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$L_{n/2}$-pinching theorems for submanifolds with parallel mean curvature in a sphere

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1. Introduction.

Let M^n be an n-dimensional oriented closed minimal submanifold in the unit sphere $S^{n+p}(1)$. We denote the square of the length of the second fundamental form by S. It is well known [**10**] that if $S<n/(2-1/p)$ on M, then $S\equiv 0$ and hence M is isometric to the unit sphere $S^n(1)$. Further discussions in this direction have been carried out by many other authors ([**4**], [**7**], [**12**], etc.), but all these results have pointwise condition for S. It seems to be interesting to study the L_q-pinching condition for S. By using eigenvalue estimate, Shen [**9**] proved the following

THEOREM A. *Let $M^n \to S^{n+1}(1)$ be an oriented closed embedded minimal hypersurface with* $\mathrm{Ric}_M \geqq 0$. *If $\int_M S^{n/2}<C_1'(n)$, where $C_1'(n)$ is a positive universal constant, then M is a totally geodesic hypersurface.*

By using Gauss-Bonnet Theorem and a generalized Simons' inequality, Lin and Xia [**6**] proved the following

THEOREM B. *Let M^{2m} be an even dimensional oriented closed minimal submanifold in $S^{2m+p}(1)$. If the Euler characteristic of M is not greater than two, and $\int_M S^m<C_2'(m, p)$, where $C_2'(m, p)$ is a positive universal constant depending on m and p, then M is totally geodesic.*

In the present paper, we will study the $L_{n/2}$-pinching problem for n-dimensional compact submanifolds with parallel mean curvature in the unit sphere $S^{n+p}(1)$. Now we define our pinching constants as follows

$$\alpha(n, H)=\frac{2na(n)}{C^2(n)[(a(n)b(n, H))^{1/2}+(1+a(n))^{1/2}(2+b(n, H))^{1/2}]^2}, \tag{1.1}$$

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where $a(n)=(n+2)(n-2)^2[2n(n-1)]^{-2}$, $b(n, H)=(n-2)^2H^2[2(n-1)(1+H^2)]^{-1}$, $H=$ mean curvature of M, $C(n)=2^n(1+n)^{1+1/n}(n-1)^{-1}\sigma_n^{-1}$ and $\sigma_n=$volume of the unit ball in $\boldsymbol{R}^n$.

$$\alpha_1(n)=\frac{2na(n)}{C^2(n)[(a(n)d(n))^{1/2}+(1+a(n))^{1/2}(2+d(n))^{1/2}]^2}, \tag{1.2}$$

where $d(n)=(n-2)^2(2n-2)^{-1}$.

$$\beta(n, p)=\frac{(p-1)n(n+2)(n-2)^2}{(2p-3)C^2(n)[4n^2(n-1)^2+(n+2)(n-2)^2]}, \tag{1.3}$$

$$\gamma(n, p)=\frac{pn(n+2)(n-2)^2}{(2p-1)C^2(n)[4n^2(n-1)^2+(n+2)(n-2)^2]}, \tag{1.4}$$

$$C_1(n, p, H, \chi(M))=\begin{cases}\alpha^{n/2}(n, H), & \text{for } H\neq 0, n\geqq 3 \text{ and } p=1,\\ \min\{\alpha^{n/2}(n, H), \beta^{n/2}(n, p)\}, & \text{for } H\neq 0, n\geqq 3 \text{ and } p\geqq 2,\\ \gamma^{n/2}(n, p), & \text{for } H=0 \text{ and } n\geqq 3,\\ 8\pi+4\pi|\chi(M)|, & \text{for } n=2,\end{cases} \tag{1.5}$$

where $\chi(M)$ is the Euler characteristic of M.

$$C_2(n, p)=\begin{cases}\alpha_1^{n/2}(n), & \text{for } H\neq 0,\ n\geqq 3 \text{ and } p=1,\\ \min\{\alpha_1^{n/2}(n), \beta^{n/2}(n, p)\}, & \text{for } H\neq 0,\ n\geqq 3 \text{ and } p\geqq 2,\\ \gamma^{n/2}(n, p), & \text{for } H=0 \text{ and } n\geqq 3,\\ 16\pi, & \text{for } n=2 \text{ and the genus } g(M)=0,\\ 8\pi, & \text{for } n=2 \text{ and the genus } g(M)\geqq 1.\end{cases} \tag{1.6}$$

By using a different argument, we obtain the following

THEOREM 1. *Let M^n be an n-dimensional oriented closed submanifold with parallel mean curvature in $S^{n+p}(1)$. If $\int_M(S-nH^2)^{n/2}<C_1(n, p, H, \chi(M))$, then M is a totally umbilical submanifold, and hence M is isometric to the sphere $S^n(1/\sqrt{1+H^2})$.*

THEOREM 2. *Let M^n be an oriented closed submanifold with parallel mean curvature in $S^{n+p}(1)$. If $\int_M(S-nH^2)^{n/2}<C_2(n, p)$, then M is totally umbilical.*

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2. Preliminaries.

