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A general method of axiomatizing fragments

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The purpose of this paper is to present a general method of axiomatizing fragments of first order theories. Smorynski [**6**] presented a *new semi-model-theoretic* method of axiomatizing fragments. Our method in this paper is *proof-theoretic*. For the sake of contrast, we prove several results in [**6**] by our method. The method in this paper is a generalization of one in [**9**], and a development of one in [**3**] and [**4**]. Motohashi [**4**] introduced approximation theory of uniqueness conditions by existence conditions, and gave an axiomatization theorem for intuitionistic theories with equality which are axiomatized by uniqueness conditions and existence conditions. We modify the notion of approximations in [**4**], and give an axiomatization theorem for any intuitionistic theories with equality. We introduce the notion of *inference forms*, which plays an important role in our discussion. Inference forms are figures expressing inference rules such that first order logistics are formalized by sets of inference forms. For a given fragment of a theory, by choosing a suitable set of inference forms which formalizes the theory, we can construct a series of axioms which axiomatizes the fragment.

§1 is a preliminary section in which we introduce several basic notations and notions. In §2, we introduce the notion of inference forms, and give a cut elimination theorem for logistics formalized by sets of inference forms. The proofs in the following sections are based on the cut elimination theorem. In §3, first, we introduce the notion of approximations which is a modification of one in [**4**]. Then, we give several theorems for axiomatizing fragments by approximations. In §4, we prove several axiomatization results, some of which are proved in [**6**]. §5 is preparatory to axiomatization theorems in §6. In §6, we give an axiomatization theorem for classical theories with equality and an axiomatization theorem for intuitionistic theories with equality.

§1. Preliminary.

We are concerned with Gentzen-type systems: A *logistic* (or a *theory*) consists of initial sequents and inference figures. In *intuitionistic* logistics, the succedent of each sequent consists of at most one formula.

We use the propositional symbols $\curlyvee$ and $\curlywedge$, that say the true sentence and the false sentence, respectively, and every logistic concerned has the following inference figures:

$$\frac{\curlyvee,\ \Gamma\rightarrow\Delta}{\Gamma\rightarrow\Delta} \qquad \frac{\Gamma\rightarrow\Delta,\ \curlywedge}{\Gamma\rightarrow\Delta}.$$

In [2], variables are distinguished into two kinds: *free* variables and *bound* variables. However, in this paper, variables are not to be so. The adjectives "free" and "bound" modify not variables, but occurrences of variables.

Each variant of a formula is identified with the formula (for *variant*, see [5] p. 35).

A *substitution* is a figure of the form $\binom{x_1 \cdots x_n}{t_1 \cdots t_n}$, where $x_1, \cdots, x_n$ are pairwise distinct variables and $t_1, \cdots, t_n$ are arbitrary terms. For each figure A, $A\binom{x_1 \cdots x_n}{t_1 \cdots t_n}$ denotes the figure which results from replacing all free occurrences of $x_1, \cdots, x_n$ in A by $t_1, \cdots, t_n$, respectively. We agree that whenever $A\binom{x_1 \cdots x_n}{t_1 \cdots t_n}$ appears, all the variables occurring in terms $t_1, \cdots, t_n$ are free at the places of the free occurrences of $x_1, \cdots, x_n$, respectively, in A. For each figure A and for each substitution θ, the figure $A\theta$ is called an *instance* of A.

An *n-ary predicate* is a figure of the form $\lambda x_1 \cdots x_n.\mathfrak{A}$, where $\mathfrak{A}$ is a formula and $x_1, \cdots, x_n$ are pairwise distinct variables. A predicate $\lambda x_1 \cdots x_n.\mathfrak{A}$ is *closed* if no other variables than $x_1, \cdots, x_n$ occur free in $\mathfrak{A}$. A predicate $\lambda x_1 \cdots x_n.\mathfrak{A}$ is *atomic* if the formula $\mathfrak{A}$ is atomic.

If A is an n-ary predicate of the form $\lambda x_1 \cdots x_n.\mathfrak{A}$, then $A(t_1, \cdots, t_n)$ denotes the formula $\mathfrak{A}\binom{x_1 \cdots x_n}{t_1 \cdots t_n}$.

For a sequence $\bar{x}$ of pairwise distinct n variables of the form $x_1, \cdots, x_n$, $\forall\bar{x}$ and $\exists\bar{x}$ denote $\forall x_1 \cdots \forall x_n$ and $\exists x_1 \cdots \exists x_n$, respectively. If $n=0$ then $\forall\bar{x}$ and $\exists\bar{x}$ are empty.

If $\bar{s}$ and $\bar{t}$ are sequences of terms $s_1, \cdots, s_n$ and $t_1, \cdots, t_n$, respectively, then $\bar{s}=\bar{t}$ denotes the conjunction $s_1=t_1\wedge \cdots \wedge s_n=t_n$.

For a set $\mathcal{S}$ of symbols, a figure is *$\mathcal{S}$-free* if it has no occurrences of symbols in $\mathcal{S}$. For a set $\mathcal{P}$ of predicate symbols, a formula is *$\mathcal{P}$-atomic* if it is an atomic formula whose predicate symbol belongs to $\mathcal{P}$. A formula is *$\mathcal{P}$-positive* if it has no negative occurrences of $\mathcal{P}$-atomic formulas. A formula is *$\mathcal{P}$-negative* if it has no positive occurrences of $\mathcal{P}$-atomic formulas. For a trio $(\mathcal{P}, \mathcal{N}, \mathcal{F})$ of mutually disjoint sets of predicate symbols, a formula is a *$(\mathcal{P}, \mathcal{N}, \mathcal{F})_+$-formula* if it is $\mathcal{P}$-positive, $\mathcal{N}$-negative, and $\mathcal{F}$-free. A formula is a *$(\mathcal{P}, \mathcal{N}, \mathcal{F})_-$-formula* if it is a $(\mathcal{N}, \mathcal{P}, \mathcal{F})_+$-formula. A sequent is a *$(\mathcal{P}, \mathcal{N}, \mathcal{F})$-clause* if the antecedent consists of $\mathcal{P}\cup\mathcal{F}$-atomic formulas and the succedent consists of $\mathcal{N}\cup\mathcal{F}$-atomic formulas. A sequent is $(\mathcal{P}, \mathcal{N}, \mathcal{F})$*-sequent* if the antecedent consists of $\mathcal{P}\cup\mathcal{F}$-atomic formulas and $(\mathcal{P}, \mathcal{N}, \mathcal{F})_-$-formulas, and the succedent consists of $\mathcal{N}\cup\mathcal{F}$-atomic formulas and $(\mathcal{P}, \mathcal{N}, \mathcal{F})_+$-formulas. A $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-sequent is *simple* if the succedent consists of only $(\mathcal{P}, \mathcal{N}, \mathcal{F})_+$-formulas. A $(\emptyset, \emptyset, \mathcal{F})$-clause is called an $\mathcal{F}$*-clause,* and a $(\emptyset, \emptyset, \mathcal{F})$-sequent is called an $\mathcal{F}$*-sequent.*

For a set $\mathcal{H}$ of formulas, the $\mathcal{H}$*-part of a theory* $\boldsymbol{T}$ is the set of all the $\boldsymbol{T}$-provable formulas belonging to $\mathcal{H}$. If $\mathcal{H}$ is the set of all the $\mathcal{S}$-free formulas, then the $\mathcal{H}$-part of a theory $\boldsymbol{T}$ is called the $\mathcal{S}$*-free part* of $\boldsymbol{T}$. If $\mathcal{H}$ is the set of all the $(\mathcal{P}, \mathcal{N}, \mathcal{F})_+$-formulas, then the $\mathcal{H}$-part of a theory $\boldsymbol{T}$ is called the $(\mathcal{P}, \mathcal{N}, \mathcal{F})_+$*-part* of $\boldsymbol{T}$. The $\mathcal{H}$-part of a theory $\boldsymbol{S}$ is *axiomatized by a series of axioms* $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \cdots$ *in a theory* $\boldsymbol{T}$ if $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \cdots$ are $\boldsymbol{S}$-provable sentences in $\mathcal{H}$ and the $\mathcal{H}$-part of $\boldsymbol{S}$ is identical with the $\mathcal{H}$-part of the theory obtained from $\boldsymbol{T}$ by adding the axioms $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \cdots$.

For a logistic $\boldsymbol{L}$, a *quasi-$\boldsymbol{L}$-proof figure* is a proof figure whose inference figures are some in $\boldsymbol{L}$ and whose initial sequents are arbitrary.

§ 2. Inference forms.

A *universal condition* is a figure of the form $\forall\bar{x}(\Gamma\to\Delta)$, where $\Gamma\to\Delta$ is a sequent. An *existensial condition* is a figure of the form $\exists\bar{y}(\Pi)$, where Π is a finite sequence of formulas. Note that $\forall\bar{x}$ and $\exists\bar{y}$ may be empty.

For each universal condition $\forall\bar{x}(\mathfrak{A}_1, \cdots, \mathfrak{A}_m\to\mathfrak{B}_1, \cdots, \mathfrak{B}_n)$, $\forall\bar{x}(\mathfrak{A}_1, \cdots, \mathfrak{A}_m\to\mathfrak{B}_1, \cdots, \mathfrak{B}_n)^\natural$ denotes the formula $\curlywedge$, $\forall\bar{x}(\mathfrak{B}_1\vee\cdots\vee\mathfrak{B}_n)$, $\forall\bar{x}\neg(\mathfrak{A}_1\wedge\cdots\wedge\mathfrak{A}_m)$, or $\forall\bar{x}(\mathfrak{A}_1\wedge\cdots\wedge\mathfrak{A}_m\supset\mathfrak{B}_1\vee\cdots\vee\mathfrak{B}_n)$, according as $m=0$ and $n=0$, $m=0$ and $n\neq0$, $m\neq0$ and $n=0$, or $m\neq0$ and $n\neq0$. For each existensial condition $\exists\bar{y}(\mathfrak{C}_1, \cdots, \mathfrak{C}_k)$, $\exists\bar{y}(\mathfrak{C}_1, \cdots, \mathfrak{C}_k)^\natural$ denotes the formula $\curlyvee$ or $\exists\bar{y}(\mathfrak{C}_1\wedge\cdots\wedge\mathfrak{C}_k)$, according as $k=0$ or not. If Σ is a sequence $F_1, \cdots, F_n$ of conditions, then $\Sigma^\natural$ denotes the sequence $F_1^\natural, \cdots, F_n^\natural$.

A universal condition $\forall x_1\cdots\forall x_n(\Gamma\to\Delta)$, is a *variant* of a universal condition $\forall y_1\cdots\forall y_n(\Gamma'\to\Delta')$ if $\Gamma\to\Delta$ is $(\Gamma'\to\Delta')\binom{y_1\ \cdots\ y_n}{x_1\ \cdots\ x_n}$, and $x_1, \cdots, x_n$ do not occur free in $\Gamma'\to\Delta'$. An existensial condition $\exists x_1\cdots\exists x_n(\Pi)$ is a *variant* of an existensial condition $\exists y_1\cdots\exists y_n(\Pi')$ if Π is $(\Pi')\binom{y_1\ \cdots\ y_n}{x_1\ \cdots\ x_n}$, and $x_1, \cdots, x_n$ do not occur free in Π'. Each variant of a condition is identified with the condition.

