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Application of the theory of KM_2O-Langevin equations to the linear prediction problem for the multi-dimensional weakly stationary time series

Dedicated to Professor Hiroshi Tanaka on his sixtieth birthday

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§1. Introduction.

Let $X=(X(n); n\in\mathbb{Z})$ be a d-dimensional weakly stationary time series on a probability space $(\Omega, \mathcal{B}, P)$ with expectation vector 0 and covariance matrix function R. Wiener and Masani ([16], [17], [3]) have developed a theory of the linear prediction problem for the time series X. By the innovation method, they have introduced an innovation process $\boldsymbol{\varepsilon}_+=(\varepsilon_+(n); n\in\mathbb{Z})$ by

$$\varepsilon_+(n)=X(n)-P_{M_{-\infty}^{n-1}(X)}X(n), \tag{1.1}$$

where $P_{M_{-\infty}^{n-1}(X)}$ stands for the projection operator on the past subspace $\boldsymbol{M}_{-\infty}^{n-1}(\boldsymbol{X})$ of $L^2(\Omega, \mathcal{B}, P)$ defined by

$$\boldsymbol{M}_{-\infty}^{n-1}(\boldsymbol{X}) = \text{the closed subspace generated by } \{X_j(l);\, l\leqq n-1, 1\leqq j\leqq d\}. \tag{1.2}$$

We denote by $V_+\in M(d;\mathbb{R})$ the prediction error matrix:

$$V_+=E(\varepsilon_+(0)^t\varepsilon_+(0)). \tag{1.3}$$

The process $\boldsymbol{X}$ is said to be purely nondeterministic if and only if $\bigcap_{n\in Z}\boldsymbol{M}_{-\infty}^{n}(X)=0$ and to be of full rank if and only if the rank of the matrix V_+ is d. It has been in [16] characterized that $\boldsymbol{X}$ is purely nondeterministic and of full rank if and only if it has a spectral density matrix function Δ such that

$$\log(\det(\Delta))\in L^1(-\pi,\pi) \tag{1.4}$$

and then proved that

$$\det V_+=\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\log(\det(\Delta(\theta)))d\theta\right]. \tag{1.5}$$

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Furthermore in [17] and [3] they have derived a generating function which, in the one-dimensional case, corresponds to the outer function of the spectral density function: the main purpose of [17] was to obtain certain algorithms computing the generating function, the linear predictor and the prediction error matrix in terms of the spectral density matrix function under the conditions that X is purely nondeterministic and of full rank and the eigenvalues of the spectral density matrix function are essentially bounded above and away from zero; the latter condition has been weakened in [3]. For that purpose, they have used the theorem of alternating projections due to von Neuman [14]. However, it seems that they have not succeeded in obtaining the computable algorithm which is fit for the application to other science.

Differently from Wiener-Masani's method in [16], [17] and [3], we shall apply the theory of KM_2O-Langevin equations to the linear prediction problem for the process X which is purely nondeterministic and of full rank and to solve some unsettled problems in [17] and [3]. We shall find that, in addition to the forward innovation process $\boldsymbol{\varepsilon}_+$ and the forward prediction error matrix V_+, the backward innovation process $\boldsymbol{\varepsilon}_-=(\varepsilon_-(n)\,;\,n\in\mathbb{Z})$ and the backward prediction error matrix $V_-\in M(d\,;\,\mathbb{R})$ will be indispensable to obtain certain computable algorithm for the linear predictor which is fit for the application to other science:

$$\varepsilon_-(n)=X(n)-P_{\boldsymbol{M}_{n+1}^{\infty}(X)}X(n)\,; \tag{1.6}$$

$$V_-=E(\varepsilon_-(0)^t\varepsilon_-(0)). \tag{1.7}$$

By using the so-called innovation method, we have in [7] constructed the theory of KM_2O-Langevin equations with finite delay drift term for the multi-dimensional weakly stationary time series. Some relations which hold between both the delay and fluctuation coefficients in KM_2O-Langevin equations play important roles in this theory. In the field of systems, control and information engineerings, they have been known as LD-algorithm for the one-dimensional case and LWWR-algorithm for the multi-dimensional case in the model fitting of AR-Langevin equations with finite degree ([2], [1], [15], [18]). A fundamental feature of the theory of KM_2O-Langevin equations lies in a comprehension that such algorithms can be understood as a kind of fluctuation-dissipation theorem ([7]). As the application of the theory of KM_2O-Langevin equations to data analysis, we are going to develop a new project of the stationary, causal and prediction analysis ([11], [10], [13]).

Now we shall explain the contents of this paper. In § 2 we shall recall and rearrange a part of the Wiener-Masani's theory in [16]. The theory of KM_2O-Langevin equations will be introduced in § 3 according to [7]. In particular, we shall rearrange the KM_2O-Langevin data associated with the process X

which consists of the triplet of the forward and backward KM_2O-Langevin delay functions, the forward and backward KM_2O-Langevin partial correlation functions, and the forward and backward KM_2O-Langevin fluctuation functions. By taking certain scaling limit of the forward (resp. backward) KM_2O-Langevin equation, we shall in §4 derive a forward (resp. backward) AR(∞)-Langevin equation which governs the time evolution of the time series $\boldsymbol{X}$. §5 treats the global behaviour of the KM_2O-Langevin partial correlation (resp. delay) functions by using the algorithms mentioned above. We shall in §6 obtain two kinds of prediction formulae for the linear predictor in terms of the innovation process $\boldsymbol{\varepsilon}_+$ and the process $\boldsymbol{X}$, respectively. In the final §7, we shall give a concrete representation theorem for the generating function in the Wiener-Masani's theory ([**17**] and [3]) which, according to the nomenclature in the one-dimensional time series, will be called in this paper an outer matrix function of the spectral density matrix function Δ of the time series $\boldsymbol{X}$.

This paper has been announced in [8]. The method used in this paper can be effectively applied to the non-linear prediction problem for one-dimensional strictly stationary time series in order to resolve the open problem in Masani and Wiener ([4]) which will be appeared in the future ([12]). Furthermore in [**13**] we shall develop the prediction analysis based upon non-linear causal analysis in the part (III) of our new project.

§2. The Wiener-Masani's theory.

