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An approach to the characteristic free Dutta multiplicity

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Introduction.

In "Algèbre locale. Multiplicités" [20], Serre conjectured:

CONJECTURE. *Let X be a connected regular scheme and Y, Z closed irreducible subschemes of X. Then, for each irreducible component W of $Y \cap Z$,*

(S1) $\operatorname{codim}(W, X) \leqq \operatorname{codim}(Y, X) + \operatorname{codim}(Z, X)$,

(S2) *if* $\operatorname{codim}(W, X) < \operatorname{codim}(Y, X) + \operatorname{codim}(Z, X)$, *then*

$$\sum_i (-1)^i l_{\mathcal{O}_{W,X}}(\operatorname{Tor}_i^{\mathcal{O}_{W,X}}(\mathcal{O}_{W,Y}, \mathcal{O}_{W,Z})) = 0\,,$$

(S3) *if* $\operatorname{codim}(W, X) = \operatorname{codim}(Y, X) + \operatorname{codim}(Z, X)$, *then*

$$\sum_i (-1)^i l_{\mathcal{O}_{W,X}}(\operatorname{Tor}_i^{\mathcal{O}_{W,X}}(\mathcal{O}_{W,Y}, \mathcal{O}_{W,Z})) > 0\,.$$

In [20] Serre proved (S1) in general and (S2), (S3) in the case where the regular local ring $\mathcal{O}_{W,X}$ is unramified, i.e., either $\mathcal{O}_{W,X}$ contains a field or the square of its maximal ideal does not contain p, where $p > 0$ is the characteristic of the residue class field $\mathcal{O}_{W,X}/\mathcal{M}_{W,X}$. Furthermore Roberts [17], Gillet and Soulé [9] independently solved (S2) affirmatively. (Roberts proved (S2) under a weaker condition ([15], [16], [17]) using the intersection theory. Dutta, Hochster and MacLaughlin [6] constructed the following important example:

Put $A = k[[x, y, z, w]]/(xy - zw)$ (k is a field), and $M = A/(x, z)$. Then there exists an A-module N such that $l_A(N) = 15$, $pd_A N = 3$ and $\sum_i (-1)^i l_A(\operatorname{Tor}_i^A(M, N)) = -1$.
We can explain this phenomenon in terms of localized Chern characters as in Example 18.3.14 in [8].)

(S1) is a remarkable result which enables us to estimate the minimum of the dimension of the intersection of two closed irreducible subschemes when they actually intersect. It is expected that such an inequality holds even under a weaker condition. (By the intersection theorem due to Roberts [18], we have

$$\dim M + \operatorname{depth} N \leqq \operatorname{depth} A$$

for any Noetherian local ring A and any finitely generated A-modules M and

N such that $l_A(M \otimes_A N) < \infty$ and $pd_A N < \infty$.)

When $\boldsymbol{G}.$ (resp. $\boldsymbol{H}.$) is the minimal $\mathcal{O}_{W,X}$-free resolution of $\mathcal{O}_{W,Y}$ (resp. $\mathcal{O}_{W,Z}$), it follows from the argument in [**17**] that

$$\sum_i (-1)^i l_{\mathcal{O}_{W,X}}(\mathrm{Tor}_i^{\mathcal{O}_{W,X}}(\mathcal{O}_{W,Y}, \mathcal{O}_{W,Z}))$$

$$= \mathrm{ch}_{\mathrm{Spec}(\mathcal{O}_{W,W})}^{\mathrm{Spec}(\mathcal{O}_{W,Y})}(\boldsymbol{H}. \otimes_{\mathcal{O}_{W,X}} \mathcal{O}_{W,Y}) \cap [\mathrm{Spec}(\mathcal{O}_{W,Y})]$$

$$= \mathrm{ch}_{\mathrm{Spec}(\mathcal{O}_{W,W})}^{\mathrm{Spec}(\mathcal{O}_{W,Z})}(\boldsymbol{G}. \otimes_{\mathcal{O}_{W,X}} \mathcal{O}_{W,Z}) \cap [\mathrm{Spec}(\mathcal{O}_{W,Z})]$$

in the case of $\mathrm{codim}(W, X) = \mathrm{codim}(Y, X) + \mathrm{codim}(Z, X)$, where $\mathrm{ch}_*^*(*)$ is the localized Chern character and $[\mathrm{Spec}(\mathcal{O}_{W,Y})]$ and $[\mathrm{Spec}(\mathcal{O}_{W,Z})]$ are cycles in the Chow groups. So, in order to prove (S3), it seems to be crucial to calculate $\mathrm{ch}_{\mathrm{Spec}(A/\mathfrak{m})}^{\mathrm{Spec}(A)}(\boldsymbol{F}.) \cap [\mathrm{Spec}(A)]$ for a Noetherian local ring $(A, \mathfrak{m})$ and a bounded A-free complex $\boldsymbol{F}.$ which is exact except for $\{\mathfrak{m}\}$. Such an invariant is called the Dutta multiplicity (see [**4**], [**15**], [**16**], [**18**], [**19**]), which is the main theme of the present paper. The Dutta multiplicity is a natural generalization of the usual multiplicity (see Definition 2.3 and Remark 2.5).

The next section is devoted to defining the Dutta multiplicity and discussing its basic properties. In section 3 we will prove (S3) in a special case using some results on the Dutta multiplicity and give an algebraic description to this multiplicity in the case where the dimension of the given local ring is less than or equal to 3. Section 4 is devoted to arguing the difference between the Dutta multiplicity and the alternative sum of the lengths of the homology modules of a given perfect complex. Furthermore we will prove

THEOREM 4.3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and $\boldsymbol{F}.$ a perfect A-complex of length d with support $\{\mathfrak{m}\}$. Suppose that one of the following conditions is satisfied:*

(0) *$(A, \mathfrak{m})$ is a Gorenstein ring.*

(1) *$d \leqq 2$ and A is equi-dimensional.*

(2) *$(A, \mathfrak{m})$ is normal with $d \leqq 4$ and the canonical class $\mathrm{cl}(K_A)$ is torsion in the divisor class group $\mathrm{Cl}(A)$.*

(3) *$d \leqq 3$ and $\mathrm{A}_{d-1}\,\mathrm{Spec}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} = (0)$.*

(4) *There exists a regular local ring $(S, \mathfrak{n})$ and a finite free S-complex $\boldsymbol{G}.$ such that A is a homomorphic image of S and $\boldsymbol{G}. \otimes_S A$ is isomorphic to $\boldsymbol{F}.$.*

Then $\sum_{i=0}^d (-1)^i l_A(H_i(\boldsymbol{F}.)) = \sum_{i=0}^d (-1)^i l_A(H_i(\boldsymbol{F}.^*[-d]))$ holds.

We also give an example satisfying $\sum_{i=0}^d (-1)^i l_A(H_i(\boldsymbol{F}.)) \neq \sum_{i=0}^d (-1)^i l_A(H_i(\boldsymbol{F}.^*[-d]))$. The last section is devoted to proving

THEOREM 5.2. *Let $(A, \mathfrak{m})$ be a normal Noetherian local ring of dimension 3, and*

$$F.: 0 \longrightarrow F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

a minimal self-dual perfect A-complex with support $\{\mathfrak{m}\}$. *Assume the following three conditions.*

(T1) $\mathrm{rank}_A F_0 = 1$.

(T2) $\mu_A(I_1(d_1)) = \mathrm{rank}_A F_1$.

(T3) *All the Koszul relations of* d_1 *are contained in* $d_2(F_2)$.

Then, $\mathbf{D}_A(\boldsymbol{F}.) = l_A(H_0(\boldsymbol{F}.)) - l_A(H_1(\boldsymbol{F}.)) > 0$.

The idea for proving this is to use the structure theorem of Gorenstein ideals of codimension 3 due to Buchsbaum and Eisenbud [2]. (When we show the positivity of the Dutta multiplicity of the perfect complex $\boldsymbol{F}.$ of length 3, we may assume that $\boldsymbol{F}.$ is self-dual by Remark 3.6.)

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2. Definition and basic properties.

Throughout this paper we assume that all Noetherian local rings are homomorphic images of regular local rings and p denotes a prime integer. For an A-module M, $l_A(M)$ (resp. $pd_A(M)$) denotes the length (resp. the projective dimension) of M.

This section will be devoted to defining the Dutta multiplicity (see [4] in the case of characteristic p and [18] in the general case) and arguing basic properties on the Dutta multiplicity.

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d.

DEFINITION 2.1. A complex $\boldsymbol{F}.$ is said to be *perfect* when all F_i's are finitely generated free modules such that $F_0 \neq 0$, $F_i = 0$ for $i < 0$ and $i \gg 0$. For a perfect complex $\boldsymbol{F}.$, we define the *support* of $\boldsymbol{F}.$ by

$$\bigcup_i \mathrm{supp}(H_i(\boldsymbol{F}.)) \subseteq \mathrm{Spec}(A)$$

and denote it by $\mathrm{supp}(\boldsymbol{F}.)$. $\boldsymbol{F}.$ is called a perfect complex of *length* n when it is perfect with $n = \max\{i \mid F_i \neq 0\}$.

REMARK 2.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and

$$\boldsymbol{F}.: 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

a perfect complex with support $\{\mathfrak{m}\}$. Then the new intersection theorem (Roberts [18]) guarantees the inequality $n \geqq d$.

When A is Cohen-Macaulay and n is equal to d, $H_i(\boldsymbol{F}.)$ vanishes for every $i > 0$. In general, for any minimal perfect complex

$$\boldsymbol{F}. : 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0\,,$$

with support $\{\mathfrak{m}\}$ such that $F_n \neq 0$ (*minimal* means that all the boundary maps of $\boldsymbol{F}.\otimes_A A/\mathfrak{m}$ vanish), the property so-called the *depth sensitivity* holds ([**3**]), i.e., for any finitely generated A-module M,

$$\operatorname{depth} M = n - \max\{i \mid H_i(\boldsymbol{F}.\otimes M) \neq 0\}\,.$$

Definition 2.3. For a Noetherian local ring $(A, \mathfrak{m})$ of dimension d and a perfect A-complex $\boldsymbol{F}.$ with support $\{\mathfrak{m}\}$, the rational number

$$\mathrm{ch}^{\mathrm{Spec}(A)}_{\mathrm{Spec}(A/\mathfrak{m})}(\boldsymbol{F}.)\cap[\mathrm{Spec}(A)]$$

is called the *Dutta multiplicity* of $\boldsymbol{F}.$ and denoted by $\mathbf{D}_A(\boldsymbol{F}.)$. (See Roberts [**18**].)

