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Groups associated with unitary forms of Kac-Moody algebras

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Introduction.

The groups associated with Kac-Moody algebras, the Kac-Moody groups, were constructed and studied in Peterson-Kac [**9**]. These groups are, so to speak, "infinite dimensional Chevalley groups", and inherit many good properties of finite dimensional Chevalley groups. For example, they have BN-pairs, the corresponding flag varieties are defined, and their Bruhat decompositions are given. Moreover, among the Bruhat cells, the same closure relation holds, in a certain topology, as in the case of finite-dimensional semisimple algebraic groups. (See [**9**] and also Tits [**10**].)

But, it seems to us that this type of construction has some disadvantage for applications to the representation theory. In fact, the exponential map is not defined on the whole Lie algebra, but only for certain generators of algebra (elements of Cartan subalgebra and root vectors of real roots). In other words, there do not exist the 1-parameter subgroups which correspond to root vectors of imaginary roots. (See §1 and §4 for details.) As a consequence, even if we could differentiate a representation of a Kac-Moody group, there should be so much difficulties to examine if the differential gives a representation of its Lie algebra or not.

Motivated by these observations, we attempted to construct groups associated with Kac-Moody algebras in such a way that the exponential map is defined for every element of algebras. This is partially achieved in this paper. We construct such groups corresponding to certain real forms, called unitary forms, of complex Kac-Moody algebras with symmetrizable generalized Cartan matrices (GCM). For finite dimensional complex semisimple Lie algebras, unitary forms are nothing but compact real forms.

We now explain the construction. Let $\mathfrak{g}$ be a complex Kac-Moody algebra with symmetrizable GCM, $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{k}$ the unitary form of $\mathfrak{g}$. Let $\Lambda\in\mathfrak{h}^*$ be a dominant integral element and $L(\Lambda)$ the irreducible highest weight module for $\mathfrak{g}$ with highest weight Λ. It was proved in [**6**, Theorem 1] that $L(\Lambda)$ has a canonical pre-Hilbert space structure under which each element of $\mathfrak{k}$ acts on $L(\Lambda)$ as an antisymmetric operator. We prove in § 3 the existence of an absorbing subset B of $\mathfrak{g}$ with the following property. For any $x \in B$ and $v \in L(\Lambda)$ the series $\sum_{m=0}^{\infty}(m!)^{-1}x^m v$ is absolutely convergent. The linear map $\exp x : L(\Lambda) \ni v \to \sum_{m=0}^{\infty}(m!)^{-1}x^m v \in H(\Lambda)$ from $L(\Lambda)$ to $H(\Lambda)$ is extended to a unitary operator on $H(\Lambda)$ if $x \in B \cap \mathfrak{k}$, where $H(\Lambda)$ is the completion of $L(\Lambda)$. The map exp is naturally extended to a map from $\mathfrak{k}$ into the group $U(\Lambda)$ of all unitary operators on $H(\Lambda)$ equipped with the strong operator topology. Our group K^{Λ} associated with the unitary form $\mathfrak{k}$ is defined as the closed subgroup of $U(\Lambda)$ generated by $\exp \mathfrak{k}$.

We define a subgroup H^{Λ} of K^{Λ} by $H^{\Lambda} = \exp(\mathfrak{h} \cap \mathfrak{k})$. In case where $\mathfrak{g}$ is finite-dimensional, H^{Λ} is nothing but a maximal torus of K^{Λ}. Even when $\mathfrak{g}$ is not finite-dimensional, H^{Λ} is compact in many good cases. We remark here, for any $x \in \mathfrak{k}$ and $v \in L(\Lambda)$, the function $f(t) = (\exp tx)v$ of real variable t with values in $H(\Lambda)$ is differentiable, and its differential at $t=0$ is equal to xv, the original action of x on v (see § 3.3).

Let G_0 be the Kac-Moody group in [**9**] associated with $\mathfrak{g}$, K_0 the subgroup of G_0 corresponding to the unitary form $\mathfrak{k}$, and π_{Λ} the representation of G_0 on $L(\Lambda)$ (see § 4 for detail). The groups associated with $\mathfrak{g}$ larger than G_0 had been defined by Garland [**3**] in case of affine type, and by Tits [**10**] in general case, and the existence of the largest one was proved by Mathieu [**7**]. Roughly speaking, these groups are formal extensions of G_0 in the direction of positive roots (but not in the direction of negative roots). Their subgroups corresponding to the unitary form $\mathfrak{k}$ remain to be the same, not extended from K_0.

By [**9**, Corollary 4], $\pi_{\Lambda}(K_0)$ is contained in K^{Λ}. H^{Λ} normalizes $\pi_{\Lambda}(K_0)$ and so $H^{\Lambda} \cdot \pi_{\Lambda}(K_0)$ is a subgroup of K^{Λ}. The group $H^{\Lambda} \cdot \pi_{\Lambda}(K_0)$ contains exponentials of generators, as a Lie algebra, of $\mathfrak{k}$, but does not contain all exponentials $\exp x$ $(x \in \mathfrak{k})$. Therefore, the following problem naturally arises, which is discussed in § 4.

PROBLEM. *Is the closure of $H^{\Lambda} \cdot \pi_{\Lambda}(K_0)$ equal to K^{Λ}?*

This paper is organized as follows. In § 1, we recall some preliminary facts about Kac-Moody algebras. In § 2, we first construct certain Hermitian forms on highest weight modules. In case the highest weights are dominant integral, they give the pre-Hilbert space structure described above. Then, we estimate the norm of the action of root vectors on the pre-Hilbert spaces.

§ 3 is the main part of this paper, where the groups in title are constructed as described above. In § 4, the relations between our groups and Kac-Moody groups, treated in [**3**], [**7**], [**9**] and [**10**], are studied. In § 5, we study the relations between the groups K^{Λ} in § 3 and the groups constructed on lowest weight modules in the same manner as in § 3. For every dominant integral

A, they are mutually isomorphic and the lowest and highest weight modules are contragredient to each other as representations of the group.

NOTATIONS. We denote by $\boldsymbol{C}$ the complex number field, $\boldsymbol{R}$ the real number field, $\boldsymbol{Q}$ the rational number field, and $\boldsymbol{Z}$ the ring of rational integers. For an ordered set $(S, \leqq)$ and $s \in S$, we define subsets $S_{>s}$ and $S_{\geqq s}$ of S by

$$S_{>s} = \{t \in S \mid t > s\},$$
$$S_{\geqq s} = \{t \in S \mid t \geqq s\}.$$

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§ 1. Preliminaries.

In this section, we recall primary facts in the theory of Kac-Moody algebras briefly, needed in the succeeding sections. For detailed accounts, see [5] for example.

1.1. Kac-Moody algebras. Let n be a positive integer and $A=(a_{ij})_{1\leq i,j\leq n}$ be a generalized Cartan matrix (GCM). This means that A satisfies the following conditions:

i) $a_{ij} \in \boldsymbol{Z}$,
ii) $a_{ij} \leqq 0$ if $i \neq j$, and $a_{ii}=2$,
iii) $a_{ij} = 0$ if and only if $a_{ji}=0$.

For a field k with characteristic 0, we denote by $\mathfrak{g}_k=\mathfrak{g}_k(A)$ the Kac-Moody algebra over k associated with A, which is constructed as follows.

Let $\mathfrak{h}_k=\mathfrak{h}_k(A)$ be a $(2n-\mathrm{rank}(A))$-dimensional vector space over k. Let $\Pi=\{\alpha_1, \cdots, \alpha_n\}$ and $\Pi^\vee=\{\alpha_1^\vee, \cdots, \alpha_n^\vee\}$ be linearly independent subsets in $\mathfrak{h}_k^*=\mathrm{Hom}_k(\mathfrak{h}_k, k)$ and in $\mathfrak{h}_k$, respectively, such that

$$\alpha_j(\alpha_i^\vee) = a_{ij} \quad (i, j=1, \cdots, n).$$

We denote by $\tilde{\mathfrak{g}}_k$ the Lie algebra generated by $\mathfrak{h}_k$ and $2n$ symbols $\tilde{e}_1, \cdots, \tilde{e}_n$, $\tilde{f}_1, \cdots, \tilde{f}_n$ under the following relations

i) $[h, h'] = 0 \quad (h, h' \in \mathfrak{h}_k)$,
ii) $[h, \tilde{e}_i] = \alpha_i(h)\tilde{e}_i$ and $[h, \tilde{f}_i] = -\alpha_i(h)\tilde{f}_i \quad (h \in \mathfrak{h}_k,\ 1 \leqq i \leqq n)$,
iii) $[\tilde{e}_i, \tilde{f}_j] = 0$ if $i \neq j$,
iv) $[\tilde{e}_i, \tilde{f}_i] = \alpha_i^\vee$.