An *inference form* is a figure of the form $[\Phi\,;\Sigma\to\Theta\,;\Psi]$, where Φ and Ψ are finite sequences of formulas, Σ is a finite sequence of universal conditions, and Θ is a finite sequence of existensial conditions. An inference form $[\Phi\,;\forall\bar{x}_1(\Gamma_1\to\Delta_1), \cdots, \forall\bar{x}_m(\Gamma_m\to\Delta_m)\to\Theta\,;\Psi]$ is *intuitionistic* if each Δ_i consists of at most one formula, Ψ consists of at most one formula, and whenever Θ is not empty, Ψ is empty.

For a trio $(\mathscr{P}, \mathscr{N}, \mathscr{F})$ of mutually disjoint sets of predicate symbols, an inference form

$$[\Phi\,;\ \forall\bar{x}_1(\Gamma_1\to\Delta_1),\ \cdots,\ \forall\bar{x}_m(\Gamma_m\to\Delta_m)\to\exists\bar{y}_1(\Pi_1),\ \cdots,\ \exists\bar{y}_n(\Pi_n)\,;\ \Psi]$$

is a $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-*inference form* if each formula in Φ is atomic or $\mathscr{P}\cup\mathscr{N}\cup\mathscr{F}$-free, each formula in $\Gamma_1, \cdots, \Gamma_m, \Pi_1, \cdots, \Pi_n$ is $\mathscr{P}\cup\mathscr{F}$-atomic or $\mathscr{P}\cup\mathscr{N}\cup\mathscr{F}$-free, each formula in $\Delta_1, \cdots, \Delta_m$ is $\mathscr{N}\cup\mathscr{F}$-atomic or $\mathscr{P}\cup\mathscr{N}\cup\mathscr{F}$-free, and Ψ is empty. Moreover, if each formula in $\Delta_1, \cdots, \Delta_m$ is $\mathscr{P}\cup\mathscr{N}\cup\mathscr{F}$-free, then the $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-inference form is said to be *simple*.

A $(\varnothing, \varnothing, \mathscr{F})$-inference form is called an $\mathscr{F}$-*inference form*.

Let I be an inference form of the following form:

$$[\Phi\,;\ \forall\bar{x}_1(\Gamma_1\to\Delta_1),\ \cdots,\ \forall\bar{x}_m(\Gamma_m\to\Delta_m)\to\exists\bar{y}_1(\Pi_1),\ \cdots,\ \exists\bar{y}_n(\Pi_n)\,;\ \Psi].$$

Then an *I-rule* is an inference figure of the form

$$\frac{\Gamma_1, \Gamma\to\Delta, \Delta_1\ \cdots\ \Gamma_m, \Gamma\to\Delta, \Delta_m\quad \Pi_1, \Pi\to\Lambda\ \cdots\ \Pi_n, \Pi\to\Lambda}{\Phi, \Gamma, \Pi\to\Delta, \Lambda, \Psi}$$

where each variable in $\bar{x}_1, \cdots, \bar{x}_m, \bar{y}_1, \cdots, \bar{y}_n$ is not free in the lower sequent, i.e. it satisfies the *eigenvariable condition*. When we are concerned with an intuitionistic logistic, I is intuitionistic and Δ is empty, and then the I-rule is called an *intuitionistic I-rule*. Each occurrence of a formula in Φ or Ψ in the I-rule is called a *principal formula* of the I-rule. If an occurrence of a formula $\mathfrak{A}$ is a principal formula of the I-rule, then the formula $\mathfrak{A}$ itself also called a *principal formula* of the I-rule. Similarly, when we mention a cut formula, it means a formula or an occurrence of a formula according to the context. Note that if I is a $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-inference form, then all the upper sequents of an I-rule are $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-sequents if and only if the lower sequent is a $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-sequent, and if I is simple, then all the upper sequents of an I-rule are simple $(\mathscr{P}, \mathscr{N}, \mathscr{F})$ sequents if and only if the lower sequent is a simple $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-sequent.

For a set $\mathcal{S}$ of inference forms, an $\mathcal{S}$-*rule* is an I-rule for some instance I of an inference form in $\mathcal{S}$.

We confuse often a universal condition of the form $(\to\mathfrak{A})$ and an existensial condition of the form $(\mathfrak{B})$ with the formula $\mathfrak{A}$ and the formula $\mathfrak{B}$, respectively. For example, we may write $[\mathfrak{A}\supset\mathfrak{B}\,;\ \mathfrak{A}\to\mathfrak{B}\,;\ ]$ instead of $[\mathfrak{A}\supset\mathfrak{B}\,;\ (\to\mathfrak{A})\to(\mathfrak{B})\,;\ ]$.

For a classical logistic $\boldsymbol{L}$ and a set $\mathcal{S}$ of inference forms, $\boldsymbol{L}[\mathcal{S}]$ denotes the classical logistic obtained from $\boldsymbol{L}$ by adding all the $\mathcal{S}$-rules. For an intuitionistic logistic $\boldsymbol{L}$ and a set $\mathcal{S}$ of inference forms, $\boldsymbol{L}[\mathcal{S}]$ denotes the intuitionistic logistic obtained from $\boldsymbol{L}$ by adding all the intuitionistic $\mathcal{S}$-rules.

Let $\mathcal{E}$ be the set of all the inference forms

$$[\ ;\ \to s=s\,;\ ],\quad [\ ;\ s=t, \boldsymbol{A}(s)\to\ ;\ \boldsymbol{A}(t)]\quad\text{and}\quad [\boldsymbol{A}(s)\,;\ s=t\to\boldsymbol{A}(t):\ ]$$

such that s and t are terms and A is an atomic unary predicate. For each logistic $\boldsymbol{L}$, $\boldsymbol{L}_e$ denotes the logistic $\boldsymbol{L}[\mathcal{E}]$.

An inference figure *holds* in a logistic $\boldsymbol{L}$ if whenever all the upper sequents are $\boldsymbol{L}$-provable, the lower sequent is $\boldsymbol{L}$-provable. An inference form I holds in a logistic $\boldsymbol{L}$ if all I-rules hold in $\boldsymbol{L}$.

Let $\boldsymbol{L}$ be the logistic $\boldsymbol{LK}$, $\boldsymbol{LJ}$, $\boldsymbol{LK}_e$ or $\boldsymbol{LJ}_e$. Let $\mathcal{S}$ be a set of inference forms, and P an $\boldsymbol{L}[\mathcal{S}]$-proof figure. An occurrence $\hat{\mathfrak{A}}$ of a formula $\mathfrak{A}$ in P is *$\mathcal{S}$-free* if there is no bundle through $\hat{\mathfrak{A}}$ and an occurrence of $\mathfrak{A}$ which is a principal formula of an $\mathcal{S}$-rule (for *bundle*, see [8] p. 78). A cut in P is *$\mathcal{S}$-free* if both cut formulas are $\mathcal{S}$-free. A cut is *atomic* if the cut formula is atomic.

By the same way as Gentzen's proof of Hauptsatz, we have the following cut elimination theorem:

THEOREM 2.1. *Let $\boldsymbol{L}$ be the logistic $\boldsymbol{LK}$, $\boldsymbol{LJ}$, $\boldsymbol{LK}_e$, or $\boldsymbol{LJ}_e$. Let $\mathcal{S}$ be a set of $(\mathcal{P}\cup\mathcal{N}\cup\mathcal{F})$-inference forms. Then each $\boldsymbol{L}[\mathcal{S}]$-provable sequent is $\boldsymbol{L}[\mathcal{S}]$-provable without $\mathcal{S}$-free cut, and atomic cut.*

§ 3. Approximations.

Let $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{F}$ be mutually disjoint sets of predicate symbols.

A function A from the set of $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-clauses to the set of $(\mathcal{P}, \mathcal{N}, \mathcal{F})_-$-formulas is a *$(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximation* if all the variables occurring free in $A(\Gamma\to\Delta)$ occur in $\Gamma\to\Delta$ for each $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-clause $\Gamma\to\Delta$, and the outer most logical symbol of $A(\Gamma\to)$ is $\neg$ for each finite sequence Γ of $\mathcal{P}\cup\mathcal{F}$-atomic formulas. If the formula $A(\Gamma\to)$ has the form $\neg\mathfrak{A}$, then we write the formula $\mathfrak{A}$ $A(\Gamma)$. A $(\emptyset, \emptyset, \mathcal{F})$-approximation is called an *$\mathcal{F}$-approximation.*

Let A be a $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximation, and $\boldsymbol{S}$ and $\boldsymbol{T}$ theories.

A satisfies the initial condition in $\boldsymbol{T}$ if, for each $\mathcal{P}$-atomic formula $\mathfrak{A}$, each $\mathcal{N}$-atomic formula $\mathfrak{B}$, and each $\mathcal{F}$-atomic formula $\mathfrak{C}$, the sequents $A(\mathfrak{A})\to\mathfrak{A}$, $\mathfrak{B}\to A(\to\mathfrak{B})$, and $\to A(\mathfrak{C}\to\mathfrak{C})$ are $\boldsymbol{T}$-provable.

A is *commutative for substitution on $\boldsymbol{T}$* if, for each substitution θ, and for each $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-clause $\Gamma\to\Delta$, the formulas $A(\Gamma\to\Delta)\theta\equiv A(\Gamma\theta\to\Delta\theta)$ and $A(\Gamma)\theta\equiv A(\Gamma\theta)$ are $\boldsymbol{T}$-provable.

A is *monotone on $\boldsymbol{T}$* if the sequents $A(\Gamma\to\Delta)\to A(\Pi\to\Lambda)$, $A(\Pi)\to A(\Gamma)$ and $A(\Gamma)\supset A(\Pi\to\Lambda)\to A(\Pi\to\Lambda)$ are $\boldsymbol{T}$-provable for each pair of $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-clauses $\Gamma\to\Delta$ and $\Pi\to\Lambda$ such that all the formulas in Γ occur in Π and all the formulas in Δ occur in Λ.

A is *transitive on $\boldsymbol{S}$* if the sequents $\Gamma, A(\Gamma\to\Delta)\to\Delta$ and $\Gamma\to A(\Gamma)$ are $\boldsymbol{S}$-provable for each $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-clause $\Gamma\to\Delta$.

A is *adequate for $(\boldsymbol{S}, \boldsymbol{T})$* if A satisfies the initial conditions in $\boldsymbol{T}$, and is commutative for substitution on $\boldsymbol{T}$, monotone on $\boldsymbol{T}$ and transitive on $\boldsymbol{S}$.

The $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-approximation B defined below is said to be *basic*: For each $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-clause $\Gamma\to\Delta$,

$$B(\Gamma\to\Delta)=\begin{cases} \curlyvee & \text{if there is an } \mathscr{F}\text{-atomic formula occurring in both } \Gamma \text{ and } \Delta, \\ (\Gamma'\to\Delta')^{\natural} & \text{otherwise,}\end{cases}$$

where Γ' and Δ' are the subsequences obtained from Γ and Δ, respectively, by deleting all the $\mathscr{F}$-atomic formulas.

