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Periodic behavior of solutions to parabolic-elliptic free boundary problems

Dedicated to Professor Niro Yanagihara on his 60th birthday

By Nobuyuki KENMOCHI and Masahiro KUBO

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1. Introduction.

In the papers [**14**, **15**, **16**], parabolic-elliptic variational inequalities, formulated in a domain $\Omega\subset \boldsymbol{R}^N$ $(N\geqq 1)$, with some time-dependent constraints have been considered. The existence and uniqueness theorems were there established with some results on asymptotic stability of solutions. In some cases the constraint is an obstacle imposed at a time-dependent part of the boundary Γ of Ω. We deal here with a mathematical model of a parabolic-elliptic free boundary problem which arises in the flow with saturation and unsaturation in porous media, when the water level of the reservoir changes periodically in time. For related papers concerning the analysis of saturated-unsaturated flows in porous media we refer to [**1**, **2**, **3**, **5**, **6**, **7**, **8**, **9**, **10**, **11**, **17**, **18**].

The problem is formulated as follows. Assume that for each $t\in \boldsymbol{R}$, Γ consists of disjoint three parts $\Gamma_D(t)$, $\Gamma_N(t)$ and $\Gamma_U(t)$, i.e., $\Gamma=\Gamma_D(t)\cup\Gamma_N(t)\cup\Gamma_U(t)$ (disjoint union). Correspondingly, for a given interval $J=(t_0, \infty)$ we define non-cylindrical subsets $\Sigma_\nu(J)$, $\nu=D, N, U$, of $\Sigma(J)=J\times\Gamma$ by

$$\Sigma_\nu(J)=\bigcup_{t\in J}\{t\}\times\Gamma_\nu(t), \qquad \nu=D, N, U.$$

The problem is described in the following system:

$$\rho(v)_t-\Delta v=f \qquad \text{in } Q(J)=J\times\Omega, \tag{1.1}$$

$$v=g_D \qquad \text{on } \Sigma_D(J), \tag{1.2a}$$

$$\partial_n v=q_N \qquad \text{on } \Sigma_N(J), \tag{1.2b}$$

$$v\leqq g_U,\quad \partial_n v\leqq q_U,\quad (\partial_n v-q_U)(v-g_U)=0 \qquad \text{on } \Sigma_U(J) \tag{1.2c}$$

and the initial condition

$$\rho(v(t_0, \cdot))=u_0 \qquad \text{for } x\in\Omega,$$

where v is the unknown, $\partial_n v$ denotes the outward normal derivative of v on Γ, $\rho: \boldsymbol{R}\to\boldsymbol{R}$ is a non-decreasing Lipschitz continuous function and f, g_D, q_N, g_U, q_U are given data. The corresponding periodic problem is the system consisting of (1.1), (1.2) and the periodic condition

$$v(t+T, x) = v(t, x) \qquad \text{for } (t, x)\in Q(J),$$

where T is a given positive number.

In this literature we know some results (cf. Kenmochi-Kubo [**12**]) about the global boundedness of solutions and the existence of periodic solutions. However we have noticed no further investigation on the periodic behavior of solutions. In this paper, under suitable assumptions on the periodicity of the data we shall show that there exists one and only one periodic (weak) solution and that the periodic solution is asymptotically stable and is characterized by the global boundedness of the trajectory.

NOTATIONS. In general, for a (real) Banach space V we denote by $|\cdot|_V$ the norm in V, by V^* the dual space and by $(\cdot, \cdot)_{V^*,V}$ the natural duality pairing between V^* and V. In particular, if V is a Hilbert space, then we denote by $(\cdot, \cdot)_V$ the inner product in V.

We use the symbol "$\to$" or "lim" to indicate strong convergence in various Banach spaces, unless otherwise stated.

Let V be a Hilbert space and φ be a proper (i.e. $-\infty<\varphi\leqq\infty$, $\varphi\not\equiv\infty$ on V) l.s.c. (lower semicontinuous) and convex function on V. Then, we denote by $\partial\varphi$ the subdifferential of φ in V. The subdifferential $\partial\varphi$ is a (multi-valued) operator in V which is defined by

$$\partial\varphi(z) = \{z^*\in V;\ (z^*, y-z)_V\leqq\varphi(y)-\varphi(z) \text{ for all } y\in V\}.$$

The domains of φ and $\partial\varphi$ are respectively the sets

$$D(\varphi) = \{z\in V;\ \varphi(z)<\infty\}$$

and

$$D(\partial\varphi) = \{z\in V;\ \partial\varphi(z)\neq\emptyset\}.$$

We refer to Brézis [**4**] for details of subdifferentials and for general theory of monotone operators.

Throughout this paper, let Ω be a bounded domain in $\boldsymbol{R}^N$ $(N\geqq 1)$ with regular boundary $\Gamma=\partial\Omega$, and employ the following conventions:

$$Q = \boldsymbol{R}\times\Omega, \qquad \Sigma = \boldsymbol{R}\times\Gamma;$$

$$u\vee v = \sup\{u, v\}, \qquad u\wedge v = \inf\{u, v\},$$

$$u^+ = u\vee 0, \qquad u^- = -(u\wedge 0) \qquad \text{for } u, v\in L^1(\Omega);$$

$$H = L^2(\Omega), \qquad (\cdot, \cdot) = (\cdot, \cdot)_H;$$

$$X = H^1(\Omega), \qquad a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx .$$

Sometimes, for $u, v \in L^1(\Omega)$, we write "$u \geqq v$ on Ω" for "$u \geqq v$ a.e. on Ω". Also, for the trace space $H^{1/2}(\Gamma)$ of $H^1(\Omega)$ and its dual $H^{-1/2}(\Gamma)$ we denote by $\langle \cdot, \cdot \rangle_\Gamma$ the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, i.e.

$$\langle \cdot, \cdot \rangle_\Gamma = (\cdot, \cdot)_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} .$$

We denote by $d\Gamma$ the usual surface measure on Γ.

2. Variational formulation and main results.

The problem is discussed under the following assumptions (A.1), (A.2) and (A.3) on ρ, $\Gamma_\nu(t)$ ($\nu = D, N, U$) and functions g_D, q_N, g_U and q_U on Σ:

(A.1) $\rho(\cdot): \boldsymbol{R} \to \boldsymbol{R}$ is a non-decreasing Lipschitz continuous function.

(A.2) (*geometric condition*) For each $t \in \boldsymbol{R}$, there exists a C^1-diffeomorphism $\Theta(t, \cdot) = (\theta^1(t, \cdot), \cdots, \theta^N(t, \cdot)): \bar{\Omega} \to \bar{\Omega}$ such that

(a) $\Theta(0, \cdot)$ is the identity on $\bar{\Omega}$;

(b) $\Gamma_\nu(t) = \Theta(t, \Gamma_\nu(0))$, $\nu = D, N, U$, where for all $t \in \boldsymbol{R}$, $\Gamma_D(t)$, $\Gamma_N(t)$, $\Gamma_U(t)$ are mutually disjoint measurable subsets of Γ and $\Gamma = \Gamma_D(t) \cup \Gamma_N(t) \cup \Gamma_U(t)$;

(c) $(\partial/\partial x_j)\theta^i$, $(\partial/\partial t)\theta^i$, $(\partial^2/\partial t \partial x_j)\theta^i$ are continuous on $\boldsymbol{R} \times \bar{\Omega}$ for $i, j = 1, 2, \cdots, N$;

(d) $\mathrm{meas}_\Gamma\{\bigcap_{t \in \boldsymbol{R}} \Gamma_D(t)\} > 0$.