There exists the largest proper ideal $\mathfrak{r}$ of $\tilde{\mathfrak{g}}_k$ intersecting with $\mathfrak{h}_k$ trivially. The Kac-Moody algebra $\mathfrak{g}_k$ is defined by

$$\mathfrak{g}_k = \tilde{\mathfrak{g}}_k/\mathfrak{r}.$$

Let $\tilde{p}$ be the canonical projection from $\tilde{\mathfrak{g}}_k$ onto $\mathfrak{g}_k$. We put

$$e_i = \tilde{p}(\tilde{e}_i) \quad \text{and} \quad f_i = \tilde{p}(\tilde{f}_i) \qquad (i=1, \cdots, n).$$

Since $\tilde{p}|\mathfrak{h}_k$ is injective, we identify $\mathfrak{h}_k$ with $\tilde{p}(\mathfrak{h}_k)$. Then, $\mathfrak{g}_k$ is generated by $\mathfrak{h}_k, e_1, \cdots, e_n, f_1, \cdots, f_n$. We call $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$ the Chevalley generators and $\mathfrak{h}_k$ the Cartan subalgebra of $\mathfrak{g}_k$.

Let $\Delta=\Delta(A)$ be the root system of $(\mathfrak{g}_k, \mathfrak{h}_k)$, $\Delta_+=\Delta_+(A)$ the set of positive roots corresponding to the simple system Π of Δ, and $\mathfrak{g}_k=\mathfrak{h}_k\oplus\sum_{\alpha\in\Delta}^{\oplus}\mathfrak{g}_k^{\alpha}$ the root space decomposition. We put $\mathfrak{n}_{\pm,k}=\mathfrak{n}_{\pm,k}(A)=\sum_{\alpha\in\Delta_+}\mathfrak{g}_k^{\pm\alpha}$. Let ω_k be the involutive automorphism on $\mathfrak{g}_k$ defined by $\omega_k(e_i)=-f_i$ $(i=1, \cdots, n)$ and $\omega_k(h)=-h$ for all $h\in\mathfrak{h}_k$. Note that if k' is an extension field of k, then we see that $\mathfrak{g}_{k'}=k'\otimes_k\mathfrak{g}_k$, $\mathfrak{h}_{k'}=k'\otimes_k\mathfrak{h}_k$ and so on.

We denote by $W=W(A)$ the Weyl group of $\mathfrak{g}_k$. Let $\Delta^{\mathrm{re}}=\Delta^{\mathrm{re}}(A)$ be the union of W-orbits in $\mathfrak{h}_k^*$ through Π, and $\Delta^{\mathrm{im}}=\Delta^{\mathrm{im}}(A)$ the complement of Δ^{re} in Δ. Each element of Δ^{re} (resp. Δ^{im}) is called a real (resp. imaginary) root.

Throughout this paper, we assume that A is symmetrizable, that is, there exists a diagonal matrix D with entries in $\boldsymbol{Q}_{>0}$ such that DA is a symmetric matrix.

REMARK. For a symmetrizable GCM A, it was proved in [2] that the ideal $\mathfrak{r}$ is generated by $(\mathrm{ad}\,\tilde{e}_i)^{1-a_{ij}}\cdot\tilde{e}_j$ and $(\mathrm{ad}\,\tilde{f}_i)^{1-a_{ij}}\cdot\tilde{f}_j$ $(i\neq j)$.

Since A is assumed to be symmetrizable, there exist standard invariant bilinear forms on $\mathfrak{g}_k$ (see [5, §3] and [8, §3]). We fix one of them and denote it by $(\cdot|\cdot)_k$. This form is non-degenerate and symmetric, and moreover, it holds that for any $\alpha, \beta\in\Delta\cup\{0\}$,

$$(\mathfrak{g}_k^{\alpha}|\mathfrak{g}_k^{\beta}) = 0 \quad \text{if } \alpha+\beta\neq 0,$$

where $\mathfrak{g}_k^0=\mathfrak{h}_k$. In particular, $(\cdot|\cdot)_k|\mathfrak{h}_k\times\mathfrak{h}_k$ is non-degenerate, and so it defines a linear isomorphism ν_k from $\mathfrak{h}_k$ onto $\mathfrak{h}_k^*=\mathrm{Hom}_k(\mathfrak{h}_k, k)$. We denote by the same notation $(\cdot|\cdot)_k$ the induced bilinear forms on $\mathfrak{h}_k$ and on $\mathfrak{h}_k^*$. These bilinear forms on $\mathfrak{h}_k$ and $\mathfrak{h}_k^*$ are both invariant under the action of W, and ν_k is W-equivariant.

Let $\alpha\in\Delta$. For any $x\in\mathfrak{g}_k^{\alpha}$ and any $y\in\mathfrak{g}_k^{-\alpha}$, there holds that

$$[x, y] = (x|y)_k\nu_k^{-1}(\alpha). \tag{1.1.1}$$

Moreover, it holds that

$$\alpha\in\Delta^{\mathrm{re}} \text{ if and only if } (\alpha|\alpha)_k>0, \tag{1.1.2}$$

$$\alpha\in\Delta^{\mathrm{im}} \text{ if and only if } (\alpha|\alpha)_k\leq 0. \tag{1.1.3}$$

Here, $(\cdot|\cdot)_k$ is $\boldsymbol{Q}$-valued on $\mathfrak{h}_{\boldsymbol{Q}}^*$ and Δ is contained in $\mathfrak{h}_{\boldsymbol{Q}}^*$. So, the inequalities in (1.1.2) and (1.1.3) have a sense.

In this paper, the required properties of $(\cdot|\cdot)_k$ are (1.1.1), (1.1.2) and (1.1.3) only.

1.2. Unitary forms. From now on, we concentrate on the case where $k=\boldsymbol{R}$ or $k=\boldsymbol{C}$. If $k=\boldsymbol{C}$, then the subscript k should be omitted, that is, we write $\mathfrak{g}$ for $\mathfrak{g}_{\boldsymbol{C}}$ and $\mathfrak{h}$ for $\mathfrak{h}_{\boldsymbol{C}}$ etc.

We denote by ω_0 the conjugate linear extension of $\omega_{\boldsymbol{R}}$ to $\mathfrak{g}$. Let $\mathfrak{k}$ be the set of fixed points of ω_0. Then, $\mathfrak{k}$ is a real Lie subalgebra of $\mathfrak{g}$ and we have $\mathfrak{g}=\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{k}$. We call $\mathfrak{k}$ the unitary form of $\mathfrak{g}$. If $\mathfrak{g}$ is finite-dimensional, then $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$.

We define a sesquilinear form $(\cdot|\cdot)_0$ on $\mathfrak{g}$ by

(1.2.1) $$(x|y)_0=-(x|\omega_0(y))$$

for $x, y\in\mathfrak{g}$. $(\cdot|\cdot)_0$ is a Hermitian form and its restriction to $\mathfrak{n}_++\mathfrak{n}_-$ is positive definite (cf. [**6**, Theorem 1]). By invariance of $(\cdot|\cdot)$ under $\mathfrak{g}$, the form $(\cdot|\cdot)_0$ has the following property. For any $x, y, z\in\mathfrak{g}$, it holds that

(1.2.2) $$([x, y]|z)_0=-(y|[\omega_0(x), z])_0.$$

In particular, the restriction of $(\cdot|\cdot)_0$ to $\mathfrak{k}$ is invariant. We call the property (1.2.2) the contravariance of the form $(\cdot|\cdot)_0$.

§ 2. Irreducible highest weight modules with dominant integral highest weights.

This section is devoted to the construction of contravariant Hermitian forms on highest weight modules with highest weights $\lambda\in\mathfrak{h}_{\boldsymbol{R}}^*$.

At first, we recall the definition of Verma module $M(\lambda)$ as a quotient of $U(\mathfrak{g})$ by a certain left ideal L_λ, and then we define a contravariant Hermitian form on $U(\mathfrak{g})$. It is proved that the kernel of the form contains the left ideal L_λ and so it induces a Hermitian form on $M(\lambda)$. By an easy argument, we see that this form induces the unique (up to constant factors) contravariant form $(\cdot|\cdot)_\lambda$ on the irreducible quotient $L(\lambda)$ of $M(\lambda)$.

