J. Math. Soc. Japan
Vol. 42, No. 3, 1990

On the jump-diffusion approximation of stochastic difference equations driven by a mixing sequence

By Tsukasa FUJIWARA

(Received Feb. 16, 1989)

§ 1. Introduction.

In this paper, we will establish a limit theorem for a sequence of stochastic processes determined by random difference equations. A feature of our work is that the limiting process is a Markov process with jumps whereas the pre-limiting processes are non-Markovian.

There has been extensive works concerning the problem of approximating non-Markovian process by diffusion. See, for example, Kesten and Papanicolau [**5**] and Khas'minskii [**6**]. Our aim is to extend these works to jump-diffusions (strong Markov processes with jumps) and it will give us some new aspects of limit theorems. In particular, we are interested in the problem how the continuous part and the jump part of limiting process come out. Recently, such extension has been studied by several authors. For example, Jacod and Shiryaev [**4**] gives a comprehensive survey of limit theorems in which they treat the weak convergence of semimartingales. Their standpoint of view is that the convergence of characteristics of semimartingales implies that of semimartingales, and their results include very general limit theorems. But we emphasize that the result of this paper is not contained in theirs, because our setting of problem is concerned with some mixing property which does not yield the convergence of characteristics in the sense of [**4**]. See Remark 3 in § 2.

Now, the problem we will discuss is formulated as follows. For an $\boldsymbol{R}^e$-valued array $\{\xi_{n,k}; n, k \in \boldsymbol{N}\}$ of random variables and a $d \times e$-matrix valued function C on $\boldsymbol{R}^d$, we consider the stochastic difference equation:

$$(1.1)\qquad \begin{cases} \varphi_{n,k} - \varphi_{n,k-1} = C(\varphi_{n,k-1})(\xi_{n,k} - a_n), & k = 1, 2, \cdots, \\ \varphi_{n,0} = x_0 \in \boldsymbol{R}^d, \end{cases}$$

where we set $a_n = E[\xi_{n,1} I_{(0,1]}(|\xi_{n,1}|)]$ and I_A denotes the indicator function of a set A.

Let $\{j_n\}_n$ be a positive sequence diverging to infinity. Define a sequence of stochastic processes $\{\varphi_n\}_n$ by

$$(1.2)\qquad \varphi_n(t) = \varphi_{n,[j_n t]}$$

for $t \in [0, \infty)$. Here, $[t]$ is the integral part of t. Then, each φ_n can be regarded as a random variable with values in the Skorohod space $D_d =: D([0, \infty), \boldsymbol{R}^d)$ (the space of all right continuous functions from $[0, \infty)$ to $\boldsymbol{R}^d$ with lefthand limits, equipped with the so-called Skorohod topology). Our problem is to show the weak convergence of this sequence $\{\varphi_n\}_n$. In this problem, our standpoint of discussion is that the weak convergence of the driving noise processes

$$\xi_n(t) = \sum_{k=1}^{[j_n t]} (\xi_{n,k} - a_n) \tag{1.3}$$

to a Lévy process implies that of the system processes φ_n to a jump-diffusion. For the purpose, as a condition for $\{\xi_{n,k}\}$, we will adopt the strongly uniform mixing condition which is considered to be an intermediate between the independence and the uniform mixing condition. See § 2 about the definition. Under this mixing condition, Samur showed the weak convergence of $\{\xi_n\}_n$ of (1.3) to a Lévy process in [7] and [8], which is an extension of famous Kolmogorov-Gnedenko's results for independent and identically distributed arrays. Our aim is to extend his results for the driving noise processes to those for the system processes. But, his method is not effective on our case, because it depends deeply on the fact that the limiting processes have independent increments. So we will apply the so-called martingale method, which goes as follows. We show the tightness of a sequence of stochastic processes, and then characterize any limit process by showing that the law of it is the unique solution of a martingale problem. We will see that this method is applicable to our jump-diffusions as well as diffusions.

Next, we explain the context of this paper. In § 2, we will give the definition of some notions and notations to formulate our results. We will also give the definition of strongly uniform mixing array, and then we will state our results. Main theorem is Theorem 1, which states that the sequence $\{\varphi_n\}_n$ of (1.2) converges weakly in $D([0, \infty), \boldsymbol{R}^d)$ to the solution of a stochastic differential equation of jump type characterized by the setting of the theorem. To prove Theorem 1, it is convenient to show the weak convergence of the joint processes $\tilde{\varphi}_n = (\varphi_n, \xi_n)$ instead of φ_n. Theorem 2 which states on this $\{\tilde{\varphi}_n\}_n$ is a rewriting of Theorem 1 and we will give a proof only for Theorem 2. Theorem 3 is a supplementary result to main theorem. In § 3, we will give a proof of Theorem 2. Since it is long, we will divide it into several steps. In § 4, we will give example of the arrays of random variables which satisfy the conditions in our theorems.

The author would like to express sincere gratitude to Professor Hiroshi Kunita for his kindhearted advices.

§ 2. Statement of results.

In this section, we give the definition of some basic notions and notations which will be necessary and then we state our results.

We first define some properties of driving noise processes of (1.1). For each $n \in \boldsymbol{N}$, let $\{\xi_{n,k}\,;\, k \in \boldsymbol{N}\}$ be an array of $\boldsymbol{R}^e$-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. We say that this array is stationary if for each $n \in \boldsymbol{N}$ the joint law of $(\xi_{n,1}, \cdots, \xi_{n,k})$ under P is equal to that of $(\xi_{n,1+l}, \cdots, \xi_{n,k+l})$ for all $k, l \in \boldsymbol{N}$.

We next introduce a kind of measure for the dependence between $\xi_{n,k}$'s. We say that an array $\{\xi_{n,k}\}$ of random variables satisfies the strongly uniform mixing condition with the rate function ψ (ψ-mixing) if

$$\text{(2.1)}\quad \psi(k) = \sup_{n\in N}\sup_{l\in N}\sup\left\{\left|\frac{P(A\cap B)}{P(A)P(B)}-1\right|;\ A\in\mathcal{F}^n_{1,l}, B\in\mathcal{F}^n_{l+k,\infty}, \text{and } P(A)P(B)>0\right\}$$

converges to 0 as $k\to\infty$, where we set $\mathcal{F}^n_{k,l}=\sigma[\xi_{n,j}\,;\, k\leqq j\leqq l]$ for $1\leqq k\leqq l\leqq\infty$. As far as we know, this mixing condition was first introduced in Blum-Hanson-Koopman [2] (in [2], they call it $*$-mixing) and it was used in Samur [7] and [8] to show the convergence theorems in which the limits have infinite divisible laws.

On the other hand, to represent jump processes (for example Lévy processes or the jump part of Markov processes) by stochastic integrals, we often use point processes. It is well-known that for a σ-finite measure ν on $(\boldsymbol{R}^e\setminus\{0\}, \mathcal{B}(\boldsymbol{R}^e\setminus\{0\}))$ satisfying $\int_{\boldsymbol{R}^e\setminus\{0\}}\min\{|z|^2, 1\}\nu(dz)<\infty$ there exists a stationary Poisson point process $\{p(t)\}$ on $\boldsymbol{R}^e\setminus\{0\}$ with the intensity measure ν. See Ikeda-Watanabe [**3**, Chapter I, Theorem 9]. We denote its (compensated) counting measure by $N_p(dudz)$ ($\tilde{N}_p(dudz)=N_p(dudz)-du\nu(dz)$, respectively). See [**3**, pp. 42~44, 59~63] for the precise definition of the above notions and the stochastic integrals with respect to the point processes. We also refer to [**3**] for the theory of stochastic differential equations of jump type which will appear in the statement of our theorem.

We give miscellaneous notations in the following. For a σ-finite measure ν on $\boldsymbol{R}^e\setminus\{0\}$, we put $C(\nu)=\{r>0\,;\,\nu(\{z\,;\,|z|=r\})=0\}$. We denote by $C^r(\boldsymbol{R}^d, \boldsymbol{R}^e)$ the space of all functions from $\boldsymbol{R}^d$ into $\boldsymbol{R}^e$ possessing continuous derivatives of order up to and including r. In the case of $e=1$, we denote it by $C^r(\boldsymbol{R}^d)$. We also denote by $C^r_b(\boldsymbol{R}^d, \boldsymbol{R}^e)$ the space of all functions of class $C^r(\boldsymbol{R}^d, \boldsymbol{R}^e)$ possessing bounded m-th derivatives for all $m\leqq r$. We denote $\boldsymbol{R}^d\otimes\boldsymbol{R}^e$ the set of all real $d\times e$ matrices, which is identified with $\boldsymbol{R}^{d\times e}$. We denote by $C_0(\boldsymbol{R}^e)$ the space of all bounded continuous functions defined on $\boldsymbol{R}^e$ which are 0 around $0\in\boldsymbol{R}^e$ and have a limit at the infinity.

For an array $\{\xi_{n,k}\}$ on $\boldsymbol{R}^e$, we set, for simplicity, $\xi_{n,k,\delta}=\xi_{n,k}I_{(0,\delta]}(|\xi_{n,k}|)$ and $\xi_{n,k}^{(\delta)}=\xi_{n,k}I_{(\delta,\infty)}(|\xi_{n,k}|)$. We denote by $\xi_{n,k}^p$ the p-th component of $\xi_{n,k}$ for $p=1, 2, \cdots, e$.

Now, let us state main theorem in this paper. We first introduce assumptions for the theorem.

Let $\{\xi_{n,k}\,;\ n,\ k\in\boldsymbol{N}\}$ be an $\boldsymbol{R}^e\setminus\{0\}$-valued, stationary, and strongly uniform mixing array of random variables which satisfies the following conditions (A. I), (A. II), and (A. III).

(A. I): For some sequence $\{j_n\,;\ n\in\boldsymbol{N}\}$ of positive integers diverging to infinity, it holds:

$$j_n\int_{\boldsymbol{R}^e} f(z)P(\xi_{n,1}\in dz)\longrightarrow\int_{\boldsymbol{R}^e} f(z)\nu(dz) \qquad \text{as } n\to\infty \tag{2.2}$$

for all $f\in C_0(\boldsymbol{R}^e)$, where ν is a σ-finite measure on $(\boldsymbol{R}^e\setminus\{0\},\ \mathscr{B}(\boldsymbol{R}^e\setminus\{0\}))$ satisfying $\int_{\boldsymbol{R}^e\setminus\{0\}}\min\{|z|^2,\ 1\}\nu(dz)<\infty$.

(A. II): There exists a sequence $\{\delta_k\in C(\nu)\,;\ k\in\boldsymbol{N},\ \delta_k\downarrow 0 \text{ as } k\uparrow\infty\}$ which satisfies the following conditions (a) and (b):

(a)

$$\sup_{n\in\boldsymbol{N}} j_n E[|\xi_{n,1,\delta_0}|^2]<\infty\,. \tag{2.3}$$

(b) The limits:

$$V_0^{p,q}=\lim_{k\to\infty}\lim_{n\to\infty} j_n E[\eta_{n,1,\delta_k}^p\eta_{n,1,\delta_k}^q] \tag{2.4}$$

and

$$V_1^{p,q}=\lim_{k\to\infty}\lim_{n\to\infty} j_n\sum_{l=2}^{j_n} E[\eta_{n,1,\delta_k}^p\eta_{n,l,\delta_k}^q] \tag{2.5}$$

exist for all $p,\ q=1, \cdots, e$, where $\eta_{n,l,\delta_k}=\xi_{n,l,\delta_k}-E[\xi_{n,l,\delta_k}]$.