(A.3) (*compatibility condition*) There exist functions g, $\bar{q}_N$, $\bar{q}_U$ in $W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}; X) \cap L^\infty_{\mathrm{loc}}(\boldsymbol{R}; H^2(\Omega))$ such that for all $t \in \boldsymbol{R}$

(a) $g(t, x) = \begin{cases} g_D(t, x) & \text{for a.e. } x \in \Gamma_D(t), \\ g_U(t, x) & \text{for a.e. } x \in \Gamma_U(t), \end{cases}$

(b) $\bar{q}_N(t, x) = q_N(t, x)$ for a.e. $x \in \Gamma_N(t)$,

(c) $\bar{q}_U(t, x) = q_U(t, x)$ for a.e. $x \in \Gamma_U(t)$.

Now we give a weak formulation of the problem in variational sense. To this end, we define a closed convex subset $K(t)$ of X for each $t \in \boldsymbol{R}$ by

$$K(t) = \{z \in X; z = g(t) \text{ a.e. on } \Gamma_D(t),\ z \leqq g(t) \text{ a.e. on } \Gamma_U(t)\}$$

DEFINITION 2.1. (i) Let $J = [t_0, t_1]$, $-\infty < t_0 < t_1 < \infty$, and $f \in L^2(J; H)$. Then a function $v: J \to X$ is called *a weak solution of* $P(f)$ *on* J, if $v \in L^2(J; X)$, $v(t) \in K(t)$ for a.e. $t \in J$, $\rho(v) \in W^{1,2}(J; H)$ and

$$(\rho(v)'(t)-f(t),\ v(t)-z)+a(v(t),\ v(t)-z) \tag{2.1}$$
$$\leqq\int_{\Gamma_{N(t)}} q_N(t,\ x)(v(t,\ x)-z(x))d\Gamma+\int_{\Gamma_{U(t)}} q_U(t,\ x)(v(t,\ x)-z(x))d\Gamma,$$
$$\text{for any } z\in K(t) \text{ and a.e. } t\in J,$$

where $\rho(v)'=\rho(v)_t$ and $f(t)=f(t,\ \cdot)$.

(ii) Let J' be any interval in $\boldsymbol{R}$ and $f\in L^2_{\mathrm{loc}}(J';H)$. Then a function $v: J'\to X$ is called *a weak solution of* $P(f)$ *on* J', if v is a weak solution of $P(f)$ on every compact subinterval J of J' in the above sense.

(iii) Let $J'=[t_0,\ t_1]$ or $[t_0,\ \infty)$, $f\in L^2_{\mathrm{loc}}(J';H)$ and $u_0\in H$. Then a function $v: J'\to X$ is *a weak solution of the Cauchy problem* $CP(f,\ u_0)$ *on* J' *for* $P(f)$ *associated with the initial condition*

$$\rho(v(t_0,\ \cdot))=u_0 \qquad in\ H, \tag{2.2}$$

if v is a weak solution of $P(f)$ on J' and (2.2) is satisfied.

(iv) Let T be a positive number and $f\in L^2_{\mathrm{loc}}(\boldsymbol{R};H)$. Then, a function $v: \boldsymbol{R}\to X$ is called *a* T*-periodic weak solution of* $P(f)$, if it is a weak solution of $P(f)$ on $\boldsymbol{R}$ and

$$v(t+T,\ \cdot)=v(t,\ \cdot) \quad \text{in } X \qquad \text{for any } t\in\boldsymbol{R}.$$

REMARK 2.1. As was seen in Kenmochi-Pawlow [**15**; Lemma 3.1], the variational inequality (2.1) implies that

$$\rho(v)_t-\Delta v=f \qquad \text{in } Q \text{ (in the distribution sense)},$$

$$\langle\partial_n v(t),\ v(t)-g(t)\rangle_\Gamma=\int_{\Gamma_{N(t)}} q_N(t,\ x)(v(t,\ x)-g(t,\ x))d\Gamma \tag{2.3}$$
$$+\int_{\Gamma_{U(t)}} q_U(t,\ x)(v(t,\ x)-g(t,\ x))d\Gamma, \quad \text{for a.e. } t\in J,$$

and

$$\langle\partial_n v(t),\ \eta\rangle_\Gamma\geqq\int_{\Gamma_{N(t)}} q_N(t,\ x)\eta(x)d\Gamma+\int_{\Gamma_{U(t)}} q_U(t,\ x)\eta(x)d\Gamma, \tag{2.3$'$}$$
$$\text{for any } \eta\in K_0(t) \text{ and a.e. } t\in J,$$

where

$$K_0(t)=\{\eta\in X;\ \eta=0 \text{ a.e. on } \Gamma_D(t),\ \eta\leqq 0 \text{ a.e. on } \Gamma_U(t)\}. \tag{2.4}$$

We easily understand that the boundary conditions (1.2a), (1.2b) and (1.2c) are satisfied in the sense of (2.3), (2.3)$'$ and (2.3)$''$ mentioned below:

$$v(t)\in K(t) \qquad \text{for any } t\in J. \tag{2.3$''$}$$

The main results are stated in the following theorems.

THEOREM 2.1. *Suppose that* (A.1), (A.2) *and* (A.3) *hold together with the following condition* (A.4) *for a positive number* T*:*

(A.4) $\Gamma_\nu(t+T)=\Gamma_\nu(t)$, $\nu=D, N, U$, *for any* $t\in\boldsymbol{R}$, *and* $g_D(t+T, x)=g_D(t, x)$, $q_N(t+T, x)=q_N(t, x)$, $g_U(t+T, x)=g_U(t, x)$, $q_U(t+T, x)=q_U(t, x)$ *for* a.e. $(t, x)\in\Sigma$.

Further, let $f\in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}; H)$ *and suppose*

$$f(t+T, x)=f(t, x) \qquad \textit{for } \text{a.e. } (t, x)\in Q\,. \tag{2.5}$$

Then there exists one and only one weak solution ω *of* $P(f)$ *on* $\boldsymbol{R}$ *such that the trajectory* $\{\omega(t)\,; t\in\boldsymbol{R}\}$ *is bounded in* H, *and moreover this solution* ω *is a unique* T*-periodic weak solution of* $P(f)$ *on* $\boldsymbol{R}$.

THEOREM 2.2. *Assume that all the assumptions of Theorem* 2.1 *are satisfied. Then, for any weak solution* v *of* $P(f)$ *on* $[t_0, \infty)$, $t_0\in\boldsymbol{R}$, *we have*

$$\rho(v(t, \cdot))-\rho(\omega(t, \cdot))\longrightarrow 0 \qquad \textit{in } H \textit{ and weakly in } X \textit{ as } t\to\infty\,, \tag{2.6}$$

where ω *is the* T*-periodic weak solution of* $P(f)$ *on* $\boldsymbol{R}$.