Let M^n be an n-dimensional compact manifold immersed in an $(n+p)$-dimensional unit sphere $S^{n+p}(1)$. We will always take M to be oriented, and make use of the following convention on the range of indices:

$$1 \leqq A, B, C, \cdots \leqq n+p, \quad 1 \leqq i, j, k, \cdots \leqq n, \quad n+1 \leqq \alpha, \beta, \gamma, \cdots \leqq n+p.$$

We choose a local field of orthonormal frames $e_1, e_2, \cdots, e_{n+p}$ in $S^{n+p}(1)$ such that, restricted to M, the vectors $e_1, e_2, \cdots, e_n$ are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the field of dual frames and the connection 1-forms of $S^{n+p}(1)$ respectively. Restricting these forms to M, we have

$$\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega_j, \qquad h_{ij}^\alpha = h_{ji}^\alpha, \tag{2.1}$$

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{2.2}$$

$$R_{\alpha\beta kl} = \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta), \tag{2.3}$$

$$B = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \qquad \xi = \frac{1}{n} \sum_{\alpha,i} h_{ii}^\alpha e_\alpha, \tag{2.4}$$

where R_{ijkl}, $R_{\alpha\beta kl}$, B and ξ are the curvature tensor, the normal curvature tensor, the second fundamental form and the mean curvature vector of M respectively. We define

$$S = \|B\|^2, \qquad H = \|\xi\|, \qquad H^\alpha = (h_{ij}^\alpha)_{n\times n}. \tag{2.5}$$

M is called a submanifold with parallel mean curvature if ξ is parallel in the normal bundle of M. In particular, M is called minimal if $\xi=0$ identically.

When $\xi \neq 0$, we choose e_{n+1} such that $e_{n+1} // \xi$, $\operatorname{tr} H^{n+1} = nH$ and $\operatorname{tr} H^\beta = 0$, $n+2 \leqq \beta \leqq n+p$. The following propositions will be used in the proof of our theorems.

PROPOSITION 1. *Let M^n be a submanifold with parallel mean curvature in $S^{n+1}(1)$. Denote $\sum_{i,j}(h_{ij}^{n+1})^2$ and $\sum_{i,j,\beta\neq n+1}(h_{ij}^\beta)^2$ by S_H and S_I respectively. Then*

(i) *If $H=0$, then*

$$\frac{1}{2}\Delta S \geqq \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 - \left(2 - \frac{1}{p}\right) S^2 + nS,$$

(ii) *If $H \neq 0$, then*

$$\frac{1}{2}\Delta S_H \geqq \sum_{i,j,k} (h_{ijk}^{n+1})^2 + (S_H - nH^2)\left[n + 2nH^2 - S - \frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S-nH^2}\right],$$

$$\frac{1}{2}\Delta S_I \geqq \sum_{\substack{i,j,k\\ \beta\neq n+1}}(h^\beta_{ijk})^2+nH\sum_{\beta\neq n+1}\mathrm{tr}\,[H^{n+1}(H^\beta)^2]-\sum_{\beta\neq n+1}[\mathrm{tr}\,(H^{n+1}H^\beta)]^2$$
$$+nS_I-\Big(2-\frac{1}{p-1}\Big)S_I^2\,, \qquad \text{for } p\neq 1\,.$$

PROOF. (i) If $H=0$, see [**10**, §5.3].

(ii) If $H\neq 0$, then $\omega_{n+1,\alpha}=0$, for all α. This together with (2.3) and the structure equations of $S^{n+p}(1)$ (see [**12**, I, p. 348]) implies

$$R_{n+1\alpha kl}=0\,, \qquad \text{for all } \alpha,\ k,\ l\,. \tag{2.6}$$

Following [**12**, I, p. 351], we have

$$\Delta h^\alpha_{ij}=\sum_{k,m}h^\alpha_{km}R_{mijk}+\sum_{k,m}h^\alpha_{im}R_{mkjk}+\sum_{k,\beta}h^\beta_{ki}R_{\beta\alpha jk}\,. \tag{2.7}$$

It follows from (2.2), (2.6) and (2.7) that

$$\frac{1}{2}\Delta S_H=\sum_{i,j,k}(h^{n+1}_{ijk})^2+nS_H-S_H^2-n^2H^2+nH\,\mathrm{tr}\,(H^{n+1})^3$$
$$-\sum_{\beta\neq n+1}[\mathrm{tr}\,(H^{n+1}H^\beta)]^2\,. \tag{2.8}$$