Let $\boldsymbol{X}=(X(n)\,;\,n\in\mathbb{Z})$ be a d-dimensional weakly stationary time series on a probability space $(\Omega, \mathcal{B}, P)$ with expectation vector 0 and covariance matrix function R:

$$E(X(n))=0 \qquad (n\in\mathbb{Z})\,; \tag{2.1}$$

$$E(X(m)\,{}^tX(n))=R(m-n) \qquad (m,\, n\in\mathbb{Z})\,. \tag{2.2}$$

We introduce a notation which will be often used in this paper. For any d-dimensional square-integrable stochastic process $\boldsymbol{Y}=({}^t(Y_1(n), Y_2(n), \cdots, Y_d(n))\,;\,n\in\mathbb{Z})$ defined on the probability space $(\Omega, \mathcal{B}, P)$, we define, for any $m, n, -\infty\leqq n\leqq m\leqq\infty$, a real closed subspace $\boldsymbol{M}_n^m(\boldsymbol{Y})$ of $L^2(\Omega, \mathcal{B}, P)$ by

$$\boldsymbol{M}_n^m(\boldsymbol{Y})\equiv \text{the closed linear hull of } \{Y_j(k)\,;\,k\in\mathbb{Z},\ n\leqq k\leqq m,\ 1\leqq j\leqq d\}\,. \tag{2.3}$$

Furthermore $P_{\boldsymbol{M}_n^m(\boldsymbol{Y})}$ denotes the projection operator on the closed subspace $\boldsymbol{M}_n^m(\boldsymbol{Y})$.

We shall recall and rearrange a part of the Wiener-Masani's theory for the linear prediction problem of the time series $\boldsymbol{X}$ ([**16**]). By the innovation method, they have introduced an innovation process $\boldsymbol{\varepsilon}_+=(\varepsilon_+(n)\,;\,n\in\mathbb{Z})$ by

(2.4) $$\varepsilon_+(n) = X(n) - P_{M_{-\infty}^{n-1}(X)} X(n)$$

and defined the prediction error matrix $V_+ \in M(d\,;\mathbb{R})$ by

(2.5) $$V_+ = E(\varepsilon_+(0)^t \varepsilon_+(0)).$$

It is easy to see that the time series $\boldsymbol{\varepsilon}_+$ is a white noise with covariance matrix function V_+, that is, for any $m, n \in \mathbb{Z}$,

(2.6) $$E(\varepsilon_+(m)^t \varepsilon_+(n)) = \delta_{mn} V_+ .$$

The process $\boldsymbol{X}$ is said to be purely nondeterministic if and only if

(2.7) $$\bigcap_{n\in \mathbf{Z}} \boldsymbol{M}_{-\infty}^{n}(\boldsymbol{X}) = 0$$

and Wiener and Masani have proved

THEOREM 2.1 ([**16**]). *The condition* (2.7) *is equivalent to the following causal condition:*

(2.8) $$\boldsymbol{M}_{-\infty}^{n}(\boldsymbol{X}) = \boldsymbol{M}_{-\infty}^{n}(\boldsymbol{\varepsilon}_+) \qquad (n \in \mathbb{Z}).$$

Furthermore, $\boldsymbol{X}$ is said to be of full rank if and only if the rank of the matrix V_+ is d. Wiener and Masani have proved

THEOREM 2.2 ([**16**]). *The time series $\boldsymbol{X}$ is purely nondeterministic and of full rank if and only if it has a spectral density matrix function Δ such that*

(2.9) $$\log(\det(\Delta)) \in L^1(-\pi, \pi).$$

Then the following formula holds:

(2.10) $$\det V_+ = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(\det(\Delta(\theta)))d\theta\right].$$

We call the time series $\boldsymbol{\varepsilon}_+$ and the matrix V_+ the forward innovation process and the forward prediction error matrix, respectively. For further study, we need to introduce the backward innovation process $\boldsymbol{\varepsilon}_- = (\varepsilon_-(n)\,; n \in \mathbb{Z})$ and the backward prediction error matrix V_- by

(2.11) $$\varepsilon_-(n) = X(n) - P_{M_{n+1}^{\infty}(X)} X(n)$$

and

(2.12) $$V_- = E(\varepsilon_-(0)^t \varepsilon_-(0)).$$

Similarly for $\boldsymbol{\varepsilon}_+$, the time series $\boldsymbol{\varepsilon}_-$ is a white noise with covariance matrix function V_-:

(2.13) $$E(\varepsilon_-(m)^t \varepsilon_-(n)) = \delta_{mn} V_- \qquad (m, n \in \mathbb{Z}).$$

COROLLARY 2.1. *We assume that the time series X is purely nondeterministic and of full rank. Then the following hold:*

$$M_n^{\infty}(X) = M_n^{\infty}(\varepsilon_-) \qquad (n \in \mathbb{Z}); \tag{2.14}$$

$$\det V_- = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(\det(\Delta(\theta)))d\theta\right]. \tag{2.15}$$

PROOF. We consider the d-dimensional weakly stationary time series $X_- \equiv (X(-n); n \in \mathbb{Z})$. When we use the notation that $\varepsilon_-(X)$ (resp. $\varepsilon_+(X_-)$) stands for the backward (resp. forward) innovation process associated with the time series X (resp. X_-), it can be seen that $M_n^{\infty}(X) = M_{-\infty}^{-n}(X_-)$ and $\varepsilon_-(X)(n) = \varepsilon_+(X_-)(-n)$. Furthermore it follows that the spectral density matrix function of X_- becomes ${}^t\Delta(\theta)$ and $\det(\Delta(\theta)) = \det({}^t\Delta(\theta))$. Therefore we can prove Corollary 2.1 from Theorems 2.1 and 2.2. Q. E. D.

§ 3. The theory of KM$_2$O-Langevin equations.