In proving the theorems, the key tool is the lattice structure of the set of all solutions to $P(f)$, which is observed in the next section.

3. Known results and some lemmas.

First let us recall some results which are directly obtained from the abstract theory of evolution equations of the form:

$$\text{(E)} \qquad \rho(v)'(t)+\partial\varphi^t(v(t))\ni f(t)\,,$$

where φ^t is a proper l.s.c. convex function on H and $\partial\varphi^t(\cdot)$ is its subdifferential in H.

As was seen in [**15**], system $\{(1.1), (1.2)\}$ can be reformulated in the form (E), where for each $t\in\boldsymbol{R}$, φ^t is given by

$$\varphi^t(z)=\begin{cases} \dfrac{1}{2}|\nabla z|_H^2-\displaystyle\int_{\Gamma_N(t)} q_N(t, x)z(x)d\Gamma-\int_{\Gamma_U(t)} q_U(t, x)z(x)d\Gamma & \text{for } z\in K(t), \\ \infty\,, \quad \text{otherwise.} \end{cases} \tag{3.1}$$

Under assumptions (A.1), (A.2) and (A.3), the following facts [A], $\cdots$, [E] are due to [**12, 14, 15, 16**].

[A] (cf. [**15**; Theorem 2.1], [**16**; Lemma 2.1]) Let $J=[t_0, t_1]$ be any compact interval in $\boldsymbol{R}$, $f\in W^{1,1}(J, H)$ and u_0 be any function in X such that $\rho(v_0)=u_0$ for some $v_0\in K(t_0)$. Then $CP(f, u_0)$ has one and only one weak solution v on J such that $v(t)\in K(t)$ for any $t\in J$, $v\in L^\infty(J; H)$ and

$$\varphi^t(v(t)) = \min_{\substack{z\in K(t)\\ \rho(z)=\rho(v(t,\cdot))}} \varphi^t(z) \qquad \text{for any } t\in J.$$

[B] (cf. [**14**; Theorem 1.2], [**15**; Lemma 5.4]) Let J' be any interval in $\boldsymbol{R}$, $f_1, f_2\in L^2_{\mathrm{loc}}(J', H)$, and let v_1 and v_2 be any weak solutions of $P(f_1)$ and $P(f_2)$ on J', respectively. Then,

$$|[\rho(v_1(t))-\rho(v_2(t))]^+|_{L^1(\Omega)} \leqq |[\rho(v_1(s))-\rho(v_2(s))]^+|_{L^1(\Omega)} + \int_s^t |[f_1(\tau)-f_2(\tau)]^+|_{L^1(\Omega)} d\tau \qquad \text{for any } s,\ t\in J' \text{ with } s\leqq t.$$

[C] (cf. [**14**; Theorem 1.3, 1.4]) Let $J=[t_0, t_1]$, $\{f_n\}$ be a bounded sequence in $W^{1,1}(J; H)$ and $\{v_{0,n}\}$ be a sequence in $K(t_0)$ which is bounded in X. Let v_n be the weak solution of $CP(f_n, u_{0,n})$, with $u_{0,n}=\rho(v_{0,n})$, on J for each n. If $f_n\to f$ in $L^2(J; H)$ and $u_{0,n}\to u_0$ in H. Then v_n converges to the weak solution v of $CP(f, u_0)$ on J as $n\to\infty$ in the sense that

$$\rho(v_n) \longrightarrow \rho(v) \qquad \text{in } C(J; H) \text{ and weakly in } W^{1,2}(J; H)$$

and

$$v_n \longrightarrow v \qquad \text{weakly* in } L^\infty(J; X)$$

as $n\to\infty$.

Since ρ is non-decreasing on $\boldsymbol{R}$, the inverse ρ^{-1} is a maximal monotone graph in $\boldsymbol{R}\times\boldsymbol{R}$, where ρ^{-1} is defined as a multi-valued function in general. We now take a proper l.s.c. convex function j on $\boldsymbol{R}$ such that $\partial j=\rho^{-1}$, where ∂j is the subdifferential of j in $\boldsymbol{R}$; note here that by the Lipschitz continuity of ρ we can take j so as to satisfy

$$j(r) \geqq C(j)|r|^2, \qquad r\in\boldsymbol{R},$$

for some positive constant $C(j)$. Besides we define the proper l.s.c. convex function $\hat{j}$ on H by

$$\hat{j}(z) = \begin{cases} \int_\Omega j(z(x))dx & \text{if } z\in H \text{ and } j(z)\in L^1(\Omega), \\ \infty, & \text{otherwise.} \end{cases}$$

Clearly,

$$\hat{j}(z) \geqq C(j)|z|^2_H, \qquad r\in\boldsymbol{R}.$$

[D] (cf. [**12**; Theorem 1.1], [**14**; Theorem 1.1]) In addition to (A.1), (A.2) and (A.3), suppose that (A.4) holds together with (2.5) for $f\in W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}; H)$ and $T>0$. Let J' be any interval of the form $[t_0, \infty)$ in $\boldsymbol{R}$ and v be a weak solution of $P(f)$ on J. Then v has the bounds

$$\sup_{t\geqq t_0} \hat{j}(\rho(v(t))) \leqq L(\hat{j}(\rho(v(t_0))),\ |f|_{W^{1,1}(t_0,t_0+T; H)})$$

and

$$\sup_{s \leq t \leq s+T} (t-s)|v(t)|_X^2 \leqq L(\hat{j}(\rho(v(s))), |f|_{W^{1,1}(t_0, t_0+T; H)}) \qquad \text{for any } s \geqq t_0 ,$$

where $L(\cdot, \cdot): \boldsymbol{R}_+ \times \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is a non-decreasing function; of course L depends on the quantities in conditions (A.1), $\cdots$, (A.4), but not on the initial time t_0.

[E] (cf. [**12**; Theorem 3.2]) Under the same assumptions as in [D], $P(f)$ has at least one T-periodic weak solution on $\boldsymbol{R}$.

We show below that the set of all solutions to $P(f)$ has an order structure.

