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Global Sebastiani-Thom theorem for polynomial maps

By András NÉMETHI

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1. Introduction.

There are several methods for studying the topological type of affine hypersurfaces. Kouchnirenko and M. Oka have investigated in [**Ko**], [**O2**], [**O3**], [**O4**] the relation between Newton boundary and the topology of the generic fiber, Hà and Lê [**H-L**] determined the bifurcation set of polynomial maps with two variables using the Euler characteristic of the fibers. Another approach is due to Broughton [**Br1**], [**Br2**] who have introduced and studied the class of "tame" polynomials. His results have been extended by the author for the larger class of "quasitame" polynomials [**Ne**].

In this note we establish a Sebastiani-Thom type result. More precisely: Let $g: \boldsymbol{C}^n \to \boldsymbol{C}$ and $h: \boldsymbol{C}^m \to \boldsymbol{C}$ be polynomial maps with bifurcation sets Λ_g resp. Λ_h. We consider the sum-map $f: \boldsymbol{C}^n \times \boldsymbol{C}^m \to \boldsymbol{C}$, $f(x, y)=g(x)+h(y)$. We prove the following

THEOREM.

a) *The bifurcation set of f is contained in $\Lambda_g+\Lambda_h$.*

b) *The generic fiber of f is homotopic equivalent with the join space of the generic fibers of the polynomial maps g and h.*

c) *The global algebraic monodromy of f (around all the bifurcation points) is induced by the join of the global geometric monodromies of g and h. (In particular it can be determined in terms of the global algebraic monodromies of g and h).*

This result extends the results of Sebastiani-Thom [**Se-T**] and K. Sakamoto [**Sa1**], [**Sa2**] (in the local case) and M. Oka [**O1**] in the special case of weighted homogeneous polynomials. The proof is based on a new technique which applies to the general (global) case of the polynomials (without $\boldsymbol{C}^*$-action).

We are indebted to the referee for suggesting us the proof of Theorem 3.2 which is more natural and simple than our original proof.

2. The generic fiber.

Let $g: \boldsymbol{C}^n \to \boldsymbol{C}$ and $h: \boldsymbol{C}^m \to \boldsymbol{C}$ be polynomial maps. Then there exists a finite set $\Lambda_g=\{c_1, \cdots, c_t\}$ (respectively $\Lambda_h=\{d_1, \cdots, d_s\}$) such that $g: \boldsymbol{C}^n - g^{-1}(\Lambda_g) \to \boldsymbol{C}-\Lambda_g$ (respectively $h: \boldsymbol{C}^m - h^{-1}(\Lambda_h) \to \boldsymbol{C}-\Lambda_h$) is a C^∞ locally trivial fibration (see [**Ve**], [**Br1**], [**H-L**]).

We define $f: \boldsymbol{C}^n \times \boldsymbol{C}^m \to \boldsymbol{C}$ by $f(x, y)=g(x)+h(y)$ and $\Lambda_f=\Lambda_g+\Lambda_h=\{c_i+d_j \mid c_i \in \Lambda_g, d_j \in \Lambda_h\}$. If we introduce the set $L_e=\{(c, d) \in \boldsymbol{C} \times \boldsymbol{C} \mid c+d=e\}$ for all $e \in \boldsymbol{C}$, and the map $u=g \times h: \boldsymbol{C}^n \times \boldsymbol{C}^m \to \boldsymbol{C} \times \boldsymbol{C}$, $u(x, y)=(g(x), h(y))$, then $f^{-1}(e)=u^{-1}(L_e)$. Hence the study of the polynomial f is in strong connection with the study of the map u and the mutual position of the line L_e and the set $\Lambda=(\boldsymbol{C} \times \Lambda_h) \cup (\Lambda_g \times \boldsymbol{C})$. If $e \notin \Lambda_f$, then let $\{C_i\}_{i=\overline{1,t}}=\{(c_i, e-c_i)\}_{i=\overline{1,t}}=L_e \cap (\Lambda_g \times \boldsymbol{C})$ and $\{D_j\}_{j=\overline{1,s}}=\{(e-d_j, d_j)\}_{j=\overline{1,s}}=L_e \cap (\boldsymbol{C} \times \Lambda_h)$.

