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Harmonic metrics, harmonic tensors, and Gauss maps

By Bang-yen CHEN and Tadashi NAGANO*)

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§1. Introduction.

If $\phi:(M, g)\to(N, \tilde{g})$ is a harmonic immersion ([4] or Section 3), then the identity map $1_M:(M, g)\to(M, \phi^*\tilde{g})$ is harmonic (Proposition 3.1). Thus it would be natural to study the Riemannian metrics G such that $1_M:(M, g)\to(M, G)$ is harmonic. We say that G is then a harmonic metric with respect to the given Riemannian metric g. We will study the space $\mathcal{H}_g$ of these G in Section 2. We will show that $\mathcal{H}_g$ is a very large star set in the space, $\mathcal{S}$, of the symmetric covariant tensor fields of degree 2 on M. (Proposition 2.2 and Theorem 2.5). In Section 2 we will also introduce the concept of *(relative) harmonic tensors* and obtain many fundamental results of such tensors. In connection with Berger-Ebin decomposition of $\mathcal{S}$ ([1] or Lemma 2.4), we will show that the space of harmonic tensors in $\mathcal{S}$ is linearly isomorphic with the space, $\ker\delta$, of the co-closed tensor fields in $\mathcal{S}$ by the Einstein tensor field in the relativity theory for $\dim M\neq 2$ (Theorem 2.5, see Theorem 2.6 for $\dim M=2$). Moreover, we will relate harmonic tensors with holomorphic and geodesic vector fields on M (Theorems 2.11 and 2.12). Also we will obtain a volume-decreasing phenomenon in $T_g\mathcal{H}_g\cap\ker\delta$ (Theorem 2.10) and that $T_g\mathcal{H}_g\cap(\ker\delta)^\perp$ is the nullity space for the harmonic map $1_M:(M, g)\to(M, g)$ (Proposition 2.13).

In Section 3 we will obtain a necessary and sufficient condition for an immersion between two Riemannian manifolds to be harmonic. And we will obtain some simple applications of this result. Under the assumption that ϕ is an immersion into the euclidean space $(E^m, \tilde{g}_0)$ and the Gauss map $\Gamma: M\to(Q_{m-2}, \tilde{G}_0)$ is also an immersion we will show that (i) $\phi:(M, \phi^*\tilde{g}_0)\to(E^m, \tilde{g}_0)$ is harmonic if and only if $\phi:(M, \Gamma^*\tilde{G}_0)\to(E^m, \tilde{g}_0)$ is harmonic (Theorem 4.1) and (ii) the Gauss map $\Gamma:(M, \Gamma^*\tilde{G}_0)\to(Q_{m-2}, \tilde{G}_0)$ is harmonic and $1_M:(M, \phi^*\tilde{g}_0)\to(M, \Gamma^*\tilde{G}_0)$ is affine if and only if $(M, \phi^*\tilde{g}_0)$ is of constant curvature and either (a) $(M, \phi^*\tilde{g}_0)$ is immersed in a hypersphere of E^m as a minimal surface or (b) $(M, \phi^*\tilde{g}_0)$ is immersed into an open portion of the product surface of two planar circles (Theorem 5.8).

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For general results and known results on harmonic maps, please refer to Eells and Lemaire [4].

§ 2. Geometry of identity maps and harmonic tensors.

We consider an n-dimensional connected manifold M covered by systems of coordinates (x^h) and with a Riemannian metric $g=g_{ji}dx^j dx^i$, where the indices $h, i, j, k, \cdots$ run over the range $1, 2, \cdots, n$. We denote the Christoffel symbols with respect to g by

$$\Gamma_{ji}^h=\frac{g^{ha}}{2}\{\partial_j g_{ia}+\partial_i g_{ja}-\partial_a g_{ji}\}$$

and the covariant derivative of a tensor, say T_{ji}, by

$$\nabla_k T_{ji}=\partial_k T_{ji}-\Gamma_{kj}^a T_{ai}-\Gamma_{ki}^a T_{ja}\,,$$

where ∂_k denotes the partial differentiation with respect to x^k. We denote by $K_{kji}{}^h$ and R_{ji} the curvature tensor and the Ricci tensor, respectively.

