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On affine hypersurfaces with parallel nullity

By Katsumi NOMIZU*) and Barbara OPOZDA**)

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Affine differential geometry for hypersurfaces in the classical sense of Blaschke is based on the hypothesis that the given hypersurface is nondegenerate; quote from [**B**, p. 104]: Für parabolisch gekrümmte Flächen ("Torsen", $LN-M^2=0$) versagt die Grundform. In relative geometry (for example, see [**S**]) and in the study of affine immersions [**N-P1**], [**N-P2**], the nondegeneracy condition is often important, although a few results (for example, Berwald's theorem [**N-P2**], Radon's theorem [**O**]) have been established under a somewhat weaker assumption on the rank of the fundamental form h.

In this paper, we examine a general condition weaker than nondegeneracy under which geometry of a given hypersurface can be reduced to the classical situation. We start with an immersion $f: M^n \to R^{n+1}$. *For an arbitrary choice of a transversal vector field ξ, consider the condition that the kernel T^0 of h be parallel relative to the connection ∇ induced by ξ. It turns out that this condition is independent of the choice of ξ.* Under this condition of parallel nullity and under a completeness assumption which is also intrinsic, we shall show that f is globally a cylinder immersion of the form $M^n=M^r \times L$, $f=f_1 \times f_0$, where $f_1: M^r \to R^{r+1}$ is a nondegenerate hypersurface, L is a leaf of T^0, and f_0 is a connection-preserving map of L onto R^{n-r}, where R^{r+1} and R^{n-r} are affine subspaces in R^{n+1} that are mutually transversal. Such a representation is unique up to equiaffine transformation. Thus the geometry of M^n is completely determined by that of a profile nondegenerate hypersurface M^r in R^{r+1} that is itself uniquely determined up to equiaffine equivalence. For later applications we include additional information on transversal vector fields.

1. Preliminaries.

Let $f: M^n \to R^{n+1}$ be a connected hypersurface immersed in the affine space R^{n+1} provided with a fixed determinant function (volume element). Around

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each point of M^n let ξ be an arbitrarily chosen transversal vector field. As usual, we write

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X, Y)\xi \tag{I}$$

and

$$D_X \xi = -f_*(SX) + \tau(X)\xi\,, \tag{II}$$

where X, Y are vector fields on M^n, ∇ is the induced connection on M^n, h the affine fundamental form, S the shape operator, and τ the transversal connection form, all depending on the chosen ξ. The following lemma is standard.

LEMMA 1. *If we change ξ to another transversal vector field $\bar{\xi}=(f_*Z+\xi)/\lambda$, where Z is a certain vector field on M^n and λ a positive function, then the induced connection, the affine fundamental form, the transversal connection form, and the shape operator change as follows:*

$$\bar{h} = \lambda h\,; \tag{1}$$

$$\bar{\nabla}_X Y = \nabla_X Y - h(X, Y)Z\,; \tag{2}$$

$$\bar{\tau} = \tau + \eta - d(\log \lambda)\,, \tag{3}$$

where η is the 1*-form such that $\eta(X)=h(X, Z)$ for all X;*

$$\bar{S}X = [SX - \bar{\nabla}_X Z + \tau(X)Z + h(X, Z)Z]/\lambda\,. \tag{4}$$

By virtue of (1) we see that the rank of $\bar{h}$ at a point x is the same as that of h at x. We call it the rank of f at x. We also see that the null space $\{X: h(X, Y)=0$ for all $Y\}$ at x is the same as the null space of $\bar{h}$ at x. This null space of h is denoted by $T^0(x)$. We shall say that T^0 is parallel relative to ∇ if, for any curve from x to y, parallel translation along the curve maps $T^0(x)$ onto $T^0(y)$. In this case, the dimension of $T^0(x)$ remains constant on M^n. In general, it is known that a differentiable distribution, say T^0, is parallel if and only if for any vector field $Y \in T^0$ we have $\nabla_X Y \in T^0$ for every vector field X.