We shall recall the theory of KM$_2$O-Langevin equations from [7]. Let $X = (X(n); n \in \mathbb{Z})$ be the same time series as in § 2. In this section, we treat the case where the covariance function R has a spectral density matrix function Δ defined on $[-\pi, \pi)$:

$$R(n) = \int_{-\pi}^{\pi} e^{-in\theta}\Delta(\theta)d\theta \qquad (n \in \mathbb{Z}). \tag{3.1}$$

Then we define, for each $n \in \mathbb{N}$, two block Toeplitz matrices $T_n^+, T_n^- \in M(nd; \mathbb{R})$ by

$$T_n^{\pm} = \begin{pmatrix} R(0) & R(\pm 1) & \cdots & R(\pm(n-1)) \\ R(\mp 1) & R(0) & \cdots & R(\pm(n-2)) \\ \vdots & \vdots & \ddots & \vdots \\ R(\mp(n-1)) & R(\mp(n-2)) & \cdots & R(0) \end{pmatrix}. \tag{3.2$_\pm$}$$

It is to be noted ([7]) that

$${}^tR(n) = R(-n) \qquad (n \in \mathbb{Z}), \tag{3.3}$$

$$T_n^+, T_n^- \in GL(nd; \mathbb{R}) \qquad (n \in \mathbb{N}), \qquad \text{and} \tag{3.4}$$

$$T_1^+ = T_1^- = R(0). \tag{3.5}$$

According to the method of innovation, we introduce the d-dimensional forward (resp. backward) KM$_2$O-Langevin force $\nu_+ = (\nu_+(n); n \in \mathbb{N}^*)$ (resp. $\nu_- = (\nu_-(l); l \in -\mathbb{N}^*)$) as follows:

$$\nu_+(n) = X(n) - P_{M_0^{n-1}(X)}X(n) \qquad (n \in \mathbb{N}^*); \tag{3.6$_+$}$$

$$\nu_-(l) = X(l) - P_{M^0_{l+1}(X)}X(l) \qquad (l \in -\mathbb{N}^*), \tag{3.6$_-$}$$

where $\mathbb{N}^* = \{0, 1, 2, \cdots\}$, $-\mathbb{N}^* = \{0, -1, -2, \cdots\}$, and $\boldsymbol{M}_0^{-1}(\boldsymbol{X}) = \boldsymbol{M}_1^0(\boldsymbol{X}) = \{0\}$.

For each $n \in \mathbb{N}^*$, let $V_+(n)$ (resp. $V_-(n)$) be the covariance matrix of $\nu_+(n)$ (resp. $\nu_-(-n)$). We call the function $V_+(\cdot)$ (resp. $V_-(\cdot)$) the forward (resp. backward) KM_2O-Langevin fluctuation function. The following causal relation holds among $\boldsymbol{X}$, $\boldsymbol{\nu}_+$ and $\boldsymbol{\nu}_-$:

Causal relation ([7]).

$$\nu_+(0) = \nu_-(0) = X(0). \tag{3.7}$$

$$E(\nu_\pm(\pm m)\,{}^t\nu_\pm(\pm n)) = \delta_{mn}V_\pm(n) \qquad (m, n \in \mathbb{N}^*). \tag{3.8$_\pm$}$$

$$\boldsymbol{M}_0^n(\boldsymbol{X}) = \boldsymbol{M}_0^n(\boldsymbol{\nu}_+) \qquad (n \in \mathbb{N}^*). \tag{3.9$_+$}$$

$$\boldsymbol{M}_{-n}^0(\boldsymbol{X}) = \boldsymbol{M}_{-n}^0(\boldsymbol{\nu}_-) \qquad (n \in \mathbb{N}^*). \tag{3.9$_-$}$$

Let the system $\{\gamma_+(n, k), \gamma_-(n, k), \delta_+(m), \delta_-(m), V_+(l), V_-(l)\,;\, l \in \mathbb{N}^*, m, n \in \mathbb{N}, 1 \leq k < n\}$ of $M(d\,;\,\mathbb{R})$ be the KM_2O-Langevin data associated with the process $\boldsymbol{X}$. We know that $\boldsymbol{X}$ satisfies the forward (resp. backward) KM_2O-Langevin equation (3.10$_+$) (resp. (3.10$_-$)):

KM_2O-Langevin equations ([7]).

$$X(\pm n) = -\sum_{k=1}^{n-1}\gamma_\pm(n, k)X(\pm k) - \delta_\pm(n)X(0) + \nu_\pm(\pm n) \qquad (n \in \mathbb{N}). \tag{3.10$_\pm$}$$

We call the function $\gamma_+(\cdot, *)$ (resp. $\gamma_-(\cdot, *)$) the forward (resp. backward) KM_2O-Langevin delay function associated with the process $\boldsymbol{X}$. The function $\delta_+(\cdot)$ (resp. $\delta_-(\cdot)$) is said to be the forward (resp. backward) KM_2O-Langevin partial correlation function associated with the process $\boldsymbol{X}$.

REMARK 3.1. The forward KM_2O-Langevin equation (3.10$_+$) is a discrete analogue to the $(\alpha, \beta, \gamma, \delta)$-Langevin equation derived by T. Miyoshi ([**5**], [**6**]).

Concerning the relation between the Toeplitz matrices and the KM_2O-Langevin fluctuation functions, we can use the KM_2O-Langevin equations to see that

$$\det T_n^\pm = \prod_{k=0}^{n-1}\det V_\pm(k) \qquad (n \in \mathbb{N}). \tag{3.11$_\pm$}$$

If follows from (3.4) and (3.11$_\pm$) that

$$V_+(n), V_-(n) \in GL(d\,;\,\mathbb{R}) \qquad (n \in \mathbb{N}^*). \tag{3.12}$$

The fluctuation-dissipation theorem (**FDT**) we have stated in § 1 is the following:

FDT ([2], [1], [15], [18], [7]). *For any n, $k \in \mathbb{N}$, $n>k$,*

$$\gamma_{\pm}(n, k) = \gamma_{\pm}(n-1, k-1) + \delta_{\pm}(n)_{\mp}\gamma(n-1, n-k-1); \tag{$3.13_{\pm}$}$$

$$V_{\pm}(n) = (I - \delta_{\pm}(n)\delta_{\mp}(n))V_{\pm}(n-1); \tag{$3.14_{\pm}$}$$

$$\delta_{-}(n)V_{+}(n-1) = V_{-}(n-1){}^{t}\delta_{+}(n); \tag{3.15}$$

$$\delta_{-}(n)V_{+}(n) = V_{-}(n){}^{t}\delta_{+}(n), \tag{3.16}$$

where we put

$$\gamma_{+}(m, 0) = \delta_{+}(m) \quad \textit{and} \quad \gamma_{-}(m, 0) = \delta_{-}(m) \qquad (m \in \mathbb{N}). \tag{3.17}$$