Let $\lambda\in\mathfrak{h}_{\boldsymbol{R}}^*$ be dominant integral. Then, it was proved in [**6**, Theorem 1] that the form $(\cdot|\cdot)_\lambda$ is positive definite. In the last of this section, we estimate the norm of the action of each root vector of a root α, as a linear map from a weight space $L(\lambda)_\mu$ into $L(\lambda)_{\mu+\alpha}$ with respect to the restriction of $(\cdot|\cdot)_\lambda$. Since any weight space is finite-dimensional, this norm is finite. Using this estimation, we see that all the elements of $\mathfrak{g}$ have finite norms as operators on $L(\lambda)$ if and only if $\dim\mathfrak{g}$ is finite.

2.1. Irreducible highest weight modules. At first, we recall the definition of Verma modules for $\mathfrak{g}$. Let $\lambda\in\mathfrak{h}^*$, and L_λ be the left ideal of $U(\mathfrak{g})$ generated by $\{h-\lambda(h)\mid h\in\mathfrak{h}\}$ and $\mathfrak{n}_+$. Here, for a Lie algebra $\mathfrak{a}$, $U(\mathfrak{a})$ is the universal enveloping algebra of $\mathfrak{a}$. The Verma module $M(\lambda)$ is defined by

$$M(\lambda)=U(\mathfrak{g})/L_\lambda, \tag{2.1.1}$$

where $U(\mathfrak{g})$ is considered as a $U(\mathfrak{g})$-module by left multiplications.

We denote by $L(\lambda)$ a unique irreducible quotient of $M(\lambda)$ and by $P(\lambda)$ the set of all weights of $L(\lambda)$.

For each $\lambda\in\mathfrak{h}^*$, we define a subset $S(\lambda)$ of the index set $\{1,\cdots,n\}$ of the GCM $A=(a_{ij})_{1\leq i,j\leq n}$ by

$$\begin{aligned} S(\lambda)=\{i\mid (\lambda|\alpha_i)\neq 0, &\text{ or there exists a sequence } i_0,\cdots,i_p=i \\ &\text{such that } (\lambda|\alpha_{i_0})\neq 0 \text{ and } a_{i_q i_{q+1}}\neq 0 \text{ for all } q=0,\cdots,p-1\}. \end{aligned} \tag{2.1.2}$$

Let $A_1=(a_{ij})_{i,j\in S(\lambda)}$ and $A_2=(a_{ij})_{i,j\in\{1,\cdots,n\}-S(\lambda)}$. Then, we have

$$\begin{cases} \mathfrak{g}=\mathfrak{g}(A_1)\oplus\mathfrak{g}(A_2) & \text{(direct sum of ideals)}, \\ \mathfrak{h}=\mathfrak{h}(A_1)\oplus\mathfrak{h}(A_2). \end{cases} \tag{2.1.3}$$

Moreover, we see that $\mathfrak{g}(A_2)$ (resp. $\mathfrak{n}_\pm(A_1)$) acts trivially (resp. faithfully) on $L(\lambda)$, and that $L(\lambda)$ is irreducible as a $\mathfrak{g}(A_1)$-module. These facts imply that, in case we treat $L(\lambda)$, we may assume $S(\lambda)$ to be equal to the whole set $\{1,\cdots,n\}$.

2.2. Contravariant Hermitian form. For any $x\in\mathfrak{g}$, we put

$$x^*=-\omega_0(x). \tag{2.2.1}$$

$\mathfrak{g}\ni x\to x^*\in\mathfrak{g}$ is an involutive conjugate linear antiautomorphism.

Let V be a $\mathfrak{g}$-module, and $\langle\cdot,\cdot\rangle$ a sesquilinear form on V. If $\langle\cdot,\cdot\rangle$ satisfies the following property, then we call it a contravariant form. For all $v, w\in V$, $x\in U(\mathfrak{g})$

$$\langle xv, w\rangle=\langle v, x^*w\rangle, \tag{2.2.2}$$

where $U(\mathfrak{g})\ni x\to x^*\in U(\mathfrak{g})$ is the unique antiautomorphism induced by (2.2.1).

Denote by P the projection from $U(\mathfrak{g})$ onto $U(\mathfrak{h})$ with respect to the decomposition $U(\mathfrak{g})=U(\mathfrak{h})\oplus(U(\mathfrak{g})\mathfrak{n}_++\mathfrak{n}_-U(\mathfrak{g}))$. Since $\mathfrak{h}$ is abelian, we can identify $U(\mathfrak{h})$ with the polynomial ring $\boldsymbol{C}[\mathfrak{h}^*]$ on $\mathfrak{h}^*$ canonically. For each $\lambda\in\mathfrak{h}^*$, put

$$\langle x\mid y\rangle_\lambda=P(y^*x)(\lambda). \tag{2.2.3}$$

Clearly, $\langle\cdot|\cdot\rangle_\lambda$ is a contravariant sesquilinear form. Moreover, we have

LEMMA 2.2.1. *If $\lambda\in\mathfrak{h}^*_{\boldsymbol{R}}$, then $\langle\cdot|\cdot\rangle_\lambda$ is a Hermitian form.*

PROOF. By definition, $*$ is equal to the identity on $U(\mathfrak{h}_R)$ and $U(\mathfrak{h})= \boldsymbol{C}\otimes_R U(\mathfrak{h}_R)$. So, $U(\mathfrak{h})$ is $*$-invariant and the restriction of $*$ to $U(\mathfrak{h})$ is equal to the conjugation with respect to $U(\mathfrak{h}_R)$. Therefore, we have

$$P(x^*) = P(x)^*, \tag{2.2.4}$$

$$y^*(\lambda) = \overline{y(\lambda)} \tag{2.2.5}$$

for all x, $y\in U(\mathfrak{h})$ and all $\lambda\in\mathfrak{h}_R^*$. The lemma is clear from (2.2.4) and (2.2.5). Q.E.D.

LEMMA 2.2.2. *Let $\lambda\in\mathfrak{h}_R^*$. The left ideal L_λ of $U(\mathfrak{g})$ is contained in*

$$\mathrm{Ker}\langle\cdot|\cdot\rangle_\lambda = \{x\in U(\mathfrak{g}) \mid \langle y|x\rangle_\lambda=0 \ \textit{ for all } y\in U(\mathfrak{g})\}.$$

PROOF. By definition, $P(y^*x)=0$ for all $x\in\mathfrak{n}_+$ and $y\in U(\mathfrak{g})$. Hence $\mathfrak{n}_+\subset\mathrm{Ker}\langle\cdot|\cdot\rangle_\lambda$.

For any $h\in\mathfrak{h}$ and any $x\in\mathfrak{n}_+$, we have

$$xh = hx+[x, h] \in U(\mathfrak{g})\mathfrak{n}_+,$$

and so it holds that

$$(\mathfrak{n}_-U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{n}_+)U(\mathfrak{h}) \subset \mathfrak{n}_-U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{n}_+. \tag{2.2.6}$$

Therefore, we have

$$P(y^*x) = P(y^*)x \tag{2.2.7}$$

for all $x\in U(\mathfrak{h})$ and $y\in U(\mathfrak{g})$.

Let $h\in\mathfrak{h}$. It follows from (2.2.7) that

$$P(y^*(h-\lambda(h)))(\lambda) = P(y^*)(\lambda)\cdot(h-\lambda(h))(\lambda) = 0.$$

for all $y\in U(\mathfrak{g})$. Hence $\{h-\lambda(h)\mid h\in\mathfrak{h}\}\subset\mathrm{Ker}\langle\cdot|\cdot\rangle_\lambda$.

$\mathrm{Ker}\langle\cdot|\cdot\rangle_\lambda$ is a left ideal of $U(\mathfrak{g})$ by its contravariance and contains generators of left ideal L_λ as proved above. Therefore, the lemma holds. Q.E.D.

Let $\lambda\in\mathfrak{h}_R^*$. By Lemma 2.2.2, $\langle\cdot|\cdot\rangle_\lambda$ defines a contravariant Hermitian form on $M(\lambda)$, which will be denoted by $(\cdot|\cdot)_\lambda$.

LEMMA 2.2.3. $\mathrm{Ker}(\cdot|\cdot)_\lambda$ *is the largest proper submodule of* $M(\lambda)$.

PROOF. Since $(\cdot|\cdot)_\lambda$ is contravariant, $\mathrm{Ker}(\cdot|\cdot)_\lambda$ is a submodule of $M(\lambda)$. Let v_0 be the canonical image in $M(\lambda)=U(\mathfrak{g})/L_\lambda$ of 1 in $U(\mathfrak{g})$. By definition, $(v_0|v_0)_\lambda=1$, and so the submodule $\mathrm{Ker}(\cdot|\cdot)_\lambda$ is proper.