(The map $\mathrm{ch}^{\mathrm{Spec}(A)}_{\mathrm{Spec}(A/\mathfrak{m})}(\boldsymbol{F}.) : \mathrm{A}_*\,\mathrm{Spec}(A)_{\boldsymbol{Q}} \to \mathrm{A}_*\,\mathrm{Spec}(A/\mathfrak{m})_{\boldsymbol{Q}} = \boldsymbol{Q}$ is the localized Chern character determined by $\boldsymbol{F}.$ (see Fulton [8]) and $\mathrm{A}_*(-)_{\boldsymbol{Q}}$ is the rational Chow group. Furthermore $[\mathrm{Spec}(A)]$ is an element of $\mathrm{A}_d\,\mathrm{Spec}(A)_{\boldsymbol{Q}}$ defined by

$$[\mathrm{Spec}(A)] = \sum_{\substack{\mathfrak{p}\in\mathrm{Spec}(A)\\ \dim A/\mathfrak{p}=d}} l_{A_\mathfrak{p}}(A_\mathfrak{p})\cdot[\mathrm{Spec}(A/\mathfrak{p})]\,,$$

where the above sum runs over all prime ideals of coheight d.)

In order to calculate the intersection multiplicities of modules, it is very crucial to investigate the Dutta multiplicities of perfect complexes. For example, see Roberts [**15**], [**17**].

Remark 2.4. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and $\boldsymbol{F}.$ a perfect A-complex with support $\{\mathfrak{m}\}$. Since localized Chern characters are compatible with proper push-forwards (see Fulton [8]), we have

$$\mathrm{ch}^{\mathrm{Spec}(A)}_{\mathrm{Spec}(A/\mathfrak{m})}(\boldsymbol{F}.)\cap[\mathrm{Spec}(A/\mathfrak{p})] = \mathrm{ch}^{\mathrm{Spec}(A/\mathfrak{p})}_{\mathrm{Spec}(A/\mathfrak{m})}(\boldsymbol{F}.\otimes_A A/\mathfrak{p})\cap[\mathrm{Spec}(A/\mathfrak{p})]$$

for any prime ideal $\mathfrak{p}$. So, it holds

$$\begin{aligned}\mathbf{D}_A(\boldsymbol{F}.) &= \mathrm{ch}^{\mathrm{Spec}(A)}_{\mathrm{Spec}(A/\mathfrak{m})}(\boldsymbol{F}.)\cap[\mathrm{Spec}(A)]\\ &= \sum_{\dim A/\mathfrak{p}=d} l_{A_\mathfrak{p}}(A_\mathfrak{p})\cdot(\mathrm{ch}^{\mathrm{Spec}(A)}_{\mathrm{Spec}(A/\mathfrak{m})}(\boldsymbol{F}.)\cap[\mathrm{Spec}(A/\mathfrak{p})])\\ &= \sum_{\dim A/\mathfrak{p}=d} l_{A_\mathfrak{p}}(A_\mathfrak{p})\cdot(\mathrm{ch}^{\mathrm{Spec}(A/\mathfrak{p})}_{\mathrm{Spec}(A/\mathfrak{m})}(\boldsymbol{F}.\otimes_A A/\mathfrak{p})\cap[\mathrm{Spec}(A/\mathfrak{p})])\\ &= \sum_{\dim A/\mathfrak{p}=d} l_{A_\mathfrak{p}}(A_\mathfrak{p})\cdot\mathbf{D}_{A/\mathfrak{p}}(\boldsymbol{F}.\otimes_A A/\mathfrak{p})\,.\end{aligned}$$

We note that $\boldsymbol{F}.\otimes_A A/\mathfrak{p}$ is a perfect $A/\mathfrak{p}$-complex with support $\{\mathfrak{m}/\mathfrak{p}\}$.

Therefore, when we calculate the Dutta multiplicity, we may assume that the given ring $(A, \mathfrak{m})$ is an integral domain like the usual multiplicity.

The next remark implies that the notion of the Dutta multiplicity is a natural generalization of the usual multiplicity.

REMARK 2.5. Let $x_1, \cdots, x_d$ be a system of parameters of a Noetherian local ring $(A, \mathfrak{m})$. Denote by $\boldsymbol{K}.(\underline{x}, A)$ the Koszul complex determined by $x_1, \cdots, x_d$. By Corollary 18.1.2 and Example 18.3.12 in [8], we have

$$\begin{aligned}\mathbf{D}_A(\boldsymbol{K}.(\underline{x}, A)) &= \mathrm{ch}_{\mathrm{Spec}(A/\mathfrak{m})}^{\mathrm{Spec}(A)}(\boldsymbol{K}.(\underline{x}, A)) \cap [\mathrm{Spec}(A)] \\ &= \sum_i (-1)^i l_A(H_i(\boldsymbol{K}.(\underline{x}, A))) \\ &= e_{(\underline{x})}(A),\end{aligned}$$

where $e_{(\underline{x})}(A)$ is the usual multiplicity of A along the parameter ideal $(\underline{x})=(x_1, \cdots, x_d)$. So, when $\{x_1, \cdots, x_d\}$ is a minimal reduction of $\mathfrak{m}$, the Dutta multiplicity $\mathbf{D}_A(\boldsymbol{K}.(\underline{x}, A))$ coincides with the usual multiplicity $e_{\mathfrak{m}}(A)$. Furthermore if $\{x_1, \cdots, x_d\}$ is a minimal reduction both of $\mathfrak{m}$ and $\mathfrak{m}/\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of coheight d, then we obtain the following famous formula on the usual multiplicity:

$$\begin{aligned}e_{\mathfrak{m}}(A) &= \mathrm{ch}_{\mathrm{Spec}(A/\mathfrak{m})}^{\mathrm{Spec}(A)}(\boldsymbol{K}.(\underline{x}, A)) \cap [\mathrm{Spec}(A)] \\ &= \sum_{\dim A/\mathfrak{p}=d} l_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) \cdot (\mathrm{ch}_{\mathrm{Spec}(A/\mathfrak{m})}^{\mathrm{Spec}(A/\mathfrak{p})}(\boldsymbol{K}.(\underline{x}, A/\mathfrak{p})) \cap [\mathrm{Spec}(A/\mathfrak{p})]) \\ &= \sum_{\dim A/\mathfrak{p}=d} l_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) \cdot e_{\mathfrak{m}/\mathfrak{p}}(A/\mathfrak{p}).\end{aligned}$$

In order to calculate the Dutta multiplicity, we may assume that the given local ring $(A, \mathfrak{m})$ is complete and the residue class field $A/\mathfrak{m}$ is algebraically closed as follows.

PROPOSITION 2.6. *Let $C \to D$ be a faithfully flat extension of regular local rings such that* $\dim C = \dim D$. *Suppose that $A=C/I$ is a Noetherian local ring with the maximal ideal* $\mathfrak{m}$. *Then for a perfect A-complex $\boldsymbol{F}.$ with support* $\{\mathfrak{m}\}$, $\mathbf{D}_{D/ID}(\boldsymbol{F}.\otimes_A D/ID) = \mathbf{D}_A(\boldsymbol{F}.) \cdot l_{D/ID}(D/\mathfrak{m}D)$.

PROOF. First note that $\boldsymbol{F}.\otimes_A D/ID$ is a perfect D/ID-complex such that its support is only at the maximal ideal of D/ID and $l_{D/ID}(D/\mathfrak{m}D) < \infty$.

For the simplicity of notation we put $B = D/ID$. Though $A \to B$ and $A/\mathfrak{m} \to B/\mathfrak{m}B$ are not necessarily of finite type, we can define the flat pull-backs $\mathrm{A}_* \mathrm{Spec}(B) \to \mathrm{A}_* \mathrm{Spec}(A)$ and $\mathrm{A}_* \mathrm{Spec}(B/\mathfrak{m}B) \to \mathrm{A}_* \mathrm{Spec}(A/\mathfrak{m})$. (Generalize the results of Section 1.7 in [8].) Then the diagram

$$\begin{array}{ccc}
A_* \operatorname{Spec}(A)_Q & \xrightarrow{\mathrm{ch}^{\operatorname{Spec}(A)}_{\operatorname{Spec}(A/\mathfrak{m})}(F.)} & A_* \operatorname{Spec}(A/\mathfrak{m})_Q \\
\downarrow & & \downarrow \\
A_* \operatorname{Spec}(B)_Q & \xrightarrow{\mathrm{ch}^{\operatorname{Spec}(B)}_{\operatorname{Spec}(B/\mathfrak{m}B)}(F.\otimes_A B)} & A_* \operatorname{Spec}(B/\mathfrak{m}B)_Q
\end{array}$$

is commutative. (Generalize Theorem 18.1 in [8].)

So, we have $\mathbf{D}_B(\boldsymbol{F}.\otimes_A B)=\mathbf{D}_A(\boldsymbol{F}.)\cdot l_B(B/\mathfrak{m}B)$. Q. E. D.

Therefore, when we investigate the Dutta multiplicities of perfect complexes such that their supports are at the maximal ideal of a given Noetherian local ring, we may assume that the given ring is a complete local domain such that its residue class field is algebraically closed.

In the case of positive characteristic, we can express the Dutta multiplicity by a purely algebraic method (see Dutta [4]) as follows.