Immediately, we have the following corollary:

COROLLARY 3.1. *Let $\boldsymbol{L}$ be the logistic $\boldsymbol{LK}$ or $\boldsymbol{LJ}$. Let B be the basic $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-approximation. Then B is adequate for $(\boldsymbol{L}, \boldsymbol{L})$.*

Let A be a $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-approximation. Let $\Gamma\to\Delta$ be a $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-sequent of the form

$$\Gamma_1, \mathfrak{A}_1, \Gamma_2, \mathfrak{A}_2, \cdots, \Gamma_m, \mathfrak{A}_m, \Gamma_{m+1}\to\Delta_1, \mathfrak{B}_1, \Delta_2, \mathfrak{B}_2, \cdots, \Delta_n, \mathfrak{B}_n, \Delta_{n+1},$$

where $\Gamma_1, \Gamma_2, \cdots, \Gamma_m, \Gamma_{m+1}$ are sequences of $\mathscr{P}\cup\mathscr{F}$-atomic formulas, $\Delta_1, \Delta_2, \cdots, \Delta_n, \Delta_{n+1}$ are sequences of $\mathscr{N}\cup\mathscr{F}$-atomic formulas, $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_m$ are $(\mathscr{P}, \mathscr{N}, \mathscr{F})_-$-formulas, and $\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_n$ are $(\mathscr{P}, \mathscr{N}, \mathscr{F})_+$-formulas. Then $A(\Gamma\to\Delta)$ denotes the formula

$$(\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_m\to A(\Gamma_1, \Gamma_2, \cdots, \Gamma_{m+1}\to\Delta_1, \Delta_2, \cdots, \Delta_{n+1}), \mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_n)^{\natural},$$

or

$$(\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_m, A(\Gamma_1, \Gamma_2, \cdots, \Gamma_{m+1})\to\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_n)^{\natural}$$

according as some Δ_j is not empty, or all Δ_j are empty. Moreover, $A(\Gamma)$ denotes the formula

$$\mathfrak{A}_1\wedge\mathfrak{A}_2\wedge\cdots\wedge\mathfrak{A}_m\wedge A(\Gamma_1, \Gamma_2, \cdots, \Gamma_{m+1}).$$

We have the following corollary:

COROLLARY 3.2. *Let A be a $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-approximation which is commutative for substitution on $\boldsymbol{T}$. Then for each $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-sequent $\Gamma\to\Delta$ and each pair of terms s and t, the sequents $A\left(\Gamma\binom{x}{s}\right)\to\exists xA(\Gamma)$, $\forall xA(\Gamma)\to A\left(\Gamma\binom{x}{t}\right)$, $A\left(\Gamma\binom{x}{s}\to\Delta\binom{x}{s}\right)\to\exists xA(\Gamma\to\Delta)$ and $\forall xA(\Gamma\to\Delta)\to A\left(\Gamma\binom{x}{t}\to\Delta\binom{x}{t}\right)$ are $\boldsymbol{T}$-provable. Moreover, if $\boldsymbol{T}$ satisfies equality axioms, then the sequents $s=t$, $A\left(\Gamma\binom{x}{s}\to\Delta\binom{x}{s}\right)\to A\left(\Gamma\binom{x}{t}\to\Delta\binom{x}{t}\right)$ and $s=t$, $A\left(\Gamma\binom{x}{s}\right)\to A\left(\Gamma\binom{x}{t}\right)$ are $\boldsymbol{T}$-provable.*

Let $\bar{A}$ be a series $A_0, A_1, A_2, \cdots$ of $(\mathscr{P}, \mathscr{N}, \mathscr{F})$-approximations, and let $\boldsymbol{S}$ and $\boldsymbol{T}$ be theories.

$\bar{A}$ satisfies the initial condition in $\boldsymbol{T}$ if each A_k satisfies the initial condition

in $\boldsymbol{T}$.

$\bar{A}$ is *commutative for substitution on* $\boldsymbol{T}$, if each A_k is commutative for substitution on $\boldsymbol{T}$.

$\bar{A}$ is *monotone on* $\boldsymbol{T}$ if A_k is monotone on $\boldsymbol{T}$, and the sequents $A_k(\Gamma\rightarrow\Delta)\rightarrow A_{k+1}(\Gamma\rightarrow\Delta)$ and $A_{k+1}(\Gamma)\rightarrow A_k(\Gamma)$ are $\boldsymbol{T}$-provable for each number k and for each $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-clause $\Gamma\rightarrow\Delta$.

$\bar{A}$ is *transitive on* $\boldsymbol{S}$ if each A_k is transitive on $\boldsymbol{S}$.

$\bar{A}$ is *adequate for* $(\boldsymbol{S}, \boldsymbol{T})$ if $\bar{A}$ satisfies the initial condition in $\boldsymbol{T}$, and is commutative for substitution on $\boldsymbol{T}$, monotone on $\boldsymbol{T}$ and transitive on $\boldsymbol{S}$.

For a set $\mathcal{S}$ of $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-inference forms, $\bar{A}$ *satisfies* $\mathcal{S}$-rules in $\boldsymbol{T}$ if for each number h and each $\mathcal{S}$-rule

$$\frac{\Gamma_1, \Gamma\rightarrow\Delta, \Delta_1 \cdots \Gamma_m, \Gamma\rightarrow\Delta, \Delta_m \quad \Pi_1, \Pi\rightarrow\Lambda \cdots \Pi_n, \Pi\rightarrow\Lambda}{\Phi, \Gamma, \Pi\rightarrow\Delta, \Lambda}$$

such that the upper sequents are $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-sequents, there is a number k such that the following inference figure holds in $\boldsymbol{T}$:

$$\frac{\rightarrow A_h(\Gamma_1, \Gamma\rightarrow\Delta, \Delta_1)\cdots\rightarrow A_h(\Gamma_m, \Gamma\rightarrow\Delta, \Delta_m) \quad \rightarrow A_h(\Pi_1, \Pi\rightarrow\Lambda)\cdots\rightarrow A_h(\Pi_n, \Pi\rightarrow\Lambda)}{\rightarrow A_k(\Phi, \Gamma, \Pi\rightarrow\Delta, \Lambda)}.$$

THEOREM 3.1. *Let* $\boldsymbol{L}$ *be the logistic* $\boldsymbol{LK}$, $\boldsymbol{LJ}$, $\boldsymbol{LK}_e$ *or* $\boldsymbol{LJ}_e$. *Let* $\boldsymbol{T}$ *be a theory on* $\boldsymbol{L}$. *Let* $\mathcal{S}$ *be a set of* $(\mathcal{P}, \mathcal{N}, \mathcal{F})$*-inference forms such that the theory* $\boldsymbol{L}[\mathcal{S}]$ *is an extension of* $\boldsymbol{T}$. *Let* $A_0, A_1, A_2, \cdots$ *be a series of* $(\mathcal{P}, \mathcal{N}, \mathcal{F})$*-approximations which is adequate for* $(\boldsymbol{L}[\mathcal{S}], \boldsymbol{T})$, *and satisfies* $\mathcal{S}$*-rules in* $\boldsymbol{T}$. *Then the* $(\mathcal{P}, \mathcal{N}, \mathcal{F})_+$*-part of* $\boldsymbol{L}[\mathcal{S}]$ *is axiomatized by the series of axioms* $A_0(\varepsilon), A_1(\varepsilon), A_2(\varepsilon), \cdots$ *in* $\boldsymbol{T}$, *where* ε *is the empty sequence of* $\mathcal{P}\cup\mathcal{F}$*-atomic formulas.*

PROOF. For each number k, the $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximation A_k is transitive on $\boldsymbol{L}[\mathcal{S}]$, hence $A_k(\varepsilon)$ is $\boldsymbol{L}[\mathcal{S}]$-provable. On the other hand, $\boldsymbol{L}[\mathcal{S}]$ is an extension of $\boldsymbol{T}$. Therefore the theory $\boldsymbol{L}[\mathcal{S}]$ is an extension of the theory $\boldsymbol{T}$ with $A_0(\varepsilon), A_1(\varepsilon), A_2(\varepsilon), \cdots$. Thus, it sufficies to show that, for each $\boldsymbol{L}[\mathcal{S}]$-provable $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-sequent $\Pi\rightarrow\Lambda$, the formula $A_k(\Pi\rightarrow\Lambda)$ is $\boldsymbol{T}$-provable for some number k. Now let $\Pi\rightarrow\Lambda$ be an $\boldsymbol{L}[\mathcal{S}]$-provable $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-sequent. By Theorem 2.1, there is an $\boldsymbol{L}[\mathcal{S}]$-proof figure of $\Pi\rightarrow\Lambda$ such that it is $\mathcal{S}$-free cut free and atomic cut free. Let P be such a proof-figure. Then each sequent in P is $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-sequent. The series of approximations is commutative for substitution and monotone on $\boldsymbol{T}$, and satisfies $\mathcal{S}$-rule and the initial condition in $\boldsymbol{T}$, hence, by induction on the number of inference figures in P, we can show that $A_k(\Pi\rightarrow\Lambda)$ is $\boldsymbol{T}$-provable for some k.

A function A from the set of finite sequences of $\mathcal{P}\cup\mathcal{F}$-atomic formulas to the set of $(\mathcal{P}, \mathcal{N}, \mathcal{F})_+$-formulas is a *simple* $(\mathcal{P}, \mathcal{N}, \mathcal{F})$*-approximation* if all variables occurring free in $A(\Gamma)$ occur in Γ for each finite sequence Γ of $\mathcal{P}\cup\mathcal{F}$-atomic formulas. A simple $(\emptyset, \emptyset, \mathcal{F})$-approximation is called a *simple $\mathcal{F}$-approximation.*

From a $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximation A, we induce the simple $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximation A_+ such that $A_+(\Gamma)=A(\Gamma)$ for each finite sequence Γ of $\mathcal{P}\cup\mathcal{F}$-atomic formulas. The induced simple $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximation A_+ is called the *simple part* of A.

The notions for $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximations defined above are redefined for simple $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximations by restricting to the simple parts.

The simple $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximation B defined below is said to be *basic*: For each sequence Γ of $\mathcal{P}\cup\mathcal{F}$-atomic formulas, $B(\Gamma)=(\Gamma')^{\natural}$, where Γ' is the subsequence obtained from Γ by deleting all the $\mathcal{F}$-atomic formulas.

COROLLARY 3.3. *Let $\boldsymbol{L}$ be the logistic $\boldsymbol{LK}$ or $\boldsymbol{LJ}$. Let B be the basic simple $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximation. Then B is adequate for $(\boldsymbol{L}, \boldsymbol{L})$.*

Adding the assumption "*$\mathcal{S}$ is a set of simple $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-inference forms*" to Theorem 3.1, we have the following version:

THEOREM 3.2. *Let $\boldsymbol{L}$ be the logistic $\boldsymbol{LK}$, $\boldsymbol{LJ}$, $\boldsymbol{LK}_e$ or $\boldsymbol{LJ}_e$. Let $\boldsymbol{T}$ be a theory on $\boldsymbol{L}$. Let $\mathcal{S}$ be a set of simple $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-inference forms such that the theory $\boldsymbol{L}[\mathcal{S}]$ is an extension of $\boldsymbol{T}$. Let $A_0, A_1, A_2, \cdots$ be a series of simple $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximations which is adequate for $(\boldsymbol{L}[\mathcal{S}], \boldsymbol{T})$, and satisfies $\mathcal{S}$-rules in $\boldsymbol{T}$. Then the $(\mathcal{P}, \mathcal{N}, \mathcal{F})_+$-part of $\boldsymbol{L}[\mathcal{S}]$ is axiomatized by the series of axioms $A_1(\varepsilon), A_2(\varepsilon), A_3(\varepsilon), \cdots$ in $\boldsymbol{T}$.*

§ 4. Axiomatizing results.