Let V be a proper submodule of $M(\lambda)$ and μ a weight of V. Then $\mu<\lambda$, and so, for any weight vector v of weight μ, there exists $x\in U(\mathfrak{n}_-)\mathfrak{n}_-$ such that $v=xv_0$. Since $(U(\mathfrak{n}_-)\mathfrak{n}_-)^*=U(\mathfrak{n}_+)\mathfrak{n}_+$ and $\mathfrak{n}_+v_0=0$, we have

$$(v|v_0)_\lambda = (v_0|x^*v_0)_\lambda = (v_0|0)_\lambda = 0.$$

Hence, V is orthogonal to v_0 and to $U(\mathfrak{g})v_0=M(\lambda)$ with respect to $(\cdot|\cdot)_\lambda$ because $(\cdot|\cdot)_\lambda$ is contravariant. This proves the lemma. Q.E.D.

By this lemma, $(\cdot|\cdot)_\lambda$ on $M(\lambda)$ induces a non-degenerate contravariant Hermitian form on $L(\lambda)$. We denote it by the same notation $(\cdot|\cdot)_\lambda$. The following proposition was proved in [**6**, Theorem 1], by means of which we can construct our groups corresponding to the unitary form $\mathfrak{k}$ in the next section.

PROPOSITION 2.2.4 [**6**, Theorem 1]. *If λ is dominant integral, then the Hermitian form $(\cdot|\cdot)_\lambda$ on $L(\lambda)$ is positive definite.*

In the rest of this paper, we always assume that $\lambda=\Lambda$ is a dominant integral element of $\mathfrak{h}_R^*$ such that $(\Lambda|\alpha)>0$ for some $\alpha\in\Delta_+$.

2.3. Estimate of the norm of the action of root vectors on $L(\Lambda)$. Let $\alpha\in\Delta_+$. Take a root vector $x_\alpha\in\mathfrak{g}^\alpha$ such that $\|x_\alpha\|_0=1$. From (1.1.1), we have

$$\text{(2.3.1)} \qquad \begin{cases} [x_\alpha, x_\alpha^*] = \nu^{-1}(\alpha), \\ [\nu^{-1}(\alpha), x_\alpha] = (\alpha|\alpha)x_\alpha, \quad [\nu^{-1}(\alpha), x_\alpha^*] = -(\alpha|\alpha)x_\alpha^*. \end{cases}$$

Hence, the subspace $\mathfrak{g}(x_\alpha)=\boldsymbol{C}x_\alpha^*+\boldsymbol{C}\nu^{-1}(\alpha)+\boldsymbol{C}x_\alpha$ is a subalgebra of $\mathfrak{g}$ such that

$$\text{(2.3.2)} \qquad \begin{cases} \mathfrak{g}(x_\alpha) \cong sl_2(\boldsymbol{C}) & \text{if } (\alpha|\alpha)\neq 0, \\ \mathfrak{g}(x_\alpha) \cong H_3 & \text{if } (\alpha|\alpha)=0, \end{cases}$$

where H_3 denotes the 3-dimensional Heisenberg algebra over $\boldsymbol{C}$, that is, H_3 is the 3-dimensional complex Lie algebra with a basis $\{a, b, c\}$ such that $[a, b]=c$ and $[c, a]=[c, b]=0$.

Note that the choice of x_α is not unique since $\dim\mathfrak{g}^\alpha$ is not necessarily equal to 1.

Let $\mu\in\mathfrak{h}^*$ and $L(\Lambda)_\mu$ and $L(\Lambda)_{\mu+\alpha}$ the weight spaces of weight μ and $\mu+\alpha$ respectively. It holds that

$$\begin{aligned} v \in L(\Lambda)_\mu\cap(x_\alpha^*L(\Lambda)_{\mu+\alpha})^\perp &\Longleftrightarrow v \in L(\Lambda)_\mu \text{ and } (v|x_\alpha^*u)_\Lambda = 0 \text{ for all } u\in L(\Lambda)_{\mu+\alpha} \\ &\Longleftrightarrow v \in L(\Lambda)_\mu \text{ and } (x_\alpha v|u)_\Lambda = 0 \text{ for all } u\in L(\Lambda)_{\mu+\alpha} \\ &\Longleftrightarrow v \in L(\Lambda)_\mu \text{ and } x_\alpha v = 0 \quad (\text{since } x_\alpha v\in L(\Lambda)_{\mu+\alpha}). \end{aligned}$$

Hence, $L(\Lambda)_\mu$ is decomposed into an orthogonal sum as

$$\text{(2.3.3)} \qquad L(\Lambda)_\mu = \{v\in L(\Lambda)_\mu \mid x_\alpha v=0\}\oplus(x_\alpha^*L(\Lambda)_{\mu+\alpha}).$$

For $v\in L(\Lambda)_\mu-\{0\}$ such that $x_\alpha v=0$, the sub-$\mathfrak{g}(x_\alpha)$-module $U(\mathfrak{g}(x_\alpha))v$ of $L(\Lambda)$ is the irreducible highest weight module with highest weight $\mu|\boldsymbol{C}\nu^{-1}(\alpha)$, because it has a non-degenerate contravariant Hermitian form $(\cdot|\cdot)_\Lambda|U(\mathfrak{g}(x_\alpha))v$.

From the above argument, we see that

PROPOSITION 2.3.1. *As a $\mathfrak{g}(x_\alpha)$-module, $L(\Lambda)$ is an orthogonal direct sum of irreducible highest weight modules of the form $U(\mathfrak{g}(x_\alpha))v$ with $v\in L(\Lambda)_\mu$ such that $x_\alpha v=0$.*

We estimate the norm of the action of $\mathfrak{g}(x_\alpha)$ on each irreducible component $U(\mathfrak{g}(x_\alpha))v$.

For $i\in\boldsymbol{Z}_{\geq 0}$, we put

$$\begin{cases} v_i=\|x_\alpha^{*i}v\|_\Lambda^{-1}x_\alpha^{*i}v & \text{if } x_\alpha^{*i}v\neq 0 \\ v_i=0 & \text{if } x_\alpha^{*i}v=0. \end{cases} \tag{2.3.4}$$

Then, $\{v_i|v_i\neq 0\}$ is an orthonormal basis of the space $U(\mathfrak{g}(x_\alpha))v$, and it holds that for $i\in\boldsymbol{Z}_{\geq 0}$ that

$$\begin{cases} x_\alpha v_i\in\boldsymbol{C}v_{i-1}, \quad x_\alpha^* v_i\in\boldsymbol{C}v_{i+1}, \\ \nu^{-1}(\alpha)v_i=(\mu-i\alpha|\alpha)v_i, \end{cases} \tag{2.3.5}$$

where $v_{-1}=0$. We define $\xi_i\in\boldsymbol{C}$ by

$$x_\alpha^* v_i=\xi_i v_{i+1}. \tag{2.3.6}$$

By contravariance of $(\cdot|\cdot)_\Lambda$, x_α is the adjoint operator of x_α^* on $L(\Lambda)$. Hence,

$$x_\alpha v_i=\overline{\xi_{i-1}}v_{i-1}, \tag{2.3.7}$$

where $\xi_{-1}=0$. (2.3.5)$\sim$(2.3.7) imply that

$$|\xi_i|^2=(i+1)((\mu|\alpha)-2^{-1}i(\alpha|\alpha)). \tag{2.3.8}$$

If we take Λ as μ and if $(a_{ij})_{i,j\in S(\Lambda)}$ is not of finite type, there exists $\alpha\in\Delta_+^{\mathrm{im}}((a_{ij})_{i,j\in S(\Lambda)})$ such that $(\Lambda|\alpha)>0$ (cf. [5, Chap. 4]), and so (2.3.8) together with (1.1.3) implies the following theorem.

THEOREM 2.3.2. *Let $\mathfrak{g}$ be a Kac-Moody algebra with a symmetrizable generalized Cartan matrix, $\mathfrak{h}$ its Cartan subalgebra and $\Lambda\in\mathfrak{h}^*$ is a dominant integral element. Under the same notations as in* § 1.1, *the following conditions* (i), (ii) *and* (iii) *are equivalent to each other.*

(i) $[\mathfrak{g},\mathfrak{g}]\subset\boldsymbol{B}(L(\Lambda))$, *the set of bounded linear operators on* $L(\Lambda)$.

(ii) $\mathfrak{g}\subset\boldsymbol{B}(L(\Lambda))$.

(iii) $S(\Lambda)=\emptyset$ *or* $(a_{ij})_{i,j\in S(\Lambda)}$ *is the Cartan matrix of a finite dimensional complex semisimple Lie algebra.*

§ 3. Groups associated with the unitary form $\mathfrak{k}$ of $\mathfrak{g}$.