REMARK 2.7. Let $(A, \mathfrak{m})$ be a d-dimensional complete local domain of characteristic $p>0$, and assume that its residue class field $A/\mathfrak{m}$ is algebraically closed. Then for a perfect complex $\boldsymbol{F}.$ with support $\{\mathfrak{m}\}$, it is known (see Roberts [18]) that

$$\mathbf{D}_A(\boldsymbol{F}.) = \lim_{n\to\infty} \frac{1}{p^{de}} \sum_{i\geq 0} (-1)^i l_A(H_i(\boldsymbol{F}.\otimes_A {}^eA)),$$

where ${}^eA=A$ is an A-module via the e-th iteration of the Frobenius map. (The invariant like the right hand side of the above equation was first discovered by Dutta [4]. An elementary proof of the rationality of the right hand side was given by Seibert [21].)

The next remark enables us to assume the normality of a given local ring when we calculate the Dutta multiplicity.

REMARK 2.8. Let $(A, \mathfrak{m})$ be a complete local domain with residue class field $A/\mathfrak{m}$ algebraically closed. Since A is excellent henselian, the normalization $A\to\bar{A}$ is finite and $\bar{A}$ is a complete local domain with residue class field $A/\mathfrak{m}$. (See [11] and [14].) Let $\boldsymbol{F}.$ be a perfect A-complex with support $\{\mathfrak{m}\}$. By the compatibility of localized Chern characters with proper push-forwards, we have

$$\begin{aligned}
\mathbf{D}_A(\boldsymbol{F}.) &= \mathrm{ch}^{\operatorname{Spec}(A)}_{\operatorname{Spec}(A/\mathfrak{m})}(\boldsymbol{F}.)\cap[\operatorname{Spec}(A)] \\
&= \mathrm{ch}^{\operatorname{Spec}(\bar{A})}_{\operatorname{Spec}(\bar{A}/\mathfrak{m}\bar{A})}(\boldsymbol{F}.\otimes_A\bar{A})\cap[\operatorname{Spec}(\bar{A})] \\
&= \mathbf{D}_{\bar{A}}(\boldsymbol{F}.\otimes_A\bar{A}).
\end{aligned}$$

3. Perfect complexes of the minimal length.

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and $\boldsymbol{F}.$ a perfect complex with support $\{\mathfrak{m}\}$. The new intersection theorem (Roberts [**18**]) implies that the length of $\boldsymbol{F}.$ is at least d. (Note that the Koszul complexes of parameter ideals are perfect complexes of length d with support $\{\mathfrak{m}\}$.) Consider the following:

CONJECTURE 3.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Suppose that $\boldsymbol{F}.$ is a minimal perfect A-complex of length d with support $\{\mathfrak{m}\}$. Then $\mathbf{D}_A(\boldsymbol{F}.)>0$.*

REMARK 3.2. The previous conjecture is true when $\boldsymbol{F}.$ is the Koszul complexes of parameter ideals because their Dutta multiplicities coincide with the usual multiplicities along the parameter ideals. If A is complete intersection, then it is known that the above conjecture is true (see Corollary 18.1.2 in [8] and Remark 2.2). Furthermore this conjecture is affirmative when $(A, \mathfrak{m})$ contains a field of positive characteristic (Roberts [**18**]). In the case where $(A, \mathfrak{m})$ is a local ring essentially of finite type over a field of characteristic zero, we can reduce this case to the case of positive characteristic. Therefore this conjecture is true when $(A, \mathfrak{m})$ is a local ring essentially of finite type over a field.

The following is an immediate corollary of some results about the positivity of the Dutta multiplicities.

PROPOSITION 3.3. *Let $(A, \mathfrak{m})$ be complete intersection, and M, N finitely generated A-modules of finite projective dimension such that $l_A(M\otimes_A N)<\infty$, $\dim M+\dim N=\dim A$ and $\dim M=\operatorname{depth} M$. Suppose that one of the following conditions is satisfied.*

- *A is essentially of finite type over a field.*
- *$p^n N=0$ for an integer n, where $p>0$ is the characteristic of the residue class field $A/\mathfrak{m}$.*

Then $\sum_i(-1)^i l_A(\mathrm{Tor}_i^A(M, N))>0$ holds. (A part of this proposition was proved by Dutta [**5**].*)*

PROOF. Put $s=\dim M$, $t=\dim N$ and $d=\dim A$. Then $t=d-s$ is equal to the projective dimension of M by the Auslander-Buchsbaum formula. Let $I=\operatorname{ann}_A(M)$ and $J=\operatorname{ann}_A(N)$. Furthermore let $\boldsymbol{F}.$ (resp. $\boldsymbol{G}.$) be the minimal A-free resolution of M (resp. N). Set $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_l\}=\{\mathfrak{p}\in\operatorname{supp}(N)\mid\dim A/\mathfrak{p}=t\}$. It follows from the argument in the proof of the vanishing theorem of intersection multiplicities due to Roberts [**17**] that

$$\sum_i (-1)^i l_A(\mathrm{Tor}_i^A(M, N)) = \sum_{j=1}^{l} l_{A_{\mathfrak{p}_j}}(N_{\mathfrak{p}_j})\cdot \mathrm{ch}_{\mathrm{Spec}(A/\mathfrak{m})}^{\mathrm{Spec}(A/J)}(\boldsymbol{F}.\otimes_A A/J)\cap[\mathrm{Spec}(A/\mathfrak{p}_j)]$$

$$= \sum_{j=1}^{l} l_{A_{\mathfrak{p}_j}}(N_{\mathfrak{p}_j})\cdot \mathrm{ch}_{\mathrm{Spec}(A/\mathfrak{m})}^{\mathrm{Spec}(A/\mathfrak{p}_j)}(\boldsymbol{F}.\otimes_A A/\mathfrak{p}_j)\cap[\mathrm{Spec}(A/\mathfrak{p}_j)]$$

$$= \sum_{j=1}^{l} l_{A_{\mathfrak{p}_j}}(N_{\mathfrak{p}_j})\cdot \mathbf{D}_{A/\mathfrak{p}_j}(\boldsymbol{F}.\otimes_A A/\mathfrak{p}_j)\,.$$

(In fact, the assumption that A is complete intersection implies $\tau_{\mathrm{Spec}(A)}([A])=[\mathrm{Spec}(A)]$ by Corollary 18.1.2 and Theorem 18.3 (5) in [8]. Therefore $\sum_i(-1)^i l_A(\mathrm{Tor}_i^A(M, N))=\mathrm{ch}_{\mathrm{Spec}(A/\mathfrak{m})}^{\mathrm{Spec}(A)}(\boldsymbol{F}.\otimes_A\boldsymbol{G}.)\cap\tau_{\mathrm{Spec}(A)}([A])=\mathrm{ch}_d(\boldsymbol{F}.\otimes_A\boldsymbol{G}.)\cap[\mathrm{Spec}(A)]$ by the Riemann-Roch formula (see Example 18.3.2 in [8]), where $\tau_{\mathrm{Spec}(A)}:\mathrm{K}_0A_{\boldsymbol{Q}}\to\mathrm{A}_*\mathrm{Spec}(A)_{\boldsymbol{Q}}$ is the Riemann-Roch map defined in Section 18 of [8]. (Recall that, for a scheme X, a closed subscheme Y of X, and a bounded locally free complex $\boldsymbol{H}.$ over X which is exact except for Y, $\mathrm{ch}_i(\boldsymbol{H}.):\mathrm{A}_*X_{\boldsymbol{Q}}\to\mathrm{A}_*Y_{\boldsymbol{Q}}$ is the map defined by $\mathrm{ch}_i(\boldsymbol{H}.)|_{\mathrm{A}_rX_{\boldsymbol{Q}}}=p_{r-i}\circ(\mathrm{ch}_Y^X|_{\mathrm{A}_rX_{\boldsymbol{Q}}})$, where $p_{r-i}:\mathrm{A}_*Y_{\boldsymbol{Q}}\to\mathrm{A}_{r-i}Y_{\boldsymbol{Q}}$ is the projection [8].) Then $\mathrm{ch}_d(\boldsymbol{F}.\otimes_A\boldsymbol{G}.)=\sum_{i+j=d}\mathrm{ch}_i(\boldsymbol{F}.\otimes_A A/J)\cdot\mathrm{ch}_j(\boldsymbol{G}.)=\sum_{i+j=d}\mathrm{ch}_j(\boldsymbol{G}.\otimes_A A/I)\cdot\mathrm{ch}_i(\boldsymbol{F}.)$ holds by Example 18.1.5 in [8]. Since $\dim A/I=s=d-t$, $\mathrm{ch}_i(\boldsymbol{F}.)\cap[\mathrm{Spec}(A)]=0$ when $i<t$. Similarly, $\mathrm{ch}_j(\boldsymbol{G}.)\cap[\mathrm{Spec}(A)]=0$ when $j<s$. On the other hand, we have $\mathrm{ch}_i(\boldsymbol{F}.\otimes_A A/J)\cdot\mathrm{ch}_j(\boldsymbol{G}.)=\mathrm{ch}_j(\boldsymbol{G}.\otimes_A A/I)\cdot\mathrm{ch}_i(\boldsymbol{F}.)$ for any i and j by the commutativity of the localized Chern characters (Roberts [17]). Therefore we get $\mathrm{ch}_d(\boldsymbol{F}.\otimes_A\boldsymbol{G}.)\cap[\mathrm{Spec}(A)]=\mathrm{ch}_t(\boldsymbol{F}.\otimes_A A/J)\cap(\mathrm{ch}_s(\boldsymbol{G}.)\cap[\mathrm{Spec}(A)])$. By Example 18.3.2 in [8], we obtain $\tau_{\mathrm{Spec}(A/J)}([N])=\mathrm{ch}_{\mathrm{Spec}(A/J)}^{\mathrm{Spec}(A)}(\boldsymbol{G}.)\cap\tau_{\mathrm{Spec}(A)}([A])=\mathrm{ch}_{\mathrm{Spec}(A/J)}^{\mathrm{Spec}(A)}(\boldsymbol{G}.)\cap[\mathrm{Spec}(A)]$. Therefore, $p_t\circ\tau_{\mathrm{Spec}(A/J)}([N])=\mathrm{ch}_s(\boldsymbol{G}.)\cap[\mathrm{Spec}(A)]$ holds, where $p_t:\mathrm{A}_*\mathrm{Spec}(A/J)_{\boldsymbol{Q}}\to\mathrm{A}_t\mathrm{Spec}(A/J)_{\boldsymbol{Q}}$ is the projection. By Theorem 18.3 (5) in [8], $p_t\circ\tau_{\mathrm{Spec}(A/J)}([N])$ coincides with $\sum_{j=1}^l l_{A_{\mathfrak{p}_j}}(N_{\mathfrak{p}_j})\cdot[\mathrm{Spec}(A/\mathfrak{p}_j)]$. Therefore

$$\mathrm{ch}_d(\boldsymbol{F}.\otimes_A\boldsymbol{G}.)\cap[\mathrm{Spec}(A)] = \mathrm{ch}_t(\boldsymbol{F}.\otimes_A A/J)\cap(p_t\circ\tau_{\mathrm{Spec}(A/J)}([N]))$$

$$= \mathrm{ch}_{\mathrm{Spec}(A/\mathfrak{m})}^{\mathrm{Spec}(A/J)}(\boldsymbol{F}.\otimes_A A/J)\cap\Big(\sum_{j=1}^{l} l_{A_{\mathfrak{p}_j}}(N_{\mathfrak{p}_j})\cdot[\mathrm{Spec}(A/\mathfrak{p}_j)]\Big)$$

is satisfied.)