Let $\{e_i\}$ be a frame diagonalizing the matrix (h^{n+1}_{ij}) such that $h^{n+1}_{ij}=\lambda_i\delta_{ij}$, $1\leqq i,\ j\leqq n$. Set $\mu_i=H-\lambda_i$, $A_k=\sum_i \lambda_i^k$ and $B_k=\sum_i \mu_i^k$. Then

$$B_1=0, \qquad B_2=A_2-nH^2, \qquad B_3=3HA_2-2nH^3-A_3\,. \tag{2.9}$$

By using Lagrange's method, we have $B_3\leqq(n-2)[n(n-1)]^{-1/2}B_2^{3/2}$. Combining (2.8), (2.9) and this, we obtain

$$\frac{1}{2}\Delta S_H\geqq\sum_{i,j,k}(h^{n+1}_{ijk})^2+B_2\Big[n+2nH^2-S-\frac{n(n-2)}{\sqrt{n(n-1)}}HB_2^{1/2}\Big]$$
$$\geqq\sum_{i,j,k}(h^{n+1}_{ijk})^2+(S_H-nH^2)\Big[n+2nH^2-S-\frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S-nH^2}\,\Big].$$

When $p\geqq 2$, it is straightforward to see from (2.2), (2.3), (2.6) and (2.7) that

$$\frac{1}{2}\Delta S_I=\sum_{\substack{i,j,k\\ \beta\neq n+1}}(h^\beta_{ijk})^2+nH\sum_{\beta\neq n+1}\mathrm{tr}\,[H^{n+1}(H^\beta)^2]-\sum_{\beta\neq n+1}[\mathrm{tr}\,(H^{n+1}H^\beta)]^2+nS_I$$
$$-\sum_{\alpha,\beta\neq n+1}\mathrm{tr}\,(H^\alpha H^\beta-H^\beta H^\alpha)^2-\sum_{\alpha,\beta\neq n+1}[\mathrm{tr}\,(H^\alpha H^\beta)]^2\,.$$

Using the same argument as in the proof of [**10**, Lemma 5.31], we get

$$\sum_{\alpha,\beta\neq n+1}\mathrm{tr}\,(H^\alpha H^\beta-H^\beta H^\alpha)^2+\sum_{\alpha,\beta\neq n+1}[\mathrm{tr}\,(H^\alpha H^\beta)]^2\leqq\Big(2-\frac{1}{p-1}\Big)S_I^2\,.$$

So

$$\frac{1}{2}\Delta S_I \geqq \sum_{\substack{i,j,k\\ \beta\neq n+1}}(h_{ijk}^{\beta})^2+nH\sum_{\beta\neq n+1}\operatorname{tr}[H^{n+1}(H^{\beta})^2]-\sum_{\beta\neq n+1}[\operatorname{tr}(H^{n+1}H^{\beta})]^2$$
$$+nS_I-\Big(2-\frac{1}{p-1}\Big)S_I^2\,.$$

PROPOSITION 2 ([**5**, Theorem 2.1]). *Let $M^n\to N^{n+p}$ be a compact submanifold with boundary. Suppose N is a simply connected and complete manifold with nonpositive sectional curvature. Then for all $f\in C^1(M)$, $f\geqq 0$ and $f|_{\partial M}=0$, f satisfies*

$$\Big(\int_M f^{n/(n-1)}\Big)^{(n-1)/n}\leqq C(n)\int_M(|\nabla f|+fH)\,.$$

3. Proof of the theorems.

LEMMA 1. *Let M^n be a submanifold with parallel mean curvature in $S^{n+p}(1)$. Set $f_\varepsilon=(S_H-nH^2+n\varepsilon^2)^{1/2}$, $g_\varepsilon=(S_I+n(p-1)\varepsilon^2)^{1/2}$ and $h_\varepsilon=(S+np\varepsilon^2)^{1/2}$.*

(i) *If $H\neq 0$, then*

$$\sum_{i,j,k}(h_{ijk}^{n+1})^2\geqq\frac{n+2}{n}|\nabla f_\varepsilon|^2\,,$$

$$\sum_{\substack{i,j,k\\ \beta\neq n+1}}(h_{ijk}^{\beta})^2\geqq\frac{n+2}{n}|\nabla g_\varepsilon|^2,\qquad \textit{for } p\neq 1\,.$$

(ii) *If $H=0$, then*

$$\sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2\geqq\frac{n+2}{n}|\nabla h_\varepsilon|^2\,.$$

PROOF. If $H\neq 0$, putting $x_{ij}^{n+1}=h_{ij}^{n+1}-H\delta_{ij}+\varepsilon\delta_{ij}$, we have

$$x_{ijk}^{n+1}=h_{ijk}^{n+1}\,, \tag{3.1}$$

$$\sum_{i,j,k}(x_{ijk}^{n+1})^2=\sum_{i,j,k}(h_{ijk}^{n+1})^2\,. \tag{3.2}$$

Let $\{e_i\}$ be a frame diagonalizing the matrix (h_{ij}^{n+1}) such that $h_{ij}^{n+1}=\lambda_i\delta_{ij}$, $1\leqq i$, $j\leqq n$. Then