This section is the main part of this paper. We show that the exponential map from $\mathfrak{k}$ into the group $U(\Lambda)$ of all unitary operators on $H(\Lambda)$, the completion of $L(\Lambda)$, can be defined by means of the usual power series expansion, where Λ is a dominant integral element of $\mathfrak{h}_R^*$. The groups in the title are defined as the closed subgroups K^Λ of $U(\Lambda)$ generated by $\exp\mathfrak{k}$, for various Λ.

These groups correspond to the unitary form $\mathfrak{k}$, and their "complexifications", corresponding to $\mathfrak{g}$ itself, are not discussed here. In our groups K^Λ, $\exp x$ is defined for every element x of $\mathfrak{k}$, contrary to the case of Kac-Moody groups where the exponential is defined only for elements of $\mathfrak{h}$ and root vectors of real roots but not even for their linear combinations in general (cf. [**9**]). It seems to us that our construction of the groups has an advantage at this point.

3.1. Definition of the exponential map. Denote by $H(\Lambda)$ the completion of the pre-Hilbert space $L(\Lambda)$. In the following, we will show that the exponential map exp from $\mathfrak{k}$ into the group of unitary operators on $H(\Lambda)$ can be defined by means of the usual power series expansion.

Let $\alpha\in\Delta_+$ and $x\in\mathfrak{g}^\alpha$. From the proof of [**6**, Theorem 1], we have the inequality

$$\|xv\|_\Lambda^2 \leqq 2^{-1}(|\Lambda+\rho|^2-|\mu+\rho|^2)\cdot\|x\|_0^2\cdot\|v\|_\Lambda^2 \qquad (\mu\in P(\Lambda),\ v\in L(\Lambda)_\mu), \tag{3.1.1}$$

where $|\lambda|^2=(\lambda|\lambda)$ for $\lambda\in\mathfrak{h}^*$ and ρ is an element of $\mathfrak{h}_R^*$ which takes the value 1 on each simple coroot.

Fix an inner product $(\cdot|\cdot)_1$ on $\mathfrak{h}$ such that

$$|(h|h')| \leqq \|h\|_1\|h'\|_1 \qquad (h,\ h'\in\mathfrak{h}). \tag{3.1.2}$$

We denote by the same notation $(\cdot|\cdot)_1$ the inner product on $\mathfrak{h}^*$ defined by $(\lambda|\lambda')_1=(\nu^{-1}(\lambda)|\nu^{-1}(\lambda'))_1$ for $\lambda,\ \lambda'\in\mathfrak{h}^*$. Then, we have

$$|\lambda(h)| = |(\nu^{-1}(\lambda)|h)| \leqq \|\nu^{-1}(\lambda)\|_1\|h\|_1 = \|\lambda\|_1\|h\|_1 \tag{3.1.3}$$

for any $\lambda\in\mathfrak{h}^*$ and any $h\in\mathfrak{h}$.

For $h,\ h'\in\mathfrak{h}$ and $y,\ y'\in\mathfrak{n}_-+\mathfrak{n}_+$, put

$$(h+y|h'+y')_1 = (h|h')_1+(y|y')_0. \tag{3.1.4}$$

Then, $(\cdot|\cdot)_1$ is an inner product on $\mathfrak{g}$ and (3.1.1) is rewritten as

$$\|xv\|_\Lambda^2 \leqq 2^{-1}(|\Lambda+\rho|^2-|\mu+\rho|^2)\cdot\|x\|_1^2\cdot\|v\|_\Lambda^2. \tag{3.1.1$'$}$$

From (1.1.1), it holds that

$$xx^* = x^*x+\|x\|_1^2\cdot\nu^{-1}(\alpha). \tag{3.1.5}$$

Hence, we have

$$\|x^*v\|_A^2 = (xx^*v\,|\,v)_A$$
$$= (x^*xv\,|\,v)_A + (\|x\|_1^2\nu^{-1}(\alpha)v\,|\,v)_A$$
$$= \|xv\|_A^2 + (\mu\,|\,\alpha)\|x\|_1^2\|v\|_A^2,$$

whence the inequality

$$\|x^*v\|_A^2 \leqq 2^{-1}(|\Lambda+\rho|^2 - |\mu+\rho|^2 + 2(\mu\,|\,\alpha))\|x^*\|_1^2\|v\|_A^2 \tag{3.1.6}$$

holds because $\|x^*\|_1 = \|x\|_1$ by definition.

From (3.1.2), there holds that

$$(|\Lambda+\rho|^2 - |\mu+\rho|^2)^{1/2} \leqq \|\mu\|_1 + \|\rho\|_1 + \|\Lambda+\rho\|_1. \tag{3.1.7}$$

On the other hand, an easy calculation implies

$$|\Lambda+\rho|^2 - |\mu+\rho|^2 + 2(\mu\,|\,\alpha) = |\Lambda+\rho|^2 - |\mu+\rho-\alpha|^2 - 2(\rho\,|\,\alpha) + (\alpha\,|\,\alpha).$$

Here, if α is an imaginary root, then it holds that

$$-2(\rho\,|\,\alpha) + (\alpha\,|\,\alpha) \leqq -2(\rho\,|\,\alpha) < 0$$

since $(\alpha\,|\,\alpha) \leqq 0$ by (1.1.3). If α is a real root, we have

$$2(\rho\,|\,\alpha)(\alpha\,|\,\alpha)^{-1} = \rho(2(\alpha\,|\,\alpha)^{-1}\nu^{-1}(\alpha)) \geqq 1$$

and so it holds that

$$-2(\rho\,|\,\alpha) + (\alpha\,|\,\alpha) \leqq 0$$

since $(\alpha\,|\,\alpha) > 0$ by (1.1.2). In any case, the inequality $-2(\rho\,|\,\alpha) + (\alpha\,|\,\alpha) \leqq 0$ holds. Therefore, we have

$$(|\Lambda+\rho|^2 - |\mu+\rho|^2 + 2(\mu\,|\,\alpha))^{1/2} \leqq \|\mu\|_1 + \|\rho\|_1 + \|\alpha\|_1 + \|\Lambda+\rho\|_1. \tag{3.1.8}$$

Put $C_\Lambda = \|\rho\|_1 + \|\Lambda+\rho\|_1$. By (3.1.3), (3.1.1′) and (3.1.6)$\sim$(3.1.8), the following inequality holds for any $\alpha \in \Delta \cup \{0\}$, $x_\alpha \in \mathfrak{g}^\alpha$, $\mu \in P(\Lambda)$ and $v \in L(\Lambda)_\mu$ (we put $\mathfrak{g}^0 = \mathfrak{h}$):

$$\|x_\alpha v\|_\Lambda \leqq (\|\mu\|_1 + \|\alpha\|_1 + C_\Lambda)\|x_\alpha\|_1\|v\|_\Lambda. \tag{3.1.9}$$

Let $x = \sum_{\alpha \in \Delta \cup \{0\}} x_\alpha \in \mathfrak{g}$, $x_\alpha \in \mathfrak{g}^\alpha$, and $\Delta(x) = \{\alpha \in \Delta \cup \{0\} \,|\, x_\alpha \neq 0\}$. For $\mu \in P(\Lambda)$ and $v \in L(\Lambda)_\mu$, we have by (3.1.9)

$$\|x^m v\|_\Lambda \leqq \sum_{(\beta_1, \cdots, \beta_m) \in \Delta(x)} m\|x_{\beta_m} \cdots x_{\beta_1} v\|_\Lambda$$
$$\leqq \sum \left\{ \prod_{j=1}^{m} (\|\mu + \beta_1 + \cdots + \beta_{j-1}\|_1 + \|\beta_j\|_1 + C_\Lambda)\|x_{\beta_j}\|_1 \right\} \cdot \|v\|_\Lambda$$
$$\leqq \sum\{\Pi(\|\mu\|_1 + C_\Lambda + \|\beta_1\|_1 + \cdots + \|\beta_j\|_1)\|x_{\beta_j}\|_1\} \cdot \|v\|_\Lambda.$$

LEMMA 3.1.1. *There holds that*

$$\|x^m v\|_\Lambda \leqq \prod_{j=1}^{m}(\|\mu\|_1+C_\Lambda+j\cdot \sup_{\beta\in\Delta(x)}\|\beta\|_1)\cdot(\#\Delta(x))^m\cdot\|x\|_1^m\cdot\|v\|_\Lambda \tag{3.1.10}$$

From this inequality, we have the following corollary.