FDT implies that both the KM_2O-Langevin delay and fluctuation functions can be recursively calculated from the KM_2O-Langevin partial correlation functions. On the other hand, the latter can be obtained from the correlation function R by the following formulae:

KM_2O-Langevin partial correlation functions ([2], [1], [15], [18], [7]).

$$\delta_{\pm}(n) = -(R(\pm n) + \sum_{k=0}^{n-2} \gamma_{\pm}(n-1, k)R(\pm(k+1)))V_{\pm}(n-1)^{-1} \quad (n \in \mathbb{N}). \tag{$3.18_{\pm}$}$$

For any m, $n \in \mathbb{N}^*$, $m \geqq n$, we define $P_+(m, n)$, $P_-(m, n)$ and $e_+(m, n)$, $e_-(m, n)$ by

$$P_{\pm}(m, n) = E[X(\pm m){}^{t}\nu_{\pm}(\pm n)]V_{\pm}(n)^{-1/2} \tag{$3.19_{\pm}$}$$

and

$$e_{+}(m, n) = E[(X(m) - P_{M_0^n(X)}X(m)){}^{t}(X(m) - P_{M_0^n(X)}X(m))], \tag{3.20_{+}}$$

$$e_{-}(m, n) = E[(X(-m) - P_{M_{-n}^0(X)}X(-m)){}^{t}(X(-m) - P_{M_{-n}^0(X)}X(-m))]. \tag{3.20_{-}}$$

We call the function $P_+(\cdot, *)$ (resp. $P_-(\cdot, *)$) the forward (resp. backward) prediction function and the function $e_+(\cdot, *)$ (resp. $e_-(\cdot, *)$) the forward (resp. backward) prediction error function. Then we know

Prediction formulae ([7]). (i) *For any m, $n \in \mathbb{N}^*$, $m \geqq n$,*

$$P_{M_0^n(X)}X(m) = \sum_{k=0}^{n} P_{+}(m, k)V_{+}(k)^{-1/2}\nu_{+}(k); \tag{3.21_{+}}$$

$$P_{M_{-n}^0(X)}X(-m) = \sum_{k=0}^{n} P_{-}(m, k)V_{-}(k)^{-1/2}\nu_{-}(-k). \tag{3.21_{-}}$$

(ii) *For any* $m, n \in \mathbb{N}^*$, $m > n$,

(3.22₊) $$P_{M_0^n(X)}X(m) = \sum_{k=0}^{n} Q_+(m, n; k)X(k);$$

(3.22₋) $$P_{M_{-n}^0(X)}X(-m) = \sum_{k=0}^{n} Q_-(m, n; k)X(-k).$$

Here the $M(d; \mathbb{R})$-valued prediction functions $P_\pm(\cdot, *)$ and $Q_\pm(\cdot, *; \star)$ can be determined by the following algorithms:

Prediction algorithms ([7]). (i) *For any* $m, k \in \mathbb{N}^*$, $m \geqq k$,

(3.23±) $$P_\pm(m, k) = \begin{cases} V_\pm(k)^{1/2} & \text{if } m = k \\ -\sum_{l=k}^{m-1} \gamma_\pm(m, l) P_\pm(l, k) & \text{if } m \geqq k+1. \end{cases}$$

(ii) *For any* $m, n, k \in \mathbb{N}^*$, $m > n \geqq k$,

(3.24±) $$Q_\pm(m, n; k) = \begin{cases} -\gamma_\pm(n+1, k) & \text{if } m = n+1 \\ \sum_{l=n+1}^{m-1} \gamma_\pm(m, l) Q_\pm(l, n; k) - \gamma_\pm(m, k) & \text{if } m \geqq n+2. \end{cases}$$

Finally the prediction error functions can be calculated by the following formulae:

Prediction error formulae ([7]). (i) *For any* $m, n \in \mathbb{N}^*$, $m > n$,

(3.25±) $$e_\pm(m, n) = \sum_{k=n+1}^{m} P_\pm(m, k)^t P_\pm(m, k).$$

(ii) *In particular, for any* $n \in \mathbb{N}^*$,

(3.26±) $$e_\pm(n+1, n) = (I - \delta_\pm(n+1)\delta_\mp(n+1)) \cdots (I - \delta_\pm(1)\delta_\mp(1)) R(0).$$

§4. AR(∞)-Langevin equations.

In the sequel, we shall assume that the time series $\boldsymbol{X}$ is purely nondeterministic and of full rank. Hence we remark from Theorem 2.2 that $\boldsymbol{X}$ has the spectral density matrix function Δ satisfying the regularity condition (2.9).

Let N be any fixed natural number. Using the unitary discrete group $(U(n); n \in \mathbb{Z})$ on the Hilbert space $\boldsymbol{M}_{-\infty}^{\infty}(\boldsymbol{X})$ defined by

(4.1) $$U(n)(X_j(m)) = X_j(m+n) \qquad (m, n \in \mathbb{Z}, 1 \leqq j \leqq d),$$

we define two d-dimensional weakly stationary processes $\boldsymbol{\varepsilon}_N^+ = (\varepsilon_N^+(n); n \in \mathbb{Z})$ and $\boldsymbol{\varepsilon}_N^- = (\varepsilon_N^-(n); n \in \mathbb{Z})$ by

(4.2±) $$\varepsilon_N^\pm(n) = U(n \mp N)\nu_\pm(\pm N).$$

Then the forward (resp. backward) KM$_2$O-Langevin equation (3.10₊) (resp. (3.10₋))

can be transformed into the following generalized forward (resp. backward) AR(N)-Langevin equation (4.3_+) (resp. (4.3_-)):

$$X(n) = -\sum_{k=1}^{N} \gamma_+(N, N-k)X(n-k) + \varepsilon_N^+(n) \qquad (n \in \mathbb{Z}); \tag{4.3_+}$$

$$X(n) = -\sum_{k=1}^{N} \gamma_-(N, N-k)X(n+k) + \varepsilon_N^-(n) \qquad (n \in \mathbb{Z}). \tag{4.3_-}$$

REMARK 4.1. The AR-Langevin equations with finite degree can be characterized in the framework of the generalized AR-Langevin equations ([**10**]).