Let G be another Riemannian metric on M. Denote by $\nabla^G, \cdots$, etc., the corresponding quantities on M for G. We say that G is a harmonic metric with respect to g if the identity map $1_M:(M, g)\to(M, G)$ is harmonic in the sense of [5]. Thus, the metric G is harmonic with respect to g if and only if $g^{ba}L_{ba}^h=0$, where $L_{ji}^h={}^G\Gamma_{ji}^h-\Gamma_{ji}^h$.

The main purpose of this section is to study the geometry of the identity map $1_M:(M, g)\to(M, G)$ such that 1_M is harmonic.

First we observe that $L=(L_{ji}^h)$ is a tensor of type (1, 2) satisfying $L_{ji}^h=L_{ij}^h$. Moreover, from the definition of Christoffel symbols, we also have

$$\nabla_k G_{ji}=G_{ja}L_{ki}^a+G_{ia}L_{kj}^a\,. \tag{2.1}$$

For the metric tensor G, if we put

$$f=\operatorname{tr}G=G_a^a=g^{ab}G_{ba}\,, \tag{2.2}$$

where "tr" denotes the trace operator with respect to g, then we find

$$\nabla_j f=2G_a^b L_{bj}^a\,. \tag{2.3}$$

We give the following *fundamental lemma* which provides an easy way to verify whether a metric G is harmonic with respect to the given one g.

LEMMA 2.1. *Let g and G be two Riemannian metrics on a manifold M. Then G is harmonic with respect to g if and only if*

$$\nabla_i f=2\omega_i\,, \tag{$*$}$$

where $f=\mathrm{tr}\,G$ *and* $\omega_i=\nabla^a G_{ai}$.

PROOF. From (2.1) we have

$$g^{ba}\nabla_b G_{ai}=G_{ci}g^{ba}L_{ba}^c+G_b^a L_{ai}^b. \tag{2.4}$$

Combining this with (2.3) we obtain the lemma.

Since (*) is a linear differential equation, Lemma 2.1 implies that if G is a Riemannian metric which is harmonic w.r.t. g, then so is $tg+(1-t)G$ for t in a neighborhood (open or not) of $[0, 1]$ on which it defines a Riemannian metric on M. If we denote by $\mathcal{M}$ the space of all Riemannian metrics on M, by $\mathcal{H}_g$ the subset of $\mathcal{M}$ given by $\mathcal{H}_g=\{G\in\mathcal{M}\mid G$ is harmonic w.r.t. $g\}$ and by $\mathcal{S}$ the space of all symmetric covariant tensor fields of degree 2, then we have the following.

PROPOSITION 2.2. *$\mathcal{H}_g$ is a star set centered at g in the vector space $\mathcal{S}$.*

Now, we introduce (relative) harmonic tensors as follows: a tensor G in $\mathcal{S}$ is called a *harmonic tensor with respect to g* if it satisfies condition (*). In this sense, the vector space of all harmonic tensors with respect to g may be thought of as "the tangent space" $T_g\mathcal{H}_g$, the tangent space to $\mathcal{H}_g$ at g. From Lemma 2.1 we also have the following.

LEMMA 2.3. *If G is a harmonic tensor with respect to g, then $g+tG$ is a harmonic Riemannian metric (with respect to g) for t in a neighborhood of* 0 *provided that M is compact.*

Let $\mathcal{T}_s^r$ denote the space of all tensors of type (r, s). We denote by δ the codifferential of $\mathcal{T}_s^r$ defined by $(\delta T)_{i_1\cdots i_s}^{j_1\cdots j_r}=-\nabla^a T_{a i_1\cdots i_s}^{j_1\cdots j_r}$. Then $\delta:\mathcal{T}_{s+1}^r\to\mathcal{T}_s^r$. We also denote by δ the restriction of δ to $\mathcal{S}$. A tensor T is called *co-closed* if $\delta T=0$. If a tensor $T\in\mathcal{S}$ is regarded as an energy-momentum tensor, the equation $\delta T=0$ is sometimes called the *conservative law of energy-momentum* (see, for instance, [8]).

We recall the following decomposition of Berger-Ebin. (Formula (3.1) in [1]).