LEMMA 3.1. *Let* $f \in W_{\mathrm{loc}}^{1,1}(\boldsymbol{R}; H)$ *and* $T>0$, *and assume that* (A.1), $\cdots$, (A.4) *hold. Let* $J'=[t_0, \infty)$ *or* $\boldsymbol{R}$, *and let* v_1, v_2 *be any solutions of* $P(f)$ *on* J'. *Assume that* $v_1, v_2 \in L^\infty(J'; H)$. *Then there exist two weak solutions* v_*, v^* *of* $P(f)$ *on* J' *such that*

$$v_*, v^* \in L^\infty(J'; H)$$

and

$$v_* \leqq v_1 \wedge v_2, \qquad v_1 \vee v_2 \leqq v^* \qquad \text{on } J' \times \Omega .$$

PROOF. We give a proof of the lemma only in the case of $J'=\boldsymbol{R}$. Let $z_n=v_1(-n, \cdot) \vee v_2(-n, \cdot)$ for each $n=1, 2, \cdots$. Then, by assumption, $\{z_n\}$ and $\{\rho(z_n)\}$ are bounded in H. Since

$$z_n \in \rho^{-1}(\rho(z_n)) = \partial \hat{j}(\rho(z_n)) ,$$

we see

$$\hat{j}(\rho(z_n)) \leqq (z_n, \rho(z_n)-z_0)+\hat{j}(z_0)$$

for a function $z_0 \in D(\hat{j})$. From this inequality it follows that $\{\hat{j}(\rho(z_n))\}$ is bounded. Therefore, denoting by w_n the weak solution of $CP(f, \rho(z_n))$ on $[-n, \infty)$, we see from [B] and [D] that

$$v_1 \vee v_2 \leqq w_n \qquad \text{on } (-n, \infty) \times \Omega ,$$

and

$$|w_n|_{L^\infty(-n+T, \infty; X)} \leqq L_1$$

for a positive constant L_1 independent of n. By the last estimates for w_n, $\{w_n(-m+T)\}_{n \geq m}$ and $\{\rho(w_n)(-m+T)\}_{n \geq m}$ are bounded in X for each $m=1, 2, \cdots$. Since $X \subset H$ with compact injection, by a standard diagonal argument it is possible to select a subsequence $\{n_k\}$ of $\{n\}$ such that $\{\rho(w_{n_k})(-m+T)\}_{k \geq 1}$ converges in H for each $m=1, 2, \cdots$. Therefore, the result [C] implies that for each m, there is a weak solution $w^{(m)}$ of $P(f)$ on $[-m+T, \infty)$ and

$$\rho(w_{n_k}) \longrightarrow \rho(w^{(m)}) \qquad \text{in } C(J; H) \text{ and weakly in } W^{1,2}(J; H)$$

as well as

$$w_{n_k} \longrightarrow w^{(m)} \qquad \text{weakly* in } L^\infty(J; X)$$

as $k\to\infty$ for every compact interval J in $[-m+T, \infty)$. Clearly, $w^{(m)}=w^{(m')}$ on $[-m+T, \infty)$, if $m'\geqq m$. Now, define a function $v^*: \boldsymbol{R}\to H$ by $v^*=w^{(m)}$ on $[-m+T, \infty)$ for $m=1, 2, \cdots$. Then, it is easily seen that v^* is a weak solution of $P(f)$ on $\boldsymbol{R}$, $v^*\in L^\infty(\boldsymbol{R}; X)$ and $v_1\vee v_2\leqq v^*$ on $Q=\boldsymbol{R}\times\Omega$. Similarly, the existence of v_* is shown. q. e. d.

LEMMA 3.2. *Let $f\in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}; H)$ and $T>0$, and suppose that* (A.1), $\cdots$, (A.4) *and* (2.5) *hold. Then we have:*

(i) *Let $t_0\in\boldsymbol{R}$ and v be a weak solution of $P(f)$ on $[t_0, \infty)$. Then $v\in L^\infty(t_0+\delta; X)$ for every positive constant δ.*

(ii) *Let v be a weak solution of $P(f)$ on $\boldsymbol{R}$. Then $v\in L^\infty(\boldsymbol{R}; H)$ implies $v\in L^\infty(\boldsymbol{R}; X)$.*

This lemma is a direct consequence of the fact [D].

4. Order property of flux.

In this section we consider the stationary problem

$$z^* \in \partial\varphi^t(z) \tag{4.1}$$

for each $t\in\boldsymbol{R}$ and $z^*\in H$, where φ^t is the convex function given by (3.1). In [**15**; Lemma 3.1] it was proved that (4.1) is equivalent to the following system (4.2), $\cdots$, (4.5):

$$-\Delta z = z^* \qquad \text{in } \Omega \text{ (in the distribution sense)}, \tag{4.2}$$

$$z \in K(t)\,, \tag{4.3}$$

$$\langle \partial_n z,\, g(t)-z\rangle_\Gamma = \langle \chi_N(t)q_N(t)+\chi_U(t)q_U(t),\, g(t)-z\rangle_\Gamma\,, \tag{4.4}$$

$$\langle \partial_n z,\, w\rangle_\Gamma \geqq \langle \chi_N(t)q_N(t)+\chi_U(t)q_U(t),\, w\rangle_\Gamma \quad \text{for any } w\in K_0(t)\,, \tag{4.5}$$

where $\chi_\nu(t)=\chi_\nu(t, \cdot)$ ($\nu=D, N, U$) is the characteristic function associated with the set $\Gamma_\nu(t)$, and $K_0(t)$ is the set given by (2.4). This characterization of (4.1) also shows that $\partial\varphi^t$ is single-valued in H, and if $z\in D(\partial\varphi^t)$, then $z\in X$, $\partial\varphi^t(z)=-\Delta z$ in H and hence $\partial_n z\in H^{-1/2}(\Gamma)$.

PROPOSITION 4.1. *Let $t\in\boldsymbol{R}$ and z, $\hat{z}\in D(\partial\varphi^t)$ such that $z\leqq\hat{z}$ on Ω. Then,*

$$\partial_n z \geqq \partial_n \hat{z} \qquad \text{in the sense of } H^{-1/2}(\Gamma)\,, \tag{4.6}$$

that is,

$$\langle \partial_n z,\, \eta\rangle_\Gamma \geqq \langle \partial_n \hat{z},\, \eta\rangle_\Gamma \quad \text{for any } \eta\in H^{1/2}(\Gamma) \text{ with } \eta\geqq 0 \text{ a.e. on } \Gamma. \tag{4.6$'$}$$

PROOF. For simplicity we write ϕ, Γ_ν and χ_ν ($\nu=D, N, U$) for φ^t, $\Gamma_\nu(t)$ and

$\chi_\nu(t,\cdot)$ as well as g_D, q_N, g_U and q_U for $g_D(t,\cdot)$, $q_N(t,\cdot)$, $g_U(t,\cdot)$ and $q_U(t,\cdot)$. If z and $\hat{z}$ are regular, for instance, if z, $\hat{z}\in H^2(\Omega)$, then (4.6) is obtained directly from the boundary condition {(1.2a)-(1.2c)}. In fact, in such a case, our boundary condition makes sense in the space $H^{1/2}(\Gamma)$, and it is enough to see (4.6) pointwise except on a null set of Γ. However, in general, z and $\hat{z}$ are not necessarily smooth enough, so that we employ here an approximation of the boundary condition by the penalty method. Consider the approximation ϕ^μ, $\mu>0$, of ϕ which is defined by

$$\phi^\mu(v)=\begin{cases}\dfrac{1}{2}|\nabla v|_H^2-\langle\chi_N q_N+\chi_U q_U,\, v\rangle_\Gamma+\dfrac{1}{2\mu}\displaystyle\int_\Gamma\chi_D|v-g_D|^2d\Gamma+\dfrac{1}{2\mu}\int_\Gamma\chi_U|(v-g_U)^+|^2d\Gamma & \text{for } v\in X,\\ \infty, \quad \text{otherwise.}\end{cases}$$