As a matter of convenience, we take $\delta_0=1$.

(A. III): The mixing rate function of the array $\{\xi_{n,k}\}$ satisfies

$$\sum_{k=1}^{\infty}\phi(k)^{1/2}<\infty\,. \tag{2.6}$$

Further, we suppose that the coefficient C in (1.1) satisfies

(C. I): $$C\in C_b^2(\boldsymbol{R}^d,\ \boldsymbol{R}^d\otimes\boldsymbol{R}^e)\,.$$

THEOREM 1. *Assume* (A. I), (A. II), (A. III), *and* (C. I). *Then, the process* φ_n *of* (1.2) *converges in law to the unique solution* φ *of the following stochastic differential equation:*

$$\varphi(t) = x_0 + \int_0^t C(\varphi(u))B^V(du) + \int_0^t b(\varphi(u))du + \int_0^{t+}\int_{|z|\leq 1} C(\varphi(u-))z\tilde{N}_p(dudz) + \int_0^{t+}\int_{|z|>1} C(\varphi(u-))zN_p(dudz)\,, \tag{2.7}$$

where B^V is Brownian motion with mean 0 *and covariance matrix*

$$(V^{p,q} = V_0^{p,q} + V_1^{p,q} + V_1^{q,p})_{p,q=1,\cdots,e}\,, \tag{2.8}$$

$$b^j(x) = \sum_{i=1}^{d} \sum_{p,q=1}^{e} C^{i,p}(x)V_1^{p,q}\frac{\partial C^{j,q}}{\partial x^i}(x)\,, \tag{2.9}$$

and $p(t)$ is e-dimensional stationary Poisson point process with the intensity measure ν.

We give several remarks on this theorem.

REMARK 1. Under (2.2), (2.3), and (2.6), the matrices V_0 and V_1 do not depend on the choice of $\{\delta_k\}$.

REMARK 2. If we know that the diffusion coefficient $C(x)VC(x)^*$ of the process (2.7) does not degenerate, we can give a sufficient condition weaker than (C. I). Here C^* is the transpose of a matrix C. But it often happens that $V=0$ as we will see at § 4. This is one of reasons why we adopt the condition (C. I) which is adequate for the case that the diffusion coefficient may be degenerate.

REMARK 3. When $\{\xi_{n,k}\}$ is an independent and identically distributed array (that is the case of $\phi(k)\equiv 0$), then $V=V_0$ and $b(x)\equiv 0$. In such case, we can obtain Theorem 1 similarly as in [**4**, IX. Theorem 3.21, p. 505]. But our theorem does not follow from theirs provided $\phi\neq 0$.

REMARK 4. When $C\equiv 1$, then $b(x)=0$ and the limiting process is a Lévy process starting from x_0 with the characteristics (Lévy system) $(V, 0, \nu)$. This is nothing but a main result of Samur [7] and [8]. So our proof of Theorem 1 will give another one for his results though our assumption (A. III) on the mixing rate function is slightly stronger than his.

Now, note that by the representation of (1.1) and (2.7) we can see that, roughly speaking, prelimiting processes φ_n and the limiting process φ are functionals of (φ_n, ξ_n) and (φ, ξ), respectively. Here,

$$\xi(t) = B^V(t) + \int_0^{t+}\int_{|z|\leq 1} z\tilde{N}_p(dudz) + \int_0^{t+}\int_{|z|>1} zN_p(dudz)\,.$$

Therefore, for the proof of Theorem 1, we need to show the weak convergence of $\{\xi_n\}$ as well as that of $\{\varphi_n\}$. To this end, it is sufficient to show the weak convergence of the pair $\{\tilde{\varphi}_n = (\varphi_n, \xi_n)\}$ in the product space $D_d\times D_e$. But we will give a stronger assertion: the weak convergence of $\{\tilde{\varphi}_n\}$ in D_{d+e}.

THEOREM 2. *Let* φ_n *and* ξ_n *be the processes defined in* (1.2) *and* (1.3), *respectively. Set*

$$\tilde{\varphi}_n(t) = \begin{pmatrix} \varphi_n(t) \\ \xi_n(t) \end{pmatrix}. \tag{2.10}$$

Then, under the assumptions (A. I), (A. II), (A. III), *and* (C. I), *it holds:*

$$\tilde{\varphi}_n \xrightarrow{\mathcal{L}} \tilde{\varphi} \quad \text{in } D_{d+e} \text{ as } n\to\infty , \tag{2.11}$$

(*i.e.* $\tilde{\varphi}_n$ *converges in law to* $\tilde{\varphi}$ *as* D_{d+e}*-valued random variables*) *where* $\tilde{\varphi}$ *is the unique solution of the following stochastic differential equation:*

$$\begin{aligned}\tilde{\varphi}(t) = \tilde{x}_0 &+ \int_0^t \tilde{C}(\tilde{\varphi}(u))B^V(du) + \int_0^t \tilde{b}(\tilde{\varphi}(u))du + \int_0^{t+}\int_{|z|\leq 1} \tilde{C}(\tilde{\varphi}(u-))z\tilde{N}_p(dudz) \\ &+ \int_0^{t+}\int_{|z|>1} \tilde{C}(\tilde{\varphi}(u-))zN_p(dudz).\end{aligned} \tag{2.12}$$

Here, we put $\tilde{x}_0 = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \in \boldsymbol{R}^{d+e}$ *for* $x_0 \in \boldsymbol{R}^d$, $0 \in \boldsymbol{R}^e$,

$$\tilde{C}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C(x) \\ I_e \end{pmatrix} \quad (I_e \text{ denotes the } e\times e \text{ identity matrix}), \quad \text{and}$$

$$\tilde{b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b(x) \\ 0 \end{pmatrix} \quad \text{for } x\in\boldsymbol{R}^d,\ y\in\boldsymbol{R}^e .$$

We will give a proof of this Theorem 2 in the next section because Theorem 2 clearly implies Theorem 1.

Finally, we give a generalization of Theorem 1. We extend (1.1) to the following one:

$$\begin{cases} \varphi_{n,k} - \varphi_{n,k-1} = C_n(\varphi_{n,k-1})(\xi_{n,k} - a_n) + (1/j_n)B_n(\varphi_{n,k-1}) \\ \varphi_{n,0} = x_0 , \end{cases}$$

and set $\varphi_n(t) = \varphi_{n,[j_n t]}$.

We introduce assumptions on the coefficients.

(C_n. I): $C_n \in C^2(\boldsymbol{R}^d, \boldsymbol{R}^d \otimes \boldsymbol{R}^e)$ for all $n\in\boldsymbol{N}$ and $\sup_n \sup_{x\in\boldsymbol{R}^d} |C_n(x)| < \infty$. Further, there exists a function $C \in C_b^2(\boldsymbol{R}^d, \boldsymbol{R}^d\otimes\boldsymbol{R}^e)$ such that

$$\sup_{|x|\leq N}\{|C_n(x) - C(x)| + |C_n^{(1)}(x) - C^{(1)}(x)| + |C_n^{(2)}(x) - C^{(2)}(x)|\} \longrightarrow 0 \quad \text{as } n\to\infty,$$

for each $N>0$. Here, we denote by $C^{(i)}$ the derivative of order i.

(C_n. II): $B_n \in C^1(\boldsymbol{R}^d, \boldsymbol{R}^d)$ for all $n\in\boldsymbol{N}$ and $\sup_n \sup_{x\in\boldsymbol{R}^d} |B_n(x)| < \infty$. Further, there exists a function $B \in C_b^1(\boldsymbol{R}^d, \boldsymbol{R}^d)$ such that

$$\sup_{|x|\leq N}\{|B_n(x) - B(x)| + |B_n^{(1)}(x) - B^{(1)}(x)|\} \longrightarrow 0 \quad \text{as } n\to\infty ,$$

for each $N>0$.

THEOREM 3. *Under the assumptions* (A. I), (A. II), (A. III), (C_n. I), *and* (C_n. II), *it holds:*

$$\varphi_n \xrightarrow{\mathcal{L}} \varphi \qquad in\ D_d\ as\ n \to \infty,$$

where φ *is the unique solution of the following stochastic differential equation:*

$$\varphi(t) = x_0 + \int_0^t C(\varphi(u)) B^V(du) + \int_0^t (B+b)(\varphi(u)) du + \int_0^{t+}\int_{|z| \leqq 1} C(\varphi(u-)) z \tilde{N}_p(dudz)$$
$$+ \int_0^{t+}\int_{|z|>1} C(\varphi(u-)) z N_p(dudz).$$

We will not give a proof of this theorem because it is similar to that of Theorem 2.

§ 3. Proof of Theorem 2.

3-1. Preliminaries and ϕ-mixing property. In the subsequent sections, we will give a proof of Theorem 2. It is organized in three parts. The first step is to establish Theorem 2 for the localized and truncated processes of $\{\tilde{\varphi}_n\}_n$, which are uniformly bounded and have uniformly bounded jumps. We define them in Section 3-2 below. To complete the first step, we show the tightness of them in Section 3-3 and then we characterize any limiting process in Section 3-4. The second step is to remove the restriction of localization in the first step. Section 3-5 contains the step. The final step is to remove the restriction of truncation in the second step, which completes a proof of Theorem 2. This step is discussed in Section 3-6.

Before we proceed to the proof of Theorem 2, we give a lemma on the strongly uniform mixing property, which will be used frequently in the proof.

LEMMA 3.1. *Set* $m<l<k$.

(1) *For an* $\mathcal{F}^n_{1,l}$ ($\mathcal{F}^n_{k,\infty}$)*-measurable integrable function* X (Y, *respectively*), *it holds:*

$$|E[XY]-E[X]E[Y]| \leqq \phi(k-l)E[|X|]E[|Y|]. \tag{3.1}$$

(2) *Let* X *be an* $\mathcal{F}^n_{1,m}$*-measurable integrable function,* Y *be an* $\mathcal{F}^n_{l,l}$*-measurable integrable function, and* Z *be an* $\mathcal{F}^n_{k,\infty}$*-measurable integrable function with* $E[Z]=0$. *Then it holds:*

$$|E[X(YZ-E[YZ])]| \leqq \sqrt{2}(\phi^*+1)\phi(k-l)^{1/2}\phi(l-m)^{1/2} \times E[|X|]E[|Y|]E[|Z|], \tag{3.2}$$

where we put $\phi^* = \phi(1)+1$.

PROOF OF LEMMA 3.1. (1) From the definition of rate function ϕ, it is obvious that $|P(A\cap B)-P(A)P(B)| \leqq \phi(k-l)P(A)P(B)$ for $A \in \mathcal{F}^n_{1,l}$ and $B \in \mathcal{F}^n_{k,\infty}$.

We can easily extend this to (3.1) by the simple function approximation.