According to the definition of the KM_2O-Langevin forces and ($4.2_\pm$), we have

$$\varepsilon_N^+(n) = X(n) - P_{M_{n-N}^{n-1}(X)} X(n) \qquad (n \in \mathbb{Z}); \tag{4.4_+}$$

$$\varepsilon_N^-(n) = X(n) - P_{M_{n+1}^{n+N}(X)} X(n) \qquad (n \in \mathbb{Z}). \tag{4.4_-}$$

Hence, by noting (2.4) and (2.11), we can let $N \to \infty$ in ($4.4_\pm$) to obtain the following

THEOREM 4.1.

$$\operatorname*{l.i.m.}_{N\to\infty} \varepsilon_N^\pm(n) = \varepsilon_\pm(n) \qquad \textit{for any } n \in \mathbb{Z}. \tag{$4.5_\pm$}$$

Concerning the global behaviour of the KM_2O-Langevin fluctuation functions, we have

THEOREM 4.2.

$$\lim_{n\to\infty} V_\pm(n) = V_\pm. \tag{$4.6_\pm$}$$

PROOF. According to the definition of the forward KM_2O-Langevin force, it follows from the weak stationarity of the process $\boldsymbol{X}$ that, for any $n \in \mathbb{N}$,

$$V_+(n+1) = E[(X(0) - P_{M_{-n-1}^{-1}(X)} X(0))^t (X(0) - P_{M_{-n-1}^{-1}(X)} X(0))].$$

Therefore, letting $n \to \infty$ in the above, we find from (2.4), (2.5), (2.11) and (2.12) that (4.6_+) holds. In the same way, we can prove (4.6_-). Q. E. D.

By virtue of Theorem 2.2, the matrices $V_\pm$ are non-singular and so we can define two d-dimensional time series $\boldsymbol{\xi}_\pm = (\xi_\pm(n); n \in \mathbb{Z})$ by

$$\xi_\pm(n) = V_\pm^{-1/2} \varepsilon_\pm(n). \tag{$4.7_\pm$}$$

It at once follows from (2.6) and (2.13) that two time series $\boldsymbol{\xi}_\pm$ are standard white noises:

$$E(\xi_\pm(m)^t \xi_\pm(n)) = \delta_{mn} I. \tag{$4.8_\pm$}$$

Immediately from (2.8), (2.14) and (4.7$_\pm$), we have

THEOREM 4.3. *There exist the following causal relations among* $\boldsymbol{X}$ *and* $\boldsymbol{\xi}_\pm$*:*

$$\boldsymbol{M}_{-\infty}^{n}(\boldsymbol{X}) = \boldsymbol{M}_{-\infty}^{n}(\boldsymbol{\xi}_+) \qquad (n\in\mathbb{Z})\,; \tag{4.9$_+$}$$

$$\boldsymbol{M}_{n}^{\infty}(\boldsymbol{X}) = \boldsymbol{M}_{n}^{\infty}(\boldsymbol{\xi}_-) \qquad (n\in\mathbb{Z})\,. \tag{4.9$_-$}$$

Moreover, by letting $N\to\infty$ in the generalized AR(N)-Langevin equations (4.3$_\pm$), we find from Theorems 4.1 and 4.2 that

THEOREM 4.4. *The time series* $\boldsymbol{X}$ *satisfies the following two kinds of stochastic difference equations:*

$$X(n) = -\operatorname*{l.i.m.}_{N\to\infty}\sum_{k=1}^{N-1}\gamma_+(N,\, N-k)X(n-k)+V_+^{1/2}\xi_+(n) \qquad (n\in\mathbb{Z})\,; \tag{4.10$_+$}$$

$$X(n) = -\operatorname*{l.i.m.}_{N\to\infty}\sum_{k=1}^{N-1}\gamma_-(N,\, N-k)X(n+k)+V_-^{1/2}\xi_-(n) \qquad (n\in\mathbb{Z})\,. \tag{4.10$_-$}$$

DEFINITION 4.1. We call equation (4.10$_+$) (resp. (4.10$_-$)) the forward (resp. backward) AR(∞)-Langevin equation associated with the time series $\boldsymbol{X}$.

We remark that equations (4.10$_\pm$) give the concrete representations for the inclusion $\supset$ in the causal relations (4.9$_\pm$).

§ 5. The global behaviour of KM$_2$O-Langevin partial correlation (resp. delay) functions.

In order to obtain the concrete form of the limit of coefficients in AR(∞)-Langevin equations (4.10$_\pm$), we shall study the global behaviour for both the KM$_2$O-Langevin partial correlation functions and the KM$_2$O-Langevin delay functions.

LEMMA 5.1.

$$\lim_{n\to\infty}\delta_\pm(n) = 0\,.$$

PROOF. We claim that

$$\text{the sequence } (\delta_+(n)\,;\, n\in\mathbb{N}) \text{ is bounded.} \tag{5.1}$$

By virtue of (3.14$_\pm$) and (3.15) in **FDT**,

$$V_\pm(n+1)-V_\pm(n) = -\delta_\pm(n+1)V_\pm(n)\,{}^t\delta_\pm(n+1) \tag{5.2$_\pm$}$$

and so, for any $x\in\mathbb{R}^d$,

$$\begin{aligned}((V_+(n+1)-V_+(n))x,\, x) &= (V_+(n)\,{}^t\delta_+(n+1)x,\, {}^t\delta_+(n+1)x)\\ &\geqq (V_+(\infty)\,{}^t\delta_+(n+1)x,\, {}^t\delta_+(n+1)x)\,.\end{aligned}$$

Hence, there exists a positive constant c such that, for any $n\in\mathbb{N}$,

$$\|{}^t\delta_+(n+1)x\| \leqq c\|x\| ,$$

which yields (5.1).

Let $D_+\in M(d ; \mathbb{R})$ be any limit point of the sequence $(\delta_+(n) ; n\in\mathbb{N})$ along a subsequence $(n_k ; k\in\mathbb{N})$ converging to ∞. Letting $k\to\infty$ in (5.2_+) along the subsequence above, we find that

$$D_+V_+{}^tD_+ = 0$$

and so the non-singularity of the matrix V_+ implies that $D_+=0$, which completes the forward part. The backward part can be proved in like manner.