4.1. Apartness vs. equality (cf. [10], [6]). Let $\#$ be a binary predicate symbol, and $\mathcal{A}_0$ the set of the simple $\{\#\}$-inference forms $[x\#x:\rightarrow;\ ]$, $[x\#y;\ \rightarrow y\#x;\ ]$, $[x\#z;\ \rightarrow x\#y,\ y\#z;\ ]$, and $[\ ;(x\#y\rightarrow\ )\rightarrow x=y;\ ]$. For each positive number n, let $\mathcal{A}_n$ be the union of the set $\mathcal{A}_0$ and the singleton set of the $\{\#\}$-inference form

$$[\ ;\ \rightarrow\exists x_0\cdots\exists x_n(x_0\#x_1,\ \cdots,\ x_0\#x_n,\ x_1\#x_2,\ \cdots,\ x_1\#x_n,\ \cdots,\ x_{n-1}\#x_n);\ ].$$

Let $\mathcal{A}_\omega$ be the union $\mathcal{A}_0\cup\mathcal{A}_1\cup\mathcal{A}_2\cup\cdots$, and $\boldsymbol{AP}_n$ the intuitionistic theory $\boldsymbol{LJ}_e[\mathcal{A}_n]$ for each n with $n\leqq\omega$. Note that $\boldsymbol{AP}_0$ is a conservative extension of the theory $\boldsymbol{LJ}[\mathcal{A}_0-\{[\ ;(x\#y\rightarrow\ )\rightarrow x=y;\ ]\}]$ without equality.

Let $\mathcal{S}_0$ be the set of the following inference forms:

$$[\neg x\neq_0 y;\ \rightarrow\ ;\ x=y],$$
$$[\neg x\neq_1 y;\ \rightarrow\ ;\ x=y],$$
$$[\neg x\neq_2 y;\ \rightarrow\ ;\ x=y],$$
$$\vdots$$

where $x \neq_n y$ $(n=0, 1, 2, \cdots)$ are the abbreviations as follows:

$$x \neq_0 y \quad : \neg x=y .$$

$$x \neq_{m+1} y : \forall z(x \neq_m z \vee z \neq_m y) .$$

For each positive number n, let $\mathcal{S}_n$ be the set of the union of the set $\mathcal{S}_0$ and the set of the inference forms

$$[\ ;\rightarrow; \exists x_0 \cdots \exists x_n(x_0 \neq_h x_1 \wedge \cdots \wedge x_0 \neq_h x_n \wedge x_1 \neq_h x_2 \wedge \cdots \wedge x_1 \neq_h x_n \wedge \cdots \wedge x_{n-1} \neq_h x_n)],$$

such that $h<\omega$. Let $\mathcal{S}_\omega$ be the union $\mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots$, and $\boldsymbol{SEQ}_n^\omega$ the intuitionistic theory $\boldsymbol{LJ}_e[\mathcal{S}_n]$ for each $n \leqq \omega$.

THEOREM (van Dalen and Statman [11]). *$\boldsymbol{AP}_n$ is conservative over $\boldsymbol{SEQ}_n^\omega$ for each n with $1 \leqq n \leqq \omega$.*

PROOF. For each natural number k, let A_k be the simple $\{\#\}$-approximation such that $A_k(u_1 \# v_1, \cdots, u_m \# v_m)$ is the formula $(u_1 \neq_k v_1, \cdots, u_m \neq_k v_m)^\natural$. Then, for each number n, we can check the following facts:

(1) $\boldsymbol{AP}_n$ is an extension of $\boldsymbol{SEQ}_n^\omega$.

(2) The series $A_1, A_2, A_3, \cdots$ of simple $\{\#\}$-approximations is adequate for $(\boldsymbol{AP}_n, \boldsymbol{SEQ}_n^\omega)$, and satisfies $\mathcal{A}_n$-rules in $\boldsymbol{SEQ}_n^\omega$.

Therefore, by Theorem 3.2, for each number n and each $\{\#\}$-free formula $\mathfrak{A}$, $\mathfrak{A}$ is $\boldsymbol{AP}_n$-provable if and only if $A_k(\varepsilon) \rightarrow \mathfrak{A}$ is $\boldsymbol{SEQ}_n^\omega$-provable for some k. On the other hand, $A_k(\varepsilon)$ is the true sentence for each k. Thus $\boldsymbol{AP}_n$ is conservative over $\boldsymbol{SEQ}_n^\omega$ for each n.

4.2. Apartness vs. linear order (cf. [6]). Let $<$ be a binary predicate symbol. Let $\mathcal{L}_0$ be the union of $\mathcal{A}_0$ and the set of the inference forms $[x<x; \rightarrow\ ;]$, $[x<y, y<z; \rightarrow x<z;]$, $[x<y; \rightarrow x<z, z<y;]$, $[x \# y; \rightarrow x<y, y<x;]$ and $[x<y; \rightarrow x \# y;]$. Let $\boldsymbol{LO}_0$ be the intuitionistic theory $\boldsymbol{LJ}_e[\mathcal{L}_0]$. Moreover, let $\mathcal{D}^-$ be the set of inference forms $[\ ; \rightarrow \exists x_1 \exists x_2(x_1<x_2);]$ and $[x<y; \rightarrow \exists z(x<z, z<y);]\}$, and $\boldsymbol{DLO}^-$ the intuitionistic theory $\boldsymbol{LO}_0[\mathcal{D}^-]$.

THEOREM (Smorynski [6]). *$\boldsymbol{LO}_0$ is conservative over $\boldsymbol{AP}_0$, and $\boldsymbol{DLO}^-$ is conservative over $\boldsymbol{AP}_\omega$.*

PROOF. Let A be the simple $\{<\}$-approximation such that $A(\varepsilon)$ is the sentence $\forall x(x=x)$, and, for each nonempty finite sequence Σ of $\{<\}$-atomic formulas, $A(\Sigma)$ is the conjunction of all the disjunctions $\mathfrak{D}$ of $\{\#\}$-atomic formulas such that each variable in $\mathfrak{D}$ occurs in Σ, and the sequent $\Sigma \rightarrow \mathfrak{D}$ is $\boldsymbol{LO}_0$-provable. Then A is adequate for $(\boldsymbol{LO}_0, \boldsymbol{AP}_0)$. The sequents $A(x<x) \rightarrow$ and $A(x<y, y<z, \Sigma) \rightarrow A(x<z, \Sigma)$ are $\boldsymbol{AP}_0$-provable for each finite sequence Σ of $\{<\}$-atomic formulas. Moreover, the sequents $A(x<y, \Sigma)\to A(x<z, \Sigma)\vee A(z<y, \Sigma)$ and $x\# y$, $A(\Sigma)\to A(x<y, \Sigma)\vee A(y<x, \Sigma)$ are $\boldsymbol{AP}_0$-provable for each finite sequence Σ of $\{\#\}$-atomic formulas. Now let A_k be A for each k. Then the series $A_0, A_1, A_2, \cdots$ of simple $\{<\}$-approximations satisfies $\mathcal{L}_0$-rules in $\boldsymbol{AP}_0$, and so Theorem 3.2 says that $\boldsymbol{LO}_0$ is conservative over $\boldsymbol{AP}_0$.

If Σ is a finite sequence of $\{<\}$-atomic formulas, and z is a variable occurring in neither Σ nor $x<y$, then the sequent

$$z\# x,\ z\# y,\ z\# u_1,\ z\# u_2, \cdots, z\# u_k,\ A(\Sigma, x<y)\to A(\Sigma, x<z, z<y)$$

is $\boldsymbol{AP}_0$-provable, where $u_1, u_2, \cdots, u_k$ are all the variables occurring in Σ. Therefore the sequent $A(\Sigma, x<y)\to \exists z A(\Sigma, x<z, z<y)$ is $\boldsymbol{AP}_\omega$-provable. Thus the series of simple $\{<\}$-approximations satisfies $\mathcal{D}^-$-rules in $\boldsymbol{AP}_\omega$, and so $\boldsymbol{DLO}^-$ is conservative over $\boldsymbol{AP}_\omega$.

4.3. The equality parts of $\boldsymbol{DLO}^+$. Let $\mathcal{D}^+$ be the pair of the inference forms $[\ ;\to\exists x_1\exists x_2(x_1<x_2);\ ]$ and $[\ ;\to\forall x\forall y\exists z((x<y\supset x<z\wedge z<y)\wedge(y<x\supset y<z\wedge z<x)\wedge(x=y\supset x=z\wedge z=y));\ ]$, and let $\boldsymbol{DLO}^+$ be the intuitionistic theory $\boldsymbol{LO}_0[\mathcal{D}^+]$.

Let B be a ternary predicate symbol. Let $\mathcal{D}^*$ be the set of the following simple $\{B, <\}$-inference forms:

$$[B(x, z, y),\ x<y;\ \to(x<z,\ z<y);\ ]$$
$$[B(x, z, y),\ y<x;\ \to(y<z,\ z<x);\ ]$$
$$[B(x, z, y),\ x=y;\ \to(x=z,\ z=y);\ ]$$
$$[\ ;\ \to\exists z B(x, z, y);\ ].$$

Then $\boldsymbol{LO}_0[\mathcal{D}^*]$ is conservative over $\boldsymbol{DLO}^+$.

For each finite sequence Σ of $\{B, <, \#\}$-atomic formulas and for each $\{=\}$-clause $\Gamma\to\Delta$, the formula $(\Gamma\to\Delta)^\natural$ is called a *$\{=\}$-section of* Σ if each variable in $\Gamma\to\Delta$ occurs in Σ, no formula occur twice in Γ or in Δ, and the sequent $\Sigma^*\to(\Gamma\to\Delta)^\natural$ is $\boldsymbol{LO}_0$-provable, where Σ^* is the sequence obtained from Σ by replacing all $\{B\}$-atomic formulas $B(x, z, y)$ with $(x<z\wedge z<y)\vee(x=z\wedge z=y)\vee(y<z\wedge z<x)$.

Let $E_0, E_1, E_2, \cdots$ be the series of the simple $\{B, <, \#\}$-approximations as follows:

$E_0(\varepsilon)$ is the sentence $\forall x x=x$, and $E_0(\Sigma)$ is the conjunction of all the $\{=\}$-section of Σ for each nonempty sequence Σ of $\{B, <, \#\}$-atomic formulas. For each number n and each sequence Σ of $\{B, <, \#\}$-atomic formulas, $E_{n+1}(\Sigma)$ is the conjunction of the formulas $\forall x\forall y(\neg E_n(x\# y, \Sigma)\supset x=y)$, $\forall x\forall y\exists z E_n(B(x, z, y), \Sigma)$, $E_n(u\# v, \Sigma)$ such that $u<v$ occurs in Σ, and $E_n(u<v, \Sigma)\vee E_n(v<u, \Sigma)$

such that $u\#v$ occurs in Σ.