Then $\boldsymbol{F}.\otimes_A A/\mathfrak{p}_j$ is a perfect $A/\mathfrak{p}_j$-complex of length $t=\dim A/\mathfrak{p}_j$ with support $\{\mathfrak{m}/\mathfrak{p}_j\}$ for each j. Hence, if one of the conditions in this proposition is satisfied, the Dutta multiplicities $\mathbf{D}_{A/\mathfrak{p}_j}(\boldsymbol{F}.\otimes_A A/\mathfrak{p}_j)$ are positive for every j (see Remark 3.2). Therefore the intersection multiplicity $\sum_i(-1)^i l_A(\mathrm{Tor}_i^A(M, N))$ must be positive. Q.E.D.

In the rest of this section we will give an algebraic description to the Dutta multiplicities of perfect complexes in the case where $\dim A\leqq 3$.

PROPOSITION 3.4. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring and* $\boldsymbol{F}.$ *a perfect*

A-complex with support $\{\mathfrak{m}\}$.

1. *If* $\dim A \leqq 1$, *then* $\mathbf{D}_A(\boldsymbol{F}.)=\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))$.
2. *If* $\dim A=2$ *and* A *is equi-dimensional, then* $\mathbf{D}_A(\boldsymbol{F}.)=\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))$.
3. *If* $\dim A=3$ *and* A *is normal, then*

$$\begin{aligned}\mathbf{D}_A(\boldsymbol{F}.) &= \frac{1}{2}\Big\{\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))+\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-3]))\Big\} \\ &= \frac{1}{2}\Big\{\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))+\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.\otimes_A K_A))\Big\},\end{aligned}$$

where K_A is the canonical module of A, $\boldsymbol{F}.^*$ is the dual complex whose component of degree t is $\mathrm{Hom}_A(F_{-t}, A)$ and $\boldsymbol{F}.^*[-3]$ is the shifted complex, i.e., $(\boldsymbol{F}.^*[-3])_i=F^*_{-3+i}$. (Recall that A is assumed to be a homomorphic image of a regular local ring. Therefore A has the canonical module K_A.)

Before proving this proposition, we should investigate $\tau_{\mathrm{Spec}(A)}([K_A])$.

For a Noetherian local ring A of dimension d, we set

$$\tau_{\mathrm{Spec}(A)}([A])=\tau_d([A])+\tau_{d-1}([A])+\cdots+\tau_0([A]),$$

where $\tau_i([A])\in \mathrm{A}_i\,\mathrm{Spec}(A)_{\boldsymbol{Q}}$. Note that $\tau_d([A])$ is equal to $[\mathrm{Spec}(A)]$ by Theorem 18.3 (5) in [8].

LEMMA 3.5. *Let A be a Noetherian normal local domain of dimension d.*

(1) $\tau_{\mathrm{Spec}(A)}([K_A])\equiv\tau_d([A])-\tau_{d-1}([A])+\tau_{d-2}([A]) \mod(\bigoplus_{i=0}^{d-3}\mathrm{A}_i\,\mathrm{Spec}(A)_{\boldsymbol{Q}})$.

(2) *Put* $\mathrm{Cl}(A)_{\boldsymbol{Q}}=\mathrm{Cl}(A)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$, *where* $\mathrm{Cl}(A)$ *is the divisor class group of* A. *Then* $\mathrm{Cl}(A)_{\boldsymbol{Q}}$ *is naturally isomorphic to* $\mathrm{A}_{d-1}\,\mathrm{Spec}(A)_{\boldsymbol{Q}}$.

(3) $\mathrm{cl}(K_A)=2\cdot\tau_{d-1}([A])$ *in* $\mathrm{Cl}(A)_{\boldsymbol{Q}}=\mathrm{A}_{d-1}\,\mathrm{Spec}(A)_{\boldsymbol{Q}}$, *where* $\mathrm{cl}(K_A)$ *stands for the canonical class.*

PROOF. Take a regular local ring S and its prime ideal $\mathfrak{p}$ satisfying $S/\mathfrak{p}\cong A$. Put $n=\mathrm{ht}_S\mathfrak{p}$. Let $\boldsymbol{G}.$ be the minimal S-free resolution of A. Then we obtain $K_A=\mathrm{Ext}_S^n(A, S)=\mathrm{H}_0(\mathrm{Hom}_S(\boldsymbol{G}.^*[-n], S))$.

First we will prove (1). If A is Cohen-Macaulay, then $\boldsymbol{G}.^*[-n]$ is the minimal S-free resolution of K_A. Therefore

$$\begin{aligned}\tau_{\mathrm{Spec}(A)}([K_A]) &= \mathrm{ch}_{\mathrm{Spec}(A)}^{\mathrm{Spec}(S)}(\boldsymbol{G}.^*[-n])\cap[\mathrm{Spec}(S)] \\ &= \tau_d([A])-\tau_{d-1}([A])+\tau_{d-2}([A])-\cdots+(-1)^i\tau_{d-i}([A])+\cdots\end{aligned}$$

in this case (see Example 18.1.2 in [8]). Assume that A is not Cohen-Macaulay. Then $d\geqq 3$, because A is normal. We get $\dim \mathrm{H}_i(\boldsymbol{G}.^*[-n])\leqq d-3$ for $i\neq 0$, because $A_{\mathfrak{q}}$ is Cohen-Macaulay for every prime ideal $\mathfrak{q}$ such that $\mathrm{ht}_A\mathfrak{q}\leqq 2$. By Example 18.1.2 in [8] we obtain

$$\begin{aligned}
&\tau_d([A])-\tau_{d-1}([A])+\tau_{d-2}([A])-\cdots+(-1)^i\tau_{d-i}([A])+\cdots \\
&= \mathrm{ch}^{\mathrm{Spec}(S)}_{\mathrm{Spec}(A)}(\boldsymbol{G}.^*[-n])\cap[\mathrm{Spec}(S)] \\
&= \sum_i(-1)^i\tau_{\mathrm{Spec}(A)}(\mathrm{H}_i(\boldsymbol{G}.^*[-n])) \\
&\equiv \tau_{\mathrm{Spec}(A)}([K_A]) \quad \mathrm{mod}\,(\oplus_{i=0}^{d-3}\mathrm{A}_i\,\mathrm{Spec}(A)_{\boldsymbol{Q}}).
\end{aligned}$$

(Recall that $\tau_{\mathrm{Spec}(A)}([M])$ is contained in $\oplus_{i=0}^{j}\mathrm{A}_i\,\mathrm{Spec}(A)_{\boldsymbol{Q}}$ when $\dim M\leqq j$ by Theorem 18.3 (5) in [8].)

It is easy to check that $\mathrm{Cl}(A)_{\boldsymbol{Q}}\cong\mathrm{A}_{d-1}\mathrm{Spec}(A)_{\boldsymbol{Q}}$ by sending a divisorial ideal J of A to $[\mathrm{Spec}(A/J)]=\tau_{d-1}([A/J])$.

Let J be a divisorial ideal isomorphic to K_A. Then there exists an exact sequence

$$0\longrightarrow K_A\longrightarrow A\longrightarrow A/J\longrightarrow 0.$$

Therefore $\tau_{\mathrm{Spec}(A)}([A/J])=\tau_{\mathrm{Spec}(A)}([A])-\tau_{\mathrm{Spec}(A)}([K_A])$ holds. Hence, from (1), $\mathrm{cl}(K_A)=\tau_{d-1}([A/J])=2\cdot\tau_{d-1}([A])$ in $\mathrm{Cl}(A)_{\boldsymbol{Q}}=\mathrm{A}_{d-1}\mathrm{Spec}(A)_{\boldsymbol{Q}}$. Q. E. D.

PROOF OF PROPOSITION 3.4. If A is Artinian, we have

$$\begin{aligned}
\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.)) &= \mathrm{ch}^{\mathrm{Spec}(A)}_{\mathrm{Spec}(A/\mathrm{m})}(\boldsymbol{F}.)\cap\tau_{\mathrm{Spec}(A)}([A]) \\
&= \mathrm{ch}^{\mathrm{Spec}(A)}_{\mathrm{Spec}(A/\mathrm{m})}(\boldsymbol{F}.)\cap[\mathrm{Spec}(A)] \\
&= \mathbf{D}_A(\boldsymbol{F}.).
\end{aligned}$$

If $\dim A\geqq 1$, then $\mathrm{A}_0\mathrm{Spec}(A)_{\boldsymbol{Q}}=(0)$. So, we have $\tau_{\mathrm{Spec}(A)}([A])=\tau_1([A])=[\mathrm{Spec}(A)]$ when $\dim A=1$.