COROLLARY 3.1.2. *Fix* $0<\varepsilon<1$ *and* $v\in L(\Lambda)$. *Then, the series*

$$\sum_{m=0}^{\infty}\|(m!)^{-1}x^m v\|_\Lambda$$

converge uniformly and are bounded on the set

$$\left\{x\in\mathfrak{g}\ \middle|\ \Big(\sup_{\beta\in\Delta(x)}\|\beta\|_1\Big)\cdot(\#\Delta(x))\cdot\|x\|_1\leqq\varepsilon,\ \|x\|_1\cdot(\#\Delta(x))\leqq(1-\varepsilon)^{-1}\right\}.$$

Let $0<\varepsilon<1$ and put

$$B_\varepsilon=B_\varepsilon(A)=\left\{x\in\mathfrak{g}\ \middle|\ \Big(\sup_{\beta\in\Delta(x)}\|\beta\|_1\Big)\cdot(\#\Delta(x))\cdot\|x\|_1\leqq\varepsilon,\right.$$
$$\left.\|x\|_1\cdot(\#\Delta(x))\leqq(1-\varepsilon)^{-1}\right\} \tag{3.1.11}$$

$$B=B(A)=\bigcup_{0<\varepsilon<1}B_\varepsilon=\left\{x\in\mathfrak{g}\ \middle|\ \Big(\sup_{\beta\in\Delta(x)}\|\beta\|_1\Big)\cdot(\#\Delta(x))\cdot\|x\|_1<1\right\}. \tag{3.1.11$'$}$$

We note that B and B_ε are not convex although they are symmetric and absorbing.

By Corollary 3.1.2, for $x\in B$, we can define a linear map $\exp x$ from pre-Hilbert space $L(\Lambda)$ into its completion $H(\Lambda)$ by

$$(\exp x)v=\sum_{m=0}^{\infty}(m!)^{-1}x^m v,\qquad v\in L(\Lambda). \tag{3.1.12}$$

If $x\in B\cap\mathfrak{k}$, by the standard calculation, we see that $\exp x$ is an isometry. Hence, $\exp x$ is extended to an isometry from $H(\Lambda)$ into $H(\Lambda)$ uniquely, which will be denoted again by $\exp x$.

We prove the following two lemmas, key to define our group associated to $\mathfrak{k}$.

LEMMA 3.1.3. *For every* $x\in B\cap\mathfrak{k}$, $\exp x$ *is an unitary operator on* $H(\Lambda)$ *and we have*

$$(\exp x)^{-1}=\exp(-x).$$

PROOF. For $u, v\in L(\Lambda)$, we have

$$\begin{aligned}((\exp x)u\,|\,v)_\Lambda &= \sum_{m=0}^{\infty}(m!)^{-1}(x^m u\,|\,v)_\Lambda \\ &= \sum_{m=0}^{\infty}(m!)^{-1}(u\,|\,(-x)^m v)_\Lambda = (u\,|\,(\exp(-x))v)_\Lambda .\end{aligned}$$

Since $\exp(\pm x)$ are both isometries, the equality

$$((\exp x)u\,|\,v)_\Lambda = (u\,|\,(\exp(-x))v)_\Lambda \tag{3.1.13}$$

is also true for $u, v \in H(\Lambda)$. Hence, there holds that

$$\begin{aligned} v \in ((\exp x)H(\Lambda))^\perp &\Longrightarrow (\exp(-x))v \in H(\Lambda)^\perp \\ &\Longrightarrow (\exp(-x))v = 0 \qquad \Longrightarrow v = 0,\end{aligned}$$

whence $((\exp x)H(\Lambda))^\perp = 0$. Since $\exp x$ is an isometry, the image $(\exp x)H(\Lambda)$ is closed and so $(\exp x)H(\Lambda) = H(\Lambda)$. Therefore, $\exp x$ is an unitary operator and, from (3.1.13), the lemma holds. Q. E. D.

LEMMA 3.1.4. *For any $x, y \in B\cap\mathfrak{k}$, if $[x, y]=0$, then $\exp x$ and $\exp y$ commute. Moreover, if $x+y \in B\cap\mathfrak{k}$, then there holds the equality*

$$(\exp x)\cdot(\exp y) = \exp(x+y).$$

PROOF. Let $u, v \in L(\Lambda)$, then

$$\begin{aligned}((\exp x)(\exp y)u\,|\,v)_\Lambda &= ((\exp y)u\,|\,(\exp(-x))v)_\Lambda \\ &= \sum_{p,q=0}^{\infty}(p!\cdot q!)^{-1}(y^p u\,|\,(-x)^q v)_\Lambda \\ &= \sum_{m=0}^{\infty}\ \sum_{p\geq 0,\, q\geq 0,\, p+q=m}(p!\cdot q!)^{-1}(y^p u\,|\,(-x)^q v)_\Lambda \\ &= \sum_{m=0}^{\infty}\ \sum_{p,q}(p!\cdot q!)^{-1}(x^q y^p u\,|\,v)_\Lambda \\ &= \sum_{m=0}^{\infty}(m!)^{-1}((x+y)^m u\,|\,v)_\Lambda .\end{aligned}$$

The similar calculation leads us to the equality

$$((\exp y)(\exp x)u\,|\,v)_\Lambda = \sum_{m=0}^{\infty}(m!)^{-1}((y+x)^m u\,|\,v)_\Lambda .$$

Hence, $(\exp x)(\exp y)u - (\exp y)(\exp x)u \in L(\Lambda)^\perp = H(\Lambda)^\perp = 0$ for all $u \in L(\Lambda)$ and so for all $u \in H(\Lambda)$. Thus, the first statement is true. The second statement is clear from the above equalities. Q. E. D.

Let $U(\Lambda)$ be the group of unitary operators on $H(\Lambda)$ equipped with the strong topology of operators. We define the map exp from $\mathfrak{k}$ into $U(\Lambda)$ as follows. For any $x \in \mathfrak{k}$, there exists a positive integer m such that $m^{-1}x \in B$ and then, we put

$$\exp x = (\exp m^{-1}x)^m. \tag{3.1.14}$$

By Lemmas 3.1.3 and 3.1.4, this definition is independent of the choice of m and it holds that for any $x, y \in \mathfrak{k}$

$$(\exp x)^{-1} = \exp(-x), \tag{3.1.15}$$

$$(\exp x)(\exp y) = \exp(x+y) \quad \text{if } [x, y]=0. \tag{3.1.16}$$

3.2. Groups K^Λ and H^Λ. Denote by $K^\Lambda = K^\Lambda(A)$ the closed subgroup of $U(\Lambda)$ generated by $\exp \mathfrak{k}$, and put $H^\Lambda = H^\Lambda(A) = \exp\sqrt{-1}\mathfrak{h}_{\boldsymbol{R}}$. By (3.1.15) and (3.1.16), H^Λ is a subgroup of K^Λ.

Let $\boldsymbol{T}$ be the 1-dimensional torus. We consider the direct product group $\boldsymbol{T}^{P(\Lambda)}$, where $P(\Lambda)$ is, as before, the set of all weights of $L(\Lambda)$. For any $(c_\lambda)_{\lambda \in P(\Lambda)} \in \boldsymbol{T}^{P(\Lambda)}$, we define an element c of $U(\Lambda)$ by

$$cv = c_\lambda v \qquad (v \in L(\Lambda)_\lambda). \tag{3.2.1}$$

The map $(c_\lambda) \to c$ is an isomorphism from $\boldsymbol{T}^{P(\Lambda)}$ into $U(\Lambda)$ as topological groups. By this isomorphism, we regard $\boldsymbol{T}^{P(\Lambda)}$ as a compact subgroup of $U(\Lambda)$. It is clear that H^Λ is a subgroup of $\boldsymbol{T}^{P(\Lambda)}$:

$$H^\Lambda = \{(e^{\lambda(h)})_{\lambda \in P(\Lambda)} = \exp h \in \boldsymbol{T}^{P(\Lambda)} \mid h \in \sqrt{-1}\mathfrak{h}_{\boldsymbol{R}}\} \tag{3.2.2}$$

and that $\exp: \sqrt{-1}\mathfrak{h}_{\boldsymbol{R}} \to \boldsymbol{T}^{P(\Lambda)}$ is continuous.

Let $\Xi(\Lambda)$ be the subgroup of $\mathfrak{h}^*$ generated by $P(\Lambda)$. We have the following lemma.

Lemma 3.2.1. *The subgroup $\Xi(\Lambda)$ is expressed as*

$$\Xi(\Lambda) = \boldsymbol{Z}\Lambda + \sum_{i \in S(\Lambda)} \boldsymbol{Z}\alpha_i.$$

Proof is exactly the same as that of [**3**, § 15, Lemma (15.2)].