$$x_{ij}^{n+1}=(\lambda_i-H+\varepsilon)\delta_{ij}\,, \tag{3.3}$$

$$\sum_{i,j}(x_{ij}^{n+1})^2=f_\varepsilon^2\,, \tag{3.4}$$

$$(2f_\varepsilon|\nabla f_\varepsilon|)^2=|\nabla f_\varepsilon^2|^2=4\sum_k\Big(\sum_i x_{ii}^{n+1}x_{iik}^{n+1}\Big)^2$$
$$\leqq 4\Big[\sum_i(x_{ii}^{n+1})^2\Big]\Big[\sum_{i,k}(x_{iik}^{n+1})^2\Big]=4f_\varepsilon^2\Big[\sum_{i,k}(x_{iik}^{n+1})^2\Big]\,. \tag{3.5}$$

On the other hand, we have

$$\sum_{i,j,k}(x_{ijk}^{n+1})^2 \geqq 2\sum_{i\neq k}(x_{iik}^{n+1})^2+\sum_{i,k}(x_{iik}^{n+1})^2. \tag{3.6}$$

For each fixed k, we have

$$\begin{aligned}\sum_i (x_{iik}^{n+1})^2 &= \sum_{i\neq k}(x_{iik}^{n+1})^2+(\sum_i x_{iik}^{n+1}-\sum_{i\neq k}x_{iik}^{n+1})^2\\ &= \sum_{i\neq k}(x_{iik}^{n+1})^2+(\sum_{i\neq k}x_{iik}^{n+1})^2\\ &\leqq \sum_{i\neq k}(x_{iik}^{n+1})^2+(n-1)\sum_{i\neq k}(x_{iik}^{n+1})^2. \end{aligned}\tag{3.7}$$

Combining (3.2), (3.5), (3.6) and (3.7), we obtain

$$\sum_{i,j,k}(h_{ijk}^{n+1})^2 \geqq \frac{n+2}{n}\sum_{i,k}(x_{iik}^{n+1})^2 \geqq \frac{n+2}{n}|\nabla f_\varepsilon|^2.$$

If $H\neq 0$ and $p\geqq 2$, we put $x_{ij}^\beta=h_{ij}^\beta+\varepsilon\delta_{ij}$, $n+2\leqq\beta\leqq n+p$. By using the argument above, we obtain

$$|\nabla(g_\varepsilon^\beta)^2|^2 \leqq \frac{4n}{n+2}(g_\varepsilon^\beta)^2[\sum_{i,j,k}(h_{ijk}^\beta)^2], \tag{3.8}$$

where $g_\varepsilon^\beta=[\sum_{i,j}(x_{ij}^\beta)^2]^{1/2}$. From (3.8) we have

$$\begin{aligned}|\nabla g_\varepsilon^2| &\leqq \sum_{\beta\neq n+1}|\nabla(g_\varepsilon^\beta)^2|\\ &\leqq 2\sqrt{\frac{n}{n+2}}\sum_{\beta\neq n+1}[(g_\varepsilon^\beta)^2\sum_{i,j,k}(h_{ijk}^\beta)^2]^{1/2}\\ &\leqq 2\sqrt{\frac{n}{n+2}}(g_\varepsilon^2)^{1/2}[\sum_{\substack{i,j,k\\ \beta\neq n+1}}(h_{ijk}^\beta)^2]^{1/2}.\end{aligned}$$

It follows that

$$\sum_{\substack{i,j,k\\ \beta\neq n+1}}(h_{ijk}^\beta)^2 \geqq \frac{n+2}{n}|\nabla g_\varepsilon|^2.$$

If $H=0$, by repeating the arguments again, we get

$$\sum_{\alpha,i,j,k}(h_{ijk}^\alpha)^2 \geqq \frac{n+2}{n}|\nabla h_\varepsilon|^2.$$

LEMMA 2. *Let M^n be a closed Riemannian manifold. If the following Sobolev inequality holds for any compact subdomain $D\subset M$*

$$\left(\int_D f^{n/(n-1)}\right)^{(n-1)/n} \leqq C(n)\int_D(|\nabla f|+Hf),\quad \text{for all } f\in C^1(D),\ f\geqq 0 \text{ and } f|_{\partial D}=0,$$

then for all $f\in C^1(M)$, $f\geqq 0$, f satisfies

$$\left(\int_M f^{n/(n-1)}\right)^{(n-1)/n} \leqq C(n)\int_M(|\nabla f|+Hf).$$

PROOF. Let $B(p, \varepsilon)$ and $B(p, 2\varepsilon)$ be geodesic balls centered at p with radius ε and 2ε respectively. For any $f\in C^1(M)$, $f\geqq 0$, we put $f_\varepsilon=\chi_\varepsilon f$, where χ_ε is a smooth cut-off function, and

$$\chi_\varepsilon=\begin{cases} 0, & \text{for } x\in B_\varepsilon, \\ 1, & \text{for } x\in M\backslash B_{2\varepsilon}, \\ \chi_\varepsilon(x)\in[0,1], \text{ and } |\nabla\chi_\varepsilon|\leqq \dfrac{1}{\varepsilon}, & \text{for } x\in B_{2\varepsilon}\backslash B_\varepsilon. \end{cases}$$