(2) We follow essentially H. Watanabe [**11**, Lemma 4]. From (3.1), it holds:

$$\begin{aligned}|E[X(YZ-E[YZ])]| &\leqq \phi(l-m)E[|X|]E[|YZ-E[YZ]|]\\ &\leqq 2\phi(l-m)\phi^* E[|X|]E[|Y|]E[|Z|].\end{aligned}$$

Similarly, we have:

$$|E[XYZ]| \leqq \phi(k-l)\phi^* E[|X|]E[|Y|]E[|Z|] \quad \text{and}$$
$$|E[YZ]| \leqq \phi(k-l)E[|Y|]E[|Z|]$$

because of $E[Z]=0$. Therefore it holds:

$$\begin{aligned}|E[X(YZ-E[YZ])]|^2 &= |E[X(YZ-E[YZ])]|\,|E[XYZ]-E[X]E[YZ]|\\ &\leqq 2\phi(l-m)\phi^*\phi(k-l)(\phi^*+1)E[|X|]^2E[|Y|]^2E[|Z|]^2.\end{aligned}$$

This yields (3.2). □

3-2. Localized and truncated processes. For each $N>0$, let r_N be a smooth function on $\boldsymbol{R}^d$ such that

$$r_N(x)=\begin{cases}1 & \text{if } |x|\leqq N\\ 0 & \text{if } |x|\geqq N+1,\end{cases}$$

and let q_N be a smooth function on $\boldsymbol{R}^e$ with the same property as above. Set $C_N(x)=r_N(x)C(x)$ and for each $M\in C(\nu)$ and $N>0$, define

$$\begin{cases}\varphi_{n,k}^{M,N}-\varphi_{n,k-1}^{M,N}=C_N(\varphi_{n,k-1}^{M,N})(\xi_{n,k,M}-a_n) & \text{for } k=1,2,\cdots\\ \varphi_{n,0}^{M,N}=x_0.\end{cases} \tag{3.3}$$

We may assume that $N\geqq 2|x_0|$ because we will take $N\to\infty$. Similarly, define

$$\begin{cases}\zeta_{n,k}^{M,N}-\zeta_{n,k-1}^{M,N}=q_N(\zeta_{n,k-1}^{M,N})(\xi_{n,k,M}-a_n) & \text{for } k=1,2,\cdots\\ \zeta_{n,0}^{M,N}=0.\end{cases} \tag{3.4}$$

Set

$$\tilde\varphi_n^{M,N}(t)=\begin{pmatrix}\varphi_n^{M,N}(t)\\ \xi_n^{M,N}(t)\end{pmatrix}\equiv\begin{pmatrix}\varphi_{n,[j_nt]}^{M,N}\\ \zeta_{n,[j_nt]}^{M,N}\end{pmatrix}. \tag{3.5}$$

We call this process the localized and truncated process of $\tilde\varphi_n$. Note that $\tilde\varphi_n^{M,N}$ is uniformly bounded and has uniformly bounded jumps.

3-3. Tightness of $\{\tilde\varphi_n^{M,N}\}_n$. To get the tightness of the sequence $\{\tilde\varphi_n^{M,N}\}_n$, we will show that the sequence satisfies Kolmogorov-Chentsov's criterion.

PROPOSITION 1. *For each $T>0$, there exists a constant $K>0$ such that*

$$E[|\tilde\varphi_n^{M,N}(t)-\tilde\varphi_n^{M,N}(s)|^2|\tilde\varphi_n^{M,N}(s)-\tilde\varphi_n^{M,N}(r)|^2]\leqq K|t-r|^2 \tag{3.6}$$

for all $0\leqq r\leqq s\leqq t\leqq T$, and

(3.7) $$E[|\tilde{\varphi}_n^{M,N}(t)-x_0|^2] \leqq Kt$$

for all $t \leqq T$.

The above proposition implies the tightness of $\{\tilde{\varphi}_n^{M,N}\}_n$ since $\tilde{\varphi}_n^{M,N}(0)=\tilde{x}_0$.

PROOF OF PROPOSITION 1. Before going to the proof, we give a few remarks to make the notations simple. First we take $j_n=n$ throughout § 3. Next we put

$$\tilde{C}_N\binom{x}{y}=\binom{C_N(x)}{q_N(y)}$$

for $x \in \boldsymbol{R}^d$ and $y \in \boldsymbol{R}^e$. Then $\tilde{\varphi}_n^{M,N}$ of (3.5) is represented as:

(3.8) $$\tilde{\varphi}_n^{M,N}(t)=\tilde{x}_0+\sum_{k=1}^{[nt]} \tilde{C}_N(\tilde{\varphi}_{n,k-1}^{M,N})(\xi_{n,k,M}-a_n),$$

where we set $\tilde{\varphi}_{n,k}^{M,N}=\binom{\varphi_{n,k}^{M,N}}{\zeta_{n,k}^{M,N}}$.

Since it is similar to the equation (3.3), we may replace $\tilde{C}_N$ and $\tilde{\varphi}_n^{M,N}$ by C_N and $\varphi_n^{M,N}$ respectively, because we do not need any changes of discussion. In the sequel, we omit superscripts M and N in $\varphi_n^{M,N}$ and $\varphi^{M,N}$, and subscript N in C_N. We will also give proofs of (3.6) and (3.7) only in 1-dimensional case, because we are able to follow easily the proofs in multidimensional case.

For fixed $M>1$, put $\eta_{n,k}=\xi_{n,k,M}-E[\xi_{n,k,M}]$ and $b_n=E[\xi_{n,k,M}^{(1)}]=E[\xi_{n,1,M}^{(1)}]$. Then we have:

$$\begin{aligned}
|\varphi_n(t)-\varphi_n(s)|^2 &= \Big|\sum_{k=[ns]+1}^{[nt]} C(\varphi_{n,k-1})(\eta_{n,k}+b_n)\Big|^2 \\
&\leqq 2\sum_{k=[ns]+1}^{[nt]} C(\varphi_{n,k-1})^2(\eta_{n,k})^2+4\sum_{l<k} C(\varphi_{n,l-1})\eta_{n,l}C(\varphi_{n,k-1})\eta_{n,k} \\
&\quad +2\Big\{\sum_{k=[ns]+1}^{[nt]} C(\varphi_{n,k-1})b_n\Big\}^2 =: 2I_1(n)+4I_2(n)+2I_3(n).
\end{aligned}$$

In the above, $\sum_{l<k}$ denotes the summation over (l, k) such that $[ns]+1 \leqq l<k \leqq [nt]$. We will use this abbreviation later, too. We will show the following lemma.

LEMMA 3.2. *Put* $\Phi_n(s)=|\varphi_n(s)-\varphi_n(r)|^2$. *Then it holds:*

(3.9) $$|E[I_i(n)\Phi_n(s)]| \leqq K\frac{[nt]-[ns]}{n}E[\Phi_n(s)]$$

for $i=1, 2, 3$, *where K is a constant which does not depend on n, r, s, and t.*

PROOF OF LEMMA 3.2. We first consider $I_1(n)$. Note that from assumptions (2.2) and (2.3) it holds

(3.10) $$\sup_{n,k} nE[|\eta_{n,k}|^2] =: K_1 < \infty.$$

Hence, we have:

$$E[I_1(n)\Phi_n(s)] = \sum_{k=[ns]+1}^{[nt]} E[C(\varphi_{n,k-1})^2(\eta_{n,k})^2\Phi_n(s)]$$
$$\leqq \phi^* \sum_{k=[ns]+1}^{[nt]} E[C(\varphi_{n,k-1})^2\Phi_n(s)]E[|\eta_{n,k}|^2]$$
$$\leqq \phi^*\|C\|^2 E[\Phi_n(s)]\frac{[nt]-[ns]}{n} nE[|\eta_{n,k}|^2]$$
$$\leqq \phi^*\|C\|^2 K_1\frac{[nt]-[ns]}{n} E[\Phi_n(s)].$$

Here, we denote by $\|C\|$ the supremum norm of C.

For $I_2(n)$, we divide it into the sum:

$$\sum_{l<k} C(\varphi_{n,l-1})^2\eta_{n,l}\eta_{n,k}+\sum_{l<k} C(\varphi_{n,l-1})\eta_{n,l}\{C(\varphi_{n,k-1})-C(\varphi_{n,l-1})\}\eta_{n,k}\,.$$

As for the first term, from Lemma 3.1 (1), we obtain:

$$\text{(3.11)}\qquad |E[\sum_{l<k} C(\varphi_{n,l-1})^2\eta_{n,l}\eta_{n,k}\Phi_n(s)]|$$
$$\leqq \sum_{l<k}\phi(k-l)E[|C(\varphi_{n,l-1})^2\eta_{n,l}\Phi_n(s)|]E[|\eta_{n,k}|]$$
$$\leqq \|C\|^2\phi^*\Big\{\sum_{k=1}^{\infty}\phi(k)\Big\}\frac{[nt]-[ns]}{n} nE[|\eta_{n,1}|^2]E[\Phi_n(s)]$$
$$\leqq \|C\|^2\phi^*\Big\{\sum_{k=1}^{\infty}\phi(k)\Big\}K_1\frac{[nt]-[ns]}{n}E[\Phi_n(s)].$$

On the second term, from the mean value theorem and (3.8) we have for each k:

$$\sum_{l=[ns]+1}^{k-1} C(\varphi_{n,l-1})\eta_{n,l}\{C(\varphi_{n,k-1})-C(\varphi_{n,l-1})\}$$
$$=\sum_{l=[ns]+1}^{k-1} C(\varphi_{n,l-1})\eta_{n,l}\sum_{j=l}^{k-1}\{C(\varphi_{n,j})-C(\varphi_{n,j-1})\}$$
$$=\sum_{l=[ns]+1}^{k-1}\sum_{j=l}^{k-1} C(\varphi_{n,l-1})\eta_{n,l}\varphi^*_{n,j}C(\varphi_{n,j-1})(\eta_{n,j}+b_n)$$

$\Big($where we put $\varphi^*_{n,j}=\int_0^1 C'(\varphi_{n,j-1}+v(\varphi_{n,j}-\varphi_{n,j-1}))dv\Big)$

$$=\sum_{j=[ns]+1}^{k-1}\{\varphi_{n,j}-\varphi_n(s)-\sum_{l=[ns]+1}^{j} C(\varphi_{n,l-1})b_n\}\varphi^*_{n,j}C(\varphi_{n,j-1})(\eta_{n,j}+b_n)\,.$$

Note that the summand is $\mathcal{F}^n_{1,j}$-measurable and the absolute value is dominated by $K_2|\eta_{n,j}+b_n|$ for some constant K_2 which does not depend on n and j, because of the uniform boundedness of $\{\varphi_{n,j}\}$ and the estimate:

$$\max_{j=[ns]+1,\cdots,[nt]}\Big|\sum_{l=[ns]+1}^{j} C(\varphi_{n,l-1})b_n\Big| \leqq T\|C\|\sup_n n|b_n| =: T\|C\|K_3<\infty.$$

It follows from the fact that $\lim_{n\to\infty}n|b_n|=\Big|\int_{1\leq|z|\leq M} z\nu(dz)\Big|<\infty$. Therefore it holds:

$$|E[\sum_{l<k} C(\varphi_{n,l-1})\eta_{n,l}\{C(\varphi_{n,k-1})-C(\varphi_{n,l-1})\}\eta_{n,k}\Phi_n(s)]| \tag{3.12}$$

$$\leqq \sum_{j<k} K_2\phi(k-j)\phi^* E[|\eta_{n,j}+b_n|]E[|\eta_{n,k}|]E[\Phi_n(s)]$$

$$\leqq K_2\phi^*\Big\{\sum_{k=1}^{\infty}\phi(k)\Big\}\frac{[nt]-[ns]}{n}\Big(\sup_n n\{3E[|\eta_{n,1}|^2]+2|b_n|^2\}\Big)E[\Phi_n(s)]$$

$$\leqq K_2K_4\phi^*\Big\{\sum_{k=1}^{\infty}\phi(k)\Big\}\frac{[nt]-[ns]}{n}E[\Phi_n(s)],$$

where $K_4=:\sup_n n\{3E[|\eta_{n,1}|^2]+2|b_n|^2\}$, which is finite. From (3.11) and (3.12), we get the conclusion.