LEMMA 2.4. *Let (M, g) be a compact Riemannian manifold. Then every symmetric tensor $S\in\mathcal{S}$ is the unique sum $S=T+\mathcal{L}_v g$ for some symmetric tensor $T\in\mathcal{S}$ and some vector field v such that* (1) $\delta T=0$ *and* (2) *T is orthogonal to $\mathcal{L}_v g$, that is, $\int_M T^{ab}(\nabla_a v_b+\nabla_b v_a)=0$, where $\mathcal{L}$ denotes the Lie derivative.*

The following two theorems describe the space of harmonic tensors (with respect to g).

THEOREM 2.5. *Let (M, g) be a Riemannian manifold of dimension $\neq 2$. Then we have the following linear isomorphism of vector spaces;*

$$\{G\in\mathcal{S}\mid G \text{ is harmonic w.r.t. } g\}\cong\ker\delta\equiv\{T\in\mathcal{S}\mid\delta T=0\}.$$

The isomorphism is given by the map $\varphi: G\mapsto G-(\mathrm{tr}\,G/2)g$.

PROOF. For any tensor G in $\mathcal{S}$ which is harmonic w.r.t. g, it is easy to verify that $\varphi(G)$ lies in $\ker\delta$. Thus φ is well-defined. Moreover, for any $T\in\ker\delta$, $\nabla^a T_{ai}=0$, let $G=T+(\mathrm{tr}\,T/(2-n))g$, then it is easy to verify that G is harmonic w.r.t. g and $\varphi(G)=T$. Thus φ is onto.

If G is harmonic w.r.t. g and $\varphi(G)=0$, then we have $G=(f/2)g$, where $f=\mathrm{tr}\,G$. From this we may prove that $G=f=0$. Thus φ is also one-to-one.

(Q.E.D.)

If M is 2-dimensional, the linear map φ is not injective, but still surjective. In this case we have the following decomposition of harmonic tensors in $\mathcal{S}$:

THEOREM 2.6. *If* $\dim M=2$, *then we have*

$$\{G\in\mathcal{S} \mid G \textit{ is harmonic w.r.t. } g\}=\{\lambda g\,|\,\lambda\in C^\infty(M)\}\oplus\{T\in\ker\delta\,|\,\mathrm{tr}\,T=0\}.$$

PROOF. Since M is 2-dimensional, λg is harmonic w.r.t. g for any function λ on M. Moreover, $\mathrm{Im}\,\varphi=\{T\in\ker\delta \mid \mathrm{tr}\,T=0\}$. It is easy to check that $\ker\varphi=\{\lambda g \mid \lambda\in C^\infty(M)\}$, where φ is the linear map given as in Theorem 2.5. (Q.E.D.)

REMARK 2.1. Theorems 2.5 and 2.6 say more than that the space of all harmonic Riemannian metrics (with respect to g) is infinite-dimensional.

Suppose M is 2-dimensional. By passing, if necessary, to the twofold covering surface, we may assume that M is orientable. It is then possible to choose a system of isothermal coordinates $\{x^1, x^2\}$ covering M. The metric tensor g on M has the following form;

$$g=E\{(dx^1)^2+(dx^2)^2\}.$$

LEMMA 2.7. *Let* (M, g) *be a Riemannian surface. Then a tensor* T *in* $\mathcal{S}$ *is a harmonic tensor with respect to* g *if and only if*

$$f_T=\frac{T_{11}-T_{22}}{2}+\sqrt{-1}\,T_{12} \tag{2.5}$$

is a holomorphic function in $z=x^1+\sqrt{-1}\,x^2$ *for any set of isothermal coordinates* x^1, x^2, *where* $T_{ij}=T(\partial/\partial x^i, \partial/\partial x^j)$.

