^{n+1}$ be its limit map. Then $\Phi(V)\subset\partial\mathcal{A}\subset\boldsymbol{C}^{n+1}-\{O\}$. Hence $\pi\circ\Phi$ is well defined and the sequence $\{f^{j_\nu}=\pi\circ F^{j_\nu}\}$ converges to $\pi\circ\Phi$ uniformly on compact sets. This proves that $\{f^j|V\}$ is a normal family. Thus $p\in\Omega'$. □

REMARK. In the one dimensional case, the equality $\Omega'=\Omega''$ has been proved as the corollary to the theorem that the Julia set $\boldsymbol{P}^1-\Omega'$ is the closure of the set of repelling periodic points (see [**M**, §11]).

THEOREM 2.3. *The Fatou set Ω is Stein, hence all Fatou components are also Stein.*

PROOF. In general, an open set $\Omega\neq\boldsymbol{P}^n$ in $\boldsymbol{P}^n$ is Stein if and only if it is pseudoconvex ([**F**], [**T**], [**U1**]). To Show that Ω is pseudoconvex, it suffices to prove that $\mathcal{H}=\pi^{-1}(\Omega)$ is pseudoconvex. This is shown by the following lemma. □

LEMMA 2.4. *Let h be a plurisubharmonic function on $\boldsymbol{C}^m$ and let*

$$\mathcal{H}=\{x\in\boldsymbol{C}^m\,|\,h \text{ is pluriharmonic in a neighborhood of } x\}.$$

Then $\mathcal{H}$ is pseudoconvex (if it is non-empty).

PROOF. To show the pseudoconvexity of $\mathcal{H}$ it suffices to prove the following assertion:

Let Δ be a polydisk with respect to a local coordinate system in $\boldsymbol{C}^m$:

$$\Delta\cong\{x=(z_1,\cdots,z_m)|\,|z_i|<1,\ i=1,\cdots,m\}$$

and let V be a "Hartogs figure" in Δ:

$$V=\{x\in\Delta \mid \text{ either } r<|z_1|<1 \text{ or } \max_{i=2,\cdots,m}|z_i|<r'\}\,, \qquad (0\leqq r,\ r'\leqq 1).$$

Then $V\subseteqq\mathcal{H}$ implies $\Delta\subseteqq\mathcal{H}$.

To prove this assertion, suppose $V\subseteqq\mathcal{H}$. Then $h|V$ is pluriharmonic and hence it is a real part of a holomorphic function on the simply connected domain V, which is continued analytically to Δ. Therefore there is a pluriharmonic function $\hat{h}$ on Δ such that $\hat{h}|V=h|V$. Put $u(x)=h(x)-\hat{h}(x)$ for $x\in\Delta$. Then u is plurisubharmonic and $u|V\equiv 0$. Considered as a function of the single variable z_1, the function u is subharmonic on $|z_1|<1$ and vanishes identically on $r<|z_1|<1$. Hence, by the maximum principle, we have $u\leqq 0$ on Δ. Since u takes the value 0 in Δ, we conclude that $u\equiv 0$, again by the maximum principle. Thus $h\equiv\hat{h}$ is pluriharmonic on Δ and $\Delta\subseteqq\mathcal{H}$. □

For the definition and fundamental properties of Kobayashi pseudodistance, we refer the readers to Kobayashi [**K**]. A complex manifold M is called Kobayashi hyperbolic if the Kobayashi pseudodistance on M is a (non-degenerate) distance. Here we will use the following facts: (1) If M is a bounded domain in $\boldsymbol{C}^m$, then M is Kobayashi hyperbolic. (2) If M is Kobayashi hyperbolic and $\alpha: N\to M$ is an injective holomorphic map, then N is Kobayashi hyperbolic. (3) M is Kobayashi hyperbolic if there is a covering manifold $\tilde{M}$ (unramified and without relative boundary) of M which is Kobayashi hyperbolic.

THEOREM 2.5. *The Fatou set Ω is Kobayashi hyperbolic.*

PROOF. It suffices to show that each Fatou component U is Kobayashi hyperbolic. Let p_0 be a point in U and choose V and $s: V\to\boldsymbol{C}^{n+1}$ as in Proposition 2.1. Since s is unique up to a constant factor, this map s can by continued analytically along all curves in U. Therefore the analytic continuation of s defines a covering manifold $\alpha:\tilde{U}\to U$ of U, and a holomorphic map $\tilde{s}:\tilde{U}\to\boldsymbol{C}^{n+1}$ such that $\pi\circ\tilde{s}=\alpha$. Since $\tilde{s}$ is injective and the image $\tilde{s}(\tilde{U})\subset\partial\mathcal{A}$ is bounded, it follows that $\tilde{U}$ is Kobayashi hyperbolic. Consequently U is also Kobayashi hyperbolic. □

This result can be slightly strengthened as follows: A complex manifold M is said to be Carathéodory hyperbolic if there exists a covering manifold $\tilde{M}$ of M such that the Carathéodory pseudodistance on $\tilde{M}$ is a (non-degenerate) distance. Since the Carathéodory pseudodistance on $\tilde{U}$ is a distance, we have the following theorem.