Finally, we consider $I_3(n)$. Since

$$\Big|\sum_{k=[ns]+1}^{[nt]} C(\varphi_{n,k-1})b_n\Big| \leqq \frac{[nt]-[ns]}{n}\|C\|K_3,$$

it holds:

$$E\Big[\Big|\sum_{k=[ns]+1}^{[nt]} C(\varphi_{n,k-1})b_n\Big|^2\Phi_n(s)\Big] \leqq T(K_3)^2\|C\|^2\Big(\frac{[nt]-[ns]}{n}\Big)E[\Phi_n(s)].$$

We have now completed the proof of Lemma 3.2. □

We continue the proof of Proposition 1. By Lemma 3.2, we get the estimate:

$$E[|\varphi_n(t)-\varphi_n(s)|^2\Phi_n(s)] \leqq K_5\frac{[nt]-[ns]}{n}E[\Phi_n(s)]$$

for some constant $K_5>0$, Clearly, we also have:

$$E[\Phi_n(s)] = E[|\varphi_n(s)-\varphi_n(r)|^2] \leqq K_5\frac{[ns]-[nr]}{n}. \tag{3.13}$$

These results yield

$$E[|\varphi_n(t)-\varphi_n(s)|^2|\varphi_n(s)-\varphi_n(r)|^2] \leqq (K_5)^2\Big(\frac{[nt]-[nr]}{n}\Big)^2,$$

which implies (3.6).

(3.7) is obtained from (3.13) taking $s=r$ and $r=0$. We have completed the proof of Proposition 1. □

3-4. Characterization of limiting process. To show the identification of any limit measure of $\{\tilde{P}_n^{M,N}=$the law of $\tilde{\varphi}_n^{M,N}\}_n$, we establish a proposition.

PROPOSITION 2. *Let $\tilde{P}^{M,N}$ be any limit measure of $\{\tilde{P}_n^{M,N}\}_n$. Define*

$$\tilde{\mathcal{L}}^{M,N}f(\tilde{x}) = \frac{1}{2}\sum_{i,j=1}^{d+e}\Big(\tilde{C}_N(\tilde{x})V\tilde{C}_N(\tilde{x})^*\Big)^{i,j}\frac{\partial^2 f}{\partial\tilde{x}^i\partial\tilde{x}^j}(\tilde{x})+\sum_{j=1}^{d+e}\tilde{b}_N^j(\tilde{x})\frac{\partial f}{\partial\tilde{x}^j}(\tilde{x}) \tag{3.14}$$

$$+\int_{|z|\leqq M}\Big\{f(\tilde{x}+\tilde{C}_N(\tilde{x})z)-f(\tilde{x})-\sum_{j=1}^{d+e}(\tilde{C}_N(\tilde{x})z)^j I_{\{|z|\leqq 1\}}\frac{\partial f}{\partial\tilde{x}^j}(\tilde{x})\Big\}\nu(dz)$$

and

$$\tilde{b}_N^j(\tilde{x}) = \sum_{i=1}^{d+e} \sum_{p,q=1}^{e} \tilde{C}_N^{i,p}(\tilde{x}) V_1^{p,q} \frac{\partial \tilde{C}_N^{j,q}}{\partial \tilde{x}^i}(\tilde{x}).$$

Set $f(\tilde{x}) = \exp(i\tilde{u}\cdot\tilde{x})$ *where* $\tilde{x}$, $\tilde{u} \in \boldsymbol{R}^{d+e}$, *and* $\tilde{u}\cdot\tilde{x} = \sum_{j=1}^{d+e} \tilde{u}^j \cdot \tilde{x}^j$. *Then,*

$$M_f(t) = f(\tilde{\varphi}(t)) - f(\tilde{x}_0) - \int_0^t \tilde{\mathcal{L}}^{M,N} f(\tilde{\varphi}(u))du \tag{3.15}$$

is a $(D_{d+e}, \mathcal{D}_t, \tilde{P}^{M,N})$*-martingale, where we denote by* $\mathcal{D}_t$ *the right continuous version of* $\mathcal{D}_t^0 = \sigma[\tilde{\varphi}(u); u \leqq t]$.

We can easily see that Proposition 2 implies that $M_f(t)$ of (3.15) are martingales for all bounded function of class $C^2(\boldsymbol{R}^{d+e})$. Therefore it shows that $\tilde{P}^{M,N}$ is a solution of martingale problem in the sense of [**4**]. See Definition III.2.4 and Theorem II.2.24 in [**4**]. On the other hand, we have the uniqueness of solutions of martingale problem for $\tilde{\mathcal{L}}^{M,N}$, because the corresponding stochastic differential equation:

$$\begin{aligned} \tilde{\varphi}^{M,N}(t) = \tilde{x}_0 &+ \int_0^t \tilde{C}_N(\tilde{\varphi}^{M,N}(u))B^V(du) + \int_0^t \tilde{b}_N(\tilde{\varphi}^{M,N}(u))du \\ &+ \int_0^{t+}\int_{|z|\leq 1} \tilde{C}_N(\tilde{\varphi}^{M,N}(u-))z\tilde{N}_p(dudz) + \int_0^{t+}\int_{1<|z|\leq M} \tilde{C}_N(\tilde{\varphi}^{M,N}(u-))zN_p(dudz) \end{aligned} \tag{3.16}$$

has the unique solution process. See [**4**, Theorem III. 2.26, 2.32 and 2.33]. Hence, we conclude that any limit $\tilde{P}^{M,N}$ is equal to the law of the solution $\tilde{\varphi}^{M,N}$ of (3.16), and this yields that

$$\tilde{\varphi}_n^{M,N} \xrightarrow{\mathcal{L}} \tilde{\varphi}^{M,N} \quad \text{in } D_{d+e} \quad \text{as } n\to\infty. \tag{3.17}$$

PROOF OF PROPOSITION 2. As in the proof of Proposition 1, we give a proof in 1-dimensional case and omit the superscripts M, N, and $\sim$.

For the limiting measure P $(=\tilde{P}^{M,N})$, set $J(\varphi) = \{t \geqq 0;\, P(\Delta\varphi(t) = \varphi(t) - \varphi(t-) \neq 0) > 0\}$, which is at most countable. To prove this proposition, it suffices to show:

$$E[\{M_f(t) - M_f(s)\}\Psi(\varphi(u_1), \cdots, \varphi(u_m))] = 0, \tag{3.18}$$

for all s, $t \in J(\varphi)^c$, $m \in \boldsymbol{N}$, $u_i \in J(\varphi)^c$ $(i=1, \cdots, m)$, $0 \leqq u_1 \leqq \cdots \leqq u_m \leqq s$, and bounded continuous functions $\Psi : \boldsymbol{R}^m \to \boldsymbol{R}$. Here, $J(\varphi)^c$ denotes the complement of the set $J(\varphi)$. It follows from the right continuity of paths and the property: $\mathcal{D}_s \subset \mathcal{D}_{s'-} \subset \mathcal{D}_{s'}^0$ for $s < s'$, where $\mathcal{D}_{s'-} = \sigma[\bigcup_{u<s'} \mathcal{D}_u^0]$.

In the sequel, we may assume that the law of φ_n converges weakly to P $(=\tilde{P}^{M,N})$. Now for $s<t$, we have by Taylor's expansion and (3.8),

$$f(\varphi_n(t)) - f(\varphi_n(s)) = \sum_{k=[ns]+1}^{[nt]} \{f(\varphi_{n,k}) - f(\varphi_{n,k-1})\} =: S_1(n) + S_2(n) + S_3(n), \tag{3.19}$$

where we put

$$S_1(n)=\sum_{k=[ns]+1}^{[nt]} f'(\varphi_{n,k-1})C(\varphi_{n,k-1})\eta_{n,k}\,, \tag{3.20}$$

$$S_2(n)=\sum_{k=[ns]+1}^{[nt]} f'(\varphi_{n,k-1})C(\varphi_{n,k-1})b_n\,, \tag{3.21}$$

$$\begin{aligned}S_3(n)=\sum_{k=[ns]+1}^{[nt]}\int_0^1\!\!\int_0^1 \alpha f''(\varphi_{n,k-1}+\alpha\beta C(\varphi_{n,k-1})\\ \times(\eta_{n,k}+b_n))d\alpha d\beta C(\varphi_{n,k-1})^2(\eta_{n,k}+b_n)^2.\end{aligned} \tag{3.22}$$

For some time, we will concentrate on $S_1(n)$. Again by applying Taylor's expansion to $f'C$ and by (3.8), we get similarly:

$$\begin{aligned}S_1(n)&=\sum_{k=[ns]+1}^{[nt]}\{\{f'C(\varphi_{n,k-1})-f'C(\varphi_n(s))\}\eta_{n,k}+f'C(\varphi_n(s))\eta_{n,k}\}\\
&=\sum_{k=[ns]+1}^{[nt]}\left\{\sum_{l=[ns]+1}^{k-1}\{f'C(\varphi_{n,l})-f'C(\varphi_{n,l-1})\}\eta_{n,k}+f'C(\varphi_n(s))\eta_{n,k}\right\}\\
&=\sum_{l<k}(f'C)'C(\varphi_{n,l-1})\eta_{n,l}\eta_{n,k}+\sum_{l<k}(f'C)'C(\varphi_{n,l-1})b_n\eta_{n,k}\\
&\quad+\sum_{l<k}\int_0^1\!\!\int_0^1\alpha(f'C)''(\varphi_{n,l-1}+\alpha\beta(\varphi_{n,l}-\varphi_{n,l-1}))d\alpha d\beta(\varphi_{n,l}-\varphi_{n,l-1})^2\eta_{n,k}\\
&\quad+\sum_{k=[ns]+1}^{[nt]}f'C(\varphi_n(s))\eta_{n,k}\,.\end{aligned}$$

As for these terms, the following lemma holds.