LEMMA 2. *The condition that T^0 is parallel relative to ∇ is independent of the choice of transversal vector field.*

PROOF. Suppose T^0 is parallel relative to ∇. For any curve x_t, $0 \leqq t \leqq 1$, and for any ∇-parallel $Y_t \in T^0$, we have by (2)

$$\bar{\nabla}_t Y_t = \nabla_t Y_t - h(X_t, Y_t)U = \nabla_t Y_t = 0\,,$$

where X_t is the tangent vector field of x_t. Thus Y_t is $\bar{\nabla}$-parallel. This means that T^0 is $\bar{\nabla}$-parallel.

From now on, we assume that our hypersurface satisfies the condition of parallel nullity (that is, T^0 is parallel relative to ∇). The distribution T^0 being parallel, it is integrable and totally geodesic. We say that T^0 is complete if each leaf L of T^0 is complete relative to ∇, that is, every ∇-geodesic in L extends infinitely for its affine parameter. In this regard we have

LEMMA 3. *On each leaf L of T^0 the induced connection ∇ is the same for any choice of ξ. In particular, the property that T^0 is complete is independent of the choice of ξ.*

PROOF. If X, Y are vector fields on L, then we have $\bar{\nabla}_X Y = \nabla_X Y - h(X, Y)U = \nabla_X Y$. Thus two connections ∇ and $\bar{\nabla}$ coincide on L.

From (I), we easily get

LEMMA 4. *For every leaf L of T^0, $f(L)$ is a totally geodesic submanifold in R^{n+1}. If T^0 is complete, then $f(L)$ is an entire affine subspace of dimension $s = \dim T^0$; f actually gives a connection-preserving diffeomorphism of L onto the affine subspace $f(L)$. Moreover, for two distinct leaves L_1 and L_2 of T^0, $f(L_1)$ and $f(L_2)$ are affine subspaces which are D-parallel in R^{n+1}.*

REMARK 1. If the connection ∇ induced by some transversal vector field ξ is complete and if T^0 is parallel, then T^0 is complete.

REMARK 2. If an affine hypersurface $f: M^n \to R^{n+1}$ has the property that $\nabla h = 0$ for some choice of transversal vector field, then it obviously satisfies the condition of parallel nullity.

REMARK 3. For an affine hypersurface $f: M^n \to R^{n+1}$, the Gauss equation implies that for each point $x \in M^n$ we have

$$T^0(x) \subset \bigcap_{X, Y \in T_x(M^n)} \ker R(X, Y).$$

The two subspaces coincide if the rank of S is >1 at x. If rank $S > 1$ everywhere and if $\nabla R = 0$, then it follows that T^0 is parallel.

We add the following facts for later use. Assume that two transversal vector fields ξ and $\bar{\xi}$ coincide mod T^0, that is, $\bar{\xi} = \xi + f_*(Z)$, where $Z \in T^0$. Then from Lemma 1 we see that

$$\bar{h} = h \quad \text{and} \quad \bar{\tau} = \tau$$

$$\bar{S} = S \bmod T^0 \quad \text{and} \quad \bar{\nabla} = \nabla \bmod T^0,$$

that is, $\bar{\nabla}_X Y - \nabla_X Y \in T^0$ for all vector fields X, Y. Now using these facts it is easy to establish the following.

LEMMA 5. *Assume that $\bar{\xi} = \xi \bmod T^0$. Then we have*

(5) $$\overline{\nabla h} = \nabla h\,,$$

(6) $$\bar{R} = R \bmod T^0\,,$$

that is, $\bar{R}(X, Y)W - R(X, Y)W \in T^0$ *for all* X, Y, W.

Moreover, if ξ *satisfies* $ST^0 \subset T^0$, *then*

(7) $$\overline{\nabla S} = \nabla S \mod T^0\,,$$

(8) $$\overline{\nabla R} = \nabla R \mod T^0\,.$$

2. Global cylinder representation of a hypersurface M^n.

We now prove the following theorem.

THEOREM. *Let* $f: M^n \to R^{n+1}$ *be a connected hypersurface such that its affine fundamental form* h *has parallel kernel* T^0. *Assume that* T^0 *is complete. Then we can express* $f: M^n \to R^{n+1}$ *as follows:* $M^n = M^r \times L$, $f = f_1 \times f_0$, *where* $f_1: M^r \to R^{r+1}$ *is a connected nondegenerate hypersurface and* f_0 *is a connection-preserving map of a leaf* L *of* T^0 *onto* R^{n-r}, *and* $R^{n+1} = R^{r+1} \times R^{n-r}$. *Such a representation is unique up to equiaffine transformation of* R^{n+1} *so that a nondegenerate profile hypersurface* M^r *is determined uniquely up to equiaffine transformation of* R^{r+1}.