Q. E. D.

LEMMA 5.2.

$$\lim_{n\to\infty}\gamma_\pm(n, k) = 0 \qquad \textit{for any fixed } k\in\mathbb{N} .$$

PROOF. We show only the forward part, because the backward part can be similarly proved. We claim that there exists a positive constant c_k such that

$$\|\gamma_+(n, n-k)\| \leqq c_k \qquad \text{for any } n\geqq k . \tag{5.3}$$

By multiplying ${}^t\nu_+(n-m)$ $(1\leqq m\leqq k)$ in the forward KM_2O-Langevin equation (3.10_+) from the right, we have

$$\begin{aligned} &E(X(n){}^t\nu_+(n-m)) \\ &= -\sum_{l=1}^{m-1}\gamma_+(n, n-l)E[X(n-l){}^t\nu_+(n-m)]-\gamma_+(n, n-m)V_+(n-m) . \end{aligned} \tag{5.4}$$

Since if follows from (3.8_+) and Theorem 4.2 that, for any $l\in\mathbb{Z}$,

$$\|E(X(l){}^t\nu_+(n-m))\| \leqq \|R(0)\|\|V_+(n-m)\| \leqq \|R(0)\|^2 ,$$

we can apply this and Theorem 4.2 to (5.4) to observe that (5.3) can be proved by mathematical induction.

Therefore, we can conclude from (3.13_+) in **FDT**, Lemma 5.1 and (5.3) that the forward part holds by mathematical induction. Q. E. D.

We are now in a position to state the main theorem for the global behaviour of the KM_2O-Langevin delay functions $\gamma_+(\cdot, *)$, $\gamma_-(\cdot, *)$.

THEOREM 5.1. *The limits $\gamma_\pm(k)\equiv\lim_{n\to\infty}\gamma_\pm(n, n-k)$ exist for any $k\in\mathbb{N}^*$ and they satisfy the following recursive relations:*

$$\gamma_\pm(k) = -\{E[X(0){}^t\xi_\pm(\mp k)]-\sum_{l=1}^{k-1}\gamma_\pm(l)E[X(0){}^t\xi_\pm(\mp(k-l))]\}V_\pm{}^{-1/2} .$$

PROOF. We prove only the forward part. According to the definition of the forward $\mathrm{KM_2O}$-Langevin force, for any l, $n\in\mathbb{N}^*$, $0\leqq l\leqq k-1<n$,

$$E(X(n-l)^t\nu_+(n-k))=E(X(0)^t\varepsilon_{n-k}^+(l-k)).$$

Substituting these relations into (5.4) with $m=k$, we have, for any $n\in\mathbb{N}$, $n\geqq k$,

$$E(X(0)^t\varepsilon_{n-k}^+(-k))=-\sum_{l=1}^{k-1}\gamma_+(n,n-l)E[X(0)^t\varepsilon_{n-k}^+(l-k)]-\gamma_+(n,n-k)V_+(n-k).$$

Therefore, we can apply Theorem 4.1 to the equations above to observe that the forward part holds by mathematical induction. Q. E. D.

§6. The prediction formulae.

We shall show that the d-dimensional white noises $\boldsymbol{\xi}_+$ and $\boldsymbol{\xi}_-$ in Theorem 4.1 play the same role as those in the canonical representation theorem for one-dimensional purely nondeterministic weakly stationary processes and give some prediction formulae.

At first we shall study the global behaviour of the prediction functions $P_\pm(\cdot,*)$.

THEOREM 6.1. (i) *For any $k\in\mathbb{N}^*$, the limits $P_\pm(k)\equiv\lim_{n\to\infty}P_\pm(n,n-k)$ exist and they are represented in terms of the white noises:*

$$P_\pm(k)=E(X(0)^t\xi_\pm(\mp k)). \tag{6.1$_\pm$}$$

(ii) *They satisfy the following estimates:*

$$\sum_{k=0}^{\infty}P_\pm(k)^tP_\pm(k)\leqq C_\pm \tag{6.2$_\pm$}$$

with two positive definite matrices C_+, $C_-\in GL(d;\mathbb{R})$.

(iii) *Moreover, they satisfy the following recursive relations:*

$$\begin{cases} P_\pm(0)=V_\pm{}^{1/2} \\ P_\pm(k)=-\sum_{l=0}^{k-1}\gamma_\pm(k-l)P_\pm(l) \qquad (k\in\mathbb{N}). \end{cases} \tag{6.3$_\pm$}$$

PROOF. Only the forward part is proved. By (3.19$_+$) and (4.2$_+$), for any n, $k\in\mathbb{N}^*$, $n\geqq k$,

$$P_+(n,n-k)=E[X(0)^t\varepsilon_n^+(-k)]V_+(n-k)^{-1/2},$$

which, combined with Theorems 4.1 and 4.2, implies that (i) holds. Furthermore, if follows from (3.25$_+$) that, for any n, $k\in\mathbb{N}^*$, $n\geqq k$,

$$e_+(n,n-k)=\sum_{l=1}^{k}P_+(n,n-k+l)^tP_+(n,n-k+l). \tag{6.4}$$

On the other hand, we can see from (3.20_+) that there exists a positive definite matrix $C_+ \in GL(d; \mathbb{R})$ satisfying

$$e_+(n, n-k) \leqq C_+ . \tag{6.5}$$

Therefore, by virtue of (i), we can let $n \to \infty$ in (6.4) to observe that (ii) holds from (6.5).

Finally it follows from ($3.23_\pm$) that, for any $n, k \in \mathbb{N}$,

$$\begin{cases} P_\pm(n, n) = V_\pm(n)^{1/2} \\ P_\pm(n, n-k) = -\sum_{l=0}^{k-1} \gamma_\pm(n, n-k+l) P_\pm(n-k+l, n-k) . \end{cases}$$

Therefore, letting $n \to \infty$ in the equations above, we find that (iii) follows from (i), Theorems 4.2 and 5.1. Q. E. D.

REMARK 6.1. The recursive relations in Theorem 5.1, together with ($6.1_\pm$), are the same as the relations ($6.3_\pm$) in Theorem 6.1.