LEMMA 3.3. (1)

$$\limsup_{n\to\infty}\sum_{l<k}|E[(f'C)'C(\varphi_{n,l-1})\{\eta_{n,l}\eta_{n,k}-E[\eta_{n,l}\eta_{n,k}]\}\Psi_n(s)]|=0\,. \tag{3.23}$$

(2)

$$\limsup_{n\to\infty}\sum_{l<k}|E[(f'C)'C(\varphi_{n,l-1})b_n\eta_{n,k}\Psi_n(s)]|=0\,. \tag{3.24}$$

(3)

$$\begin{aligned}\limsup_{n\to\infty}\sum_{l<k}\Bigg|E\Bigg[\int_0^1\!\!\int_0^1\alpha(f'C)''(\varphi_{n,l-1}+\alpha\beta(\varphi_{n,l}-\varphi_{n,l-1}))\\ \times d\alpha d\beta C(\varphi_{n,l-1})^2(\eta_{n,l}+b_n)^2\eta_{n,k}\Psi_n(s)\Bigg]\Bigg|=0\,.\end{aligned} \tag{3.25}$$

(4)

$$\limsup_{n\to\infty}\sum_{k=[ns]+1}^{[nt]}|E[f'C(\varphi_n(s))\eta_{n,k}\Psi_n(s)]|=0\,. \tag{3.26}$$

Here, we put $\Psi_n(s)=\Psi(\varphi_n(u_1),\cdots,\varphi_n(u_m))$ for Ψ and $u_1,\cdots,u_m$ in (3.18).

PROOF OF LEMMA 3.3. (1) Note that

$$\begin{aligned}(f'C)'C(\varphi_{n,l-1})&=\{(f'C)'C(\varphi_{n,l-1})-(f'C)'C(\varphi_n(s))\}+(f'C)'C(\varphi_n(s))\\
&=\sum_{j=[ns]+1}^{l-1}\Phi_{n,j}(\eta_{n,j}+b_n)+(f'C)'C(\varphi_n(s))\,,\end{aligned}$$

where $\Phi_{n,j}=\int_0^1((f'C)'C)'(\varphi_{n,j-1}+v(\varphi_{n,j}-\varphi_{n,j-1}))dvC(\varphi_{n,j-1})$, which is uniformly bounded and $\mathcal{F}^n_{1,j}$-measurable. Then we obtain from Lemma 3.1 (2):

$$\begin{aligned}
&\sum_{l<k}|E[(f'C)'C(\varphi_{n,l-1})(\eta_{n,l}\eta_{n,k}-E[\eta_{n,l}\eta_{n,k}])\Psi_n(s)]| \\
&\leqq \sum_{j<l<k}|E[\Phi_{n,j}(\eta_{n,j}+b_n)(\eta_{n,l}\eta_{n,k}-E[\eta_{n,l}\eta_{n,k}])\Psi_n(s)]| \\
&\quad+\sum_{l<k}|E[(f'C)'C(\varphi_n(s))(\eta_{n,l}\eta_{n,k}-E[\eta_{n,l}\eta_{n,k}])\Psi_n(s)]|
\end{aligned}$$

($\sum_{j<l<k}$ denotes the summation over (j, l, k) such that $[ns]+1\leqq j<l<k\leqq[nt]$)

$$\begin{aligned}
&\leqq \sum_{j<l<k}K_0\phi(k-l)^{1/2}\phi(l-j)^{1/2}E[|\Phi_{n,j}(\eta_{n,j}+b_n)\Psi_n(s)|]E[|\eta_{n,l}|]E[|\eta_{n,k}|] \\
&\quad+\sum_{l<k}K_0\phi(k-l)^{1/2}\phi(l-[ns])^{1/2}E[|(f'C)'C(\varphi_n(s))\Psi_n(s)|]E[|\eta_{n,l}|]E[|\eta_{n,k}|] \\
&\leqq K_0\Big(\sum_{k=1}^{\infty}\phi(k)^{1/2}\Big)^2\{([nt]-[ns])\|\Phi_{n,j}\|(E[|\eta_{n,1}|^2]^{1/2}+|b_n|)+\|(f'C)'C\|\} \\
&\qquad\qquad\times E[|\eta_{n,1}|^2]\|\Psi\| \\
&\longrightarrow 0 \qquad \text{as } n\to\infty,
\end{aligned}$$

because of (3.10) and $\sup_n n|b_n|<\infty$. Hence, we get (3.23).

For (2), (3), and (4), they are easily obtained from Lemma 3.1 (1) if we note $E[\eta_{n,k}]=0$. □

Next purpose is to show what is the limit of the term in (3.23):

$$\sum_{l<k}E[(f'C)'C(\varphi_{n,l-1})\Psi_n(s)]E[\eta_{n,l}\eta_{n,k}].$$

LEMMA 3.4. *For all bounded continuous function G on $\boldsymbol{R}$, it holds:*

$$\lim_{n\to\infty}E[\sum_{l<k}G(\varphi_{n,l-1})\Psi_n(s)]E[\eta_{n,l}\eta_{n,k}]=E\Big[\int_s^t G(\varphi(u))du\Psi(s)\Big]V_1,$$

where $\Psi(s)=\Psi(\varphi(u_1),\cdots,\varphi(u_m))$ and V_1 is the constant defined in (2.5) *for* 1-*dimensional case.*

PROOF OF LEMMA 3.4. Set $w_{l,k}=E[\eta_{n,l}\eta_{n,k}]$ for $l<k$, which is equal to $w_{1,k-l+1}$ by the stationarity of $\{\xi_{n,k}\}$. Then we have:

$$\begin{aligned}
\sum_{l<k}G(\varphi_{n,l-1})w_{l,k}&=\sum_{l=[ns]+1}^{[nt]-1}G(\varphi_{n,l-1})\sum_{k=l+1}^{[nt]}w_{l,k} \\
&=\sum_{l=[ns]+1}^{[nt]-1}G(\varphi_{n,l-1})\Big\{\sum_{k=2}^{[nt]-[ns]}w_{1,k}-\sum_{k=[nt]-l+2}^{[nt]-[ns]}w_{1,k}\Big\}.
\end{aligned}$$

Here, note that $\limsup_{n\to\infty}\sum_{l=[ns]+1}^{[nt]-1}\sum_{k=[nt]-l+2}^{[nt]-[ns]}|w_{1,k}|=0$. In fact, it holds:

$$\sum_{l=[ns]+1}^{[nt]-1}\sum_{k=[nt]-l+2}^{[nt]-[ns]}|w_{1,k}| = \sum_{k=2}^{[nt]-[ns]}(k-2)|w_{1,k}|$$
$$\leqq K_1(1/n)\sum_{k=2}^{[nt]-[ns]}k\phi(k) \longrightarrow 0 \qquad \text{as } n\to\infty\,,$$

because of $\sum_{k=1}^{\infty}\phi(k)<\infty$.

Next we will show that for $s<t$ it holds:

$$\lim_{n\to\infty} n\sum_{k=2}^{[nt]-[ns]} w_{1,k} = V_1\,. \tag{3.27}$$

This follows from the facts:

$$\limsup_{n\to\infty} n\sum_{k=([nt]-[ns])\wedge n}^{([nt]-[ns])\vee n}|w_{1,k}| \leqq K_1\limsup_{n\to\infty}\sum_{k=([nt]-[ns])\wedge n}^{([nt]-[ns])\vee n}\phi(k)=0\,,$$

and

$$\lim_{n\to\infty} n\sum_{k=2}^{n} w_{1,k} = \lim_{n\to\infty} n\sum_{k=2}^{n} E[\eta_{n,1,\delta}\eta_{n,k,\delta}]$$

for all $\delta\in C(\nu)$. In the above, we denote $a\vee b=\max\{a, b\}$ and $a\wedge b=\min\{a, b\}$. The latter follows from the estimates:

$$\limsup_{n\to\infty} n\sum_{k=2}^{n}|E[\eta_{n,1}^{(\delta)}\eta_{n,k,\delta}]| \leqq \sum_{k=1}^{\infty}\phi(k)\limsup_{n\to\infty} nE[|\eta_{n,1}^{(\delta)}|]E[|\eta_{n,1,\delta}|]=0\,,$$

and

$$\limsup_{n\to\infty} n\sum_{k=2}^{n}|E[\eta_{n,1}^{(\delta)}\eta_{n,k}^{(\delta)}]| = 0$$

for each $\delta\in C(\nu)$.

Now, we will complete our proof. Since $\varphi_n(u)=\varphi_{n,l-1}$ for $u\in[(l-1)/n, l/n)$, it holds:

$$E\Big[\sum_{l=[ns]+1}^{[nt]} G(\varphi_{n,l-1})\sum_{k=2}^{[nt]-[ns]} w_{1,k}\Psi_n(s)\Big]$$
$$= E\Big[\int_{([ns]+1)/n}^{[nt]/n} G(\varphi_n(u))du\Psi_n(s)\Big]\times n\sum_{k=2}^{[nt]-[ns]} w_{1,k}$$
$$\longrightarrow E\Big[\int_s^t G(\varphi(u))du\Psi(s)\Big]V_1 \qquad \text{as } n\to\infty\,,$$

because of $\varphi_n \xrightarrow{\mathcal{L}} \varphi$ whose law is P $(=\tilde{P}^{M,N})$ and (3.27). □

By this lemma, we obtain:

$$\lim_{n\to\infty}\sum_{l<k} E[(f'C)'C(\varphi_{n,l-1})\Psi_n(s)]E[\eta_{n,l}\eta_{n,k}] = E\Big[\int_s^t (f'C)'C(\varphi(u))du\Psi(s)\Big]V_1.$$

Combining this with Lemma 3.3, we arrive at the conclusion for (3.20):

$$\lim_{n\to\infty} E\Big[\sum_{k=[ns]+1}^{[nt]} f'(\varphi_{n,k-1})C(\varphi_{n,k-1})\eta_{n,k}\Psi_n(s)\Big] = E\Big[\int_s^t (f'C)'C(\varphi(u))du\Psi(s)\Big]V_1. \tag{3.28}$$

As for $S_2(n)$ of (3.21), since $nb_n \to \int_{1\leq|z|\leq M} z\nu(dz)$, it is easy to see:

$$\begin{aligned}(3.29)\qquad & \lim_{n\to\infty} E\Big[\sum_{k=[ns]+1}^{[nt]} f'(\varphi_{n,k-1})C(\varphi_{n,k-1})b_n\Psi_n(s)\Big] \\ &= E\Big[\int_s^t\int_{1\leq|z|\leq M}(f'C)(\varphi(u))zdu\nu(dz)\Psi(s)\Big].\end{aligned}$$

Finally, we consider $S_3(n)$ of (3.22). To show the convergence of it is an essential part of establishing the jump-diffusion approximation. To this end, we first prepare the following lemma. For the simplicity, we set

$$\Phi_{n,k} = \int_0^1\int_0^1 \alpha f^{(2)}(\varphi_{n,k-1}+\alpha\beta C(\varphi_{n,k-1})(\eta_{n,k}+b_n))d\alpha d\beta,$$

which is $\mathcal{F}^n_{1,k}$-measurable and bounded by $(1/2)\|f^{(2)}\|$.