Hence $f_\varepsilon\geqq 0$, $f_\varepsilon|_{\partial B(p,\varepsilon)}=0$. By the hypothesis, we have

$$\begin{aligned} \left(\int_{M\backslash B(p,\varepsilon)} f_\varepsilon^{n/(n-1)}\right)^{(n-1)/n} &\leqq C(n)\int_{M\backslash B(p,\varepsilon)}(|\nabla f_\varepsilon|+Hf_\varepsilon) \\ &\leqq C(n)\left\{\int_{B(p,2\varepsilon)}|\nabla\chi_\varepsilon|f+\int_{M\backslash B(p,\varepsilon)}\chi_\varepsilon(|\nabla f|+Hf)\right\} \\ &\leqq C(n)\left\{\max_{x\in M} f\cdot\frac{\mathrm{Vol}(B(p,2\varepsilon))}{\varepsilon}+\int_{M\backslash B(p,\varepsilon)}\chi_\varepsilon(|\nabla f|+Hf)\right\}. \qquad (3.9) \end{aligned}$$

Suppose that $\mathrm{Ric}_M\geqq-(n-1)\tau$, $\tau>0$. By using the comparison theorem for volume (see [8, p. 13]), we get

$$\begin{aligned} V(B(p, 2\varepsilon)) &\leqq \omega_{n-1}\int_0^{2\varepsilon}[\tau^{-1/2}\sinh(\sqrt{\tau}t)]^{n-1}dt \\ &\leqq \omega_{n-1}\tau^{-n/2}\int_0^{2\varepsilon\sqrt{\tau}}\left(\frac{\sinh(2\varepsilon\sqrt{\tau})}{2\varepsilon\sqrt{\tau}}\cdot t\right)^{n-1}dt \\ &= \omega_{n-1}\tau^{-n/2}\left(\frac{\sinh(2\varepsilon\sqrt{\tau})}{2\varepsilon\sqrt{\tau}}\right)^{n-1}\frac{(2\varepsilon\sqrt{\tau})^n}{n}. \end{aligned}$$

This means that $\lim_{\varepsilon\to 0}V(B(p, 2\varepsilon))/\varepsilon=0$. Therefore, as $\varepsilon\to 0$, the assertion of the lemma follows from (3.9).

LEMMA 3. *Let M^n be a closed submanifold in N^{n+p}, $n\geqq 3$. Suppose N is a simply connected and complete manifold with nonpositive sectional curvature. Then for all $t\in\boldsymbol{R}^+$ and $f\in C^1(M)$, $f\geqq 0$, f satisfies*

$$\|\nabla f\|_2^2\geqq\frac{(n-2)^2}{4(n-1)^2(1+t)}\left[\frac{1}{C^2(n)}\|f\|_{2n/(n-2)}^2-H_0^2\left(1+\frac{1}{t}\right)\|f\|_2^2\right],$$

where $H_0=\max_{x\in M}H$.

PROOF. By Proposition 2 and Lemma 2, we have

$$\left(\int_M g^{n/(n-1)}\right)^{(n-1)/n} \leqq C(n)\int_M(|\nabla g|+Hg), \quad \text{for all } g\in C^1(M),\ g\geqq 0. \tag{3.10}$$

Substituting $g=f^{2(n-1)/(n-2)}$ into (3.10), we get

$$\left(\int_M f^{2n/(n-2)}\right)^{(n-1)/n} \leqq \frac{2(n-1)}{n-2}C(n)\int_M f^{n/(n-2)}|\nabla f|+C(n)\int_M Hf^{2(n-1)/(n-2)}.$$

By using Hölder's inequality, we obtain

$$\|f\|_{2n/(n-2)} \leqq C(n)\left[\frac{2(n-1)}{n-2}\|\nabla f\|_2+H_0\|f\|_2\right].$$

This implies

$$\|f\|^2_{2n/(n-2)} \leqq C^2(n)\left[\frac{4(n-1)^2(1+t)}{(n-2)^2}\|\nabla f\|_2^2+H_0^2\left(1+\frac{1}{t}\right)\|f\|_2^2\right]. \tag{3.11}$$

The lemma follows.