Sometimes we denote ϕ by ϕ^0. We easily see that

$$\phi^\mu(v\wedge w)+\phi^\mu(v\vee w)=\phi^\mu(v)+\phi^\mu(w) \quad \text{for any } v,\, w\in X \text{ and } \mu\geqq 0, \tag{4.7}$$

and

$$\phi^\mu \longrightarrow \phi \quad \text{on } H \text{ as } \mu\downarrow 0 \text{ in the sense on Mosco [19].} \tag{4.8}$$

For each $\mu\geqq 0$ and $\lambda>0$, (4.7) implies (cf. [13]) that the resolvent $(I+\lambda\partial\phi^\mu)^{-1}$ is order-preserving in H, i.e.

$$(I+\lambda\partial\phi^\mu)^{-1}v\leqq(I+\lambda\partial\phi^\mu)^{-1}w \quad \text{on } \Omega \tag{4.9}$$
$$\text{for any } v,\, w\in H \text{ with } v\leqq w \text{ on } \Omega.$$

Also, by [20], for each $\lambda>0$ (4.8) implies that

$$(I+\lambda\partial\phi^\mu)^{-1}v\longrightarrow(I+\lambda\partial\phi)^{-1}v \quad \text{in } H \text{ as } \mu\downarrow 0 \text{ for each } v\in H. \tag{4.10}$$

Moreover, for $\mu>0$ and $\lambda>0$, the relation $w=(I+\lambda\partial\phi^\mu)^{-1}v$ is equivalent to the system {(4.11), (4.12)}:

$$w-\lambda\Delta w=v \quad \text{in } \Omega, \tag{4.11}$$

$$-\partial_n w=-\chi_N q_N-\chi_U q_U+\frac{1}{\mu}\chi_D(w-g_D)+\frac{1}{\mu}\chi_U(w-g_U)^+ \quad \text{a.e. on } \Gamma; \tag{4.12}$$

note that problem {(4.11), (4.12)} has a unique solution w in $H^{3/2}(\Omega)$ for every $v\in H$, $\mu>0$, $\lambda>0$.

Now, let z, $\hat{z}\in D(\partial\phi)$ and suppose $z\leqq\hat{z}$ on Ω. Put

$$z_\lambda^\mu=(I+\lambda\partial\phi^\mu)^{-1}z \quad \text{and} \quad \hat{z}_\lambda^\mu=(I+\lambda\partial\phi^\mu)^{-1}\hat{z}$$

for $\mu>0$ and $\lambda>0$. Then it follows from (4.9) and (4.12) that $z_\lambda^\mu\leqq\hat{z}_\lambda^\mu$ on Ω and $\partial_n z_\lambda^\mu$, $\partial_n\hat{z}_\lambda^\mu\in L^2(\Gamma)$ with

$$\partial_n z_\lambda^\mu\geqq\partial_n\hat{z}_\lambda^\mu \quad \text{on } \Gamma. \tag{4.13}$$

Besides, by (4.10),

$$z_\lambda^\mu \longrightarrow z_\lambda = (I+\lambda\partial\phi)^{-1}z \quad \text{and} \quad \hat{z}_\lambda^\mu \longrightarrow \hat{z}_\lambda = (I+\lambda\partial\phi)^{-1}\hat{z} \quad \text{in } H$$
$$\text{as } \mu \downarrow 0 \text{ for each } \lambda>0. \tag{4.14}$$

On the other hand, from the relation $(z-z_\lambda^\mu)/\lambda \in \partial\phi^\mu(z_\lambda^\mu)$ and the fact $z_\lambda \in D(\partial\phi) \subset D(\phi)$, we have

$$\phi^\mu(z_\lambda^\mu) + \left(\frac{z-z_\lambda^\mu}{\lambda}, z_\lambda - z_\lambda^\mu\right) \leqq \phi^\mu(z_\lambda) = \phi(z_\lambda) \quad \text{for any } \lambda, \mu>0. \tag{4.15}$$

Combining (4.15) with (4.14), we see that $\{z_\lambda^\mu\}_{\mu>0}$ is bounded in X for each $\lambda>0$, so that $z_\lambda^\mu \to z_\lambda$ weakly in X as $\mu \downarrow 0$ for each $\lambda>0$. Furthermore, letting $\mu \downarrow 0$ in (4.15) yields

$$\limsup_{\mu\downarrow 0}\int_\Omega |\nabla z_\lambda^\mu|^2 dx \leqq \int_\Omega |\nabla z_\lambda|^2 dx,$$

for each $\lambda>0$. Similarly, we have $\hat{z}_\lambda^\mu \to \hat{z}_\lambda$ weakly in X as $\mu \downarrow 0$ and

$$\limsup_{\mu\downarrow 0}\int_\Omega |\nabla \hat{z}_\lambda^\mu|^2 dx \leqq \int_\Omega |\nabla \hat{z}_\lambda|^2 dx,$$

for each $\lambda>0$. Therefore, we can conclude that the convergences in (4.14) hold strongly in X. Next, it follows from (4.11) that for each $\lambda>0$,

$$\Delta z_\lambda^\mu = -\frac{z-z_\lambda^\mu}{\lambda} \longrightarrow -\frac{z-z_\lambda}{\lambda} = \Delta z_\lambda \quad \text{in } H \quad \text{as } \mu \downarrow 0$$

as well as $\Delta \hat{z}_\lambda^\mu \to \Delta \hat{z}_\lambda$ in H as $\mu \downarrow 0$. Therefore

$$\partial_n z_\lambda^\mu \longrightarrow \partial_n z_\lambda, \quad \partial_n \hat{z}_\lambda^\mu \longrightarrow \partial_n \hat{z}_\lambda \quad \text{in } H^{-1/2}(\Gamma) \quad \text{as } \mu \downarrow 0. \tag{4.16}$$

Taking account of (4.13) and (4.16), we obtain

$$\partial_n z_\lambda \geqq \partial_n \hat{z}_\lambda \quad \text{in the sense of } H^{-1/2}(\Gamma) \tag{4.17}$$

for each $\lambda>0$. Moreover, since $z, \hat{z} \in D(\partial\phi)$, we see from the general theory of maximal monotone operators (for instance see [4]) that

$$z_\lambda \longrightarrow z, \quad \hat{z}_\lambda \longrightarrow \hat{z} \quad \text{in } H, \tag{4.18}$$

$$-\Delta z_\lambda = \partial\phi(z_\lambda) = \partial\phi_\lambda(z) \longrightarrow \partial\phi(z) = -\Delta z \quad \text{in } H \tag{4.19}$$

and

$$-\Delta \hat{z}_\lambda \longrightarrow -\Delta \hat{z} \quad \text{in } H \tag{4.20}$$

as $\lambda \downarrow 0$. Here $\partial\phi_\lambda$ is the Yosida-approximation of $\partial\phi$. Furthermore, by [4; Proposition 2.11], we have

$$\phi(z_\lambda) \leqq \phi(z) \quad \text{and} \quad \phi(\hat{z}_\lambda) \leqq \phi(\hat{z}) \quad \text{for any } \lambda>0.$$

By these inequalities and (4.18), we can conclude, in the same way as before, that the convergences in (4.18) hold strongly in X. These convergences together with (4.19) and (4.20) yield that $\partial_n z_\lambda \to \partial_n z$ and $\partial_n \hat{z}_\lambda \to \partial_n \hat{z}$ in $H^{-1/2}(\Gamma)$ as $\lambda \downarrow 0$, so that we obtain (4.6) from (4.17). q.e.d.