PROOF. Let x_0 be an arbitrary but fixed point of M^n. For the leaf L through x_0 of T_0, $f(L)$ is an entire affine subspace of dimension $s = n - r$ through $o = f(x_0)$ in R^{n+1}. Call it R^s. For any point $p \in R^{n+1}$ we denote by $R^s(p)$ the s-dimensional affine subspace through p that is parallel to R^s. Again from Lemma 4 we know that if $x \in M^n$, then the image by f of the leaf $L(x)$ through x coincides with $R^s(f(x))$. Let us choose an affine subspace of dimension $r+1$, say, R^{r+1} through $f(x_0)$ that is transversal to R^s. The mapping $f: M^n \to R^{n+1}$ is then transversal to R^{r+1}. In fact, for any $x \in M^n$ such that $p = f(x) \in R^{r+1}$ we have $T_p(R^{n+1}) = T_p(R^{r+1}) + f_*(T_x(M^n))$, because $f_*(T_x(M^n))$ contains $R^s(p) = f(L(x))$, where $L(x)$ is the leaf of T^0 through x. By a well-known theorem (for example, see [**H**, p. 22]), it follows that $M^r = \{x \in M^n : f(x) \in R^{r+1}\}$ is an r-dimensional submanifold of M^n. We see that the restriction of $f: M^n \to R^{n+1}$ to M^r gives rise to a hypersurface $f_1: M^r \to R^{r+1}$; we shall show in a moment that M^r is connected. In the case where the original immersion $f: M^n \to R^{n+1}$ is an imbedding, we may think of M^r as the intersection of M^n with R^{r+1}.

Now we define a one-to-one map $\Phi: M^n \to M^r \times L$ as follows. We consider $o = f(x_0)$ as the origin of R^{n+1}, R^s, and R^{r+1}, whenever we need a reference point in each of these affine spaces. Now for any $x \in M^n$, we define

$$\Phi(x)=(y, z)\in M^r\times L_0,$$

where y, z are determined as follows. Consider $p=f(x)$. For the leaf $L(x)$ of T^0 through x, $f(L(x))$ is the affine subspace $R^s(p)$, which meets R^{r+1} at a certain unique point, say, q. Since f is one-to-one on $L(x)$, there is a unique point $y\in L(x)\subset M^n$ such that $f(y)=q$. This means $y\in M^r$. On the other hand, the vector from q to p is parallel to the vector from o to z, where z is a certain uniquely determined point of R^s. It is now easy to find the inverse map $M^r\times L\to M^n$ of Φ. Since Φ is differentiable, the existence of the projection $M^n\to M^r$ shows that M^r is connected. So we get a cylinder representation of M^n with a profile hypersurface M^r.

We have yet to prove the uniqueness of such a representation. For this purpose we use the following lemma in analytic geometry that is easy to prove.

LEMMA 6. *Let R^s be a fixed affine subspace of the affine space R^{n+1}. Suppose R^{r+1} and $\bar{R}^{r+1}$ are two affine subspaces that are transversal to R^s. We define a map F_1 of R^{r+1} onto $\bar{R}^{r+1}$ as follows: for each point $x\in R^{r+1}$, let $R^s(x)$ denote the affine subspace through x that is parallel to R^s. We let $\bar{x}$ be the uniquely determined point of intersection with $\bar{R}^{r+1}$ and set $F_1(x)=\bar{x}$. Then F_1 is an affine transformation of R^{r+1} onto $\bar{R}^{r+1}$. Moreover, F_1 is equiaffine (that is volume-preserving) if we fix a determinant function (parallel volume element) ω_{n+1} on R^{n+1} and a determinant function ω_s on R^s, and further define determinant functions ω_{r+1} and $\bar{\omega}_{r+1}$ on R^{r+1} and $\bar{R}^{r+1}$, respectively, such that $\omega_{n+1}=\omega_{r+1}\wedge\omega_s=\bar{\omega}_{r+1}\wedge\omega_s$.*