LEMMA 2.1. *There exists a C^∞ diffeomorphism $v: R^2 \to L_e$ such that $v^{-1}(C_i) \subset \boldsymbol{R} \times (0, \infty)$ $(i=\overline{1,t})$ and $v^{-1}(D_j) \subset \boldsymbol{R} \times (-\infty, 0)$ $(j=\overline{1,s})$.*

This is the consequence of the following

HOMOGENEITY LEMMA [**Mil**].

Let n_i and n_i' $(i=\overline{1,k})$ be arbitrary points of the smooth, connected manifold N. Then there exists a diffeomorphism $v_N: N \to N$ which (is smoothly isotopic to the identity and) carries n_i into n_i'. Moreover, v_N can be chosen such the set $\{x \in N: v_N(x) \neq x\}^-$ is compact.

In particular, we can choose the diffeomorphism $v: \boldsymbol{R}^2 \to L_e$ such that v is an isometry outside of a large disk, hence we have the following properties in plus:

i) there exists K_1 such that $\|v^{-1}(w)-v^{-1}(w')\| < K_1 \|w-w'\|$ for all $w, w' \in L_e$.

ii) $\|d_w(v^{-1})\| \leqq K_1$ for all $w \in L_e$.

We define the following sets in L_e: $v(\boldsymbol{R} \times [0, \infty))=\mathcal{C}$, $v(\boldsymbol{R} \times (-\infty, 0])=\mathcal{D}$, $v(\boldsymbol{R} \times \{0\})=\gamma$ and $v((c_\Gamma, 0))=\Gamma$ a point on γ. Thus $C_i \in \mathcal{C}$ $(i=\overline{1,t})$ and $D_j \in \mathcal{D}$ $(j=\overline{1,s})$. Our first result is the following

THEOREM 2.2. *The restricted map $f: \boldsymbol{C}^n \times \boldsymbol{C}^m - f^{-1}(\Lambda_f) \to \boldsymbol{C}-\Lambda_f$ is a C^∞ locally trivial fibration.*

PROOF. If we denote $\mathcal{C}_\varepsilon=v(\boldsymbol{R} \times [\varepsilon, \infty))$ and $\mathcal{D}_\varepsilon=v(\boldsymbol{R} \times (-\infty, -\varepsilon])$ then there exists a sufficiently small $\varepsilon>0$ such that $C_i \in \mathcal{C}_\varepsilon$ $(i=\overline{1,t})$ and $D_j \in \mathcal{D}_\varepsilon$ $(j=\overline{1,s})$. We define a C^∞ function $\phi: L_e \to [0, 1]$ by $\phi=\phi' \circ v^{-1}$, where $\phi'(r_1, r_2)=\phi''(r_2)$ is a C^∞ function with

$$\phi''(r_2)=\begin{cases} 1 & r_2 \leqq -\dfrac{\varepsilon}{2} \\ 0 & r_2 \geqq \dfrac{\varepsilon}{2} \end{cases}$$

Let $e \notin \Lambda_f$. Since g is trivial over $pr_1(L_e - \mathcal{C}_\varepsilon) = P$ (where pr_1 is the first projection $pr_1: \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{C}$), there exists a diffeomorphism $\psi = (\psi_1, \psi_2): g^{-1}(P) \to P \times G$ such that $\psi_1 = g$.

Consider the following application:

$$\tilde{\phi}: B_{\varepsilon'} \times P \times G \to B_{\varepsilon'} \times P \times G \text{ (where } B_{\varepsilon'} = \{a \in \boldsymbol{C}: |a-e| < \varepsilon')$$

$$\tilde{\phi}(a, c, x) = (a, c + \phi(c, e-c)(a-e), x).$$

From (i) we get that there exists K such that $\|\phi(w) - \phi(w')\| \leqq K\|w - w'\|$ for arbitrary $w, w' \in L_e$, hence for $\varepsilon' > 0$ sufficiently small $\tilde{\phi}$ is injective. Similarly, using (ii) we obtain that $\tilde{\phi}$ is local diffeomorphism for ε' small. In order to prove the surjectivity of $\tilde{\phi}$ we observe the following facts:

a) $\tilde{\phi} | B_{\varepsilon'} \times v(\{\varepsilon/2 \leqq r_2 < \varepsilon\}) \times G = id$

b) $(\tilde{\phi} | B_{\varepsilon'} \times \mathcal{D}_\varepsilon \times G)(a, c, x) = (a, c + a - e, x)$

c) $\|\tilde{\phi} - id\| \leqq \varepsilon'$.