By this lemma, we see that $\Xi(\Lambda)$ is discrete if and only if the set $\{\Lambda, \alpha_i\ (i \in S(\Lambda))\}$ is linearly independent or else $\boldsymbol{Q}$-linearly dependent. Hence, $\Xi(\Lambda)$ is always discrete if A is of affine type or is non-degenerate because in both cases the above set is linearly independent. In case where $\Xi(\Lambda)$ is discrete, H^Λ is compact as shown below.

Theorem 3.2.2. *Assume $\Xi(\Lambda)$ to be discrete. Then, H^Λ is compact, and the dual of the compact abelian group H^Λ is isomorphic to $\Xi(\Lambda)$.*

PROOF. Let $\mathcal{L}$ be the kernel of $\exp: \sqrt{-1}\mathfrak{h}_{\boldsymbol{R}} \to \boldsymbol{T}^{P(\Lambda)}$. By (3.2.2), $\mathcal{L}$ is equal to $\{h \in \sqrt{-1}\mathfrak{h}_{\boldsymbol{R}} \mid \lambda(h) \in 2\pi\sqrt{-1}\boldsymbol{Z}$ for all $\lambda \in P(\Lambda)\}$. Hence, by the Pontrjagin duality of locally compact abelian groups, we have

$$\overline{\Xi(\Lambda)} \cong (\sqrt{-1}\mathfrak{h}_{\boldsymbol{R}}/\mathcal{L})^{\wedge}, \tag{3.2.3}$$

where $\bar{\ }$ means the closure in $\mathfrak{h}^*$ and $\hat{\ }$ the dual of a locally compact abelian group. Therefore, if $\Xi(\Lambda)$ is discrete, $\sqrt{-1}\mathfrak{h}_{\boldsymbol{R}}/\mathcal{L}$ is compact. Then, $\sqrt{-1}\mathfrak{h}_{\boldsymbol{R}}/\mathcal{L}$ is isomorphic to H^{Λ} as a topological group because exp induces a continuous bijective homomorphism from $\sqrt{-1}\mathfrak{h}_{\boldsymbol{R}}/\mathcal{L}$ onto H^{Λ}, and any continuous bijection from a compact space onto a Hausdorff space is homeomorphic. Q.E.D.

3.3. Irreducibility of the action of K^{Λ} on $H(\Lambda)$. Fix a positive number $\varepsilon<1$ and recall the definition (3.1.11) of the set B_{ε} on which the series $\sum_{m=0}^{\infty}\|(m!)^{-1}x^m v\|_{\Lambda}$ converge and are bounded for any $v \in L(\Lambda)$. For any $x \in \mathfrak{k}$, there exists a positive number $\delta<1$ such that if $t \in \boldsymbol{R}$ and $|t|<\delta$ then $tx \in B_{\varepsilon}$. Hence, for any $v \in L(\Lambda)$ and $t_0 \in \boldsymbol{R}$, the map $\{t \in \boldsymbol{R} \mid |t-t_0| \leqq \delta\} \ni t \to (\exp tx)v \in H(\Lambda)$ is analytic.

By a standard calculation, we get

$$d((\exp tx)v)/dt = (\exp tx)(xv) \quad (v \in L(\Lambda)). \tag{3.3.1}$$

Making use of this formula, we can prove the irreducibility of the natural action of K^{Λ} on $H(\Lambda)$ as follows.

THEOREM 3.3.1. *The natural action of K^{Λ} on $H(\Lambda)$ is irreducible.*

PROOF. Let T be a bounded linear operator on $H(\Lambda)$ which commutes with the action of K^{Λ}. In particular, T commutes with H^{Λ} and so each weight space is invariant under T. Hence, T defines a linear operator on $L(\Lambda)$ which we denote by the same symbol T. Therefore, by (3.3.1), for any $x \in \mathfrak{k}$ and any $v \in L(\Lambda)$, both sides of the equality $(\exp tx)Tv = T(\exp tx)v$ are differentiable at $t=0$, and we get the equality $xTv = Txv$. Since any linear operator on $L(\Lambda)$ commuting with $\mathfrak{k}$ (and so with $\mathfrak{g}$) is a scalar multiples of the identity operator, the restriction $T \mid L(\Lambda)$ is a scalar operator and so is T itself. Q.E.D.

§ 4. Relations with the Kac-Moody groups.

In this section, we discuss the relation between the group K^{Λ} and the subgroup of the Kac-Moody group which should correspond to the unitary form $\mathfrak{k}$.

4.1. Definition of the Kac-Moody group. At first, we recall the definition of Kac-Moody groups corresponding to $\mathfrak{g}$ given in [**9**].

Let $\tilde{G}$ be the free product group of root spaces of real roots as additive groups. We denote by i_α the inclusion $\mathfrak{g}^\alpha \to \tilde{G}$ for each $\alpha \in \Delta^{\mathrm{re}}$, the set of real roots. Let (π, V) be an integrable representation of $\mathfrak{g}$, that is, a representation of $\mathfrak{g}$ such that $\pi(x)$ is a locally nilpotent operator for any root vector x of any real root α. (The irreducible highest weight $\mathfrak{g}$-module $L(\Lambda)$ is an example of such a representation.) Then, π is integrated to a representation of $\tilde{G}$ by

$$\pi(i_\alpha(x)) = \exp \pi(x).$$

The Kac-Moody group G_0 is the quotient of $\tilde{G}$ by the intersection of kernels of all representations of $\tilde{G}$ defined from integrable representations of $\mathfrak{g}$ in such a way as described above.

4.2. The subgroup of G_0 corresponding to the unitary form. We denote again by ω_0 the automorphism on G_0 induced by ω_0 on $\mathfrak{g}$. Let K_0 be the set of fixed points of ω_0 in G_0. Then, K_0 should be the subgroup of G_0 corresponding to the subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. It was proved in [**9**, Corollary 4] that K_0 is generated by $\exp(\mathfrak{g}(\alpha) \cap \mathfrak{k})$ $(\alpha \in \Delta_+^{\mathrm{re}})$. Here, we write $\mathfrak{g}(\alpha)$ for $\mathfrak{g}(x_\alpha)$, the subalgebra generated by x_α and x_α^*, in § 2.3 because, if $\alpha \in \Delta^{\mathrm{re}}$, then $\dim \mathfrak{g}^\alpha = 1$ and so $\mathfrak{g}(x_\alpha)$ depends only on α and not on the choice of $x_\alpha \in \mathfrak{g}^\alpha$. Hence, if we denote by π_Λ the representation of G_0 on $L(\Lambda)$, $\pi_\Lambda(K_0)$ can be regarded as a subgroup of K^Λ.

The Kac-Moody group in [**9**] is minimal in a certain sense, and larger groups had been given by Garland [**3**] in case of non-twisted affine type, the simplest class of infinite-dimensional Kac-Moody algebras, and by Tits [**10**] in general case. Existence of the largest group was proved by Mathieu [**7**]. Roughly speaking, these groups are formal extension of the group in [**9**] in the direction of positive roots (but not in the direction of negative roots). Their subgroups corresponding to $\mathfrak{k}$ remain to be the same, not extended from K_0.

4.3. Relation between K_0 and K^Λ. By definition, H^Λ normalizes $\pi_\Lambda(K_0)$ and so $H^\Lambda \cdot \pi_\Lambda(K_0)$ is a subgroup of K^Λ. Since $\mathfrak{g}(\alpha) \cap \mathfrak{k}$ $(\alpha \in \Delta^{\mathrm{re}})$ and $\sqrt{-1}\mathfrak{h}_{\boldsymbol{R}}$ generate $\mathfrak{k}$ as a Lie algebra, the following problem naturally arises.

PROBLEM 4.3.1. *Does the closure of $H^\Lambda \cdot \pi_\Lambda(K_0)$ equal K^Λ?*

If the Campbell-Hausdorff formula holds for K^Λ, the problem is clearly affirmative. Of course, it is quite possible that the formula does not hold in our infinite-dimensional case and the problem might be solved negatively. However, the formula holds formally for any Lie algebra (cf. Cartier [**1**]). To give an exact meaning to the formula, we need a suitable topology on the Lie algebra $\mathfrak{k}$ with which the convergence of the formal series will be considered.