PROOF. Let $X_i=\partial/\partial x^i$. Then the Christoffel symbols satisfy (see, for instance, p. 102 of [2])

$$\Gamma^1_{11}=\Gamma^2_{12}=-\Gamma^1_{22}=\frac{X_1E}{2E},\quad \Gamma^2_{22}=\Gamma^1_{12}=-\Gamma^2_{11}=\frac{X_2E}{2E}. \tag{2.6}$$

By direct computation we have

$$\partial_2\Big(\frac{T_{11}-T_{22}}{2}\Big)-\partial_1T_{12}=\frac{1}{2}\partial_2(T_{11}+T_{22})-(\partial_2T_{22}+\partial_1T_{12}) \tag{2.7}$$

$$=\frac{1}{2}\partial_2(E\,\mathrm{tr}\,T)-E\nabla^a T_{a2}-2\Gamma^a_{22}T_{a2}-\Gamma^a_{11}T_{a2}-\Gamma^a_{12}T_{1a}$$

$$=\frac{1}{2}\partial_2(E\,\mathrm{tr}\,T)-E\nabla^a T_{a2}-\Gamma^1_{12}(T_{11}+T_{22})$$

$$=\frac{E}{2}(\nabla_2(\mathrm{tr}\,T)-2\nabla^a T_{a2})\,.$$

Similarly, we also have

$$\partial_1\Big(\frac{T_{11}-T_{22}}{2}\Big)+\partial_2 T_{12}=\frac{E}{2}(\nabla_1(\mathrm{tr}\,T)-2\nabla^a T_{a1})\,. \tag{2.8}$$

Hence, by using Lemma 2.1, we see that the function f_T is holomorphic in $z=x^1+\sqrt{-1}\,x^2$ if and only if the symmetric tensor T is harmonic w.r.t. g.

If T in $\mathcal{S}$ is a harmonic tensor (with respect to g), then $\Phi_T=f_T dz^2$ defines a holomorphic quadratic differential on (M, g) globally. In particular, we have the following well-defined linear map

$$\Phi:\{T\in\ker\delta\mid \mathrm{tr}\,T=0\}\longrightarrow\{\text{holomorphic quadratic differentials}\}$$

defined by $\Phi: T\mapsto\Phi_T$. It is easy to verify from Lemmas 2.1 and 2.7 that Φ is one-to-one and onto. Consequently we have the following.

THEOREM 2.8. *If* $\dim M=2$ *and* M *is orientable, then the space* $\{T\in\ker\delta\mid \mathrm{tr}\,T=0\}$ *is linearly isomorphic with the space of holomorphic quadratic differentials on* M *with the natural complex structure.*

By using Theorem 2.8 and Riemann-Roch's theorem we obtain the following.

COROLLARY 2.9. *If* M *is a 2-dimensional sphere or a real projective plane, then*

$$\{G\in\mathcal{S}\mid G \text{ harmonic tensor w.r.t. } g\}=\{\lambda g\mid \lambda\in C^\infty(M)\}\,.$$

EXAMPLE 2.1. The second Bianchi identity implies that (i) the *Ricci tensor* R_{ij} of a Riemannian manifold (M, g) *is* always a *harmonic* tensor with respect to g and (ii) the Ricci tensor R is co-closed if and only if (M, g) has a constant scalar curvature.

REMARK. (Added on April 20, 1983). Recently, J. DeTurck and R. Hamilton (to be published) obtained a very good application of our result given in Example 2.1. Their result concerns uniqueness and non-existence of a Riemannian metric with a given tensor R as its Ricci form. They found that, on the Euclidean sphere S^n with the metric tensor equal to the Ricci form R, the only Riemannian metric that has R as its Ricci form is a constant scalar multiple of R and moreover, there is no metric whose Ricci form is cR for any constant $c>1$.

For a co-closed harmonic metric, we have the following interesting *volume-decreasing phenomenon.*

THEOREM 2.10. *Let G be a co-closed harmonic Riemannian metric on M with respect to g. Then we have*

(a) $\mathrm{tr}_g G$ *is constant,*

(b) *for any $c \in \boldsymbol{R}^+$, cG is also a co-closed harmonic Riemannian metric with respect to g, and*

(c) *if* $\mathrm{tr}_g G = \mathrm{tr}_g g$, *then*

(c.1) *the volume form $dV_G \leqq dV_g$ at each point of M; hence the map $1_M : (M, g) \to (M, G)$ is volume-decreasing and*

(c.2) *$dV_G = dV_g$ on M if and only if $G = g$.*

PROOF. Since G is harmonic w.r.t. g, Lemma 2.1 gives $\nabla_i f = 2\omega_i$, where $\omega_i = \nabla^a G_{ai}$ and $f = \mathrm{tr}\, G$. If G is co-closed, $\omega_i = 0$. Thus f is constant on M because M is assumed to be connected. This proves (a). Statement (b) follows easily from Theorem 2.5.