LEMMA 4. *Let M^n be a closed submanifold in $S^{n+p}(1)$. Then for all $f\in C^1(M)$, $f\geqq 0$, f satisfies*

$$\|\nabla f\|_2^2 \geqq \frac{(n-2)^2}{4(n-1)^2(1+t)}\left[\frac{1}{C^2(n)}\|f\|^2_{2n/(n-2)}-(1+H_0^2)\left(1+\frac{1}{t}\right)\|f\|_2^2\right]. \tag{3.12}$$

PROOF. We consider the composition of isometric immersions $M^n \xrightarrow{\varphi} S^{n+p}(1) \xrightarrow{i} \boldsymbol{R}^{n+p+1}$, where i is the standard isometric embedding of $S^{n+p}(1)$ into $\boldsymbol{R}^{n+p+1}$. We denote the mean curvature and the relative mean curvature of $i\circ\varphi$ by $\bar{H}$ and H_R respectively. It is easy to see that $|H_R|\leqq 1$. Therefore $\bar{H}^2=H^2+H_R^2 \leqq H_0^2+1$. By Lemma 3, (3.12) holds for all $f\in C^1(M)$, $f\geqq 0$.

LEMMA 5. *Let M^n be a closed submanifold with parallel mean curvature in $S^{n+p}(1)$, $n\geqq 3$. Suppose that $H\neq 0$ and $\|S-nH^2\|_{n/2}<\alpha(n, H)$, where $\alpha(n, H)$ is defined by* (1.1). *Then M is pseudoumbilical. In particular, if $p=1$, then M is a hypersphere in $S^{n+1}(1)$.*

PROOF. By Proposition 1 and Lemma 1, we have

$$\frac{1}{2}\Delta S_H \geqq \sum_{i,j,k}(h_{ijk}^{n+1})^2+(S_H-nH^2)\left[n+2nH^2-S-\frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S-nH^2}\right], \tag{3.13}$$

where $S_H=\sum_{i,j}(h_{ij}^{n+1})^2$.

$$\sum_{i,j,k}(h_{ijk}^{n+1})^2 \geqq \frac{n+2}{n}|\nabla f_\varepsilon|^2, \tag{3.14}$$

where $f_\varepsilon=(S_H-nH^2+n\varepsilon^2)^{1/2}$. Putting $f=(S_H-nH^2)^{1/2}$, from (3.13), (3.14) and Lemma 4, we have

$$
\begin{aligned}
0 &= \frac{1}{2}\int_M \Delta S_H \\
&\geqq \int_M \left\{ \frac{n+2}{n} |\nabla f_\varepsilon|^2 + f^2\left[n+2nH^2 - S - \frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S-nH^2}\right]\right\} \\
&\geqq \frac{(n+2)(n-2)^2}{4n(n-1)^2(1+t)}\left[\frac{1}{C^2(n)}\|f_\varepsilon\|^2_{2n/(n-2)} - (1+H^2)\left(1+\frac{1}{t}\right)\|f_\varepsilon\|_2^2\right] \\
&\quad + \int_M f^2\left\{ n+nH^2 - (S-nH^2) - \frac{1}{2}\left[\frac{n(n-2)^2H^2}{r(n-1)} + r(S-nH^2)\right]\right\}, \qquad (3.15)
\end{aligned}
$$

for all $r \in \boldsymbol{R}^+$. As $\varepsilon \to 0$, (3.15) implies

$$
\begin{aligned}
0 &\geqq \frac{(n+2)(n-2)^2}{4n(n-1)^2(1+t)}\left[\frac{1}{C^2(n)}\|f\|^2_{2n/(n-2)} - (1+H^2)\left(1+\frac{1}{t}\right)\|f\|_2^2\right] \\
&\quad + \left[n+nH^2 - \frac{n(n-2)^2H^2}{2r(n-1)}\right]\|f\|_2^2 - \left(1+\frac{r}{2}\right)\|f^2\|_{n/(n-2)}\|S-nH^2\|_{n/2} \\
&= \left[\frac{(n+2)(n-2)^2}{4n(n-1)^2C^2(n)(1+t)} - \left(1+\frac{r}{2}\right)\|S-nH^2\|_{n/2}\right]\|f^2\|_{n/(n-2)} \\
&\quad + \left[n+nH^2 - \frac{n(n-2)^2H^2}{2r(n-1)} - \frac{(n+2)(n-2)^2(1+H^2)}{4n(n-1)^2 t}\right]\|f\|_2^2 . \qquad (3.16)
\end{aligned}
$$

We take $t=t(r)=(n+2)(n-2)^2/4n^2(n-1)^2[1-(n-2)^2H^2/2r(n-1)(1+H^2)]$, $r>(n-2)^2H^2/2(n-1)(1+H^2)$. This together with (3.16) yields

$$
\left[\frac{(n+2)(n-2)^2}{4n(n-1)^2C^2(n)(1+t)} - \left(1+\frac{r}{2}\right)\|S-nH^2\|_{n/2}\right]\|f^2\|_{n/(n-2)} \leqq 0 .
$$

Therefore, under the assumption

$$
\begin{aligned}
\|S-nH^2\|_{n/2} &< \alpha(n, H) \\
&= \frac{2na(n)}{C^2(n)[(a(n)b(n,H))^{1/2} + (1+a(n))^{1/2}(2+b(n,H))^{1/2}]^2} \\
&= \max_{r > \frac{(n-2)^2H^2}{2(n-1)(1+H^2)}} \frac{(n+2)(n-2)^2}{2n(n-1)^2C^2(n)(2+r)(1+t(r))},
\end{aligned}
$$

it is easy to see that $f=0$ and M is a pseudoumbilical submanifold.