5. Lemmas.

Throughout this section we suppose that all the assumptions of Theorem 2.1 are satisfied.

LEMMA 5.1. *Let ω be a T-periodic weak solution of $P(f)$ on $\boldsymbol{R}$, and let v be a weak solution of $P(f)$ on $[t_0, \infty)$, $t_0 \in \boldsymbol{R}$ such that $\omega(t) \leqq v(t)$ (resp. $\omega(t) \geqq v(t)$) on Ω for any $t \geqq t_0$. Then there exists a weak solution $\tilde{v}$ of $P(f)$ on $\boldsymbol{R}$ such that*

$$|\tilde{v}|_{L^\infty(\boldsymbol{R};H)} \leqq |v|_{L^\infty(t_0,\infty;H)}, \tag{5.1}$$

$$\omega(t) \leqq \tilde{v}(t) \quad (\textit{resp. } \omega(t) \geqq \tilde{v}(t)) \qquad \textit{on } \Omega \textit{ for any } t \in \boldsymbol{R}, \tag{5.2}$$

$$\int_\Omega \{\rho(\tilde{v}(t, x)) - \rho(\omega(t, x))\} dx = \lim_{s\to\infty} \int_\Omega \{\rho(v(s, x)) - \rho(\omega(s, x))\} dx \qquad \textit{for any } t \in \boldsymbol{R}, \tag{5.3}$$

and

$$\partial_n \tilde{v}(t, \cdot) = \partial_n \omega(t, \cdot) \qquad \textit{in } H^{-1/2}(\Gamma) \qquad \textit{for a.e. } t \in \boldsymbol{R}. \tag{5.4}$$

PROOF. We prove the lemma only in the case of $\omega \leqq v$. Note from Lemma 3.2 and [B] that $v \in L^\infty(t_0+1, \infty; X)$ and

$$d_\infty \equiv \lim_{s\to\infty} \int_\Omega \{\rho(v(s, x)) - \rho(\omega(s, x))\} dx$$

exists. Now, choose a sequence $\{n_k\} \subset \boldsymbol{N}$ with $n_k \uparrow \infty$ (as $k \to \infty$) such that for each $l = 1, 2, \cdots$, $v(-lT + n_k T)$ weakly converges in X as $k \to \infty$. Then it follows from [C] that $v_k(t) \equiv v(t + n_k T)$ converges to some weak solution $\tilde{v}$ of $P(f)$ on $\boldsymbol{R}$ in the following sense:

$$\rho(v_k) \longrightarrow \rho(\tilde{v}) \qquad \text{in } C(J; H) \text{ and weakly in } W^{1,2}(J; H)$$

and

$$v_k \longrightarrow \tilde{v} \qquad \text{weakly* in } L^\infty(J; X)$$

as $k \to \infty$ for every compact interval J in $\boldsymbol{R}$, and (5.1) and (5.2) are satisfied. Besides, for each $t \in \boldsymbol{R}$,

$$\int_\Omega \{\rho(\tilde{v}(t, x)) - \rho(\omega(t, x))\} dx = \lim_{k\to\infty} \int_\Omega \{\rho(v(t + n_k T, x)) - \rho(\omega(t + n_k T, x))\} dx = d_\infty.$$

Thus (5.3) holds. Finally we show (5.4). Since $\rho(\tilde{v})_t - \Delta\tilde{v} = f$ and $\rho(\omega)_t - \Delta\omega = f$ in Q (cf. Remark 2.1), we infer from (5.3) that for a.e. $t \in \boldsymbol{R}$,

$$\begin{aligned} 0 = \int_\Omega \{\rho(\tilde{v})_t(t, x) - \rho(\omega)_t(t, x)\} dx &= \int_\Omega \{\Delta\tilde{v}(t, x) - \Delta\omega(t, x)\} dx \\ &= \langle \partial_n \tilde{v}(t, \cdot) - \partial_n \omega(t, \cdot), 1 \rangle_\Gamma . \end{aligned} \tag{5.5}$$

On the other hand, applying Proposition 4.1, we see that

$$\partial_n\tilde{v}(t, \cdot) \leqq \partial_n\omega(t, \cdot) \quad \text{in } H^{-1/2}(\Gamma) \text{ for a.e. } t\in \boldsymbol{R}.$$

Combining this with (5.5), we get $\partial_n\tilde{v}(t, \cdot)=\partial_n\omega(t, \cdot)$ in $H^{-1/2}(\Gamma)$ for a.e. $t\in\boldsymbol{R}$. Thus we have (5.4). q.e.d.

LEMMA 5.2. *Let ω be a T-periodic weak solution of $P(f)$ on $\boldsymbol{R}$, and let v be a weak solution of $P(f)$ on $\boldsymbol{R}$ such that $v\in L^\infty(\boldsymbol{R}; H)$ and $\omega(t)\leqq v(t)$ (resp. $\omega(t)\geqq v(t)$) on Ω for any $t\geqq\boldsymbol{R}$. Then there exists a weak solution $\tilde{v}$ of $P(f)$ on $\boldsymbol{R}$ for which* (5.2), (5.4) *and the following* (5.1)′, (5.3)′ *hold:*

$$|\tilde{v}|_{L^\infty(\boldsymbol{R}; H)} \leqq |v|_{L^\infty(\boldsymbol{R}; H)}, \tag{5.1$'$}$$

$$\int_\Omega\{\rho(\tilde{v}(t, x))-\rho(\omega(t, x))\}dx = \lim_{s\to-\infty}\int_\Omega\{\rho(v(s, x))-\rho(\omega(s, x))\}dx \quad \textit{for any } t\in\boldsymbol{R}. \tag{5.3$'$}$$

PROOF. Suppose $\omega\leqq v$. By assumption,

$$d_{-\infty} \equiv \lim_{s\to-\infty}\int_\Omega\{\rho(v(s, x))-\rho(\omega(s, x))\}dx$$

exists. Also, note that $v\in L^\infty(\boldsymbol{R}; X)$ by Lemma 3.2. Therefore there is a sequence $\{n_k\}\subset\boldsymbol{N}$ such that for each $l=1, 2, \cdots$, $v(-lT-n_kT)$ weakly converges in X as $k\to\infty$. Now, put $v_k(t)=v(t-n_kT)$. Then, just as in the proof of Lemma 5.1, we can show that v_k converges to a weak solution $\tilde{v}$ of $P(f)$ on $\boldsymbol{R}$ as $k\to\infty$ in a similar sense as in the proof of Lemma 5.1, and $\tilde{v}$ satisfies (5.1)′, (5.2), (5.3)′ and (5.4). q.e.d.