Concerning the concrete representations for the inclusion $\subset$ in ($4.9_\pm$), we shall show

THEOREM 6.2. *The time series $\boldsymbol{X}$ can be represented in terms of the standard white noises $\boldsymbol{\xi}_\pm$:*

$$X(n) = \sum_{k=-\infty}^{n} P_+(n-k)\xi_+(k) \qquad (n \in \mathbb{Z}); \tag{6.6_+}$$

$$X(n) = \sum_{k=n}^{\infty} P_-(k-n)\xi_-(k) \qquad (n \in \mathbb{Z}). \tag{6.6_-}$$

PROOF. Note that each component of the right-hand sides of ($6.6_\pm$) converges in the space $L^2(\Omega, \mathcal{B}, P)$ by (ii) in Theorem 6.1. For any fixed $n \in \mathbb{Z}$, put

$$Y \equiv X(n) - \sum_{k=-\infty}^{n} P_+(n-k)\xi_+(k) .$$

It follows from (4.9_+) that $Y \in \boldsymbol{M}_{-\infty}^{n}(\boldsymbol{\xi}_+)$. On the other hand, we see from (4.9_+) and (6.1_+) that, for any $m \in \mathbb{Z}$, $m \leqq n$,

$$E(Y\,{}^t\xi_+(m)) = E(X(0)\,{}^t\xi_+(m-n)) - P_+(n-m) = 0$$

and so $Y=0$, giving (6.6_+). In the same way, (6.6_-) can be proved. Q. E. D.

By virtue of Theorems 4.3 and 6.2, the predictor based upon the past (resp. the future) can be concretely represented in terms of white noises $\boldsymbol{\xi}_+$ and $\boldsymbol{\xi}_-$.

THEOREM 6.3.

$$P_{M_{-\infty}^{0}(X)}X(n)=\sum_{k=-\infty}^{0}P_{+}(n-k)\xi_{+}(k) \qquad (n\in\mathbb{N}). \tag{6.7$_+$}$$

$$P_{M_{0}^{\infty}(X)}X(-n)=\sum_{k=0}^{\infty}P_{-}(k+n)\xi_{-}(k) \qquad (n\in\mathbb{N}). \tag{6.7$_-$}$$

On the other hand, we shall obtain another formulae of the predictors in terms of the time series X itself.

THEOREM 6.4.

$$P_{M_{-\infty}^{0}(X)}X(n)=\underset{N\to\infty}{\mathrm{l.i.m.}}\sum_{k=0}^{N}Q_{+}(N+n,\ N;\ N-k)X(-k) \qquad (n\in\mathbb{N}). \tag{6.8$_+$}$$

$$P_{M_{0}^{\infty}(X)}X(-n)=\underset{N\to\infty}{\mathrm{l.i.m.}}\sum_{k=0}^{N}Q_{-}(N+n,\ N;\ N-k)X(k) \qquad (n\in\mathbb{N}). \tag{6.8$_-$}$$

PROOF. We prove only (6.8$_+$), because (6.8$_-$) can be proved in a similar way. According to the prediction formula (3.22$_+$) with $m=N+n$ and $n=N$, for any N, $n\in\mathbb{N}$, we have

$$P_{M_{0}^{N}(X)}X(n+N)=\sum_{k=0}^{N}Q_{+}(n+N,\ N;\ k)X(k).$$

By operating the shift operator $U(-N)$ to the equation above,

$$P_{M_{-N}^{0}(X)}X(n+N)=\sum_{k=0}^{N}Q_{+}(n+N,\ N;\ N-k)X(-k).$$

Hence, letting $N\to\infty$ in this equation, we obtain (6.8$_+$). Q. E. D.

In order to obtain the concrete form of the limit of coefficients in the prediction formulae (6.8$_\pm$), we shall study the global behaviour of the prediction functions $Q_{\pm}(\cdot,\ *;\ \star)$.

THEOREM 6.5. *The limits* $Q_{\pm}(n,\ k)\equiv\lim_{N\to\infty}Q_{\pm}(n+N,\ N;\ N-k)$ *exist for any* $n\in\mathbb{N}$, $k\in\mathbb{N}^{*}$ *and they satisfy the following recursive relations:*

$$\begin{cases} Q_{\pm}(1,\ k)=-\gamma_{\pm}(k+1) \\ Q_{\pm}(n,\ k)=\sum_{l=1}^{n-1}\gamma_{\pm}(n-l)Q_{\pm}(l,\ k)-\gamma_{\pm}(n+k) \qquad (n\geqq 2). \end{cases} \tag{6.9$_\pm$}$$

PROOF. By (3.24$_+$),

$$Q_{\pm}(1+N,\ N;\ N-k)=-\gamma_{\pm}(N+1,\ N-k).$$

Hence, it follows from Theorem 5.1 that (6.9$_\pm$) holds for $n=1$. When $n\geqq 2$, furthermore, by using (3.24$_+$) again,

$$Q_{\pm}(n+N,N;\ N-k)=\sum_{l=1}^{n-1}\gamma_{\pm}(n+N,l+N)Q_{\pm}(N+l,N;\ N-k)-\gamma_{\pm}(n+N,N-k).$$

Therefore, by virtue of Theorem 5.1, we can let $N\to\infty$ in the above to observe that the limits $Q_{\pm}(n, k)\equiv\lim_{N\to\infty} Q_{\pm}(n+N, N; N-k)$ exist and they satisfy $(6.9_{\pm})$, by mathematical induction. Q. E. D.

§7. Outer matrix functions for the process X.

We determine two $M(d; \mathbb{R})$-valued functions E_+, E_- defined on $\mathbb{Z}$ by

$$E_{\pm}(n) = \sqrt{2\pi}\chi_{\{0,1,\dots\}}(\pm n)P_{\pm}(\pm n) \tag{7.1$_\pm$}$$

and then denote by H_+, H_- their Fourier inverse transform:

$$H_{\pm} = \tilde{E}_{\pm}\,. \tag{7.2$_\pm$}$$

Immediately from Theorem 6.2, we have

$$X(n) = \frac{1}{\sqrt{2\pi}} E_+ * \xi_+(n) \qquad (n\in\mathbb{Z}); \tag{7.3$_+$}$$

$$X(n) = \frac{1}{\sqrt{2\pi}} E_- * \xi_-(n) \qquad (n\in\mathbb{Z}). \tag{7.3$_-$}$$

It can be seen from $(7.3_{\pm})$ that

$$\Delta(\theta) = H_+(\theta)H_+(\theta)^* \qquad \text{a.e. } \theta\in[-\pi, \pi); \tag{7.4$_+$}$$

$$\Delta(\theta) = H_-(\theta)H_-(\theta)^* \qquad \text{a.e. } \theta\in[-\pi, \pi). \tag{7.4$_-$}$$

In particular, it is to be noted from (2.9) and $(7.4_{\pm})$ that $H_{\pm}(\theta)$ belong to $GL(d; \mathbb{R})$ for almost all $\theta\in[-\pi, \pi)$. The function H_+ is named by the generating function in [**16**], [**17**] and [**3**]. By taking account of $(4.9_{\pm})$, $(7.3_{\pm})$ and $(7.4_{\pm})$, according to the nomenclature in the one-dimensional case, we shall give the following

DEFINITION 7.1. We call the matrix function H_+ (resp. H_-) the forward (resp. backward) outer matrix function of the spectral density matrix function Δ or the process $\boldsymbol{X}$.