PROOF OF THEOREM 1. If $H \neq 0$, $n \geqq 3$ and $p=1$, the theorem follows immediately from Lemma 5.

If $H \neq 0$ and $p \geqq 2$, by Proposition 1, we have

$$\frac{1}{2}\Delta S_I \geqq \sum_{\substack{i,j,k\\ \beta\neq n+1}}(h_{ijk}^{\beta})^2+nH\sum_{\beta\neq n+1}\operatorname{tr}[H^{n+1}(H^{\beta})^2]$$

$$-\sum_{\beta\neq n+1}[\operatorname{tr}(H^{n+1}H^{\beta})]^2+nS_I-\Big(2-\frac{1}{p-1}\Big)S_I^2, \qquad (3.17)$$

where $S_I=\sum_{i,j,\beta\neq n+1}(h_{ij}^{\beta})^2$. Since $\|S-nH^2\|_{n/2}<\alpha(n, H)$, one sees from Lemma 5 that $S_H=nH^2$, namely $H^{n+1}=HI$, where I is the unit matrix. This together with (3.17) implies

$$\frac{1}{2}\Delta S_I \geqq \sum_{\substack{i,j,k\\ \beta\neq n+1}}(h_{ijk}^{\beta})^2+\Big(2-\frac{1}{p-1}\Big)S_I\Big[-S_I+\frac{p-1}{2p-3}(n+nH^2)\Big]. \qquad (3.18)$$

Putting $g=S_I^{1/2}$, from (3.18) and Lemma 1, we obtain

$$\frac{1}{2}\Delta g^2 \geqq \frac{n+2}{n}|\nabla g_{\varepsilon}|^2+\Big(2-\frac{1}{p-1}\Big)g^2\Big[-g^2+\frac{p-1}{2p-3}(n+nH^2)\Big], \qquad (3.19)$$

where $g_{\varepsilon}=(S_I+n(p-1)\varepsilon^2)^{1/2}$. It follows from (3.19) and Lemma 4 that

$$0\geqq \frac{n+2}{n}\int_M|\nabla g_{\varepsilon}|^2+\Big(2-\frac{1}{p-1}\Big)\int_M g^2\Big[-g^2+\frac{p-1}{2p-3}(n+nH^2)\Big]$$

$$\geqq \frac{(n+2)(n-2)^2}{4n(n-1)^2(1+t)}\Big[\frac{1}{C^2(n)}\|g_{\varepsilon}\|_{2n/(n-2)}^2-(1+H^2)\Big(1+\frac{1}{t}\Big)\|g_{\varepsilon}\|_2^2\Big]$$

$$-\Big(2-\frac{1}{p-1}\Big)\|S-nH^2\|_{n/2}\|g^2\|_{n/(n-2)}+(n+nH^2)\|g\|_2^2.$$

As $\varepsilon\to 0$, this implies

$$0\geqq\Big[\frac{(n+2)(n-2)^2}{4n(n-1)^2C^2(n)(1+t)}-\Big(2-\frac{1}{p-1}\Big)\|S-nH^2\|_{n/2}\Big]\|g^2\|_{n/(n-2)}$$

$$+\Big[n+nH^2-\frac{(n+2)(n-2)^2(1+H^2)}{4n(n-1)^2t}\Big]\|g\|_2^2.$$

By taking $t=(n+2)(n-2)^2/4n^2(n-1)^2$, we have

$$\Big\{\frac{n(n+2)(n-2)^2}{C^2(n)[4n^2(n-1)^2+(n+2)(n-2)^2]}-\Big(2-\frac{1}{p-1}\Big)\|S-nH^2\|_{n/2}\Big\}\|g^2\|_{n/(n-2)}\leqq 0.$$

From the assumption

$$\|S-nH^2\|_{n/2}<\beta(n, p)=\frac{(p-1)n(n+2)(n-2)^2}{(2p-3)C^2(n)[4n^2(n-1)^2+(n+2)(n-2)^2]},$$

we see that $g=0$. Therefore $S-nH^2=0$ and M is totally umbilical.

If $H=0$ and $n\geqq 3$, by Proposition 1 and Lemma 1, we have

$$\frac{1}{2}\Delta S\geqq\frac{n+2}{n}|\nabla h_{\varepsilon}|^2+nS-\Big(2-\frac{1}{p}\Big)S^2,$$

where $h_{\varepsilon}=(S+np\varepsilon^2)^{1/2}$. By using an analogous argument, we can prove that if

$$\|S\|_{n/2} < \frac{pn(n+2)(n-2)^2}{(2p-1)C^2(n)[4n^2(n-1)^2+(n+2)(n-2)^2]},$$

then M is totally geodesic.