Next we prepare a function space

$$V = \{z\in X;\ z=0 \text{ a.e. on } \Gamma_0\},$$

where $\Gamma_0=\bigcap_{t\in\boldsymbol{R}}\Gamma_D(t)$. Since $\mathrm{meas}_\Gamma(\Gamma_0)>0$ by assumption (A.2)-(d), we see that V becomes a Hilbert space with norm

$$|z|_V = |\nabla z|_H,$$

and the corresponding inner product is given by $a(\cdot, \cdot)$. For simplicity, we denote by $(\cdot, \cdot)_*$ the duality pairing $(\cdot, \cdot)_{V^*, V}$. Let F be the duality mapping from V onto V^*. Then it is easy to see that $z^*=Fz$ if and only if

$$a(z, v) = (z^*, v)_* \quad \text{for all } v\in V.$$

Also, V^* becomes a Hilbert space, in which the inner product $(\cdot, \cdot)_{V^*}$ and the norm $|\cdot|_{V^*}$ are defined by

$$(z, \hat{z})_{V^*} = (z, F^{-1}\hat{z})_* \text{ for } z, \hat{z}\in V^* \quad \text{and} \quad |z|_{V^*} = (z, z)_{V^*}^{1/2} \text{ for } z\in V^*.$$

By identifying H with its dual space, we have $V\subset H\subset V^*$ with compact and dense imbeddings, and moreover

$$(z^*, z)_* = (z^*, z) \qquad \text{for } z^*\in H \text{ and } z\in V.$$

LEMMA 5.3. *Let J' be any interval in $\boldsymbol{R}$ and let v, w be any weak solutions of $P(f)$ on J' such that*

$$\partial_n v(t) = \partial_n w(t) \qquad \text{in } H^{-1/2}(\Gamma) \ \text{ for a.e. } t\in J'.$$

Then

$$\begin{aligned}\frac{1}{2}|\rho(v(t))-\rho(w(t))|_{V^*}^2+\int_s^t(\rho(v(\tau))-\rho(w(\tau)), v(\tau)-w(\tau))d\tau \\ = \frac{1}{2}|\rho(v(s))-\rho(w(s))|_{V^*}^2 \qquad \text{for any } s, t\in J' \text{ with } s\leq t.\end{aligned} \tag{5.6}$$

PROOF. By assumption, we observe that for a.e. $t\in J'$,

$$v(t)-w(t) \in V,$$

$$-\Delta(v(t)-w(t)) = -\rho(v(t))_t+\rho(w(t))_t \qquad \text{in } \Omega$$

and

$$\partial_n(v(t)-w(t)) = 0 \qquad \text{in } H^{-1/2}(\Gamma),$$

so that for any $\eta\in X$ and a.e. $t\in J'$

$$\begin{aligned}a(v(t)-w(t), \eta) &= \int_\Omega \nabla(v(t, x)-w(t, x))\cdot\nabla\eta(x)dx \\ &= -\int_\Omega \Delta(v(t, x)-w(t, x))\eta(x)dx = -(\rho(v)'(t)-\rho(w)'(t), \eta).\end{aligned}$$

This shows that

$$-\rho(v)'(t)+\rho(w)'(t) = F(v(t)-w(t)) \qquad \text{for a.e. } t\in J'.$$

Therefore, for a.e. $t\in J'$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\rho(v(t))-\rho(w(t))|_{V^*}^2 &= (\rho(v)'(t)-\rho(w)'(t), F^{-1}(\rho(v(t))-\rho(w(t))))_* \\ &= -(F(v(t)-w(t)), F^{-1}(\rho(v(t))-\rho(w(t))))_* \\ &= -(v(t)-w(t), \rho(v(t))-\rho(w(t)))_*,\end{aligned}$$

which yields (5.6). q.e.d.

LEMMA 5.4. *Let ω be a T-periodic weak solution of $P(f)$ on $\boldsymbol{R}$ and let v be a weak solution of $P(f)$ on $\boldsymbol{R}$ such that $v\in L^\infty(\boldsymbol{R}; H)$ and*

$$\omega(t) \leq v(t) \qquad \text{on } \Omega \ \text{ for any } t\in\boldsymbol{R} \tag{5.7}$$

or

$$\omega(t) \geq v(t) \qquad \text{on } \Omega \ \text{ for any } t\in\boldsymbol{R}. \tag{5.7$'$}$$

Then $\omega(t)=v(t)$ *in* H *for any* $t\in \boldsymbol{R}$.

PROOF. We give a proof of the lemma only in the case of (5.7). Put

$$d_{-\infty}\equiv \lim_{s\to-\infty}\int_{\Omega}\{\rho(v(s,x))-\rho(\omega(s,x))\}dx$$

and note from [B] that

$$d_{-\infty}\geqq |\rho(v(t))-\rho(\omega(t))|_{L^1(\Omega)}=\int_{\Omega}\{\rho(v(t,x))-\rho(\omega(t,x))\}dx\geqq 0 \qquad (5.8)$$
$$\text{for any } t\in \boldsymbol{R}.$$

Now, by Lemma 5.2, take a weak solution $\tilde{v}$ of $P(f)$ on $\boldsymbol{R}$ so that (5.1)′, (5.2), (5.3)′ and (5.4) hold. Then, by Lemma 5.3,

$$\begin{aligned}&\frac{1}{2}|\rho(\tilde{v}(t))-\rho(\omega(t))|_{V^*}^2+\int_s^t(\rho(\tilde{v}(\tau))-\rho(\omega(\tau)),\ \tilde{v}(\tau)-\omega(\tau))d\tau\\ &\quad=\frac{1}{2}|\rho(\tilde{v}(s))-\rho(\omega(s))|_{V^*}^2 \qquad \text{for any } s\leqq t.\end{aligned} \qquad (5.9)$$

Next, making use of [C], choose a sequence $\{m_k\}\subset \boldsymbol{N}$ with $m_k\uparrow\infty$ (as $k\to\infty$) and a weak solution v^* of $P(f)$ on $[0, T]$ so that

$$\rho(\tilde{v}(\cdot+m_kT))\longrightarrow\rho(v^*) \qquad \text{in } C(0,T;H) \text{ and weakly in } W^{1,2}(0,T;H)$$

and

$$\tilde{v}(\cdot+m_kT)\longrightarrow v^* \qquad \text{weakly* in } L^\infty(0,T;X).$$

Then, clearly, $\omega\leqq v^*$,

$$d_{-\infty}=\int_{\Omega}\{\rho(v^*(t,x))-\rho(\omega(t,x))\}dx \qquad \text{for any } t\in[0,T],$$

$$|\rho(v^*(t))-\rho(\omega(t))|_{V^*}=\text{const.}\ (=d^*) \qquad \text{for any } t\in[0,T]$$

and we obtain from (5.9) that

$$\int_0^T(\rho(v^*(\tau))-\rho(\omega(\tau)),\ v^*(\tau)-\omega(\tau))d\tau=0.$$

This shows that $\rho(v^*(t))=\rho(\omega(t))$ in H for any $t\in[0, T]$. Hence $d_{-\infty}=0$, and it follows from (5.8) that $\rho(v(t))=\rho(\omega(t))$ in H for any $t\in\boldsymbol{R}$. Consequently, we get $v=\omega$ because of the uniqueness of the solution of the Cauchy problem for $P(f)$ (cf. the fact [A]). q.e.d.