LEMMA 3.5. (1)

$$\begin{aligned}(3.30)\qquad & \limsup_{n\to\infty} \sum_{k=[ns]+1}^{[nt]} |E[\Phi_{n,k}C(\varphi_{n,k-1})^2\{(\eta_{n,k}+b_n)^2 \\ &\qquad -(\eta_{n,k,\delta})^2-(\xi^{(\delta)}_{n,k,M})^2\}\Psi_n(s)]| = 0\end{aligned}$$

for each $\delta\in C(\nu)$.

(2)

$$\begin{aligned}(3.31)\qquad & \lim_{\delta\to\infty}\limsup_{n\to\infty} \sum_{k=[ns]+1}^{[nt]} |E[\{\Phi_{n,k}-(1/2)f^{(2)}(\varphi_{n,k-1})\} \\ &\qquad \times C(\varphi_{n,k-1})^2(\eta_{n,k,\delta})^2\Psi_n(s)]| = 0.\end{aligned}$$

(3)

$$\begin{aligned}(3.32)\qquad & \lim_{\delta\to\infty}\limsup_{n\to\infty} \sum_{k=[ns]+1}^{[nt]} \Big|E\Big[\Big\{\Phi_{n,k}-\int_0^1\int_0^1 \alpha f^{(2)}(\varphi_{n,k-1} \\ &+\alpha\beta C(\varphi_{n,k-1})\xi^{(\delta)}_{n,k,M})d\alpha d\beta\Big\}\times C(\varphi_{n,k-1})^2(\xi^{(\delta)}_{n,k,M})^2\Psi_n(s)\Big]\Big| = 0.\end{aligned}$$

This lemma shows that in the limiting procedure of $\lim_\delta \limsup_n E[\{\cdots\}\Psi_n(s)]$ we can replace $\Phi_{n,k}C(\varphi_{n,k-1})^2(\eta_{n,k}+b_n)^2$ by

$$\begin{aligned}(3.33)\qquad & \frac{1}{2}f^{(2)}(\varphi_{n,k-1})C(\varphi_{n,k-1})^2(\eta_{n,k,\delta})^2 \\ &+\int_0^1\int_0^1 \alpha f^{(2)}(\varphi_{n,k-1}+\alpha\beta C(\varphi_{n,k-1})\xi^{(\delta)}_{n,k,M})d\alpha d\beta\times C(\varphi_{n,k-1})^2(\xi^{(\delta)}_{n,k,M})^2.\end{aligned}$$

PROOF OF LEMMA 3.5. (1) Since

$$\begin{aligned}&(\eta_{n,k}+b_n)^2-(\eta_{n,k,\delta})^2-(\xi^{(\delta)}_{n,k,M})^2 \\ &= E[\xi^{(\delta)}_{n,k,M}]^2-2\xi^{(\delta)}_{n,k,M}E[\xi^{(\delta)}_{n,k,M}]+2\eta^{(\delta)}_{n,k}\eta_{n,k,\delta}+2b_n\eta_{n,k}+b_n^2,\end{aligned}$$

it holds:

$$\sum_{k=[ns]+1}^{[nt]} |E[\Phi_{n,k}C(\varphi_{n,k-1})^2\{(\eta_{n,k}+b_n)^2-(\eta_{n,k,\delta})^2-(\xi_{n,k,M}^{(\delta)})^2\}\Psi_n(s)]|$$
$$\leqq \|f^{(2)}\|\|C\|^2\|\Psi\|([nt]-[ns])\{3E[|\xi_{n,1,M}^{(\delta)}|]^2$$
$$+6E[|\xi_{n,1,\delta}|]E[|\xi_{n,1,M}^{(\delta)}|]+2|b_n|E[|\eta_{n,1}|^2]^{1/2}+|b_n|^2\}.$$

Then we get (3.30) if we note that for each $\delta\in C(\nu)$

$$\lim_{n\to\infty} nE[|\xi_{n,1,M}^{(\delta)}|]^2 = \lim_{n\to\infty}\frac{1}{n}\Big(nE[|\xi_{n,1,M}^{(\delta)}|]\Big)^2 = 0\times\Big\{\int_{\delta\leq|z|\leq M}|z|\nu(dz)\Big\}^2 = 0,$$

and

$$\limsup_{n\to\infty} nE[|\xi_{n,1,\delta}|]E[|\xi_{n,1,M}^{(\delta)}|] \leqq \limsup_{n\to\infty}\Big(\frac{1}{n}\Big)^{1/2}\{nE[|\xi_{n,1,\delta}|^2]\}^{1/2}nE[|\xi_{n,1,M}^{(\delta)}|]$$
$$\leqq 0\times K^{1/2}\int_{\delta\leq|z|\leq M}|z|\nu(dz).$$

(2) Since $(1/2)f^{(2)}(\varphi_{n,k-1})=\int_0^1\int_0^1\alpha f^{(2)}(\varphi_{n,k-1})d\alpha d\beta$, by the mean value theorem, we have:

$$\sum_{k=[ns]+1}^{[nt]} |E[\{\Phi_{n,k}-(1/2)f^{(2)}(\varphi_{n,k-1})\}C(\varphi_{n,k-1})^2(\eta_{n,k,\delta})^2\Psi_n(s)]|$$
$$\leqq \|f^{(3)}\|\|C\|^3\|\Psi\|([nt]-[ns])\{E[|\eta_{n,1}||\eta_{n,1,\delta}|^2]+|b_n|E[|\eta_{n,1,\delta}|^2]\}.$$

From this, we can easily see (3.31) if we note

$$\limsup_{n\to\infty} nE[|\eta_{n,1}||\eta_{n,1,\delta}|^2] \leqq 2\delta\limsup_{n\to\infty} nE[|\eta_{n,1}|^2]^{1/2}E[|\eta_{n,1,\delta}|^2]^{1/2} \leqq K\delta$$

for some K which does not depend on δ.

(3) Again by the mean value theorem, we have:

$$\sum_{k=[ns]+1}^{[nt]}\Big|E\Big[\Big\{\Phi_{n,k}-\int_0^1\int_0^1\alpha f^{(2)}(\varphi_{n,k-1}+\alpha\beta C(\varphi_{n,k-1})\xi_{n,k,M}^{(\delta)})d\alpha d\beta\Big\}$$
$$\times C(\varphi_{n,k-1})^2(\xi_{n,k,M}^{(\delta)})^2\Psi_n(s)\Big]\Big|$$
$$\leqq \|f^{(3)}\|\|C\|^3\|\Psi\|([nt]-[ns])\{E[|\eta_{n,1,\delta}||\xi_{n,1,M}^{(\delta)}|^2]$$
$$+(E[|\xi_{n,1,M}^{(\delta)}|]+|b_n|)E[|\xi_{n,1,M}^{(\delta)}|^2]\}.$$

Then, we can obtain (3.32) if we note

$$\limsup_{n\to\infty} nE[|\eta_{n,1,\delta}||\xi_{n,1,M}^{(\delta)}|^2]$$
$$\leqq 2\delta\times\limsup_{n\to\infty} nE[|\xi_{n,1,M}^{(\delta)}|^2] \leqq 2\delta\int_{|z|\leq M}|z|^2\nu(dz). \quad \square$$

As for the first term of (3.33), we get:

LEMMA 3.6.

$$\limsup_{n\to\infty}\left|E\left[\sum_{k=[ns]+1}^{[nt]}(f^{(2)}C^2)(\varphi_{n,k-1})\{(\eta_{n,k,\delta})^2-E[(\eta_{n,k,\delta})^2]\}\Psi_n(s)\right]\right|=0$$

for $\delta\in C(\nu)$.

PROOF OF LEMMA 3.6. It can be obtained by the similar consideration in the proof of Lemma 3.3 (1) because $E[\{(\eta_{n,k,\delta})^2-E[(\eta_{n,k,\delta})^2]\}]=0$. □

Combining this lemma with the assumption (2.4), we obtain:

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\left|E\left[\sum_{k=[ns]+1}^{[nt]}(f^{(2)}C^2)(\varphi_{n,k-1})(\eta_{n,k,\delta})^2\Psi_n(s)\right.\right.$$
$$\left.\left.-\int_s^t(f^{(2)}C^2)(\varphi(u))duV_0\Psi(s)\right]\right|=0. \tag{3.34}$$

Next, as for the second term of (3.33), we prepare the following lemma.

LEMMA 3.7. *Set*

$$F(x,z)=\int_0^1\int_0^1\alpha f^{(2)}(x+\alpha\beta C(x)z)d\alpha d\beta C(x)^2z^2.$$

For this function F and each $\delta\in C(\nu)$, it holds:

$$\limsup_{n\to\infty}\sum_{k=[ns]+1}^{[nt]}|E[\{F(\varphi_{n,k-1},\xi_{n,k,M}^{(\delta)})-E[F(x,\xi_{n,k,M}^{(\delta)})]|_{x=\varphi_{n,k-1}}\}\Psi_n(s)]|=0. \tag{3.35}$$

PROOF OF LEMMA 3.7. First note that by Taylor's expansion of $f(x)=\exp(ix\cdot u)$ it holds for each $m\in\boldsymbol{N}$,

$$F(x,z)=\sum_{p=0}^{m-1}F_p(x)G_p(z)+R_m(x,z)$$

where $F_p(x)=\int_0^1\int_0^1\alpha^{p+1}\beta^p d\alpha d\beta(1/p!)f^{(p+2)}(x)C(x)^{p+2}$, $G_p(z)=z^{p+2}$, and $R_m(x,z)=\int_0^1\int_0^1\alpha^{m+1}\beta^m f^{(m)}(a)d\alpha d\beta(C(x)z)^{m+2}/m!$ for some a. Since

$$\limsup_{n\to\infty}\sup_{x\in\boldsymbol{R}^d}\sum_{k=[ns]+1}^{[nt]}|E[R_m(x,\xi_{n,k,M}^{(\delta)})]|$$
$$\leqq\limsup_{n\to\infty}\{|u|^m\|C\|^{m+2}(M^m/m!)([nt]-[ns])E[|\xi_{n,1,M}^{(\delta)}|^2]\}$$
$$\leqq|u|^m\|C\|^{m+2}(M^m/m!)T\sup_n nE[|\xi_{n,1,M}^{(\delta)}|^2]\longrightarrow 0\qquad\text{as } m\to\infty,$$

to see (3.35), it suffices to show it for $F_p(x)G_p(z)$ ($p=1,\cdots,m-1$) instead of $F(x,z)$. But it can be easily seen if we replace $(f^{(2)}C^2)(x)$ and $(\eta_{n,k,\delta})^2$ in Lemma 3.6 by $F_p(x)$ and $G_p(\xi_{n,k,M}^{(\delta)})$, respectively and if we note that

$$\lim_{n\to\infty}n|E[G_p(\xi_{n,k,M}^{(\delta)})]|=\int_{\delta\leqq|z|\leqq M}G_p(z)\nu(dz)\leqq M^p\int_{|z|\leqq M}|z|^2\nu(dz).$$

Thus we have completed our proof. □

Similarly, by the above approximation of F by $\sum_p F_p(x)G_p(z)$, we can see

$$\lim_{n\to\infty}\sum_{k=[ns]+1}^{[nt]} E[E[F(x, \xi_{n,k,M}^{(\delta)})]|_{x=\varphi_{n,k-1}}\times\Psi_n(s)] \tag{3.36}$$

$$= E\Big[\int_s^t\int_{\delta\le|z|\le M} F(\varphi(u), z)du\nu(dz)\Psi(s)\Big]$$

$$\longrightarrow E\Big[\int_s^t\int_{|z|\le M} F(\varphi(u), z)du\nu(dz)\Psi(s)\Big] \quad \text{as } \delta\in C(\nu)\downarrow 0.$$