Then the series $E_0, E_1, E_2, \cdots$ of simple $\{B, <, \#\}$-approximations is adequate for $(\boldsymbol{LO}_0[\mathcal{D}^*], \boldsymbol{LJ}_e)$, and satisfies $\mathcal{L}_0\cup\mathcal{D}^*$-rules in $\boldsymbol{LJ}_e$.

Therefore, we have the following theorem:

THEOREM 4.3. *The equality part of* $\boldsymbol{DLO}^+$ *is axiomatized by the series of axioms* $E_0(\varepsilon), E_1(\varepsilon), E_2(\varepsilon), \cdots$ *in* $\boldsymbol{LJ}_e$.

Theorem 4.3 and Theorem in [9] are solutions of Problem 1 in [6].

4.4. Equality parts of intuitionistic theory of torsion free groups (cf. [6]). Let Φ be a sequence of sentences of theory of groups with the binary function symbol $*$. Let P be a ternary predicate symbol. For a finite sequence Σ of $\{P\}$-atomic formulas, and for a pair of sequences Γ and Δ of $\{*\}$-free $\{=\}$-atomic formulas, the formula $(\Gamma\to\Delta)^\natural$ is called a *$\{=\}$-section of Σ with respect to Φ* if each variable in $\Gamma\to\Delta$ occurs in Σ, no formula occurs twice in Γ or in Δ, and the sequent $\Phi, \Sigma^*\to(\Gamma\to\Delta)^\natural$ is $\boldsymbol{LJ}_e$-provable, where Σ^* is the sequence obtained from Σ by replacing all $\{P\}$-atomic formulas $P(x, y, z)$ with $x*y=z$.

Let Γ_* be the sequence of the formulas $\forall x\forall y\forall z((x*y)*z=x*(y*z))$, $\forall x\forall y\forall u\forall v(x=x*u\wedge y=y*v\supset u=v)$ and $\forall x\forall y(x*y=y*x)$. For each number n, let $\mathfrak{T}_n^*$ be the formula $\forall x(x^{n+1}=x\supset x^2=x)$. Then Γ_*, $\forall x\forall y\exists u(x*u=y)$, $\mathfrak{T}_*^1, \mathfrak{T}_*^2, \cdots$ is the series of axioms of theory of torsion free groups.

Let TF be the simple $\{P\}$-approximation such that $TF(\varepsilon)$ is the sentence $\forall x x=x$, and for each nonempty sequence Σ of $\{P\}$-atomic formulas, $TF(\Sigma)$ is the conjunction of all the $\{=\}$-section of Σ with respect to Γ_*, $\forall x\forall y(x*y=y*x)$, $\mathfrak{T}_*^n$, where n is the length of the sequence Σ.

Theory of torsion free groups is formalized by the set of the simple $\{P\}$-inference forms $[P(x, y, u), P(u, z, s), P(y, z, v), P(x, v, t); \to s=t;]$, $[P(x, u, x), P(y, v, y); \to u=v;]$, $[P(x, y, u), P(y, x, v); \to u=v;]$, $[P(x_1, x_1, x_2), P(x_2, x_1, x_3), \cdots, P(x_n, x_1, x_{n+1}); x_{n+1}=x_1\to x_2=x_1;]$ $(n=2, 3, \cdots)$, and $[\ ; \to\exists u\exists v(P(x, u, y), P(x, y, v));]$.

Then, by the same way as 4.3, we have the following theorem:

THEOREM 4.4. *The equality part of the intuitionistic theory of torsion free groupes is axiomatized in* $\boldsymbol{LJ}_e$ *by the series of the following axioms:*

$$\forall x_1\forall y_1\exists u_1\exists v_1\forall x_2\forall y_2\exists u_2\exists v_2 \cdots \forall x_k\forall y_k\exists u_k\exists v_k TF(P(x_1, u_1, y_1), P(x_1, y_1, v_1), P(x_2, u_2, y_2), P(x_2, y_2, v_2), \cdots, P(x_k, u_k, y_k), P(x_k, y_k, v_k)) \quad (k=1,2,\cdots).$$

Theorem 4.4 is a partial solution of Problem 2 in [**6**].

4.5. A generalized Minc's theorem (cf. [**7**], [**4**], [**1**]). The following theorem is a generalized form of the theorems for the existence of choice function by Minc (unpublished), Smorynski [**7**] and Motohashi [**4**]:

THEOREM (Akaboshi [**1**]). *Let $f_1, \cdots, f_n$ be function symbols which are not necessarily mutually distinct. Let $\boldsymbol{A}$ be a closed predicate in which none of the function symbols $f_1, \cdots, f_m$ occurs. Let $\boldsymbol{T}$ be the intuitionistic theory*

$$\boldsymbol{LJ}_e[\{[\ ;\ \to\ ;\ \forall\bar{x}_1 \cdots \forall\bar{x}_m \boldsymbol{A}(\bar{x}_1, f_1(\bar{x}_1), \cdots, \bar{x}_m, f_m(\bar{x}_m))]\}].$$

Then $\{f_1, \cdots, f_m\}$-free part of $\boldsymbol{T}$ is axiomtized in $\boldsymbol{LJ}_e$ by the series of the following axioms:

$$\forall\bar{x}_1^1 \cdots \forall\bar{x}_m^1 \exists y_1^1 \cdots \exists y_m^1 \forall\bar{x}_1^2 \cdots \forall\bar{x}_m^2 \exists y_1^2 \cdots \exists y_m^2 \cdots \forall\bar{x}_1^k \cdots \forall\bar{x}_m^k \exists y_1^k \cdots \exists y_m^k$$
$$(\bigwedge\!\!\!\bigwedge\{\boldsymbol{A}(\bar{x}_1^{n_1}, y_1^{n_1}, \cdots, \bar{x}_m^{n_m}, y_m^{n_m}) \mid 1 \leqq n_1 \leqq k, \cdots, 1 \leqq n_m \leqq k\} \wedge$$
$$\bigwedge\!\!\!\bigwedge\{\bar{x}_i^n = \bar{x}_j^{n'} \supset y_i^n = y_j^{n'} \mid f_i \text{ is the same symbol as } f_j,\ 1 \leqq n \leqq n' \leqq k,$$
$$1 \leqq i \leqq j \leqq m\}) \quad (k = 1, 2, \cdots).$$

PROOF. Let $R_1, \cdots, R_m$ be new predicate symbols such that R_i is the same symbol as R_j according as f_i is the same symbol as f_j. Set $\mathcal{F} = \{f_1, \cdots, f_m\}$ and $\mathcal{R} = \{R_1, \cdots, R_m\}$. Let $\mathcal{S}$ be the set of the following simple $\mathcal{R}$-inferene forms:

$$[R_1(\bar{x}_1, y_1), \cdots, R_m(\bar{x}_m, y_m);\ \to \boldsymbol{A}(\bar{x}_1, y_1, \cdots, \bar{x}_m, y_m);\ ]$$
$$[R_1(\bar{x}_1, y), R_1(\bar{x}_1, z);\ \to y = z;\ ]$$
$$\vdots$$
$$[R_m(\bar{x}_m, y), R_m(\bar{x}_m, z);\ \to y = z;\ ]$$
$$[\ ;\ \to \exists y_1 \cdots \exists y_m (R_1(\bar{x}_1, y_1), \cdots, R_m(\bar{x}_m, y_m));\ ].$$

Then the $\mathcal{F} \cup \mathcal{R}$-free part of $\boldsymbol{LJ}_e[\mathcal{S}]$ is the $\mathcal{F}$-free part of $\boldsymbol{T}$. Let $\bar{A}$ be the series of simple $\mathcal{R}$-approximations $A_0, A_1, A_2, \cdots$ defined as follows:

For each sequence Σ of $\mathcal{R}$-atomic formulas, $A_0(\Sigma)$ is the conjunction of the formula $\forall x(x = x)$, the formulas $\boldsymbol{A}(\bar{s}_1, t_1, \cdots, \bar{s}_m, t_m)$ such that all $R_1(\bar{s}_1, t_1), \cdots, R_m(\bar{s}_m, t_m)$ occur in Σ, and the formulas $\bar{s} = \bar{s}' \supset t = t'$ such that both $R(\bar{s}, t)$ and $R(\bar{s}', t')$ occur in Σ for some R in $\mathcal{R}$, and $A_{n+1}(\Sigma)$ is the formula

$$\forall\bar{x}_1 \cdots \forall\bar{x}_m \exists y_1 \cdots \exists y_m A_n(R_1(\bar{x}_1, y_1), \cdots, R_m(\bar{x}_m, y_m), \Sigma),$$

where no variable in $\bar{x}_1, \cdots, \bar{x}_m, y_1, \cdots, y_m$ occurs in Σ.

Then the series of simple $\mathcal{R}$-approximations $A_0, A_1, A_2, \cdots$ is adequate for

($\boldsymbol{LJ}_e[\mathcal{S}]$, $\boldsymbol{LJ}_e$), and satisfies $\mathcal{S}$-rules in $\boldsymbol{LJ}_e$. Thus, by Theorem 3.2, the $\mathcal{F}$-free part of the theory $\boldsymbol{T}$ is axiomatized by the series of axioms $A_0(\varepsilon)$, $A_1(\varepsilon)$, $A_2(\varepsilon)$, $\cdots$ in $\boldsymbol{LJ}_e$. On the other hand, each $A_k(\varepsilon)$ is equivalent to the following sentence in $\boldsymbol{LJ}_e$:

$$\forall \bar{x}_1^1 \cdots \forall \bar{x}_m^1 \exists y_1^1 \cdots \exists y_m^1 \forall \bar{x}_1^2 \cdots \forall \bar{x}_m^2 \exists y_1^2 \cdots \exists y_m^2 \cdots \forall \bar{x}_1^k \cdots \forall \bar{x}_m^k \exists y_1^k \cdots \exists y_m^k$$
$$(\bigwedge\!\!\!\bigwedge \{\boldsymbol{A}(\bar{x}_1^{n_1}, y_1^{n_1}, \cdots, \bar{x}_m^{n_m}, y_m^{n_m}) \mid 1 \leqq n_1 \leqq k, \cdots, 1 \leqq n_m \leqq k\} \wedge$$
$$\bigwedge\!\!\!\bigwedge \{\bar{x}_i^n = \bar{x}_j^{n'} \supset y_i^n = y_j^{n'} \mid f_i \text{ is the same symbol as } f_j,$$
$$1 \leqq n \leqq n' \leqq k,\ 1 \leqq i \leqq j \leqq m\}).$$

This completes the proof.

§ 5. Resolving trees of inference forms.