Assume that $\dim A=2$ and A is equi-dimensional. Then we have

$$\begin{aligned}
\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.)) &= \mathrm{ch}^{\mathrm{Spec}(A)}_{\mathrm{Spec}(A/\mathrm{m})}(\boldsymbol{F}.)\cap\tau_{\mathrm{Spec}(A)}([A]) \\
&= \mathbf{D}_A(\boldsymbol{F}.)+\mathrm{ch}_1(\boldsymbol{F}.)\cap\tau_1([A])
\end{aligned}$$

because $\tau_0([A])=0$. Since $\dim A-\dim\mathrm{supp}(\boldsymbol{F}.)\geqq 2$ and A is equi-dimensional, $\mathrm{ch}_1(\boldsymbol{F}.)=0$ by the vanishing theorem of the first localized Chern character [**16**].

Next assume that $\dim A=3$ and A is normal. Suppose that $A=S/I$ and S is a regular local ring of dimension n. Let $\boldsymbol{G}.$ be a minimal S-free resolution of A. Then we have

$$\tau_{\mathrm{Spec}(A)}([A])=\mathrm{ch}^{\mathrm{Spec}(S)}_{\mathrm{Spec}(A)}(\boldsymbol{G}.)\cap[\mathrm{Spec}(S)]$$

by definition of the Riemann-Roch map $\tau_{\mathrm{Spec}(A)}$. Since A is normal, we have

$$\tau_{\mathrm{Spec}(A)}([K_A])=\tau_3([A])-\tau_2([A])+\tau_1([A])$$

by the previous lemma and the fact that $\mathrm{A}_0\mathrm{Spec}(A)_{\boldsymbol{Q}}=(0)$. Therefore,

$$\frac{1}{2}\left\{\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))+\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.\otimes_A K_A))\right\}$$

$$=\frac{1}{2}\{\mathrm{ch}_{\mathrm{Spec}(A/\mathfrak{m})}^{\mathrm{Spec}(A)}(\boldsymbol{F}.)\cap\tau_{\mathrm{Spec}(A)}([A])+\mathrm{ch}_{\mathrm{Spec}(A/\mathfrak{m})}^{\mathrm{Spec}(A)}(\boldsymbol{F}.)\cap\tau_{\mathrm{Spec}(A)}([K_A])\}$$

$$=\mathbf{D}_A(\boldsymbol{F}.)+\mathrm{ch}_1(\boldsymbol{F}.)\cap\tau_1([A])\,.$$

Furthermore by the vanishing theorem of the first localized Chern character, we have $\mathrm{ch}_1(\boldsymbol{F}.)=0$.

Furthermore, we get

$$\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.\otimes_A K_A))$$

$$=\mathrm{ch}_3(\boldsymbol{F}.)\cap\tau_3([A])-\mathrm{ch}_2(\boldsymbol{F}.)\cap\tau_2([A])+\mathrm{ch}_1(\boldsymbol{F}.)\cap\tau_1([A])$$

$$=\sum_i(-1)^i l_A(\mathrm{H}(\boldsymbol{F}.^*[-3]))\,.$$

Q. E. D.

REMARK 3.6. Let $(A, \mathfrak{m})$ be a normal Noetherian local ring of dimension d and $\boldsymbol{F}.$ a perfect A-complex of length d with support $\{\mathfrak{m}\}$. From Proposition 3.4, we have $\mathbf{D}_A(\boldsymbol{F}.)=l_A(\mathrm{H}_0(\boldsymbol{F}.))>0$ when $d\leqq 2$. So, Conjecture 3.1 is true when $d\leqq 2$ by Remark 2.4, Proposition 2.6 and Remark 2.8.

Next suppose $d=3$. Then we have

$$\mathbf{D}_A(\boldsymbol{F}.)=\frac{1}{2}\left\{\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))+\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-3]))\right\}$$

$$=\frac{1}{2}\left\{\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.\oplus\boldsymbol{F}.^*[-3]))\right\}$$

$$=\frac{1}{2}\mathbf{D}_A(\boldsymbol{F}.\oplus\boldsymbol{F}.^*[-3])\,,$$

because the perfect complex $\boldsymbol{F}.\oplus\boldsymbol{F}.^*[-3]$ is self-dual. So, in order to prove Conjecture 3.1 in the case of $d=3$, we may assume that A is normal and the given complex $\boldsymbol{F}.$ is self-dual. Furthermore, in this case, $\mathbf{D}_A(\boldsymbol{F}.)=l_A(\mathrm{H}_0(\boldsymbol{F}.))-l_A(\mathrm{H}_1(\boldsymbol{F}.))$ by the depth sensitivity (see Remark 2.2).

REMARK 3.7. Let $(A, \mathfrak{m})$ be a normal local domain of dimension 3 and $\boldsymbol{F}.$ a self-dual perfect A-complex of length 3 with support $\{\mathfrak{m}\}$. Assume that A has a maximal Cohen-Macaulay module M, i. e., finitely generated module with depth $M=3$. Then we obtain

$$l_A(\mathrm{H}_0(\boldsymbol{F}.\otimes M))=\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.\otimes M))$$

$$=\mathrm{ch}_{\mathrm{Spec}(A/\mathfrak{m})}^{\mathrm{Spec}(A)}(\boldsymbol{F}.)\cap\tau_{\mathrm{Spec}(A)}([M])$$

$$=\mathrm{rank}_A M\cdot\mathbf{D}_A(\boldsymbol{F}.)+\mathrm{ch}_2(\boldsymbol{F}.)\cap\tau_2([M])+\mathrm{ch}_1(\boldsymbol{F}.)\cap\tau_1([M]),$$

since $\tau_3([M])=\mathrm{rank}_A M\cdot\tau_3([A])$. By the self-dualness of $\boldsymbol{F}.$, $\mathrm{ch}_2(\boldsymbol{F}.)=0$ by Example 18.1.2 in [8]. The vanishing theorem of the first localized Chern character implies that $\mathrm{ch}_1(\boldsymbol{F}.)=0$. Hence $\mathbf{D}_A(\boldsymbol{F}.)>0$ in this case.

The author does not know whether the existence of maximal Cohen-Macaulay module guarantees Conjecture 3.1 in general.

4. The difference between the Dutta multiplicity and the alternative sum.

This section is devoted to investigating the difference between the Dutta multiplicity and the alternative sum of the lengths of homology modules.

First of all, we give an example satisfying $\sum_i(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))<0$, where $\boldsymbol{F}.$ is a perfect complex of length d with support $\{\mathfrak{m}\}$ over a Noetherian local ring $(A, \mathfrak{m})$ of dimension d.

EXAMPLE 4.1. Put $R=k[x, y, z, w]_{(x,y,z,w)}/(xy-zw)$ (k is a field) and $M=R/(x, z)$. By [6] there exists an R-module N such that $l_R(N)=15$, $pd_A N=3$ and

$$\sum_i(-1)^i l_A(\mathrm{Tor}_i^A(M, N))=-1\,.$$

Let t be a non-negative integer and A the idealization of M^t (see [14]). Note that A is an R-algebra and isomorphic to $R\oplus M^t$ as an R-module. Let $\boldsymbol{F}.$ be the minimal free R-resolution of N and put $\boldsymbol{G}.=\boldsymbol{F}.\otimes_R A$. It is easy to check that A is a Noetherian local ring of dimension 3 and $\boldsymbol{G}.$ is a perfect A-complex of length 3 with support $\{\mathfrak{m}\}$, where $\mathfrak{m}$ is the maximal ideal of A. Then we have

$$\begin{aligned}
&\sum_{i=0}^{3}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{G}.))\\
&=\sum_{i=0}^{3}(-1)^i l_R(\mathrm{H}_i(\boldsymbol{G}.))\\
&=\sum_{i=0}^{3}(-1)^i l_R(\mathrm{H}_i(\boldsymbol{F}.))+t\sum_{i=0}^{3}(-1)^i l_R(\mathrm{H}_i(\boldsymbol{F}.\otimes_R M))\\
&=15-t\,.
\end{aligned}$$

Therefore $\sum_{i=0}^{3}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{G}.))<0$ for $t>15$.

On the other hand, $\mathbf{D}_A(\boldsymbol{G}.)>0$ is satisfied since A is essentially of finite type over a field k (see Remark 3.2).

Next we argue about the dual complex.

REMARK 4.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and $\boldsymbol{F}.$ a perfect A-complex of length n with support $\{\mathfrak{m}\}$. By Example 18.1.2 in [8], we have

$$\mathbf{D}_A(\boldsymbol{F}.) = (-1)^{n-d}\mathbf{D}_A(\boldsymbol{F}.^*[-n]) .$$

In particular, $\mathbf{D}_A(\boldsymbol{F}.)=\mathbf{D}_A(\boldsymbol{F}.^*[-d])$ when $n=d$.

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and $\boldsymbol{F}.$ a perfect A-complex of length d with support $\{\mathfrak{m}\}$. In general $\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))$ does not coincide with $\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-d]))$ even if A is Cohen-Macaulay normal as in Example 4.5. The next theorem guarantee $\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))=\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-d]))$ in some special cases.

THEOREM 4.3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and $\boldsymbol{F}.$ a perfect A-complex of length d with support $\{\mathfrak{m}\}$. Suppose that one of the following conditions is satisfied:*

(0) *$(A, \mathfrak{m})$ is a Gorenstein ring.*

(1) *$d\leqq 2$ and A is equi-dimensional.*

(2) *$(A, \mathfrak{m})$ is normal with $d\leqq 4$ and the canonical class $\mathrm{cl}(K_A)$ is torsion in the divisor class group $\mathrm{Cl}(A)$.*

(3) *$d\leqq 3$ and $\mathrm{A}_{d-1}\mathrm{Spec}(A)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=(0)$.*

(4) *There exists a regular local ring $(S, \mathfrak{n})$ and a finite free S-complex $\boldsymbol{G}.$ such that A is a homomorphic image of S and $\boldsymbol{G}.\otimes_S A$ is isomorphic to $\boldsymbol{F}.$.*

Then $\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))=\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-d]))$ holds.