Combining this with (3.35), we obtain:

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\Big|E\Big[\sum_{k=[ns]+1}^{[nt]} F(\varphi_{n,k-1}, \xi_{n,k,M}^{(\delta)})\times\Psi_n(s) - \int_s^t\int_{|z|\le M} F(\varphi(u), z)du\nu(dz)\Psi(s)\Big]\Big| = 0. \tag{3.37}$$

Thus, from (3.34) and (3.37), we arrive at the conclusion for (3.22). That is,

$$\lim_{n\to\infty} E\Big[\sum_{k=[ns]+1}^{[nt]} \Phi_{n,k}C(\varphi_{n,k-1})^2(\eta_{n,k}+b_n)^2\Psi_n(s)\Big] \tag{3.38}$$

$$= E\Big[\Big\{\int_s^t (1/2)f^{(2)}C^2(\varphi(u))duV_0+\int_s^t\int_{|z|\le M} F(\varphi(u), z)du\nu(dz)\Big\}\Psi(s)\Big].$$

Now we go back to (3.19). Take $n\to\infty$ in it. Then, from (3.28), (3.29), and (3.38), we obtain (3.18). We have thus established Proposition 2. □

3-5. Removal of localization. We now proceed into the second step: to remove the restriction of localization. For the purpose, we define the truncated process φ_n^M of φ_n for each $n\in\boldsymbol{N}$ and $M\in C(\nu)$ by

$$\varphi_n^M(t) = \varphi_{n,[nt]}^M \tag{3.39}$$

for $t\in[0, \infty)$, where

$$\varphi_{n,k}^M - \varphi_{n,k-1}^M = C(\varphi_{n,k-1}^M)(\xi_{n,k,M} - a_n) \tag{3.40}$$

for $k=1, 2, \cdots$, and

$$\varphi_{n,0}^M = x_0.$$

We also define the truncated driving noise process ξ_n^M of ξ_n by

$$\xi_n^M(t) = \sum_{k=1}^{[nt]}(\xi_{n,k,M} - a_n) \tag{3.41}$$

and set

$$\tilde\varphi_n^M(t) = \begin{pmatrix}\varphi_n^M(t)\\ \xi_n^M(t)\end{pmatrix}. \tag{3.42}$$

Our aim in this section is to prove the following proposition for the weak convergence of $\{\tilde\varphi_n^M\}_n$.

PROPOSITION 3. *For each $M \in C(\nu)$, it holds:*

$$\tilde{\varphi}_n^M \xrightarrow{\mathcal{L}} \tilde{\varphi}^M \qquad in\ D_{d+e}\ as\ n \to \infty, \tag{3.43}$$

where $\tilde{\varphi}^M$ is the unique solution of the stochastic differential equation:

$$\begin{aligned}\tilde{\varphi}^M(t) = \tilde{x}_0 + \int_0^t \tilde{C}(\tilde{\varphi}^M(u)) B^V(du) + \int_0^t \tilde{b}(\tilde{\varphi}^M(u)) du \\ + \int_0^{t+} \int_{|z| \leq 1} \tilde{C}(\tilde{\varphi}^M(u-)) z \tilde{N}_p(dudz) + \int_0^{t+} \int_{1 \leq |z| \leq M} \tilde{C}(\tilde{\varphi}^M(u-)) z N_p(dudz).\end{aligned} \tag{3.44}$$

PROOF OF PROPOSITION 3. For $N>0$, set $S_N(\tilde{\varphi}) = \inf\{t \geqq 0, |\tilde{\varphi}(t)| \geqq N/2$ or $|\tilde{\varphi}(t-)| \geqq N/2\}$ for $\tilde{\varphi} \in D_{d+e}$. Then, it defines a $\{\mathcal{D}_t^0\}$-stopping time (i. e. $\{\tilde{\varphi} : S_N(\tilde{\varphi}) \leqq t\} \in \mathcal{D}_t^0$) and it is a lower semi-continuous function from D_{d+e} to $[0, \infty]$. Define $\mathcal{D}_{S_N}^0 = \{A \in \mathcal{D};\ A \cap \{S_N \leqq t\} \in \mathcal{D}_t^0\}$. To see (3.43), note that we can apply Stroock-Varadhan [**10**] Lemma 11.1.1 to our case D_{d+e}, instead of their case $C([0, \infty), \boldsymbol{R}^d)$. According to their lemma, it suffices to check the followings: for sufficiently large N, it holds (3.17),

$$\tilde{P}_n^{M,N} = \tilde{P}_n^M \qquad \text{on } \mathcal{D}_{S_N}^0 \quad \text{for all } n \in \boldsymbol{N}, \tag{3.45}$$

$$\tilde{P}^{M,N} = \tilde{P}^M \qquad \text{on } \mathcal{D}_{S_N}^0, \tag{3.46}$$

and

$$\lim_{N \to \infty} \tilde{P}^M(S_N \leqq t) = 0 \qquad \text{for all } t>0, \tag{3.47}$$

where $\tilde{P}_n^M$ and $\tilde{P}^M$ denote the laws of $\tilde{\varphi}_n^M$ and $\tilde{\varphi}^M$, respectively.

To see them, take $N > 2\max\{\|C\|(M+1), |x_0|\}$ for fixed M. Then, since

$$\sup_{t \in [0, S_N(\tilde{\varphi}_n^{M,N})]} |\tilde{\varphi}_n^{M,N}(t)| \leqq N, \qquad \text{and} \qquad \sup_{t \in [0, S_N(\tilde{\varphi}_n^M)]} |\tilde{\varphi}_n^M(t)| \leqq N,$$

it is easy to see that $S_N(\tilde{\varphi}_n^{M,N}) = S_N(\tilde{\varphi}_n^M)$ for all $n \in \boldsymbol{N}$ and that $\tilde{\varphi}_n^{M,N}(t) = \tilde{\varphi}_n^M(t)$ if $t \leqq S_N(\tilde{\varphi}_n^M)$. These results imply (3.45). Next we can obtain (3.46) because it holds that

$$\sup_{t \in [0, S_N(\tilde{\varphi}^{M,N})]} |\tilde{\varphi}^{M,N}(t)| \leqq N \qquad \text{and} \qquad \sup_{t \in [0, S_N(\tilde{\varphi}^M)]} |\tilde{\varphi}^M(t)| \leqq N$$

provided that $N > 2\max\{\|C\|(M+1), |x_0|\}$ as above and we have the uniqueness of solution of the stochastic differential equation (3.44) (or that of martingale problem for $\tilde{\mathcal{L}}^M$ which we define by (3.14) for $\tilde{C}$ instead of $\tilde{C}_N$).

(3.47) is an easy consequence from the fact that

$$\{\tilde{\varphi}\,;\, S_N(\tilde{\varphi}) \leqq t\} = \{\tilde{\varphi}\,;\, \sup_{u \in [0,t]} |\tilde{\varphi}(u)| \geqq N/2\}$$

and $\tilde{P}^M$ is a probability measure on the Polish space D_{d+e}. □

3-6. Completion of the proof of Theorem 2. As a preparation for removing the truncation in Proposition 3, we first show the following.

PROPOSITION 4. *For the sequence* $\{\tilde{\varphi}^M\}_M$ *defined by* (3.44), *it holds:*

$$\tilde{\varphi}^M \xrightarrow{\mathcal{L}} \tilde{\varphi} \quad \text{in } D_{d+e} \text{ as } M\in C(\nu)\to\infty, \tag{3.48}$$

where $\tilde{\varphi}$ *is the process defined by* (2.12).

PROOF OF PROPOSITION 4. First we show the tightness of $\{\tilde{\varphi}^M\}_M$. To this end, by Bass [**1**] Proposition 3.2 (which was originated by Stroock [**9**], Theorem A.1), it suffice to check that for each $f\in C_0^2(\boldsymbol{R}^{d+e})$ there exists a constant C_f (depending only on $\|f\|_2=:\|f\|+\|f^{(1)}\|+\|f^{(2)}\|$) such that $f(\alpha(t))-f(\alpha(0))-C_f t$ is a supermartingale with respect to $\tilde{P}^M$. But it is easily seen if we take

$$C_f=\left\{\|\tilde{C}\|^2\|V\|+\|\tilde{b}\|+\|\tilde{C}\|\int_{|z|\leq 1}|z|^2\nu(dz)\right\}\|f\|_2+2\|f\|\nu(|z|>1).$$

Next, by Proposition 3, we have:

$$E\left[\left\{f(\tilde{\varphi}^M(t))-f(\tilde{\varphi}^M(s))-\int_s^t \tilde{\mathcal{L}}^M f(\tilde{\varphi}^M(u))du\right\}\Psi(\tilde{\varphi}^M(u_1),\cdots,\tilde{\varphi}^M(u_m))\right]=0,$$

for all $s,\ t\in J(\tilde{\varphi})^c$, $s<t$, $m\in\boldsymbol{N}$, $u_i\in J(\tilde{\varphi})^c$, $0\leqq u_1\leqq\cdots\leqq u_m\leqq s$, and bounded continuous functions $\Psi:(\boldsymbol{R}^{d+e})^m\to\boldsymbol{R}$. Take $M\to\infty$ in it, then we can see that any weak limiting measure of $\{\tilde{\varphi}^M\}_M$ is a solution of the martingale problem for $\tilde{\mathcal{L}}$ which we define by (3.14) for $\tilde{C}$ and $M=\infty$ instead of $\tilde{C}_N$ and $M<\infty$, respectively. By the uniqueness which also follows from the one of solution of (2.12), we obtain the conclusion (3.48). □

We give another preparatory result as follows. For $K\in C(\nu)$, define

$$t_K(\tilde{\varphi})=t_K(\varphi,\xi)=\inf\{t;\ |\Delta\xi(t)|\geqq K\}$$

where $\tilde{\varphi}=(\varphi,\xi)\in D_{d+e}$, $\varphi\in D_d$, and $\xi\in D_e$. Since if $\tilde{\varphi}_n\to\tilde{\varphi}$ in D_{d+e} then we have $\sup_{u\in[0,t]}|\Delta\tilde{\varphi}_n(u)|\to\sup_{u\in[0,t]}|\Delta\tilde{\varphi}(u)|$ for all $t\in J(\tilde{\varphi})^c$, we can see that $t_K(\tilde{\varphi})$ is a lower semi-continuous function from D_{d+e} to $[0,\infty]$. For a sequence $\{T_K\in[0,\infty)\}_K$ such that $T_K\uparrow\infty$ as $K\uparrow\infty$, set

$$\tau_K(\tilde{\varphi})=t_K(\tilde{\varphi})\wedge T_K.$$

Then we obtain the following lemma which is an obvious result from the definition of $\tilde{\varphi}_n$, $\tilde{\varphi}_n^M$, and τ_K.