THEOREM 2.6. *The Fatou set Ω is Carathéodory hyperbolic.*

This contains Theorem 2.4 since Carathéodory hyperbolicity implies Kobayashi hyperbolicity in general.

REMARK. If $f: \boldsymbol{P}^n \to \boldsymbol{P}^n$ is of degree 1, i.e., if f is a projective transformation, then the set Ω is not defined. The set $\Omega' = \Omega''$ is in general neither Stein nor Kobayashi hyperbolic.

3. Critical points in Fatou sets.

Let $f: \boldsymbol{P}^n \to \boldsymbol{P}^n$ be a holomorphic map of degree $d \geqq 2$. A point $p \in \boldsymbol{P}^n$ is said to be a critical point if the rank of the differential $df(p): T_p \boldsymbol{P}^n \to T_{f(p)} \boldsymbol{P}^n$ of f at p is less than n. The set of all critical points of f is given by the equation $\det(\partial f_i / \partial x_j)_{i,j=0}^n = 0$, hence is a non-empty algebraic set of codimension 1.

A Fatou component U is said to be periodic if there is an integer $j > 0$ such that $f^j(U) = U$. The least of such j is said to be the period of U. If U is a periodic Fatou component of period m, then $U_i = f^{i-1}(U)$, $i = 1, \cdots, m$ are also periodic Fatou components and $f(U_i) = U_{i+1}$ $(i = 1, \cdots, m-1)$ and $f(U_m) = U_1$. The ordered set $\{U_i\}_{i=1}^m$ is said to be a cycle of Fatou components.

Generalizing Siegel disks and Herman rings of the one variable case, Fornaess and Sibony [**FS2**] defined the following concept:

DEFINITION. A Fatou component U is said to be a Siegel domain if there exists a subsequence $\{f^{j_\nu} | U\}$ that converges uniformly to the identity map id.: $U \to U$ on compact sets.

It is clear that a Siegel domain U is periodic. If m is its period, then the components $U_i = f^{i-1}(U)$ $(i = 1, \cdots, m)$ constitute a cycle of Siegel domains and $f | U_i : U_i \to f(U_i)$ are biholomorphic maps.

REMARK. Examples of holomorphic maps on $\boldsymbol{P}^2$ which have Siegel domains can be constructed by the method of [**U3**, Section 4] from rational functions on $\boldsymbol{P}^1$ which have Siegel disks or Herman rings.

THEOREM 3.1. *Let $\{U_i\}_{i=1}^m$ be a cycle of Fatou components and suppose that $\hat{U} = \bigcup_{i=1}^m U_i$ satisfies the following conditions*:

(i) *$\hat{U}$ contains no critical points.*

(ii) *There exist a point $p_0 \in \hat{U}$ and a subsequence $\{f^{j_\nu}(p_0)\}$ which converges to a point in $\hat{U}$.*

Then $\{U_i\}$ is a cycle of Siegel domains.

PROOF. By considering f^m in place of f, the theorem can be reduced to the case of period 1. So we assume that $m = 1$ and write $U = U_1 = \hat{U}$.

Let d_U denote the Kobayashi distance on U and we choose a positive number ε so that the ε neighborhood $V = \{p \in U \mid d_U(p, p_0) < \varepsilon\}$ of p_0 is relatively compact in U. Let q_0 be the limit point of $\{f^{j_\nu}(p_0)\}$ and let $B \subset U$ be a ball with center q_0 with respect to a local coordinate system. Let d_B denote the Kobayashi distance on B and put $W=\{q\in B \mid d_B(q, q_0)<\varepsilon/2\}$. By shifting to a subsequence, we can assume that $f^{j_\nu}|U$ is uniformly convergent on compact sets and that $f^{j_\nu}(p_0)\in W$ for all ν.

We will first show that $W\subseteq f^{j_\nu}(V)$. Let V_ν denote the connected component of $f^{-j_\nu}(B)$ that contains p_0. Since U contains no critical points, $f^{j_\nu}|U$ is a covering. Hence $f^{j_\nu}|V_\nu : V_\nu\to B$ is a biholomorphic map. Applying the decreasing property of Kobayashi distance to the inverse map $(f^{j_\nu}|V_\nu)^{-1}: B\to U$, we have

$$
\begin{aligned}
d_U((f^{j_\nu}|V_\nu)^{-1}(q), p_0) &\leqq d_W(q, f^{j_\nu}(p_0)) \\
&\leqq d_W(q, q_0)+d_W(q_0, f^{j_\nu}(p_0)) < \varepsilon
\end{aligned}
$$

for $q\in W$. This shows that $(f^{j_\nu}|V_\nu)^{-1}(W)\subseteq V$; hence $W\subseteq f^{j_\nu}(V)$.