Therefore, similarly as in the proof of Theorem 1.7 [**H**] we obtain the surjectivity of $\tilde{\phi}$.

Thus, for ε' sufficiently small $\tilde{\phi}$ is diffeomorphism.

We define the diffeomorphism $\phi^g = (1, \phi_2^g): B_{\varepsilon'} \times g^{-1}(P) \to B_{\varepsilon'} \times g^{-1}(P)$ by the diagram

$$\begin{array}{ccc} B_{\varepsilon'} \times g^{-1}(P) & \xrightarrow{(1,\ \phi_2^g)} & B_{\varepsilon'} \times g^{-1}(P) \\ \wr \downarrow (1, \psi) & & \wr \downarrow (1, \psi) \\ B_{\varepsilon'} \times P \times G & \xrightarrow{\tilde{\phi}} & B_{\varepsilon'} \times P \times G\,. \end{array}$$

Therefore $g(\phi_2^g(a, x)) = g(x) + \phi(g(x), e - g(x)) \cdot (a - e)$. $(*)$

The map $(1, \phi_2^g)$ can be extended by the identity, hence we have constructed a diffeomorphism $(1, \phi_2^g): B_{\varepsilon'} \times \boldsymbol{C}^n \to B_{\varepsilon'} \times \boldsymbol{C}^n$ such that $(*)$ holds.

In similar way we obtain a diffeomorphism $(1, \phi_2^h): B_{\varepsilon'} \times \boldsymbol{C}^m \to B_{\varepsilon'} \times \boldsymbol{C}^m$, such that $h(\phi_2^h(a, y) = h(y) + (1 - \phi)(e - h(y), h(y)) \cdot (a - e)$.

We define $\bar{\phi}: B_{\varepsilon'} \times f^{-1}(e) \to f^{-1}(B_{\varepsilon'})$ by $\bar{\phi}(a, x, y) = (\phi_2^g(a, x), \phi_2^h(a, y))$. Then $\bar{\phi}$ is diffeomorphism with $f(\phi_2^g(a, x), \phi_2^h(a, y)) = a$.

In order to determine the structure of the fiber $f^{-1}(e) = u^{-1}(L_e)$ we study the restricted map $u: u^{-1}(L_e) \to L_e$.

LEMMA 2.3. *Let* $e \notin \Lambda_f$. *Then*

a) $u: u^{-1}(L_e - \Lambda) \to L_e - \Lambda$ *is a* C^∞ *locally trivial fibration. In particular* $u^{-1}(\gamma) \approx u^{-1}(\Gamma) \times R \approx G \times H \times \boldsymbol{R}$.

b) $u^{-1}(\mathcal{D}) \approx G \times h^{-1}(pr_2 \mathcal{D}) \approx G \times \boldsymbol{C}^m$

$u^{-1}(\mathcal{C}) \approx g^{-1}(pr_1 \mathcal{C}) \times H \approx \boldsymbol{C}^n \times H$, (*where G and H are the generic fibers of the polynomials g and h and "$\approx$" means "diffeomorphic"*).

PROOF. a) Let $(c, d)\in L_e-\Lambda$. Then there exists a neighbourhood U of (c, d) in L_e such that g (resp. h) is trivial over pr_1U(resp. pr_2U). Hence there exist the diffeomorphisms:

$$\phi^g=(g, \phi_2^g): g^{-1}(pr_1U) \longrightarrow pr_1U\times G \quad \text{and}$$
$$\phi^h=(h, \phi_2^h): h^{-1}(pr_2U) \longrightarrow pr_2U\times H.$$

Then $\phi: u^{-1}(U)\to U\times G\times H$, $\phi(x, y)=((g(x), h(y)), \phi_2^g(x), \phi_2^h(y))$ is a trivialization of u over U.

b) Since g over $pr_1\mathcal{D}$ is trivial fibration, we have a diffeomorphism $(g, \phi_2^g): g^{-1}(pr_1\mathcal{D})\to pr_1\mathcal{D}\times G$. The map $(\phi_2^g, p_2): u^{-1}(\mathcal{D})\to G\times h^{-1}(pr_2\mathcal{D})$, $(\phi_2^g, p_2)(x, y)=(\phi_2^g(x), y)$ is the wanted diffeomorphism. Since $\Lambda_h\subset pr_2\mathcal{D}$ and $pr_2\mathcal{D}$ is a strong deformation retract of $\boldsymbol{C}$ $h^{-1}(pr_2\mathcal{D})\approx h^{-1}(\boldsymbol{C})=\boldsymbol{C}^m$. ■