LEMMA 3.8. *Take* K *such as* $1<K<M-1$ *for sufficiently large* M, *then it holds:*

(1)

$$\tau_K(\tilde{\varphi}_n)\leqq\tau_K(\tilde{\varphi}_n^M), \tag{3.49}$$

(2)

$$\{\tau_K(\tilde{\varphi}_n)<\tau_K(\tilde{\varphi}_n^M)\} \subset \bigcup_{k=1}^{[nT_K]}\{|\xi_{n,k}|>M\}, \tag{3.50}$$

(3)

$$\tilde{\varphi}_n(t)=\tilde{\varphi}_n^M(t) \qquad \textit{if } t<\tau_K(\tilde{\varphi}_n). \tag{3.51}$$

At last, we are ready to complete our proof of Theorem 2, that is, to show (2.11). Let t be any positive number and Φ be any bounded continuous $\mathcal{D}_t^0$-measurable function. Take M and K such as $t<T_K$ and $K-1<M$. Then, by (3.51), we have:

$$\begin{aligned} E[\Phi(\tilde{\varphi}_n)] &= E[\Phi(\tilde{\varphi}_n);\tau_K(\tilde{\varphi}_n)\leqq t]+E[\Phi(\tilde{\varphi}_n);\tau_K(\tilde{\varphi}_n)>t] \\ &= E[\Phi(\tilde{\varphi}_n);\tau_K(\tilde{\varphi}_n)\leqq t]+E[\Phi(\tilde{\varphi}_n^M);\tau_K(\tilde{\varphi}_n)>t] \\ &= E[\Phi(\tilde{\varphi}_n);\tau_K(\tilde{\varphi}_n)\leqq t]-E[\Phi(\tilde{\varphi}_n^M);\tau_K(\tilde{\varphi}_n)\leqq t]+E[\Phi(\tilde{\varphi}_n^M)]. \end{aligned} \tag{3.52}$$

We first estimate $P(\tau_K(\tilde{\varphi}_n)\leqq t)$. By (3.49) and (3.50), it follows that

$$\begin{aligned} P(\tau_K(\tilde{\varphi}_n)\leqq t) &= P(\tau_K(\tilde{\varphi}_n)=\tau_K(\tilde{\varphi}_n^M)\leqq t)+P(\tau_K(\tilde{\varphi}_n)\leqq t \text{ and } \tau_K(\tilde{\varphi}_n)<\tau_K(\tilde{\varphi}_n^M)) \\ &\leqq P(\tau_K(\tilde{\varphi}_n^M)\leqq t)+P(\tau_K(\tilde{\varphi}_n)<\tau_K(\tilde{\varphi}_n^M)) \\ &\leqq P(\tau_K(\tilde{\varphi}_n^M)\leqq t)+[nT_K]P(|\xi_{n,1}|>M). \end{aligned}$$

By the lower semicontinuity of t_K and Proposition 3, we obtain

$$\limsup_{n\to\infty} P(\tau_K(\tilde{\varphi}_n^M)\leqq t) \leqq \tilde{P}^M(\tau_K\leqq t).$$

On the other hand, by (2.2), we obtain

$$\lim_{n\to\infty}[nT_K]P(|\xi_{n,1}|>M)=T_K\nu(|z|>M).$$

Thus we have

$$\limsup_{n\to\infty} P(\tau_K(\tilde{\varphi}_n)\leqq t) \leqq \tilde{P}^M(\tau_K\leqq t)+T_K\nu(|z|>M). \tag{3.53}$$

In the sequel, we denote by $\tilde{P}$ the law of $\tilde{\varphi}$ of (2.12). By (3.52), (3.53), and Proposition 3, we obtain:

$$\begin{aligned} &\limsup_{n\to\infty}|E[\Phi(\tilde{\varphi}_n)]-E^{\tilde{P}}[\Phi]| \\ &\leqq \limsup_{n\to\infty}|E[\Phi(\tilde{\varphi}_n)]-E[\Phi(\tilde{\varphi}_n^M)]| \\ &\quad +\limsup_{n\to\infty}|E[\Phi(\tilde{\varphi}_n^M)]-E^{\tilde{P}^M}[\Phi]|+|E^{\tilde{P}^M}[\Phi]-E^{\tilde{P}}[\Phi]| \\ &\leqq 2\|\Phi\| \limsup_{n\to\infty} P(\tau_K(\tilde{\varphi}_n)\leqq t)+|E^{\tilde{P}^M}[\Phi]-E^{\tilde{P}}[\Phi]| \\ &\leqq 2\|\Phi\|\{\tilde{P}^M(\tau_K\leqq t)+T_K\nu(|z|>M)\}+|E^{\tilde{P}^M}[\Phi]-E^{\tilde{P}}[\Phi]|. \end{aligned}$$

Since we can take $M\in C(\nu)$ as large as we hope, by Proposition 4, it holds:

$$\limsup_{n\to\infty}|E[\Phi(\tilde{\varphi}_n)]-E^{\tilde{P}}[\Phi]| \leqq 2\|\Phi\|\tilde{P}(\tau_K\leqq t) \leqq 2\|\Phi\|\tilde{P}(\sup_{u\in[0,t]}|\Delta\xi(u)|\geqq K)$$

for all K such that $T_K>t$. Since we can see that under the measure $\tilde{P}$ the canonical process ξ is a Lévy process with the Lévy measure ν, it holds:

$$\tilde{P}(\sup_{u\in[0,t]}|\Delta\xi(u)|\geqq K)=1-\exp\{-t\nu(|z|\geqq K)\} \longrightarrow 0 \qquad \text{as } K\to\infty .$$

From this, we obtain:

$$\lim_{n\to\infty} E[\Phi(\tilde{\varphi}_n)] = E^{\tilde{P}}[\Phi] . \tag{3.54}$$

Since t is arbitrary, (3.54) holds also for all bounded continuous functions Φ. Thus we have established Theorem 2. □

§ 4. Example.

In this section, we give an example of the array of random variables which satisfies the conditions of our theorem.

Let $F_0(x)$ be a $[0, 1]$-valued smooth function defined on $\boldsymbol{R}^1$ which satisfies for some $0<\alpha<2$

$$\begin{cases} 1-F_0(x)\sim x^{-\alpha} & \text{as } x\to+\infty \\ F_0(x)\sim|x|^{-\alpha} & \text{as } x\to-\infty \end{cases}$$

where $F(x)\sim G(x)$ denotes $F(x)/G(x)\to 1$. Moreover, we assume that the derivative $F_0'(x)$ is an even function. Define a probability measure π on $\boldsymbol{R}^1$ by $\pi(dx) =F'(x)dx$ where we put

$$F(x) = \begin{cases} F_0(C_1^{-1/\alpha}x) & \text{on } x>0 \\ F_0(C_2^{-1/\alpha}x) & \text{on } x<0 \end{cases}$$

for some non-negative constant C_1 and C_2 such that $C_1C_2>0$.

For a constant $\beta\in(0, 1)$, put $p(x, y)=\beta p(x)p(y)+1$ where

$$p(x) = \begin{cases} \sin(C_1^{-1/\alpha}x) & \text{on } x>0 \\ \sin(C_2^{-1/\alpha}x) & \text{on } x<0 . \end{cases}$$

Since $p(x, A)=:\int_A p(x, y)\pi(dy)$ $(A\in\mathcal{B}(\boldsymbol{R}^1))$ defines a probability measure for each $x\in\boldsymbol{R}^1$, we can define a stationary Markov process $\{\xi_k ; k\in\boldsymbol{N}\}$ which has the transition probability $p(x, dy)$ and the initial law $\pi(dx)$. Then, we see that this Markov process satisfies the conditions (i) and (ii) of Theorem 5 in Blum-Hanson-Koopman [2]. Hence, by the results of [2] we can conclude that $\{\xi_k\}$ is a strongly uniform mixing process with the rate function $\phi(n)=\beta^n$.

Next, we define an array $\{\xi_{n,k}\}$ by

$$\xi_{n,k} = \frac{\xi_k}{n^{1/\alpha}} \tag{4.1}$$

for the Markov process $\{\xi_k\}$ constructed as above. Then, it holds:

$$n\int_{R^1} f(z)P(\xi_{n,1}\in dz) \longrightarrow \int_{R^1} f(z)\nu_0(dz) \qquad \text{as } n\to\infty$$

for all $f\in C_0(\boldsymbol{R}^1)$, where

$$\nu_0(dz) = \alpha\left\{C_1 I_{(0,+\infty)}(z)\frac{dz}{z^{1+\alpha}} + C_2 I_{(-\infty,0)}(z)\frac{dz}{|z|^{1+\alpha}}\right\}. \tag{4.2}$$

Moreover, it is easy to see that for $\beta\in(\alpha, 2]$

$$\lim_{\delta\to 0}\limsup_{n\to\infty} nE[|\xi_{n,1,\delta}|^{\beta}] = 0.$$

Now, consider the stochastic difference equation (1.1) for the array of (4.1). Then, by Theorem 1, the sequence $\{\varphi_n(t)=\varphi_{n,[nt]}\}_{n\in N}$ converges in law to the solution of

$$\varphi(t) = x_0 + \int_0^{t+}\int_{|z|\le 1} C(\varphi(u-))z\tilde{N}_p(dudz) + \int_0^{t+}\int_{|z|>1} C(\varphi(u-))zN_p(dzdu)$$

where $\{p(t)\}$ is the stationary Poisson point process on $\boldsymbol{R}^1\setminus\{0\}$ with the intensity measure ν_0 of (4.2). Hence, in this example the limit is a pure jump process.

References

[1] R.F. Bass, Uniqueness in law for pure jump processes, Probab. Theory and Related Fields, **79** (1988), 271-287.

[2] J.R. Blum, D.L. Hanson, and L.H. Koopman, On the strong law of large numbers for a class of stochastic processes, Z. Warh. verw. Geb., **2** (1963), 1-11.

[3] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland/Kodansha, 1981.

[4] J. Jacod and A.N. Shriyaev, Limit Theorems for Stochastic Processes, Springer, 1987.

[5] H. Kesten and G.C. Papanicolau, A limit theorem for turbulent diffusion, Comm. Math. Phys., **65** (1979), 97-128.

[6] R.Z. Khas'minskii, A limit theorem for the solution of differential equations with random right-hand sides, Theory Probab. Appl., **11** (1966), 390-406.

[7] J.D. Samur, Convergence of sums of mixing triangular arrays of random vectors with stationary rows, Ann. Probab., **12** (1984), 390-426.

[8] J.D. Samur, On the invariance principle for stationary φ-mixing triangular arrays with infinitely divisible limits, Probab. Theory Related Fields, **75** (1987), 245-259.

[9] D.W. Stroock, Diffusion processes associated with Lévy generators, Z. Warh. verw. Geb., **32** (1975), 209-244.

[10] D.W. Stroock and S.R.S. Varadhan, Multidimensional Diffusion Processes, Springer, 1979.

[11] H. Watanabe, Diffusion approximations of some difference equations, II, Hiroshima Math. J., **14** (1984), 15-34.

Tsukasa FUJIWARA
Department of Applied Science
Faculty of Engineering
Kyushu University 36
6-10-1, Hakozaki, Higashi-ku
Fukuoka 812
Japan