In 4.3, we gave the set $\mathcal{D}^*$ which is a set of $\{B, <\}$-inference forms such that $\boldsymbol{LO}_0[\mathcal{D}^*]$ is conservative over $\boldsymbol{DLO}^+$. Generally, we have the following theorems:

THEOREM 5.1. *Let $\mathcal{L}$ be a language, $\boldsymbol{L}$ the logistic $\boldsymbol{LJ}$ or $\boldsymbol{LJ}_e$ over $\mathcal{L}$, $\boldsymbol{T}$ an intuitionistic theory on $\boldsymbol{L}$, and $\mathcal{F}$ a set of predicate symbols in $\mathcal{L}$. Then there are a set $\mathcal{F}_0$ of new predicate symbols and a set $\mathcal{S}$ of $\mathcal{F}\cup\mathcal{F}_0$-inference forms such that $\boldsymbol{L}'[\mathcal{S}]$ is a conservative extension of $\boldsymbol{T}$, where $\boldsymbol{L}'$ is the extension of $\boldsymbol{L}$ obtained by adding $\mathcal{F}_0$ to the language $\mathcal{L}$.*

THEOREM 5.2. *Let $\mathcal{L}$ be a language, $\boldsymbol{L}$ the logistic $\boldsymbol{LK}$ or $\boldsymbol{LK}_e$ over $\mathcal{L}$, $\boldsymbol{T}$ a classical theory on $\boldsymbol{L}$, and ($\mathcal{P}$, $\mathcal{F}$) a pair of disjoint sets of predicate symbols in $\mathcal{L}$. Then there are a set $\mathcal{F}_0$ of new predicate symbols and a set $\mathcal{S}$ of simple ($\mathcal{P}$, $\emptyset$, $\mathcal{F}\cup\mathcal{F}_0$)-inference forms such that $\boldsymbol{L}'[\mathcal{S}]$ is a conservative extension of $\boldsymbol{T}$, where $\boldsymbol{L}'$ is the extension of $\boldsymbol{L}$ obtained by adding $\mathcal{F}_0$ to the language $\mathcal{L}$.*

The aim of this section is to prove the above theorems.

Now, let $\mathcal{L}$ be a first order language, and let $\mathcal{P}$ and $\mathcal{F}$ be disjoint sets of predicate symbols in $\mathcal{L}$. For each closed predicate $\boldsymbol{A}$ in $\mathcal{L}$, let $P_{\boldsymbol{A}}$ be a new predicate symbol with the same arity as $\boldsymbol{A}$. Let $\mathcal{F}_{\mathcal{L}}$ be the set of all the new predicate symbols $P_{\boldsymbol{A}}$. A ($\mathcal{P}$, $\emptyset$, $\mathcal{F}$)-*resolving tree* of an inference form I is a derivation tree with derivation rules for inference forms described below such that its uppermost inference forms are ($\mathcal{P}$, $\emptyset$, $\mathcal{F}\cup\mathcal{F}_{\mathcal{L}}$)-inference forms and its lowermost inference form is I. We admit only the following derivation rules:

$$\frac{[\Gamma, \Pi^{\natural}; \rightarrow; \Delta^{\natural}, \Lambda]}{[\Gamma; \Pi\rightarrow\Delta; \Lambda]},$$

where $[\Gamma; \Pi\rightarrow\Delta; \Lambda]$ is an inference form in $\mathcal{L}$.

$$\frac{[\Gamma, \mathfrak{A}, \mathfrak{B}, \Pi; \rightarrow; \Lambda]}{[\Gamma, \mathfrak{B}, \mathfrak{A}, \Pi; \rightarrow; \Lambda]}.$$

$$\frac{[\Gamma; \rightarrow; \Delta, \mathfrak{A}, \mathfrak{B}, \Lambda]}{[\Gamma; \rightarrow; \Delta, \mathfrak{B}, \mathfrak{A}, \Lambda]}.$$

$$\frac{[\neg P_A(\bar{x}), \Gamma; \rightarrow; \Lambda] \quad [P_A(\bar{x}); \rightarrow; \boldsymbol{A}(\bar{x})]}{[\neg \boldsymbol{A}(\bar{x}), \Gamma; \rightarrow; \Lambda]}.$$

$$\frac{[\mathfrak{A}, \Gamma; \rightarrow; \ ]}{[\Gamma; \rightarrow; \neg \mathfrak{A}]}.$$

$$\frac{[\Gamma; \rightarrow; \Lambda, P_{\lambda\bar{x}.\neg A(\bar{x})}(\bar{x})] \quad [\boldsymbol{A}(\bar{x}), P_{\lambda\bar{x}.\neg A(\bar{x})}(\bar{x}); \rightarrow; \ ]}{[\Gamma; \rightarrow; \Lambda, \neg \boldsymbol{A}(\bar{x})]}.$$

$$\frac{[\mathfrak{A}, \mathfrak{B}, \Gamma; \rightarrow; \Lambda]}{[\mathfrak{A} \wedge \mathfrak{B}, \Gamma; \rightarrow; \Lambda]}.$$

$$\frac{[\Gamma; \rightarrow; \Lambda, \mathfrak{A}] \quad [\Gamma; \rightarrow; \Lambda, \mathfrak{B}]}{[\Gamma; \rightarrow; \Lambda, \mathfrak{A}, \mathfrak{B}]}.$$

$$\frac{[\mathfrak{A}, \Gamma; \rightarrow; \Lambda] \quad [\mathfrak{B}, \Gamma; \rightarrow; \Lambda]}{[\mathfrak{A} \vee \mathfrak{B}, \Gamma; \rightarrow; \Lambda]}.$$

$$\frac{[\Gamma; \rightarrow; \Lambda, \mathfrak{A}, \mathfrak{B}]}{[\Gamma; \rightarrow; \Lambda, \mathfrak{A} \vee \mathfrak{B}]}.$$

$$\frac{[P_A(\bar{x}) \supset \mathfrak{B}, \Gamma; \rightarrow; \Lambda] \quad [P_A(\bar{x}); \rightarrow; \boldsymbol{A}(\bar{x})]}{[\boldsymbol{A}(\bar{x}) \supset \mathfrak{B}, \Gamma; \rightarrow; \Lambda]}.$$

$$\frac{[\mathfrak{A} \supset P_B(\bar{x}), \Gamma; \rightarrow; \Lambda] \quad [\boldsymbol{B}(\bar{x}); \rightarrow; P_B(\bar{x})]}{[\mathfrak{A} \supset \boldsymbol{B}(\bar{x}), \Gamma; \rightarrow; \Lambda]}.$$

$$\frac{[\mathfrak{A}, \Gamma; \rightarrow; \mathfrak{B}]}{[\Gamma; \rightarrow; \mathfrak{A} \supset \mathfrak{B}]}.$$

$$\frac{[\Gamma; \rightarrow; \Lambda, P_C(\bar{x})] \quad [\boldsymbol{A}(\bar{x}), P_C(\bar{x}); \rightarrow; \boldsymbol{B}(\bar{x})]}{[\Gamma; \rightarrow; \Lambda, \boldsymbol{A}(\bar{x}) \supset \boldsymbol{B}(\bar{x})]},$$

where $\boldsymbol{C}$ is the predicate $\lambda\bar{x}.(\boldsymbol{A}(\bar{x}) \supset \boldsymbol{B}(\bar{x}))$.

$$\frac{[\forall x P_A(x, \bar{x}), \Gamma; \rightarrow; \Lambda] \quad [\boldsymbol{A}(x, \bar{x}); \rightarrow; P_A(x, \bar{x})]}{[\forall x \boldsymbol{A}(x, \bar{x}), \Gamma; \rightarrow; \Lambda]}.$$

$$\frac{[\Gamma; \rightarrow; \mathfrak{A}]}{[\Gamma; \rightarrow; \forall x \mathfrak{A}]},$$

where x does not occur free in Γ.

$$\frac{[\Gamma; \rightarrow; \Lambda, P_{\lambda\bar{x}.\forall x A(x,\bar{x})}(\bar{x})] \quad [P_{\lambda\bar{x}.\forall x A(x,\bar{x})}(\bar{x}); \rightarrow; \boldsymbol{A}(x, \bar{x})]}{[\Gamma; \rightarrow; \Lambda, \forall x \boldsymbol{A}(x, \bar{x})]}.$$

$$\frac{[\mathfrak{A}, \Gamma; \to; \Lambda]}{[\exists x\mathfrak{A}, \Gamma; \to; \Lambda]},$$

where x does not occur free in Γ and Λ.

$$\frac{[\Gamma; \to; \Lambda, \exists x P_A(x, \bar{x})] \quad [P_A(x, \bar{x}); \to; A(x, \bar{x})]}{[\Gamma; \to; \Lambda, \exists x A(x, \bar{x})]}.$$

$$\frac{[\Gamma; \Pi \to \Delta;]}{[\Gamma, \Pi^{\natural}; \to; \Delta^{\natural}]},$$

where $[\Gamma; \Pi \to \Delta;]$ is a $(\mathcal{P}, \emptyset, \mathcal{F} \cup \mathcal{F}_{\mathcal{L}})$-inference form.
Immediately, we have the following lemma:

LEMMA 5.1. *For each inference form in $\mathcal{L}$, there is a $(\mathcal{P}, \emptyset, \mathcal{F})$-resolving tree of it.*

LEMMA 5.2. *Let $\boldsymbol{L}$ be the logistic $\boldsymbol{LK}$, $\boldsymbol{LJ}$, $\boldsymbol{LK}_e$ or $\boldsymbol{LJ}_e$ over $\mathcal{L}$. Let $\mathcal{S}$ be a set of inference forms in $\mathcal{L}$. For each inference form I in $\mathcal{S}$, let T_I be a $(\mathcal{P}, \emptyset, \mathcal{F})$-resolving tree of I, and $\mathcal{I}_I$ the set of uppermost $(\mathcal{P}, \emptyset, \mathcal{F} \cup \mathcal{F}_{\mathcal{L}})$-inference forms of T_I. Then $\boldsymbol{L}'[\cup\{\mathcal{I}_I \mid I \in \mathcal{S}\}]$ is a conservative extension of $\boldsymbol{L}[\mathcal{S}]$, where $\boldsymbol{L}'$ is the extension of $\boldsymbol{L}$ obtained by adding $\mathcal{F}_{\mathcal{L}}$ to the language $\mathcal{L}$.*

PROOF. Set $\mathcal{I} = \cup\{\mathcal{I}_I \mid I \in \mathcal{S}\}$. For each inference form J in $\mathcal{I}$, let K be the inference form obtained from J by substituting all the new predicate symbols P_A in J by A, then K holds in $\boldsymbol{L}[\mathcal{S}]$. Therefore, each $\boldsymbol{L}'[\mathcal{I}]$-provable formula in $\mathcal{L}$ is $\boldsymbol{L}[\mathcal{S}]$-provable. On the other hand, each inference form in $\mathcal{S}$ holds in $\boldsymbol{L}'[\mathcal{I}]$, and so $\boldsymbol{L}'[\mathcal{I}]$ is an extension of $\boldsymbol{L}[\mathcal{S}]$. Thus $\boldsymbol{L}'[\mathcal{I}]$ is a conservative extension of $\boldsymbol{L}[\mathcal{S}]$.