With this preparations we can prove that following

THEOREM 2.4. *Let f be a polynomial in $\boldsymbol{C}^n\times\boldsymbol{C}^m$ such that $f(x, y)=g(x)+h(y)$. Let $F=f^{-1}(e)$ $(e\notin\Lambda_f)$, $G=g^{-1}(c)$ $(c\notin\Lambda_g)$ and $H=h^{-1}(d)$ $(d\notin\Lambda_h)$. Then there is a homotopy equivalence between F and $G*H$ (the join of G and H).*

PROOF. From Lemma 2.3 there is a homotopy equivalence between $u^{-1}(L_e)$ and $u^{-1}(L_e-(\gamma-\Gamma))$, which is homotopic equivalent to $\boldsymbol{C}^n\times H\underset{G\times H}{\cup} G\times\boldsymbol{C}^m$ (in the disjoint union of the spaces $\boldsymbol{C}^n\times H$ and $G\times\boldsymbol{C}^m$ we identify the subspaces $G\times H\subset\boldsymbol{C}^n\times H$, $G\times H\subset G\times\boldsymbol{C}^m$). But we have the following identifications of pair of spaces $(\boldsymbol{C}^n\times H, G\times H)\sim((\mathrm{Con}\,G)\times H, G\times H)$, $(G\times\boldsymbol{C}^m, G\times H)\sim(G\times\mathrm{Con}\,H, G\times H)$ (where $\mathrm{Con}\,X$ denotes the cone over X).

If we define $X_1=\{[x, t, y]\in G*H: t\leq 1/2\}$ and $X_2=\{[x, t, y]\in G*H: t\geq 1/2\}$ then $(X_1, X_1\cap X_2)\sim(G\times\mathrm{Con}\,H, G\times H)$ and $(X_2, X_1\cap X_2)\sim((\mathrm{Con}\,G)\times H, G\times H)$.

Therefore $F\sim X_1\underset{X_1\cap X_2}{\cup} X_2=G*H$. ■

COROLLARY 2.5.

a) $$\tilde{H}_r(F)=\bigoplus_{p+q=r-1}\tilde{H}_p(G)\otimes\tilde{H}_q(H)\oplus\bigoplus_{p+q=r-2}\mathrm{Tor}(\tilde{H}_p(G), \tilde{H}_q(H))$$

(*for the proof see* [**Mi2**]).

b) *If G is n_1-connected and H is n_2-connected, then F is (n_1+n_2+2)-connected. In particular F is connected.*

c) *$\pi_1(F)$=the free group of rank $(a-1)(b-1)$, where $H_0(G)=\boldsymbol{Z}^a$ and $H_0(H)=\boldsymbol{Z}^b$.*

REMARK 2.6. The homology groups can be calculated using the Mayer—

Vietoris sequence:

$$\cdots \longrightarrow \tilde{H}_q(u^{-1}(\gamma)) \longrightarrow \tilde{H}_q(u^{-1}(\mathcal{C}))\oplus\tilde{H}_q(u^{-1}(\mathcal{D})) \longrightarrow \tilde{H}_q(F) \xrightarrow{\partial_*} \cdots$$

If we use Lemma 2.3 we obtain the following exact sequence:

$$0 \longrightarrow \tilde{H}_q(F) \xrightarrow{r_*\circ\partial_*} \tilde{H}_{q-1}(G\times H) \longrightarrow \tilde{H}_{q-1}(\boldsymbol{C}^n\times H)\oplus\tilde{H}_{q-1}(G\times\boldsymbol{C}^m) \longrightarrow 0$$

which is equivalent with Corollary 2.5.a.

(Here the isomorphism r_* is induced by the natural retraction $r: u^{-1}(\gamma)\to u^{-1}(\Gamma)$.)