Before proving this theorems we have:

REMARK 4.4. With notation as above, put $M=\mathrm{H}_0(\boldsymbol{F}.)$. When A is Cohen-Macaulay, we have

$$\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.)) = l_A(M)$$

$$\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-d])) = l_A(\mathrm{Ext}_A^d(M, A)) = l_A(M\otimes_A K_A)$$

by the local duality theorem ([**10**]) and the depth sensitivity (Remark 2.2).

PROOF OF THEOREM 4.3. By Remark 4.4, it is obvious that $\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))$ is equal to $\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-d]))$ when A is a Gorenstein ring.

Suppose $d\leqq 2$.

It is trivial when $d=0$.

We can prove $l_A(\mathrm{H}_0(\boldsymbol{F}.))-l_A(\mathrm{H}_1(\boldsymbol{F}.))=l_A(\mathrm{H}_0(\boldsymbol{F}.^*[-1]))-l_A(\mathrm{H}_1(\boldsymbol{F}.^*[-1]))$ by an elementary method in the case of $d=1$ (for example, see Appendix A in [8]).

Next assume $d=2$. Then

$$\sum_{i=0}^{2}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))-\sum_{i=0}^{2}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-2])) = 2\cdot\mathrm{ch}_1(\boldsymbol{F}.)\cap\tau_1([A]) .$$

By the vanishing theorem of the first localized Chern characters, we get $\mathrm{ch}_1(\boldsymbol{F}.)=0$ since A is equi-dimensional.

When A is normal, we obtain

$$\mathrm{cl}(K_A)=2\cdot\tau_{d-1}([A])\in \mathrm{A}_{d-1}\operatorname{Spec}(A)_{\boldsymbol{Q}}=\mathrm{Cl}(A)_{\boldsymbol{Q}}$$

by Lemma 3.5. Suppose $(A, \mathfrak{m})$ is a normal local ring of dimension 3. Then we obtain

$$\begin{aligned}&\sum_{i=0}^{3}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))-\sum_{i=0}^{3}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-3]))\\&=2\cdot\mathrm{ch}_2(\boldsymbol{F}.)\cap\tau_2([A])+2\cdot\mathrm{ch}_0(\boldsymbol{F}.)\cap\tau_0([A])\,.\end{aligned}$$

It is obvious that $\tau_0([A])=0$ since $\mathrm{A}_0\operatorname{Spec}(A)_{\boldsymbol{Q}}=(0)$. Therefore $\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))=\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-d]))$ if the canonical class $\mathrm{cl}(K_A)$ is torsion in $\mathrm{Cl}(A)$. Next suppose $(A, \mathfrak{m})$ is a normal local ring of dimension 4. Then

$$\begin{aligned}&\sum_{i=0}^{4}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))-\sum_{i=0}^{4}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-4]))\\&=2\cdot\mathrm{ch}_3(\boldsymbol{F}.)\cap\tau_3([A])+2\cdot\mathrm{ch}_1(\boldsymbol{F}.)\cap\tau_1([A])\end{aligned}$$

holds. By the assumption that $\tau_3([A])=0$ and the vanishing theorem of the first localized Chern character [**16**], we get $\sum_{i=0}^{4}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))=\sum_{i=0}^{4}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-4]))$ immediately.

In the case where $d\leqq 3$ and $\mathrm{A}_{d-1}\operatorname{Spec}(A)_{\boldsymbol{Q}}=(0)$, we can prove the equality as in the same way as in the case of (1) or (2).

Lastly assume that the condition (4) is satisfied. Since $\operatorname{supp}(\boldsymbol{G}.)\cap\operatorname{Spec}(A)=\{\mathfrak{m}\}$, $\dim\operatorname{Spec}(A)+\dim\operatorname{supp}(\boldsymbol{G}.)\leqq\dim\operatorname{Spec}(S)$ holds by [**20**]. For $j<\dim\operatorname{Spec}(A)$, we obtain

$$\mathrm{ch}_j(\boldsymbol{F}.)=\mathrm{ch}_j(\boldsymbol{G}.\otimes_S A)=0$$

by [**17**] since $j<\dim\operatorname{Spec}(S)-\dim\operatorname{supp}(\boldsymbol{G}.)$. Therefore

$$\begin{aligned}&\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))-\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-d]))\\&=2\{\mathrm{ch}_{d-1}(\boldsymbol{F}.)\cap\tau_{d-1}([A])+\mathrm{ch}_{d-3}(\boldsymbol{F}.)\cap\tau_{d-3}([A])+\cdots\}\\&=0\end{aligned}$$

is satisfied. Q. E. D.

The following example implies that $\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))$ does not always coincide with $\sum_{i=0}^{d}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.^*[-d]))$ even if $(A, \mathfrak{m})$ is a Cohen-Macaulay normal ring of dimension 3. (Such an example was discovered by Roberts.)

EXAMPLE 4.5. Let k be a field and put

$$A=(k[x_0, x_1, x_2, y_0, y_1]/(x_0x_2-x_1^2, x_0y_1-x_1y_0, x_1y_1-x_2y_0))_{(x_0, x_1, x_2, y_0, y_1)}.$$

Then there exists a finitely generated A-module M such that $pd_A M<\infty$ and $\infty>l_A(M)\neq l_A(\mathrm{Ext}_A^3(M, A))$.

We can prove $l_A(M)\neq l_A(\mathrm{Ext}_A^3(M, A))$ by using the example due to Dutta-Hochster-MacLaughlin [**6**].

5. A special case of positivity of the Dutta multiplicity.

Let $(A, \mathfrak{m})$ be a normal Noetherian local ring of dimension 3, and $\boldsymbol{F}.$ a perfect A-complex of length 3 with support $\{\mathfrak{m}\}$. We have already known in Remark 3.6 that we may assume that the given perfect complex is self-dual when we show $\mathbf{D}_A(\boldsymbol{F}.)>0$. This section is devoted to proving Theorem 5.2 which implies the positivity of the Dutta multiplicity in a special case. (Recall that to prove the positivity of the Dutta multiplicities when the dimension of the given ring is 3, we may assume that the given local ring is normal by Remark 2.8.)

Before stating the theorem, we have to define some notation.

DEFINITION 5.1. For a Noetherian ring R and an R-linear map $\phi: F\to G$ between finitely generated free R-modules F and G, we denote by $I_t(\phi)$ the ideal of R generated by all t by t minors of ϕ. (This ideal does not depend on the choices of free bases of F and G.) We put $\mathrm{rank}(\phi)=\max\{t\mid I_t(\phi)\neq 0\}$ and $I(\phi)=I_{\mathrm{rank}(\phi)}(\phi)$.

For an R-linear map $d: F\to R$ such that F is a finitely generated free R-module, $d(a)b-d(b)a\in F$ is called the *Koszul relation* of the map d determined by a and b in F. (Obviously the Koszul relations are contained in $\mathrm{Ker}(d)$.)

When $(R, \mathfrak{n})$ is a Noetherian local ring, for a finitely generated R-module M, $\mu_R(M)$ is defined to be $\dim_{R/\mathfrak{n}} M/\mathfrak{n}M$, i.e., the cardinary of any minimal generating set of M as an R-module.

THEOREM 5.2. *Let $(A, \mathfrak{m})$ be a normal Noetherian local ring of dimension 3, and*

$$\boldsymbol{F}. : 0\longrightarrow F_3\xrightarrow{d_3} F_2\xrightarrow{d_2} F_1\xrightarrow{d_1} F_0\longrightarrow 0$$

a minimal self-dual perfect A-complex with support $\{\mathfrak{m}\}$. Assume the following conditions.

(T1) $\mathrm{rank}_A F_0=1$.

(T2) $\mu_A(I_1(d_1))=\mathrm{rank}_A F_1$.

(T3) *All the Koszul relations of d_1 are contained in $d_2(F_2)$.*

Then, $\mathbf{D}_A(\boldsymbol{F}.)=l_A(\mathrm{H}_0(\boldsymbol{F}.))-l_A(\mathrm{H}_1(\boldsymbol{F}.))>0$.

REMARK 5.3. With the same notation as in Theorem 5.2, the condition (T2) is automatically satisfied if A contains a field. In fact, consider the following complex:

$$\boldsymbol{F}.' : 0 \longrightarrow F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \longrightarrow 0 .$$

If (T2) is not satisfied, we may assume $d_1(e)=0$ such that $\{e\}$ is a part of a free basis of F_1. By the minimality of $\boldsymbol{F}.$, e is not contained in $d_2(F_2)$. On the other hand, $0 \to F_3 \to F_2 \to F_0$ is exact by the depth sensitivity (Lemma 2.2) and $\mathfrak{m}^n e \subset d_2(F_2)$ for $n \gg 0$. But the improved new intersection theorem (for example, see [**7**]) implies that such a complex have the length at least $3=\dim A$. Contradiction.

The author does not know the example of a minimal self-dual A-perfect complex of length 3 with support $\{\mathfrak{m}\}$ such that the condition (T2) or (T3) is not satisfied with coefficient ring A normal of dimension 3.

The most essential point of our proof of Theorem 5.2 is the next lemma. It is a slight generalization of the structure theorem of Gorenstein ideals of codimension 3 due to Buchsbaum and Eisenbud [**2**]. The proof of the lemma is the same as in [**2**].

LEMMA 5.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and*

$$\boldsymbol{F}. : 0 \longrightarrow F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

a minimal self-dual perfect A-complex such that

(L1) $\mathrm{rank}_A F_0 = 1$,

(L2) $\mathrm{H}_i(\boldsymbol{F}.)=0$ *for* $i=2, 3$,

(L3) $\mu_A(I_1(d_1))=\mathrm{rank}_A F_1$,

(L4) *all the Koszul relations are contained in* $d_2(F_2)$.

Then by a suitable choice of free bases of F_1 and F_2, the matrix of d_2 can be alternative.