If $\boldsymbol{L}$ is $\boldsymbol{LJ}$ or $\boldsymbol{LJ}_e$, then for each intuitionic theory $\boldsymbol{T}$ on $\boldsymbol{L}$, there is a set $\mathcal{S}$ of inference forms such that $\boldsymbol{T}$ is equivalent to $\boldsymbol{L}[\mathcal{S}]$. Thus we have Theorem 5.1. If $\boldsymbol{L}$ is $\boldsymbol{LK}$ or $\boldsymbol{LK}_e$, then for each classical theory $\boldsymbol{T}$ on $\boldsymbol{L}$, there is a set $\mathcal{S}$ of $\{\forall, \supset\}$-free inference forms such that $\boldsymbol{T}$ is equivalent to $\boldsymbol{L}[\mathcal{S}]$. Thus we have Theorem 5.2, by the following obvious lemma:

LEMMA 5.3. *For each $\{\forall, \supset\}$-free inference form, there is a $(\mathcal{P}, \emptyset, \mathcal{F})$-resolving tree such that all the uppermost $(\mathcal{P}, \emptyset, \mathcal{F} \cup \mathcal{F}_{\mathcal{L}})$-inference forms are simple.*

§ 6. Axiomatization theorems.

From the discussions in the previous sections, we now reached to the conclusion that it suffices to choose a suitable series of approximations for axiomatizing a part of a given theory. In this section, we are concerned with theories satisfying equality axioms, and define a canonical series of approximations for a given theory with equality. Then we give axiomatization theorems.

For atomic formulas $P_1(\bar{t}_1), \cdots, P_n(\bar{t}_n)$ and a sequence Γ of atomic formulas, let $E(P_1(\bar{t}_1), \cdots, P_n(\bar{t}_n); \Gamma)$ denote the disjunction of the conjunctions $\bar{t}_1=\bar{s}_1\wedge \cdots \wedge\bar{t}_n=\bar{s}_n$ such that $P_1(\bar{s}_1), \cdots, P_n(\bar{s}_n)$ occur in Γ.

Let A be a $(\mathcal{P}, \mathcal{N}, \mathcal{T})$-approximation, and I a $(\mathcal{P}, \mathcal{N}, \mathcal{T})$-inference form $[\Phi; \forall\bar{x}_1(\Gamma_1\to\Delta_1), \cdots, \forall\bar{x}_m(\Gamma_m\to\Delta_m)\to\exists\bar{y}_1(\Pi_1), \cdots, \exists\bar{y}_n(\Pi_n);]$. Let Φ^* be the subsequence of Φ which consists of all the $\mathcal{P}\cap\mathcal{T}$-atomic formulas in Φ, and $\Phi^\#$ the remainder. Let $\bar{w}$ be the sequence (lined up by a fixed order) of all the variables occurring free in I. For each $(\mathcal{P}, \mathcal{N}, \mathcal{T})$-clause $\Gamma\to\Lambda$, let $A(\Gamma\to\Lambda: I)$ be the $(\mathcal{P}, \mathcal{N}, \mathcal{T})_-$-formula

$$\exists\bar{v}(E(\Phi^*\theta; \Gamma), \Phi^\#\theta, \forall\bar{x}_1A(\Gamma_1\theta, \Gamma\to\Lambda^c, \Delta_1\theta), \cdots, \forall\bar{x}_mA(\Gamma_m\theta, \Gamma\to\Lambda^c, \Delta_m\theta),$$
$$\forall\bar{y}_1A(\Pi_1\theta, \Gamma\to\Lambda), \cdots, \forall\bar{y}_nA(\Pi_n\theta, \Gamma\to\Lambda))^{\natural},$$

where θ is a substitution of the form $\binom{\bar{w}}{\bar{v}}$ such that $\bar{v}$ is a sequence of variables without repetition none of which occur in $\Gamma\to\Lambda$ and I, and Λ^c denotes the empty sequence or Λ itself, according as the logistic concerned is intuitionistic or classical. For each sequence Γ of $\mathcal{P}\cup\mathcal{T}$-atomic formulas, let $A(\Gamma: I)$ be the $(\mathcal{P}, \mathcal{N}, \mathcal{T})_+$-formula

$$\forall\bar{v}(E(\Phi^*; \Gamma), \Phi^\#\theta, \forall\bar{x}_1A(\Gamma_1\theta, \Gamma\to\Delta_1\theta), \cdots, \forall\bar{x}_mA(\Gamma_m\theta, \Gamma\to\Delta_m\theta)$$
$$\to\exists\bar{y}_1A(\Pi_1\theta, \Gamma), \cdots, \exists\bar{y}_nA(\Pi_n\theta, \Gamma))^{\natural},$$

where θ is a substitution of the form $\binom{\bar{w}}{\bar{v}}$ such that $\bar{v}$ is a sequence of variables without repetition none of which occur in Γ and I.

Let A be a $(\mathcal{P}, \mathcal{N}, \mathcal{T})$-approximation, and σ an enumerating function of a set of $(\mathcal{P}, \mathcal{N}, \mathcal{T})$-inference forms. Then we define the series of $(\mathcal{P}, \mathcal{N}, \mathcal{T})$-approximations $A_0^\sigma, A_1^\sigma, A_2^\sigma, \cdots$ as follows:

A_0^σ is A.

$A_{k+1}^\sigma(\Gamma)$ is the formula $A_k^\sigma(\Gamma: \sigma(0))\wedge \cdots \wedge A_k^\sigma(\Gamma: \sigma(k))\wedge A_k^\sigma(\Gamma)$ for each finite sequence Γ of $\mathcal{P}\cup\mathcal{T}$-atomic formulas.

$A_{k+1}^\sigma(\Gamma\to\Lambda)$ is the formula $A_{k+1}^\sigma(\Gamma)\supset(A_k^\sigma(\Gamma\to\Lambda: \sigma(0))\vee \cdots \vee A_k^\sigma(\Gamma\to\Lambda: \sigma(k)) \vee A_k^\sigma(\Gamma\to\Lambda))$ for each $(\mathcal{P}, \mathcal{N}, \mathcal{T})$-clause $\Gamma\to\Lambda$ such that Λ is not empty.

The series of $(\mathcal{P}, \mathcal{N}, \mathcal{T})$-approximations $A_0^\sigma, A_1^\sigma, A_2^\sigma, \cdots$ is called the *canonical series of* $(\mathcal{P}, \mathcal{N}, \mathcal{T})$*-approximations on* A *and* σ.

PROPOSITION 6.1. *Let* $\boldsymbol{S}$ *and* $\boldsymbol{T}$ *be theories with equality. Let* A *be a* $(\mathcal{P}, \mathcal{N}, \mathcal{T})$*-approximation,* σ *an enumerating function of a set* $\mathcal{S}$ *of* $(\mathcal{P}, \mathcal{N}, \mathcal{T})$*-inference forms, and* $\bar{A}$ *the canonical series of* $(\mathcal{P}, \mathcal{N}, \mathcal{T})$*-approximations* $A_0^\sigma, A_1^\sigma, A_2^\sigma, \cdots$ *on* A *and* σ.

(1) *If A is commutative for substitution on $\boldsymbol{T}$, then so is $\bar{A}$.*

(2) *If A is monotone on $\boldsymbol{T}$, then so is $\bar{A}$.*

(3) *If A satisfies the initial condition in $\boldsymbol{T}$, then so does $\bar{A}$.*

(4) *If A is transitive on $\boldsymbol{S}$, and if all $\mathcal{S}$-rules hold in $\boldsymbol{S}$, then $\bar{A}$ is transitive on $\boldsymbol{S}$.*

(5) *If A is commutative for substitution and monotone on $\boldsymbol{T}$, the $\bar{A}$ satisfies $\mathcal{S}$-rules in $\boldsymbol{T}$.*

PROOF. (1), (2) and (3) are straightforward.

(4) We prove that A_k^q is transitive on $\boldsymbol{S}$ by induction on k.

If $k=0$ then it is trivial. Suppose that $k>0$. If suffices to show that the sequents $\Gamma\to A_{k-1}^q(\Gamma: I)$ and $\Gamma,\ A_{k-1}^q(\Gamma\to\Lambda : I)\to\Lambda$ are $\boldsymbol{T}[\mathcal{S}]$-provable for each I in $\mathcal{S}$ and for each $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-clause $\Gamma\to\Lambda$. Let I be an inference form $[\Phi;\ \forall\bar{x}_1(\Gamma_1\to\Delta_1), \cdots, \forall\bar{x}_m(\Gamma_m\to\Delta_m)\to\exists\bar{y}_1(\Pi_1), \cdots, \exists\bar{y}_n(\Pi_n);\ ]$ in $\mathcal{S}$. Let θ be a substitution of the form $\binom{\bar{w}}{\bar{v}}$ such that $\bar{w}$ is the sequence of all the variables occurring free in I, and $\bar{v}$ is a sequence of variables without repetition none of which occur in $\Gamma\to\Lambda$ and I. We may assume that no variable in $\bar{x}_1, \cdots, \bar{x}_m$, $\bar{y}_1, \cdots, \bar{y}_n$ occurs in $\Gamma\to\Lambda$. By induction hypothesis, the sequent $\Gamma_i\theta,\ \Gamma,\ A_{k-1}^q(\Gamma_i\theta,\ \Gamma\to\Delta_i\theta)\to\Delta_i\theta$ is $\boldsymbol{S}$-provable, hence the sequent $\Gamma_i\theta,\ \Gamma,\ \forall\bar{x}_i A_{k-1}^q(\Gamma_i\theta,\ \Gamma\to\Delta_i\theta)\to\Delta_i\theta$ is $\boldsymbol{S}$-provable, and so the sequent $\Gamma_i\theta,\ \Gamma,\ \forall\bar{x}_1 A_{k-1}^q(\Gamma_1\theta,\ \Gamma\to\Delta_1\theta), \cdots, \forall\bar{x}_m A_{k-1}^q(\Gamma_m\theta,\ \Gamma\to\Delta_m\theta)\to\Delta_i\theta$ is $\boldsymbol{S}$-provable for each i with $1\leqq i\leqq m$. On the other hand, by induction hypothesis, the sequent $\Pi_j\theta,\ \Gamma\to A_{k-1}^q(\Pi_j\theta,\ \Gamma)$ is $\boldsymbol{S}$-provable, and so the sequent $\Pi_j\theta,\ \Gamma\to\exists\bar{y}_1 A_{k-1}^q(\Pi_1\theta, \Gamma)\vee\cdots\vee\exists\bar{y}_n A_{k-1}^q(\Pi_n\theta, \Gamma)$ is $\boldsymbol{S}$-provable for each j with $1\leqq j\leqq n$. Consider the quasi-$\boldsymbol{LK}[\mathcal{S}]$-proof figure

$$I\theta\text{-rule}:\ \frac{\dfrac{\Gamma_1\theta,\ \Gamma,\ \Sigma\to\Delta_1\theta\ \cdots\ \Gamma_m\theta,\ \Gamma,\ \Sigma\to\Delta_m\theta\quad \Pi_1\theta,\ \Gamma\to\mathfrak{D}\ \cdots\ \Pi_n\theta,\ \Gamma\to\mathfrak{D}}{\Phi\theta,\ \Gamma,\ \Sigma,\ \Gamma\to\mathfrak{D}}}{\Gamma\to(E(\Phi^*\theta;\ \Gamma),\ \Phi^{\#}\theta,\ \Sigma\to\mathfrak{D})^{\shortparallel}}\ \begin{array}{l}\text{several exchainges, contractions, } \mathcal{E}\text{-rules, } (\wedge\to\ )\text{-rules,}\\ (\vee\to\ )\text{-rules, and a } (\ \to\supset)\text{-rule or a } (\ \to\urcorner)\text{-rule}\end{array},$$

where Σ is the sequence $\forall\bar{x}_1 A_{k-1}^q(\Gamma_1\theta,\ \Gamma\to\Delta_1\theta), \cdots, \forall\bar{x}_m A_{k-1}^q(\Gamma_m\theta,\ \Gamma\to\Delta_m\theta)$, $\mathfrak{D}$ is the disjunction $\exists\bar{y}_1 A_{k-1}^q(\Pi_1\theta,\ \Gamma)\vee\cdots\vee\exists\bar{y}_n A_{k-1}^q(\Pi_n\theta,\ \Gamma)$, Φ^* is the subsequence of Φ which consists of all the $\mathcal{P}\cup\mathcal{F}$-atomic formulas in Φ, and $\Phi^{\#}$ is the remainder. Therefore, the sequent $\Gamma\to A_{k-1}^q(\Gamma: I)$ is $\boldsymbol{S}$-provable. Similarly, the sequent $\Gamma,\ A_{k-1}^q(\Gamma\to\Lambda : I)\to\Lambda$ is $\boldsymbol{S}$-provable.