3. The global monodromy.

Let $g: \boldsymbol{C}^n\to\boldsymbol{C}$ be a polynomial map with bifurcation set $\Lambda_g\subset\boldsymbol{C}$. Consider a large circle $S_g=\{z: |z|=R_g\}$ such that $\Lambda_g\subset\{z: |z|<R_g\}$. Then g is locally trivial fibration over S_g. We shall call the isotopy class of the characteristic map $M_g: g^{-1}(R_g)\to g^{-1}(R_g)$ of this fibration the global geometric monodromy of g. M_g induces in reduced homology the global algebraic monodromy $m_g: \tilde{H}_*(g^{-1}(R_g), \boldsymbol{Z})\to\tilde{H}_*(g^{-1}(R_g), \boldsymbol{Z})$.

EXAMPLE 3.1. If g is weighted homogeneous polynomial, possibly with negative weights, then $\Lambda_g=\{0\}$ and the global (geometric resp. algebraic) monodromy agrees with the usually (geometric resp. algebraic) monodromy of g.

Let g and h as in section 2., $G=f^{-1}(R_g)$, $H=h^{-1}(R_h)$. According to Corollary 2.5.a we define

$$(m_g*m_h)_*: \tilde{H}_*(G*H) \longrightarrow \tilde{H}_*(G*H)$$
$$(m_g*m_h)_r = \bigoplus_{p+q=r-1}(m_g)_p\otimes(m_h)_q\oplus\bigoplus_{p+q=r-2}\operatorname{Tor}((m_g)_p, (m_h)_q)$$

Obviously m_g*m_h is induced by the join of the geometric monodromies M_g*M_h.

THEOREM 3.2. *Let $f(x, y)=g(x)+h(y)$ as above. Then*

$$(m_f)_* = (m_g*m_h)_* .$$

PROOF. We can suppose $R_g=R_h=R$. By Theorem 2.2 we can take $R_f=2R$. We define the C^∞ family of C^∞ diffeomorphismes $v_\tau: \boldsymbol{C}\to L_{2Re^{2\pi i\tau}}$, $\tau\in[0, 1]$ (we identify $\boldsymbol{C}$ with $\boldsymbol{R}^2$):

$v_\tau(z)=(Re^{2\pi i\tau}(1+iz), Re^{2\pi i\tau}(1-iz))$ and we denote $\mathcal{C}_\tau=v_\tau(\boldsymbol{R}\times[0, \infty))$, $\mathcal{D}_\tau=v_\tau(\boldsymbol{R}\times(-\infty, 0])$, $\gamma_\tau=v_\tau(\boldsymbol{R}\times\{0\})$ and $\Gamma_\tau=v_\tau(0)$. Then $v_0=v_1$, $L_{2Re^{2\pi i\tau}}\cap(\Lambda_g\times\boldsymbol{C})\subset\mathcal{C}_\tau$ and $L_{2Re^{2\pi i\tau}}\cap(\boldsymbol{C}\times\Lambda_h)\subset\mathcal{D}_\tau$.

Since for each $\tau\in[0, 1]$ we have the property of decomposition as in Remark 2.6, the C^∞ family of diffeomorphismes induce the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \tilde{H}_q(F) & \xrightarrow{r_*\circ\partial_*} & \tilde{H}_{q-1}(G\times H) & \longrightarrow & \tilde{H}_{q-1}(C^n\times H)\oplus\tilde{H}_{q-1}(G\times C^m) & \longrightarrow 0 \\
& \downarrow m_f & & \downarrow m & & \downarrow & \\
0 \longrightarrow & \tilde{H}_q(F) & \xrightarrow{r_*\circ\partial_*} & \tilde{H}_{q-1}(G\times H) & \longrightarrow & \tilde{H}_{q-1}(C^n\times H)\oplus\tilde{H}_{q-1}(G\times C^m) & \longrightarrow 0
\end{array}$$

where $G\times H$ is identified with $u^{-1}(\Gamma_0)=u^{-1}(\Gamma_1)$; $r=r_0=r_1$ where $\{r_\tau\}_{\tau\in[0,1]}$ is the C^∞ family of natural retractions r_τ: $u^{-1}(\gamma_\tau)\to u^{-1}(\Gamma_\tau)$; m is the monodromy induced by $\tau\mapsto u^{-1}(\Gamma_\tau)$, which is $(M_g\times M_h)_{*,q-1}$. Therefore $m_f=m_g*m_h$. ■

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András NÉMETHI

The National Institute for
Scientific and Technical Creation
Department of Mathematics
Bd. Păcii 220, 79622 Bucharest, Romania