Such a topology has not been found for $\mathfrak{k}$ until now. The difficulty lies in the following point. If $\alpha\in\Delta^{\mathrm{im}}$, then $m\alpha$ is also a root for any non-zero integer m (see [4, §4, Lemma 14]). Therefore, for any $h\in\mathfrak{h}$ such that $\alpha(h)\neq 0$, we have

$$\sup_{\beta\in\Delta}|\beta(h)| = \infty .$$

This means that the eigenvalues of $\mathrm{ad}\,h$ are not bounded. Hence, for any vector topology of $\mathfrak{k}$, $\mathrm{ad}\,h$ is not bounded as an operator on $\mathfrak{k}$. Therefore, if $\Delta^{\mathrm{im}}\neq\emptyset$, that is, $\dim\mathfrak{k}=\infty$, there is no topology in which $\mathfrak{k}$ is a normed Lie algebra. This fact causes so much difficulty to define a suitable topology on $\mathfrak{k}$.

§5. Relations with the groups constructed on lowest weight modules.

In this section, first we define a conjugate linear bijection Ω_0 from the highest weight module $L(\Lambda)$ onto the lowest weight module $L^*(\Lambda)$ with lowest weight $-\Lambda$ which commutes with the action of $\mathfrak{k}$. Here, Λ is a dominant integral element of $\mathfrak{h}_R^*$ as before.

By making use of this bijection, it is proved that there exists a contravariant inner product on $L^*(\Lambda)$, and that the group K^Λ constructed in §3 is represented unitarily and faithfully on the completion $H^*(\Lambda)$ of the pre-Hilbert space $L^*(\Lambda)$. This representation of K^Λ can be identified, through Ω_0, with the contragredient of its natural unitary representation on $H(\Lambda)$.

5.1. Irreducible lowest weight modules. For an arbitrary $\lambda\in\mathfrak{h}^*$, we put

(5.1.1) $$M^*(\lambda) = U(\mathfrak{g})/U(\mathfrak{g})(\mathfrak{n}_- + \{h+\lambda(h) \mid h\in\mathfrak{h}\}).$$

Then, $M^*(\lambda)$ is a lowest weight module with lowest weight $-\lambda$, and has the same universality in the category of lowest weight modules as $M(\lambda)$ has in that of highest weight modules. We denote by $L^*(\lambda)$ the unique irreducible quotient of $M^*(\lambda)$, which is the unique (up to equivalence) irreducible lowest weight module with lowest weight $-\lambda$.

Let P^* be the projection from $U(\mathfrak{g})$ onto $U(\mathfrak{h})$ according to the decomposition $U(\mathfrak{g})=U(\mathfrak{h})\oplus(\mathfrak{n}_+U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{n}_-)$. We define a sesquilinear form $\langle\cdot|\cdot\rangle_\lambda^*$ on $U(\mathfrak{g})$ by

(5.1.2) $$\langle x\,|\,y\rangle_\lambda^* = P^*(y^*x)(-\lambda).$$

By the same argument as in §2.2, it is shown that, for $\lambda\in\mathfrak{h}_R^*$, the form $\langle\cdot|\cdot\rangle_\lambda^*$ is Hermitian and induces the unique (up to scalar multiples) non-degenerate contravariant Hermitian form $(\cdot|\cdot)_\lambda^*$ on $L^*(\lambda)$.

5.2. Conjugate linear $\mathfrak{k}$-homomorphism from $L(\lambda)$ onto $L^*(\lambda)$. In this subsection, we assume that $\lambda\in\mathfrak{h}_R^*$.

We denote by the same notation ω_0 the unique extension to $U(\mathfrak{g})$ of the conjugate linear automorphism ω_0 of $\mathfrak{g}$ in § 1.2. For every $h, h' \in \mathfrak{h}_R$, it holds that

$$\begin{aligned}\omega_0(h+\sqrt{-1}h'-\lambda(h+\sqrt{-1}h')) &= -h+\sqrt{-1}h'-\lambda(h)+\sqrt{-1}\lambda(h') \\ &= -h+\sqrt{-1}h'+\lambda(-h+\sqrt{-1}h').\end{aligned}$$

Hence, we have

$$\text{(5.2.1)} \qquad \begin{cases} \omega_0(\mathfrak{n}_+) = \mathfrak{n}_-, \\ \omega_0(\{h-\lambda(h) \mid h \in \mathfrak{h}\}) = \{h+\lambda(h) \mid h \in \mathfrak{h}\}. \end{cases}$$

Therefore, ω_0 defines a conjugate linear bijection Ω_0 from the highest weight module $M(\lambda)$ onto the lowest weight module $M^*(\lambda)$. It is clear that there holds

$$\text{(5.2.2)} \qquad \Omega_0(xv) = \omega_0(x)\Omega_0(v) \quad (x \in \mathfrak{g},\ v \in M(\lambda)),$$

and so Ω_0 induces a conjugate linear bijection from $L(\lambda)$ onto $L^*(\lambda)$, which we denote by Ω_0 again. This new Ω_0 satisfies (5.2.2) with $v \in L(\lambda)$.

By definition, we have

$$\text{(5.2.3)} \qquad \omega_0(\omega_0(x)^*) = x^*,$$

$$\text{(5.2.4)} \qquad P(x^*) = P(x)^*$$

for all $x \in U(\mathfrak{g})$ and

$$\text{(5.2.5)} \qquad \omega_0(y^*) = {}^t y$$

for all $y \in U(\mathfrak{h})$, where P is the projection from $U(\mathfrak{g})$ onto $U(\mathfrak{h})$ in § 2.2 and the map $U(\mathfrak{g}) \ni y \to {}^t y \in U(\mathfrak{g})$ is the unique antiautomorphism given by $\mathfrak{g} \ni y \to -y \in \mathfrak{g}$. Moreover, we have the following lemma.

LEMMA 5.2.1. *As projections from $U(\mathfrak{g})$ onto $U(\mathfrak{h})$, we have*

$$\omega_0 \cdot P \cdot \omega_0 = P^*.$$

PROOF. For any $x \in U(\mathfrak{g})$, $\omega_0(x)$ is written as

$$\omega_0(x) = y_1 + y_2$$

with $y_1 \in U(\mathfrak{h})$ and $y_2 \in \mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+$. Hence, we have

$$x = \omega_0(\omega_0(x)) = \omega_0(y_1) + \omega_0(y_2),$$

$$\omega_0(y_1) \in U(h) \text{ and } \omega_0(y_2) \in \omega_0(\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+) = \mathfrak{n}_+ U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_-.$$

Therefore, it holds that

$$P^*(x) = \omega_0(y_1) = \omega_0(P(\omega_0(x))). \qquad \text{Q.E.D.}$$

Now, we can prove the following lemma, which plays the principal role in the proof of the main result of this section.

LEMMA 5.2.2. *It holds that*

(5.2.6) $$(\Omega_0(v_1)|\Omega_0(v_2))_\lambda^* = (v_2|v_1)_\lambda \quad (v_1, v_2 \in L(\lambda)).$$

PROOF. Let $x, y \in U(\mathfrak{n}_-)$ and $v \in L(\lambda)_\lambda$ the canonical image of $1 \in U(\mathfrak{g})$. Then,

$$\begin{aligned}(\Omega_0(xv)|\Omega_0(yv))_\lambda^* &= P^*(\omega_0(y)^*\omega_0(x))(-\lambda) && \text{(by definition)}\\ &= ((\omega_0 P\omega_0)(\omega_0(y)^*\omega_0(x)))(-\lambda) && \text{(by Lemma 5.2.1)}\\ &= ({}^t(P(x^*y)))(-\lambda) && \text{(by (5.2.3)}\sim\text{(5.2.5))}\\ &= P(x^*y)(\lambda) = (yv|xv)_\lambda . && \text{Q.E.D.}\end{aligned}$$

COROLLARY 5.2.3. *If* $\lambda \in \mathfrak{h}_R^*$ *is dominant integral, then* $(\cdot|\cdot)_\lambda^*$ *is positive-definite.*

Thus, we can consider the completion $H^*(\Lambda)$ of $L^*(\Lambda)$ for the dominant integral Λ. By (5.2.2) and Lemma 5.2.2, we have the following theorem, one of our main results, which establishes the duality between highest weight modules and lowest weight modules same as in the finite-dimensional case.

THEOREM 5.2.4. *Let* $\Lambda \in \mathfrak{h}_R^*$ *be a dominant integral element. Then, the group* K^Λ *is represented faithfully and unitarily on* $H^*(\Lambda)$, *and this representation is the contragredient of the natural one on* $H(\Lambda)$.

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Added in Proof. Recently, similar evaluations as (3.1.1), (3.1.1′), (3.1.6) and (3.1.9) are given by E. R. Carrington of Rutgers University. She kindly sent me a handwritten manuscript (without title), and I am grateful to her.