We suppose that $j_{\nu+1}-j_\nu\to\infty$ $(\nu\to\infty)$ by choosing a subsequence of j_ν. We assert that the sequence $\{f^{j_{\nu+1}-j_\nu}|W\}$ converges uniformly on compact sets to the identity. For $q\in W$, choose $p\in V$ such that $q=f^{j_\nu}(p)$. Then $f^{j_{\nu+1}-j_\nu}(q)=f^{j_{\nu+1}}(p)$. Therefore

$$
\sup_{q\in W} d_U(f^{j_{\nu+1}-j_\nu}(q), q) \leqq \sup_{p\in V} d_U(f^{j_{\nu+1}}(p), f^{j_\nu}(p)).
$$

Since $\{f^{j_\nu}|V\}$ is uniformly convergent, the right-hand side converges to 0. Hence $\{f^{j_{\nu+1}-j_\nu}|W\}$ converges uniformly to the identity map. Since $\{f^{j_{\nu+1}-j_\nu}|U\}$ is a normal family by definition, this converges to the identity uniformly on compact sets in U. □

An ordered set of points $\{p_i\}_{i=1}^m$ $(m\geqq 1)$ is said to be a cycle if $f(p_i)=p_{i+1}$ $(i=0, \cdots, m-1)$, and $f(p_m)=p_1$. A cycle $\{p_i\}$ is said to be attractive if the eigenvalues of the differential $df^m(p_1)$ of f^m at p_1 are all less than 1 in absolute values. Then p_i are all contained in the Fatou set. If we denote by U_i the Fatou component that contains p_i, then $\{U_i\}$ is a cycle of Fatou components. We call the set $\hat{U}=\bigcup_{i=1}^m U_i$ the immediate basin of the attractive cycle $\{p_i\}$.

COROLLARY 3.2. *The immediate basin of an attractive cycle contains critical points.*

Now we will show the same result for basins of parabolic cycles with an additional condition in two dimensional case.

First let us recall some definitions and results in [**U2**]. Let M be a complex manifold of dimension 2 and $f: M\to M$ a surjective holomorphic map. A fixed point p_0 of f is said to be semi-attractive of type $(1, b)_1$ if there is a local coordinate system (x, y) with center p_0 such that f is expressed as

$$
(x, y)\longmapsto (x+\sum_{i+j>1} a_{ij}x^i y^j,\ by+\sum_{i+j>1} b_{ij}x^i y^j), \qquad 0<|b|<1,\ a_{20}\neq 0.
$$

(See [**U2**, Section 6].) If p_0 is a semi-attractive fixed point of type $(1, b)_1$, then there is a connected open set D (base of uniform convergence) with the following properties: (i) $f(D) \subset D$; (ii) $\{f^j | D\}$ converges uniformly to the constant map p_0; (iii) if $\{f^j\}$ converges uniformly on some neighborhood of a point $p \in M$, then there exists a j_0 such that $f^{j_0}(p) \in D$ ([**U2**, Proposition 7.2]). By the immediate basin of convergence for p_0 we mean the Fatou component U that contains D. This definition is independent of the choice of D. In fact, if D, D' are two bases of uniform convergence, then $D \cap D'$ is non-empty by the property (iii).

The following theorem can be proved in the same way as [**U2**, Theorem 10].

THEOREM 3.3. *In the above situation, suppose further that U contains no critical points of f. Then U is biholomorphic to $\boldsymbol{C}^2$.*

Let $\{p_i\}_{i=1}^m$ be a cycle of periodic points of f. It is called semi-attractive of type $(1, b)_1$ if p_1 is a semi-attractive fixed point of type $(1, b)_1$ of f^m. For such a cycle, we denote by U_i the immediate basin of convergence of p_i with respect to f^m and call the set $\bigcup_{i=1}^m U_i$ the immediate basin of the cycle $\{p_i\}$.

Since $\boldsymbol{C}^2$ is not Kobayashi hyperbolic, we have the following theorem:

THEOREM 3.4. *Let $f: \boldsymbol{P}^2 \to \boldsymbol{P}^2$ be a holomorphic map of degree $\geqq 2$. Then the immediate basin of a semi-attractive cycle of type $(1, b)_1$ contains critical points.*

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