PROOF. Denote by $(\boldsymbol{G}., d_G)$ the tensor complex $\boldsymbol{F}. \otimes_A \boldsymbol{F}.$.

At first we construct a chain map $m : \boldsymbol{G}. \to \boldsymbol{F}.$ satisfying the following three conditions.

(C1) $m|_{\boldsymbol{F}. \otimes F_0}$ and $m|_{F_0 \otimes \boldsymbol{F}.}$ are isomorphisms.

(C2) For any i and j, $m(a \otimes b)=(-1)^{ij} m(b \otimes a)$ holds, where $a \in F_i$ and $b \in F_j$.

(C3) $m(c \otimes c)=0$ for any $c \in F_1$.

By choosing a generator of F_0, identify F_0 and A. Then we have $\boldsymbol{F}. \otimes F_0 = \boldsymbol{F}.$ and $F_0 \otimes \boldsymbol{F}. = \boldsymbol{F}.$. We define $m|_{\boldsymbol{F}. \otimes F_0}$ and $m|_{F_0 \otimes \boldsymbol{F}.}$ by these identifications. (Then it is clear that (C1) and (C2) is satisfied when either i or j is equal to 0.) Let $\{e_1, \cdots, e_t\}$ be a free basis of F_1 (set $t=\mathrm{rank}_A F_1$) and put $d_1(e_l)=c_l \in A$ for $l=$

$1, \cdots, t$. Consider the following diagram.

$$\begin{array}{ccccc} F_1 \otimes F_1 & \xrightarrow{\mu} & \wedge^2 F_1 & \xrightarrow{\Delta} & F_1 \otimes F_1 \\ & & & & \downarrow{\scriptstyle d_1 \otimes 1} \\ & & F_2 & \xrightarrow{d_2} & F_1 \end{array}$$

where μ (resp. Δ) is the multiplication (resp. diagonalization), i.e., $\mu(a \otimes b) = a \wedge b$ (resp. $\Delta(a \wedge b) = a \otimes b - b \otimes a$) for $a, b \in F_1$. For integers i and j such that $1 \leqq i < j \leqq t$, we have $((d_1 \otimes 1) \circ \Delta)(e_i \wedge e_j) = c_i e_j - c_j e_i \in d_2(F_2)$ by the assumption (L4). So there exists an A-linear map $\psi : \wedge^2 F_1 \to F_2$ which makes the above diagram commutative, i.e., $d_2 \circ \psi = (d_1 \otimes 1) \circ \Delta$. Put $m|_{F_1 \otimes F_1} = \psi \circ \mu$. Then $m \circ d_G|_{F_1 \otimes F_1} = d_2 \circ m|_{F_1 \otimes F_1}$ because

$$\begin{aligned} m \circ d_G(e_i \otimes e_j) &= m(c_i \otimes e_j - e_i \otimes c_j) \\ &= c_i e_j - c_j e_i \\ &= ((d_1 \otimes 1) \circ \Delta \circ \mu)(e_i \otimes e_j) \\ &= (d_2 \circ m)(e_i \otimes e_j) . \end{aligned}$$

Furthermore $m(c \otimes c) = 0$ for any $c \in F_1$ since $m|_{F_1 \otimes F_1}$ is factored by $\mu : F_1 \otimes F_1 \to \wedge^2 F_1$. Therefore (C2) and (C3) are satisfied when $i = j = 1$.

For $a \in F_2$ and $b \in F_1$, we have

$$\begin{aligned} (m \circ d_G)(a \otimes b) &= m(d_2(a) \otimes b + a \otimes d_1(b)) \\ &= m(d_2(a) \otimes b) + d_1(b) a \\ (m \circ d_G)(b \otimes a) &= m(d_1(b) \otimes a - b \otimes d_2(a)) \\ &= d_1(b) a - m(b \otimes d_2(a)) \\ &= m(d_2(a) \otimes b) + d_1(b) a . \end{aligned}$$

Since $\mathrm{H}_2(\boldsymbol{F}.) = 0$, we can construct $m|_{F_2 \otimes F_1}$ and $m|_{F_1 \otimes F_2}$ such that $d_3 \circ m = m \circ d_G$ and $m(a \otimes b) = m(b \otimes a)$ for any $a \in F_2$ and any $b \in F_1$.

Put $m|_{G_i} = 0$ for $i = 4, 5, 6$. Since d_3 is injective, m is a chain map satisfying (C1), (C2) and (C3).

$m|_{G_3} : G_3 \to F_3$ consists of the following four maps,

$$\begin{aligned} m|_{F_3 \otimes F_0} &: F_3 \otimes F_0 \longrightarrow F_3 \\ m|_{F_2 \otimes F_1} &: F_2 \otimes F_1 \longrightarrow F_3 \\ m|_{F_1 \otimes F_2} &: F_1 \otimes F_2 \longrightarrow F_3 \\ m|_{F_0 \otimes F_3} &: F_0 \otimes F_3 \longrightarrow F_3 . \end{aligned}$$

For A-modules L, M and N, it holds that $\mathrm{Hom}_A(L\otimes_A M, N)=\mathrm{Hom}_A(L, \mathrm{Hom}_A(M, N))$. Hence we obtain the following maps,

$$s_3 : F_3 \longrightarrow \mathrm{Hom}_A(F_0, F_3)$$
$$s_2 : F_2 \longrightarrow \mathrm{Hom}_A(F_1, F_3)$$
$$s_1 : F_1 \longrightarrow \mathrm{Hom}_A(F_2, F_3)$$
$$s_0 : F_0 \longrightarrow \mathrm{Hom}_A(F_3, F_3)\,,$$

from $m|_{F_3\otimes F_0}$, $m|_{F_2\otimes F_1}$, $m|_{F_1\otimes F_2}$ and $m|_{F_0\otimes F_3}$ respectively. We will show that the following diagram is commutative:

$$\begin{array}{ccc} F_3 & \xrightarrow{d_3} & F_2 \\ \downarrow{\scriptstyle s_3} & & \downarrow{\scriptstyle s_2} \\ \mathrm{Hom}_A(F_0, F_3) & \xrightarrow{d_1^*} & \mathrm{Hom}_A(F_1, F_3)\,. \end{array}$$

We have only to show that $((s_2\circ d_3)(a))(b)=((d_1^*\circ s_3)(a))(b)$ in F_3 for any $a\in F_3$ and any $b\in F_1$. Obviously $((d_1^*\circ s_3)(a))(b)=(s_3(a))(d_1(b))=m(a\otimes d_1(b))$ by definition of s_3. Furthermore $((s_2\circ d_3)(a))(b) = (s_2(d_3(a)))(b) = m(d_3(a)\otimes b)$. Since $0 = m\circ d_G(a\otimes b)=m(d_3(a)\otimes b)-m(a\otimes d_1(b))$, we have got $s_2\circ d_3=d_1^*\circ s_3$.

Since $m|_{F_3\otimes F_0}$ is an isomorphism, so is s_3. Next we will show that s_2 is also an isomorphism. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_A(F_3, A) & \xleftarrow{d_3^*} & \mathrm{Hom}_A(F_2, A) \\ \uparrow{\scriptstyle s_3^*} & & \uparrow{\scriptstyle s_2^*} \\ \mathrm{Hom}_A(\mathrm{Hom}_A(F_0, F_3), A) & \xleftarrow{(d_1^*)^*} & \mathrm{Hom}_A(\mathrm{Hom}_A(F_1, F_3), A)\,. \end{array}$$

We have only to show that s_2^* is an isomorphism. From the assumption of the self-dualness of $\boldsymbol{F}.$ and (L3), both of the following two exact sequences

$$0 \longleftarrow \mathrm{H}_0(\boldsymbol{F}.) \longleftarrow \mathrm{Hom}_A(F_3, A) \xleftarrow{d_3^*} \mathrm{Hom}_A(F_2, A)$$

$$0 \longleftarrow \mathrm{H}_0(\boldsymbol{F}.) \longleftarrow \mathrm{Hom}_A(\mathrm{Hom}_A(F_0, F_3), A) \xleftarrow{(d_1^*)^*} \mathrm{Hom}_A(\mathrm{Hom}_A(F_1, F_3), A)$$

are initial parts of the minimal free resolution of $\mathrm{H}_0(\boldsymbol{F}.)$. So, s_2^* must be an isomorphism.

Fixing a generator of F_3, identify $\mathrm{Hom}_A(F_1, F_3)$ and F_1^*. Since $s_2: F_2\to F_1^*$ is an isomorphism, we can choose a free basis $\{e_1, \cdots, e_t\}$ (resp. $\{g_1, \cdots, g_t\}$) of F_1 (resp. F_2) such that $s_2(g_i)=e_i^*$ for $i=1, \cdots, t$, where $\{e_1^*, \cdots, e_t^*\}$ is the dual basis of F_1^*. Note that $e_j^*(d_2(g_i))=(s_2(g_j))(d_2(g_i))=m(g_j\otimes d_2(g_i))$ for any i

and j. So, for integers i and j such that $i\neq j$, we have

$$\begin{aligned}(d_3\circ e_j^*)(d_2(g_i)) &= (d_3\circ m)(g_j\otimes d_2(g_i))\\ &= m(d_2(g_j)\otimes d_2(g_i))\\ &= -m(d_2(g_i)\otimes d_2(g_j))\\ &= -(d_3\circ e_i^*)(d_2(g_j))\,.\end{aligned}$$

Since d_3 is injective, $e_j^*(d_2(g_i))=-e_i^*(d_2(g_j))$. Furthermore, for $i=1,\cdots,t$,

$$(d_3\circ e_i^*)(d_2(g_i)) = m(d_2(g_i)\otimes d_2(g_i)) = 0\,.$$

So, $e_i^*(d_2(g_i))=0$. Therefore the matrix of d_2 under the free bases is alternative. Q.E.D.

Before proving Theorem 5.2, we have to recall the properties of *pfaffian ideals*.