6. Proof of the theorems.

PROOF OF THEOREM 2.1. By virtue of [E] in Section 2, $P(f)$ has at least one T-periodic weak solution ω on $\boldsymbol{R}$. Let v be any weak solution of $P(f)$ on $\boldsymbol{R}$ such that $v\in L^\infty(\boldsymbol{R};H)$. Then, by Lemma 3.1, there are weak solutions v_*

and v^* of $P(f)$ on $\boldsymbol{R}$ such that v_*, $v^* \in L^\infty(\boldsymbol{R}; H)$ and

$$v_* \leqq \omega \wedge v, \qquad \omega \vee v \leqq v^* \qquad \text{on } Q\,.$$

It follows from Lemma 5.4 that $v_* = \omega = v^*$, which shows that $v = \omega$. Thus Theorem 2.1 has been proved. q.e.d.

Under the same assumptions of Theorem 2.2 we prove:

LEMMA 6.1. *Let ω be the T-periodic weak solution of $P(f)$ on $\boldsymbol{R}$ and let v be any weak solution of $P(f)$ on $[t_0, \infty)$, $t_0 \in \boldsymbol{R}$, such that*

$$\omega \leqq v \quad \text{on } (t_0, \infty) \times \Omega \qquad \text{or} \qquad \omega \geqq v \quad \text{on } (t_0, \infty) \times \Omega\,.$$

Then

$$\rho(v(t)) - \rho(\omega(t)) \longrightarrow 0 \qquad \text{in } L^1(\Omega) \text{ as } t \to \infty\,. \tag{6.1}$$

PROOF. By Lemma 5.1, there exists a weak solution $\tilde{v}$ of $P(f)$ on $\boldsymbol{R}$ for which (5.1), $\cdots$, (5.4) hold. But Lemma 5.4 implies $\tilde{v} = \omega$, so that

$$\lim_{t \to \infty} \int_\Omega \{\rho(v(t, x)) - \rho(\omega(t, x))\} dx = 0\,.$$

This shows (6.1), since $\rho(v) \geqq \rho(\omega)$ or $\rho(v) \leqq \rho(\omega)$ on $(t_0, \infty) \times \Omega$. q.e.d.

PROOF OF THEOREM 2.2. Let ω be the T-periodic weak solution of $P(f)$ on $\boldsymbol{R}$, and v be any weak solution of $P(f)$ on $[t_0, \infty)$. First, we note from Lemma 3.1 that there are weak solutions v_* and v^* of $P(f)$ on $[t_0, \infty)$ such that

$$v_* \leqq \omega \wedge v, \quad \omega \vee v \leqq v^* \qquad \text{on } (t_0, \infty) \times \Omega\,.$$

Applying Lemma 6.1, we see that

$$\rho(v_*(t)) - \rho(\omega(t)) \longrightarrow 0 \qquad \text{in } L^1(\Omega) \tag{6.2}$$

and

$$\rho(v^*(t)) - \rho(\omega(t)) \longrightarrow 0 \qquad \text{in } L^1(\Omega) \tag{6.3}$$

as $t \to \infty$. Since

$$\rho(v_*) - \rho(\omega) \leqq \rho(v) - \rho(\omega) \leqq \rho(v^*) - \rho(\omega) \qquad \text{on } (t_0, \infty) \times \Omega\,,$$

it follows from (6.2) and (6.3) that

$$\rho(v(t)) - \rho(\omega(t)) \longrightarrow 0 \qquad \text{in } L^1(\Omega) \text{ as } t \to \infty\,. \tag{6.4}$$

Noting that $\rho(v)$, $\rho(\omega) \in L^\infty(t_0+1, \infty; X)$, we conclude from (6.4) that $\rho(v(t)) - \rho(\omega(t)) \to 0$ weakly in X as $t \to \infty$. Thus (2.6) has been proved. q.e.d.

References

[1] H.W. Alt and S. Luckhaus, Quasilinear elliptic-parabolic differential equations, Math. Z., **183** (1983), 311-341.

[2] H. W. Alt, S. Luckhaus and A. Visintin, On nonstationary flow through porous media, Ann. Mat. Pura Appl., **136** (1984), 303-316.
[3] M. Bertsch and J. Hulshof, Regularity results for an elliptic-parabolic free boundary problem, Trans. Amer. Math. Soc., **297** (1986), 337-350.
[4] H. Brézis, Operateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert, North-Holland, Amsterdam-London, 1973.
[5] E. DiBenedetto and A. Friedman, Periodic behavior for the evolutionary dam problem and related free boundary problems, Comm. Partial Diff. Equations, **11** (1986), 1297-1377.
[6] C. J. van Duyn and L. A. Peletier, Nonstationary filtration in partially saturated porous media, Arch. Rational Mech. Anal., **78** (1982), 173-198.
[7] A. Fasano and M. Primicerio, Liquid flow in partially saturated porous media, J. Inst. Math. Applics., **23** (1979), 503-517.
[8] U. Hornung, A parabolic-elliptic variational inequality, Manuscripta Math., **39** (1982), 155-172.
[9] J. Hulshof, An elliptic-parabolic free boundary problem: continuity of the interface, Proc. Roy. Soc. Edinburgh, **106** (1987), 327-339.
[10] J. Hulshof, Bounded weak solutions of an elliptic-parabolic Neumann problem, Trans. Amer. Math. Soc., **303** (1987), 211-227.
[11] J. Hulshof and L. A. Peletier, An elliptic-parabolic free boundary problem, Nonlinear Anal., **10** (1986), 1327-1346.
[12] N. Kenmochi and M. Kubo, Periodic solutions to a class of nonlinear variational inequalities with time-dependent constraints, Funkcial. Ekvac., **30** (1987), 333-349.
[13] N. Kenmochi and Y. Mizuta, Potential theoretic properties of the gradient of a convex function on a functional space, Nagoya Math. J., **59** (1975), 199-215.
[14] N. Kenmochi and I. Pawlow, A class of nonlinear elliptic-parabolic equations with time-dependent constraints, Nonlinear Anal., **10** (1986), 1181-1202.
[15] N. Kenmochi and I. Pawlow, Parabolic-elliptic free boundary problems with time-dependent obstacles, Japan J. Appl. Math., **5** (1988), 87-121.
[16] N. Kenmochi and I. Pawlow, Asymptotic behavior of solutions to parabolic-elliptic variational inequalities, to appear in Nonlinear Anal., 1989.
[17] D. Kröner, Parabolic regularization and behaviour of the free boundary for unsaturated flow in a porous medium, J. Reine Angew. Math., **348** (1984), 180-196.
[18] D. Kröner and J. F. Rodrigues, Global behaviour for bounded solutions of a porous media equation of elliptic-parabolic type, J. Math. Pures Appl., **64** (1985), 105-120.
[19] U. Mosco, Convergence of convex sets and of solutions of variational inequalities, Adv. in Math., **3** (1969), 510-585.
[20] J. Watanabe, Approximation of nonlinear problems of a certain type, Numer. Appl. Anal., **1** (1979), 147-163.

Nobuyuki KENMOCHI
Department of Mathematics
Faculty of Education
Chiba University
Chiba 260
Japan

Masahiro KUBO
Department of Mathematics
Faculty of Science and Engineering
Saga University
Saga 840
Japan