The aim of this section is to obtain a concrete representation for the outer matrix functions H_+, H_-. Let $(E(\theta); \theta\in[-\pi, \pi))$ be the resolution of the identity associated with the unitary discrete group $(U(n); n\in\mathbb{Z})$ acting on the Hilbert space $\boldsymbol{M}_{-\infty}^{\infty}(\boldsymbol{X})$:

$$X(n) = \int_{-\pi}^{\pi} e^{-in\theta}dE(\theta)X(0)\,. \tag{7.5}$$

It is to be noted that the spectral density matrix function Δ can be represented as

(7.6) $$\Delta(\theta)d\theta = d(E(\theta)X(0),\ X(0)).$$

LEMMA 7.1.

$$\xi_{\pm}(n) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-in\theta} H_{\pm}(\theta)^{-1} dE(\theta)X(0) \qquad (n\in\mathbb{Z}).$$

PROOF. By Theorem 4.3, we can find two $M(d\,;\mathbb{C})$-valued functions $F_{\pm} = (F^{\pm}_{jk})_{1\leq j,\,k\leq d}$ defined on $[-\pi,\ \pi)$ satisfying

(7.7$_{\pm}$) $$F^{\pm}_{jk} \in L^2([-\pi,\ \pi),\ \Delta_{kk}(\theta)d\theta) \qquad (1\leqq j,\ k\leqq d);$$

(7.8$_{\pm}$) $$\xi_{\pm}(0) = \int_{-\pi}^{\pi} F_{\pm}(\theta)dE(\theta)X(0).$$

However, since it follows from (2.4), (2.11) and (4.7$_{\pm}$) that

(7.9$_{\pm}$) $$U(n)\xi_{\pm}(0) = \xi_{\pm}(n) \qquad \text{for any } n\in\mathbb{Z},$$

we can see that

(7.10$_{\pm}$) $$\xi_{\pm}(n) = \int_{-\pi}^{\pi} e^{-in\theta}F_{\pm}(\theta)dE(\theta)X(0) \qquad (n\in\mathbb{Z}).$$

By substituting (7.10$_{\pm}$) into (7.3$_{\pm}$), we have

$$X(n) = \sqrt{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\tilde{E}_{\pm}(\theta)F_{\pm}(\theta)dE(\theta)X(0) \qquad \text{for any } n\in\mathbb{Z}.$$

Therefore, by virtue of the uniqueness of Fourier transform, it follows from (7.2$_{\pm}$) and (7.5) that

$$F_{\pm} = \frac{1}{\sqrt{2\pi}} H_{\pm}^{-1},$$

which, together with (7.10$_{\pm}$), completes the proof of Lemma 7.1. Q. E. D.

For each $n\in\mathbb{N}$, two $M(d\,;\mathbb{R})$-valued functions F^+_n, F^-_n are defined on $[-\pi, \pi)$:

(7.11$_{\pm}$) $$F^{\pm}_n(\theta) = I + \sum_{l=1}^{n-1}\gamma_{\pm}(n,\ n-l)e^{il\theta}.$$

We are now in a position to prove the main theorem of this section.

THEOREM 7.1. (i) *The following limits exist in* $L^2(-\pi,\ \pi)$:

$$\underset{n\to\infty}{\text{l.i.m.}}\ F^{\pm}_n H_{\pm} = \frac{1}{\sqrt{2\pi}} V_{\pm}(\infty)^{1/2}.$$

(ii) *There exists a subsequence* $(n_k)_{k=1}^{\infty}$ *converging to* ∞ *such that*

$$H_{\pm}(\theta)^{-1} = \sqrt{2\pi}V_{\pm}(\infty)^{-1/2}\lim_{k\to\infty}\sum_{l=1}^{n_k-1}(I+\gamma_{\pm}(n_k,\ n_k-l)e^{il\theta}) \quad a.\,e.\ \theta\in[-\pi,\ \pi).$$

PROOF. (ii) immediately follows from (i). Put

$$Y(n) \equiv X(0)+\sum_{l=1}^{n-1}\gamma_+(n,\ n-l)X(-l)-V_+(\infty)^{1/2}\xi_+(0).$$

According to Theorem 4.4,

$$\lim_{n\to\infty} E(Y(n)^t Y(n)) = 0\,. \tag{7.12}$$

On the other hand, by virtue of Lemma 7.1, it follows from the spectral representation (7.5) and ($7.11_\pm$) that

$$Y(n) = \int_{-\pi}^{\pi}\left\{F_n^+(\theta)-\frac{1}{\sqrt{2\pi}}V_+(\infty)^{1/2}H_+(\theta)^{-1}\right\}dE(\theta)X(0).$$

Hence, we find from (7.4_+) that

$$\begin{aligned}
&E(Y(n)^t Y(n))\\
&=\int_{-\pi}^{\pi}\left\{F_n^+(\theta)-\frac{1}{\sqrt{2\pi}}V_+(\infty)^{1/2}H_+(\theta)^{-1}\right\}\Delta(\theta)\left\{F_n^+(\theta)-\frac{1}{\sqrt{2\pi}}V_+(\infty)^{1/2}H_+(\theta)^{-1}\right\}^* d\theta\\
&=\int_{-\pi}^{\pi}\left\{F_n^+(\theta)H_+(\theta)-\frac{1}{\sqrt{2\pi}}V_+(\infty)^{1/2}\right\}\left\{F_n^+(\theta)H_+(\theta)-\frac{1}{\sqrt{2\pi}}V_+(\infty)^{1/2}\right\}^* d\theta\,.
\end{aligned}$$

Therefore, combining this with (7.12), we have the forward part of (i). The backward part can be similarly proved. Q.E.D.

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