Now suppose $\bar{\Phi}: M^{n+1}\to\bar{M}^r\times\bar{L}$ is another cylinder representation, where $\bar{f}_1:\bar{M}^r\to\bar{R}^{r+1}$ is nondegenerate hypersurface of $\bar{R}^{r+1}$ and $\bar{f}_0:\bar{L}\to\bar{R}^s$ is a connection-preserving map of a leaf $\bar{L}$ of T^0 onto an affine subspace $\bar{R}^s$ transversal to $\bar{R}^{r+1}$. We may assume, without loss of generality, that $L=\bar{L}$, $R^s=\bar{R}^s$, and $f_0=\bar{f}_0$. Then we get an equiaffine transformation $F_1: R^{r+1}\to\bar{R}^{r+1}$ in the manner of Lemma 6. Combining this with the identity map: $R^s\to\bar{R}^s$ we get an equiaffine transformation, denoted by F, of R^{n+1} onto itself. It is now clear that $F_1(M^r)=\bar{M}^r$ and $\bar{\Phi}=F\circ\Phi$. This completes the proof of the theorem.

COROLLARY. *Under the assumption of the theorem, we can find a unique transversal vector field ξ for M^n with the following properties:*

1) *ξ is D-parallel in the direction of T^0; the affine shape operator vanishes on T^0.*

2) *The restriction of ξ to a profile hypersurface M^r coincides with the affine normal of the nondegenerate hypersurface M^r.*

Such ξ is unique once a profile hypersurface is chosen.

REMARK 4. If we do not assume the completeness for T^0, then for any point x_0 of M^n we can get a local cylinder decomposition of a neighborhood U of x_0 in the form $V\times W$, where U is a nondegenerate hypersurface in R^{r+1} and W is an open subset of R^s.

We add some more information on the relationship between the geometry of M^n and that of M^r. Continuing the notation in the proof of the theorem, we define a distribution T^1 by

$$T_x^1 = f_* x^{-1}(R^{r+1}) \qquad \text{for each } x\in M^n,$$

where R^{r+1} is now considered as the vector subspace instead of the affine space R^{r+1} through $f(x_0)$. This distribution is obviously integrable. We denote by π the projection of the vector space R^{n+1} onto R^{r+1} (parallel to the subspace R^s). We also denote by the same symbol the projection of TM onto T^1 parallel to T^0 so that $f_*\circ\pi=\pi\circ f_*$. Let ξ be a transversal vector field to f. We define $\bar{\xi}=\pi\circ\xi$. Then $\bar{\xi}$ is also transversal to f and equal to ξ mod T^0. By the formulas preceding Lemma 5 and by those in Lemma 5 we have

PROPOSITION.

$$\bar{h} = h, \qquad \bar{\tau} = \tau, \qquad \bar{S} = \pi\circ S, \qquad \bar{\nabla}_X Y = \pi(\nabla_X Y)$$

$$\bar{R}(X, Y)W = \pi(R(X, Y)W),$$

$$(\bar{\nabla}_X \bar{S})(Y) = \pi(\nabla_X S)(Y),$$

and

$$(\bar{\nabla}_W \bar{R})(X, Y)W = \pi((\nabla_W R)(X, Y)V),$$

where X, Y, V, W are vector fields belonging to T^1; for the last two identities we need to assume that ξ satisfies the condition $ST^0\subset T^0$ in Lemma 5. Moreover, the same relations hold if $\bar{\nabla}$ is considered the connection on M^r (that is, the restriction to M^r).

REMARK 5. If ξ is assumed to be equiaffine, then certainly all the identities in Lemma 5 hold. Moreover, ξ is parallel relative to D along T^0.

Combining Remarks 3, 5 and the last identity in the proposition we obtain

COROLLARY. *Assume ξ is an equiaffine transversal vector field to a hypersurface $f: M^n\to R^{n+1}$ such that the induced connection satisfies $\nabla R=0$. If rank $S>1$ everywhere, then M^n is locally a cylinder $M^r\times R^s$ and $\bar{\nabla}$ on M^r is locally symmetric.*

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Katsumi NOMIZU
Department of Mathematics
Brown University
Providence, RI 02912
USA

Barbara OPOZDA
Instytut Matematyki, UJ
ul. Reymonta 4
30-059 Kraków
Poland