DEFINITION 5.5. For a $2l$ by $2l$ alternative matrix $M=(m_{ij})$, we denote by $pf_{2l}(M)$ the square root of $\det(A)$ and it is called the *$2l$-order pfaffian* of M. (It is well-known that we have

$$pf_{2l}(M) = \frac{1}{2^l\cdot l\,!}\sum_{\sigma\in\mathfrak{S}_{2l}}(\mathrm{sgn}\,\sigma)m_{\sigma(1)\sigma(2)}\cdots m_{\sigma(2l-1)\sigma(2l)}\,,$$

where the above sum runs over all permutations in the $2l$-th symmetric group $\mathfrak{S}_{2l}$. Note that the right hand side of the above equation is defined over an arbitrary commutative ring.)

For an n by n alternative matrix N and a sequence of integers $1\leqq i_1<\cdots<i_{2l}\leqq n$, denote by $pf_{2l}(i_1,\cdots,i_{2l})$ the $2l$-order pfaffian of the alternative submatrix of N consists of the i_1-th row, $\cdots$, the i_{2l}-th row and the i_1-th column, $\cdots$ the i_{2l}-th column of N. Denote by $Pf_{2l}(N)$ the ideal

$$(pf_{2l}(i_1,\cdots,i_{2l})\mid 1\leqq i_1<\cdots<i_{2l}\leqq n)$$

and call it the *pfaffian ideal* of order $2l$. (It is well-known that

$$I_{2l}(N)\subset Pf_{2l}(N)\subset\sqrt{I_{2l}(N)}$$

for any integer l and for any alternative matrix N over any commutative ring.)

REMARK 5.6. Let $C=(c_{ij})$ be a $2l+1$ by $2l+1$ alternative matrix. Then for each $i=1,\cdots,2l+1$, we have

$$\sum_{j=1}^{2l+1}(-1)^{j+1}pf_{2l}(1,\cdots,\hat{j},\cdots,2l+1)\cdot c_{ji}=0\,,$$

where $pf_{2l}(1,\cdots,\hat{j},\cdots,2l+1)$ stands for $pf_{2l}(1,\cdots,j-1,j+1,\cdots,2l+1)$. They are relations on pfaffians of degree 1. (See [13] for the details.)

REMARK 5.7. Let $\boldsymbol{Z}$ be the ring of integers and $X=(x_{ij})$ the generic $2l+1$ by $2l+1$ alternative matrix. For the simplicity of notation, we denote by $pf_{2l}(\underline{k})$ the $2l$-order pfaffian $pf_{2l}(1, \cdots, \hat{k}, \cdots, 2l+1)$. Put $R=\boldsymbol{Z}[x_{ij}]_{1\leqq i<j\leqq 2l+1}$. Denote by M the 1 by $2l+1$ matrix

$$(pf_{2l}(\underline{1}) \quad -pf_{2l}(\underline{2}) \quad \cdots \quad (-1)^{k+1}pf_{2l}(\underline{k}) \quad \cdots \quad pf_{2l}(\underline{2l+1}))$$

and put $N={}^tM$. Consider the following sequence of R-linear maps:

$$\begin{array}{ccccccccccc} \boldsymbol{L}.: 0 & \longrightarrow & R & \xrightarrow{N} & R^{2l+1} & \xrightarrow{X} & R^{2l+1} & \xrightarrow{M} & R & \longrightarrow & 0\,. \\ & & \| & & \| & & \| & & \| & & \\ & & L_3 & & L_2 & & L_1 & & L_0 & & \end{array}$$

By Remark 5.6, $MX=0$ and $XN=0$. So, $\boldsymbol{L}.$ is a complex. It is known that $\boldsymbol{L}.$ is the minimal free resolution of $R/Pf_{2l}(X)$ ([2]). It is easy to check that $I_{2l+1}(X)=0$ and $I_1(M)=I_1(N)=\sqrt{I_{2l}(X)}=Pf_{2l}(X)$ since $Pf_{2l}(X)$ is a prime ideal (see [12]). So, the complex $\boldsymbol{L}.$ has the depth sensitivity ([3]) with respect to $2l+1$ by $2l+1$ alternative matrices, i.e., for any Noetherian ring A and any $2l+1$ by $2l+1$ alternative matrix (a_{ij}) over A,

$$3-\max\{i \mid \mathrm{H}_i(\boldsymbol{L}.\otimes_R A)\neq 0\} = \mathrm{grade}(Pf_{2l}((a_{ij})))$$

holds, where A is regarded to be an R-algebra by the ring homomorphism $\phi: R\to A$ defined by $\phi(x_{ij})=a_{ij}$ for $1\leqq i<j\leqq 2l+1$.

We now start to prove Theorem 5.2.

PROOF OF THEOREM 5.2. Put $t=\mathrm{rank}_A(F_1)$. Then it is easy to see that $\mathrm{rank}(d_3)=\mathrm{rank}(d_1)=1$, $\mathrm{rank}(d_2)=t-1$ and $\sqrt{I(d_1)}=\sqrt{I(d_2)}=\sqrt{I(d_3)}=\mathfrak{m}$ because A is an integral domain. Since A is normal, $\mathrm{H}_i(\boldsymbol{F}.)=0$ for $i=2, 3$. So, by a suitable choice of free bases of F_1 and F_2, we may assume that the matrix standing for d_2 is alternative by Lemma 5.4. Therefore $\mathrm{rank}(d_2)$ must be even. Set $t=2l+1$.

Let (a_{ij}) be the $2l+1$ by $2l+1$ alternative matrix corresponding to d_2. Then $\sqrt{I(d_2)}=\sqrt{Pf_{2l}((a_{ij}))}=\mathfrak{m}$. So, $\mathrm{grade}(Pf_{2l}((a_{ij})))\geqq 2$. Therefore $\mathrm{H}_i(\boldsymbol{L}.\otimes_R A)=0$ for $i=2, 3$ (see Remark 5.7). On the other hand $0\to F_3\to F_2\to F_1$ is exact. Hence by a suitable choice of a generator of F_3, we may assume that the $2l+1$ by 1 matrix standing for d_3 is ${}^t(pf_{2l}(\underline{1}) \quad -pf_{2l}(\underline{2}) \quad \cdots \quad pf_{2l}(\underline{2l+1}))$. Next consider $\boldsymbol{F}.^*$. By the same argument as above, we may assume that the 1 by $2l+1$ matrix corresponding to d_1 is $(pf_{2l}(\underline{1}) \quad -pf_{2l}(\underline{2}) \quad \cdots \quad pf_{2l}(\underline{2l+1}))$ by a suitable choice of a generator of F_0. Therefore we obtain $\boldsymbol{F}.\cong\boldsymbol{L}.\otimes_R A$.

Let $Y=(y_{ij})$ be the $2l+1$ by $2l+1$ generic alternative matrix over A. Put

$$B=A[y_{ij}\,|\,1\leqq i<j\leqq 2l+1]_{(\mathfrak{m},\,(y_{ij}|1\leqq i<j\leqq 2l+1))}\,.$$

Since $\dim B=3+l(2l+1)$ and $I=Pf_{2l}((y_{ij}))\cdot B$ is a prime ideal of height 3, we have $\dim B/I=l(2l+1)$. (Note that A is universally catenary because it is a homomorphic image of a regular local ring.)

We give the R-algebra structure to B (resp. B-algebra structure to A) via $\phi\colon R\to B$ defined by $\phi(X)=Y$ (resp. $\xi\colon B\to A$ defined by $\xi(Y)=(a_{ij})$). Put $\boldsymbol{H}.=\boldsymbol{L}.\otimes_R B$. Note that $\boldsymbol{F}.=\boldsymbol{L}.\otimes_R A=\boldsymbol{H}.\otimes_B A$. By the depth sensitivity ([3]), $\boldsymbol{H}.$ is the minimal free resolution of B/I.

Let $\boldsymbol{K}.$ be the Koszul complex over B defined by $\{y_{ij}-a_{ij}\,|\,1\leqq i<j\leqq 2l+1\}$. It is obvious that $\boldsymbol{K}.$ is the minimal B-free resolution of A.

Consider the following double complex $\boldsymbol{H}.\otimes_B\boldsymbol{K}.$:

$$\begin{array}{ccccc}
H_3\otimes K_{l(2l+1)} & \longrightarrow & \cdots & \longrightarrow & H_3\otimes K_0\\
\downarrow & & \vdots\quad\vdots & & \downarrow\\
H_2\otimes K_{l(2l+1)} & \longrightarrow & \cdots & \longrightarrow & H_2\otimes K_0\\
\downarrow & & \vdots\quad\vdots & & \downarrow\\
H_1\otimes K_{l(2l+1)} & \longrightarrow & \cdots & \longrightarrow & H_1\otimes K_0\\
\downarrow & & \vdots\quad\vdots & & \downarrow\\
H_0\otimes K_{l(2l+1)} & \longrightarrow & \cdots & \longrightarrow & H_0\otimes K_0\,.
\end{array}$$

Then we have

$$\sum_{i=0}^{3}(-1)^i l_A(\mathrm{H}_i(\boldsymbol{F}.))=\sum_i(-1)^i l_{B/I}(\boldsymbol{K}.\otimes_B B/I)$$

by the argument on the spectral sequences. Since $\dim B/I=l(2l+1)$ and $B/(I+(y_{ij}-a_{ij}\,|\,1\leqq i<j\leqq 2l+1))=A/Pf_{2l}((a_{ij}))$, $\{\overline{y_{ij}-a_{ij}}\,|\,1\leqq i<j\leqq 2l+1\}$ is a system of parameter of B/I. ($\overline{y_{ij}-a_{ij}}$ means the image of $y_{ij}-a_{ij}$ in B/I.)

Therefore we obtain

$$l_A(\mathrm{H}_0(\boldsymbol{F}.))-l_A(\mathrm{H}_1(\boldsymbol{F}.))=\sum_i(-1)^i l_{B/I}(\boldsymbol{K}.\otimes_B B/I)=e_{(\overline{y_{ij}-a_{ij}}|1\leqq i<j\leqq 2l+1)}(B/I)>0.$$

We have completed the proof of Theorem 5.2. Q.E.D.

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