(5) Let J be an instance $I\theta$ of a $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-inference form I in $\mathcal{S}$. Suppose that $I=\sigma(h_0)$ and I has the form

$$[\Phi;\ \forall x_1(\Gamma_1\to\Delta_1), \cdots, \forall x_m(\Gamma_m\to\Delta_m)\to\exists y_1(\Pi_1), \cdots, \exists y_n(\Pi_n):\ ].$$

Let

$$\frac{\Gamma_1\theta,\ \Gamma\to\Delta,\ \Delta_1\theta\ \cdots\ \Gamma_m\theta,\ \Gamma\to\Delta,\ \Delta_m\theta\quad \Pi_1\theta,\ \Pi\to\Lambda\ \cdots\ \Pi_n\theta,\ \Pi\to\Lambda}{\Phi\theta,\ \Gamma,\ \Pi\to\Delta,\ \Lambda}$$

be a J-rule whose upper sequents are $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-sequents. Suppose that the formulas $A_h^q(\Gamma_1\theta, \Gamma\to\Delta, \Delta_1\theta), \cdots, A_h^q(\Gamma_m\theta, \Gamma\to\Delta, \Delta_m\theta), A_h^q(\Pi_1\theta, \Pi\to\Lambda), \cdots, A_h^q(\Pi_n\theta, \Pi\to\Lambda)$ are $\boldsymbol{T}$-provable for some number h. Let k be a number such that $k>h$ and $k>h_0$. By (2), $\bar{A}$ is monotone on $\boldsymbol{T}$. Therefore, the formulas $A_k^q(\Gamma_1\theta, \Phi\theta, \Gamma, \Pi\to\Delta, \Delta_1\theta), \cdots, A_k^q(\Gamma_m\theta, \Phi\theta, \Gamma, \Pi\to\Delta, \Delta_m\theta), A_k^q(\Pi_1\theta, \Phi\theta, \Gamma, \Pi\to\Delta, \Lambda), \cdots, A_k^q(\Pi_n\theta, \Phi\theta, \Gamma, \Pi\to\Delta, \Lambda)$ are $\boldsymbol{T}$-provable. Now let Γ^* be the subsequence of the sequence $\Phi\theta, \Gamma, \Pi$ which consists of all the $\mathcal{P}\cup\mathcal{F}$-atomic formulas in the sequence and $\Gamma^\#$ the remainder, and let Λ^* be the subsequence of the sequence Δ, Λ which consists of all the $\mathcal{N}\cup\mathcal{F}$-atomic formulas in the sequence Δ, Λ, and $\Lambda^\#$ the remainder. Then $E(\Phi^*\theta; \Gamma^*)$ is $\boldsymbol{T}$-provable. By (1), $\bar{A}$ is commutative for substitution on $\boldsymbol{T}$. Hence the sequent $\Gamma^\#, A_k^q(\Gamma^*: I)\to\Delta, \Lambda$ or the sequent $\Gamma^\#\to A_k^q(\Gamma^*\to\Lambda^*: I), \Lambda^\#$ is $\boldsymbol{T}$-provable, according as all the formulas in Δ, Λ are $(\mathcal{P}, \mathcal{N}, \mathcal{F})_+$ or not. Thus, by the definition of A_{k+1}^q, $A_{k+1}^q(\Phi\theta, \Gamma, \Pi\to\Delta, \Lambda)$ is $\boldsymbol{T}$-provable.

THEOREM 6.1. *Let $\boldsymbol{T}$ be an intuitionistic theory with equality. Let $\mathcal{S}$ be a set of intuitionistic $\mathcal{F}$-inference forms, and σ an enumerating function of $\mathcal{S}$. Suppose that the theory $\boldsymbol{LJ}_e[\mathcal{S}]$ is an extension of $\boldsymbol{T}$. Let A be an $\mathcal{F}$-approximation which is adequate on $(\boldsymbol{LJ}_e[\mathcal{S}], \boldsymbol{T})$ and $A_0, A_1, A_2, \cdots$ the canonical series of $\mathcal{F}$-approximations on A and σ. Then the $\mathcal{F}$-free part of $\boldsymbol{LJ}_e[\mathcal{S}]$ is axiomatized by the series of axioms $A_0(\varepsilon), A_1(\varepsilon), A_2(\varepsilon), \cdots$ in $\boldsymbol{T}$.*

PROOF. By Proposition 6.1, the canonical series of $\mathcal{F}$-approximations is adequate on $(\boldsymbol{LJ}_e[\mathcal{S}], \boldsymbol{T})$ and satisfies $\mathcal{S}$-rules in $\boldsymbol{T}$. Hence Theorem 3.1 implies the conclusion.

Corollary 3.1, Theorem 5.1 and Theorem 6.1 imply the following axiomatization theorem for intuitionistic theories with equality:

THEOREM 6.2. *Let $\boldsymbol{T}$ be an intuitionistic theory with equality, and $\mathcal{F}$ a set of predicate symbols. Then there are a set $\mathcal{F}_0$ of new predicate symbols and an enumerating function σ of a set of $\mathcal{F}\cup\mathcal{F}_0$-inference forms such that the $\mathcal{F}$-free part of $\boldsymbol{T}$ is axiomatized by the series of axioms $B_0(\varepsilon), B_1(\varepsilon), B_2(\varepsilon), \cdots$ in $\boldsymbol{LJ}_e$, where $B_0, B_1, B_2, \cdots$ is the canonical series of $\mathcal{F}\cup\mathcal{F}_0$-approximations on the basic $\mathcal{F}\cup\mathcal{F}_0$-approximation and the enumerating function σ.*

Let A be a simple $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximation, and let σ be an enumerating function of a set of simple $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-inference forms. Then we define the series of simple $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximations $A_0^q, A_1^q, A_2^q, \cdots$ as follows:

A_0^q is A;

$A_{k+1}^{\sigma}(\Gamma)$ is the formula $A_k^{\sigma}(\Gamma: \sigma(0)) \wedge \cdots \wedge A_k^{\sigma}(\Gamma: \sigma(k)) \wedge A_k^{\sigma}(\Gamma)$ for each finite sequence Γ of $\mathcal{P} \cup \mathcal{F}$-atomic formulas.

The series of simple $(\mathcal{P}, \mathcal{N}, \mathcal{F})$-approximations $A_0^{\sigma}, A_1^{\sigma}, A_2^{\sigma}, \cdots$ is called the *canonical series of simple* $(\mathcal{P}, \mathcal{N}, \mathcal{F})$*-approximations on* A *and* σ.

Then we have the simple part version of Proposition 6.1. Hence Theorem 3.2 implies the following theorem:

THEOREM 6.3. *Let* $\boldsymbol{T}$ *be a classical theory with equality. Let* $\mathcal{S}$ *be a set of simple* $(\mathcal{P}, \mathcal{N}, \mathcal{F})$*-inference forms, and* σ *an enumerating function of* $\mathcal{S}$. *Suppose that the theory* $\boldsymbol{LK}_e[\mathcal{S}]$ *is an extension of* $\boldsymbol{T}$. *Let* A *be a simple* $(\mathcal{P}, \mathcal{N}, \mathcal{F})$*-approximation which is adequate on* $(\boldsymbol{LK}_e[\mathcal{S}], \boldsymbol{T})$, *and* $A_0, A_1, A_2, \cdots$ *the canonical series of simple* $(\mathcal{P}, \mathcal{N}, \mathcal{F})$*-approximations on* A *and* σ. *Then the* $(\mathcal{P}, \mathcal{N}, \mathcal{F})_+$*-part of* $\boldsymbol{LK}_e[\mathcal{S}]$ *is axiomatized by the series of axioms* $A_0(\varepsilon), A_1(\varepsilon), A_2(\varepsilon), \cdots$ *in* $\boldsymbol{T}$.

Now, let $\mathfrak{A}^{\mathcal{N}}$ denote the formula obtained from a formula $\mathfrak{A}$ by replacing all occurrences of $\mathcal{N}$-atomic formulas $\mathfrak{B}$ by $\neg\mathfrak{B}$. Then, it is obvious that a formula $\mathfrak{A}$ is $(\mathcal{P} \cup \mathcal{N}, \emptyset, \mathcal{F})_+$ if and only if the formula $\mathfrak{A}^{\mathcal{N}}$ is $(\mathcal{P}, \mathcal{N}, \mathcal{F})_+$, and that a sequent $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_m \rightarrow \mathfrak{A}$ is $\boldsymbol{LK}_e$-provable if and only if the sequent $\mathfrak{A}_1^{\mathcal{N}}, \mathfrak{A}_2^{\mathcal{N}}, \cdots, \mathfrak{A}_n^{\mathcal{N}} \rightarrow \mathfrak{A}^{\mathcal{N}}$ is $\boldsymbol{LK}_e$-provable.

Thus Corollary 3.3, Theorem 5.2 and Theorem 6.3 imply the following axiomatization theorem for classical theories with equality:

THEOREM 6.4. *Let* $\boldsymbol{T}$ *be a classical theory with equality, and* $\mathcal{P}$, $\mathcal{N}$, *and* $\mathcal{F}$ *mutually disjoint sets of predicate symbols. Then there are a set* $\mathcal{F}_0$ *of new predicate symbols and an enumerating function* σ *of a set of simple* $(\mathcal{P} \cup \mathcal{N}, \emptyset, \mathcal{F} \cup \mathcal{F}_0)$*-inference forms such that the* $(\mathcal{P}, \mathcal{N}, \mathcal{F})_+$*-part of* $\boldsymbol{T}$ *is axiomatized by the series of axioms* $B_0(\varepsilon)^{\mathcal{N}}, B_1(\varepsilon)^{\mathcal{N}}, B_2(\varepsilon)^{\mathcal{N}}, \cdots$ *in* $\boldsymbol{LK}_e$, *where* $B_0, B_1, B_2, \cdots$ *is the canonical series of simple* $(\mathcal{P} \cup \mathcal{N}, \emptyset, \mathcal{F} \cup \mathcal{F}_0)$*-approximations on the basic simple* $(\mathcal{P} \cup \mathcal{N}, \emptyset, \mathcal{F} \cup \mathcal{F}_0)$*-approximation and the enumerating function* σ.

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