If $n=2$, it follows from [**2**, Theorem 2.1 on p. 106] that M is one of the following surfaces

(i) minimal surfaces in $S^{2+p}(1)$,

(ii) minimal surfaces in $S^{1+p}(1/\sqrt{1+H^2}) \subset S^{2+p}(1)$, $p \geqq 2$, or

(iii) surfaces with constant mean curvature in $S^3(1/\sqrt{c}) \subset S^{2+p}(1)$.

From the Gauss equation, we have

$$\int_M (2+4H^2-S) = 2\int_M K_M = 8\pi(1-g(M)),$$

where $g(M)$ is the genus of M. Hence

$$\int_M (S-2H^2) = 2(1+H^2)V(M)+8\pi(g(M)-1). \tag{3.20}$$

If M is a minimal surface with genus zero in $S^{2+p}(1)$ and $\int_M S < 16\pi$, it is clear from (3.20) that

$$V(M) < 12\pi. \tag{3.21}$$

From (3.21) and [**1**, Theorem 5.5], we see that M is a great 2-sphere in $S^{n+p}(1)$.

If M is a minimal surface with genus zero in $S^{1+p}(1/\sqrt{1+H^2})$ and $\int_M (S-nH^2) < 16\pi$, by a similar argument, we can show that M is a great 2-sphere in $S^{1+p}(1/\sqrt{1+H^2})$. Hence M is a small 2-sphere in $S^{2+p}(1)$.

If M is a surface with constant mean curvature in $S^3(1/\sqrt{c})$ and $g(M)=0$, then M is a 2-sphere in $S^{2+p}(1)$ (see [**12**, I, Theorem 5]).

If $g(M) \geqq 1$, it is easy to see from [**2**, Theorem 3.2 on p. 220]

$$V(M) \geqq \frac{4\pi}{1+H^2}.$$

Substituting into (3.20), we have

$$\int_M (S-2H^2) \geqq 8\pi g(M) \geqq 8\pi.$$

Therefore, if $n=2$ and $\int_M (S-2H^2) < 8\pi + 8\pi|g(M)-1|$, then M is totally umbilical. This completes the proof of Theorem 1.

PROOF OF THEOREM 2. It is easy to prove that

$$\alpha(n, H) \geqq \max_{r > \frac{(n-2)^2}{2(n-1)}} \frac{(n+2)(n-2)^2}{2nC^2(n)(n-1)^2(2+r)(1+\bar{t}(r))} = \alpha_1(n),$$

where $\bar{t}(r)=(n+2)(n-2)^2/4n^2(n-1)^2[1-(n-2)^2/2(n-1)r]$. This implies

$$C_1(n, p, H, \chi(M)) \geqq C_2(n, p).$$

Therefore the assertion follows from Theorem 1.

THEOREM 3. *Let M^n be an oriented closed submanifold with parallel mean curvature in $S^{n+p}(1)$. Suppose M is not totally umbilical. Then*

$$\int_M (S-nH^2)^{n/2} \geqq C_2(n, p),$$

$$\int_M S^{n/2} \geqq C_2(n, p)+\omega_n\left(\frac{nH^2}{1+H^2}\right)^{n/2},$$

where $\omega_n = Area(S^n)$.

PROOF. The first inequality follows immediately from Theorem 2.

Since $S-nH^2 \geqq 0$, we know that $S^{n/2} \geqq (S-nH^2)^{n/2}+(nH^2)^{n/2}$. On the other hand, we see from [**2**, Theorem 3.2 on p. 220] that

$$V(M) \geqq \frac{\omega_n}{(1+H^2)^{n/2}}.$$

Therefore

$$\begin{aligned}\int_M S^{n/2} &\geqq \int_M (S-nH^2)^{n/2}+\int_M (nH^2)^{n/2} \\ &\geqq C_2(n, p)+n^{n/2}H^n V(M) \\ &\geqq C_2(n, p)+\omega_n\left(\frac{nH^2}{1+H^2}\right)^{n/2}.\end{aligned}$$

Let $f: M \to G_{n+1, p}$ be the Gauss map associated to the immersion $\varphi: M \to S^{n+p}$. The energy density is given by $e(f)=\|df\|^2$. By a direct computation, one sees that $e(f)=S$. If f is harmonic, then M is minimal (see [**3**, Theorem 2]). Hence we get the following

THEOREM 4. *Let M^n be an oriented closed submanifold in $S^{n+p}(1)$. If the Gauss map $f: M \to G_{n+1, p}$ is harmonic, then either f is constant or $\int_M e^{n/2}(f) \geqq C_2(n, p)$.*

REMARK. When $n=2$ and $g(M)=0$, the pinching constant $C_2(n, p)$ is the best possible. For an example, we consider the standard minimal immersion $M=S^2(\sqrt{3}) \to S^4(1)$. Then $S=4/3$, $V(M)=12\pi$ and $\int_M S=16\pi=C_2(n, p)$.

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