For the statement (c), denote by $\lambda_1, \cdots, \lambda_n$ the eigenvalues of (G_j^i), where $G_j^i = g^{ia} G_{aj}$. We have

$$\Big(\prod_{i=1}^{n} \lambda_i\Big)^{1/n} \leqq \frac{1}{n} \sum_{i=1}^{n} \lambda_i = \frac{1}{n} \mathrm{tr}\, G = 1 \,.$$

Thus the volume form dV_G for G satisfies $dV_G \leqq dV_g$. If $dV_G = dV_g$ on M, we have $\lambda_1 = \cdots = \lambda_n$. Thus $G = g$. The converse of this is trivial. (Q.E.D.)

In [**14**], Yano proved that a vector field $v = (v^i)$ on a compact Kaehlerian manifold is holomorphic if and only if $g^{ba} \nabla_b \nabla_a v^h + R_a{}^h v^a = 0$. In [**15**], Yano and Nagano used this equation to introduce the following.

DEFINITION 2.1. A vector field $v = (v^i)$ on a Riemannian manifold (M, g) is called a *geodesic vector field* if $g^{ba} \nabla_b \nabla_a v^h + R_a{}^h v^a = 0$.

In the following we shall give a *geometric characterization* for geodesic vector fields in terms of harmonic tensors. In order to do so we assume that the tensor G is orthogonal to $\ker\delta$; that is, $G = \mathcal{L}_v g$ for some vector field v (cf. Lemma 2.4). Thus we have $\nabla_j v_i + \nabla_i v_j = G_{ji}$. Hence we get

$$\nabla_k \nabla_j v_i + \nabla_k \nabla_i v_j = \nabla_k G_{ji} \,. \tag{2.9}$$

From this we find

$$\nabla_j \nabla_k v_i + \nabla_j \nabla_i v_k = \nabla_j G_{ki} \,, \tag{2.10}$$

$$\nabla_i \nabla_j v_k + \nabla_i \nabla_k v_j = \nabla_i G_{jk} \,. \tag{2.11}$$

From (2.9), (2.10) and (2.11) we obtain

$$\nabla_k \nabla_j v_i + \nabla_j \nabla_k v_i - K_{kij}{}^a v_a - K_{jik}{}^a v_a = \nabla_k G_{ji} + \nabla_j G_{ki} - \nabla_i G_{jk}.$$

By taking contraction on k and j, this gives

$$2\nabla^a\nabla_a v_i+2R_i{}^a v_a=2\nabla^a G_{ai}-\nabla_i f\,, \tag{2.12}$$

where $f=\operatorname{tr} G$. Combining this with Lemma 2.1 we obtain the following characterization of geodesic vector fields.

THEOREM 2.11. *A vector field v on a Riemannian manifold (M, g) is a geodesic vector field if and only if $G=\mathcal{L}_v g$ is a harmonic tensor (with respect to g).*

From this we obtain immediately the following.

THEOREM 2.12. *A vector field v on a compact Kaehlerian manifold (M, J, g) is holomorphic if and only if $G=\mathcal{L}_v g$ is a harmonic tensor (with respect to g).*

We recall that the second variation for the harmonic map $1_M:(M, g)\to(M, g)$ is given by [13]

$$\int\langle\nabla v, \nabla v\rangle-R(v, v)$$

for vector fields v on M. The nullity of the harmonic map 1_M is defined as the dimension of the space of the vector fields v such that

$$\int\langle\nabla v, \nabla u\rangle-R(v, u)=0$$

for any vector field u of compact support. Thus by (2.12) we have the following.

PROPOSITION 2.13. *Let (M, g) be a Riemannian manifold. Then the nullity of the harmonic map $1_M:(M, g)\to(M, g)$ is equal to the dimension of* $(\ker\delta)^{\perp}=\{G\in\mathcal{S}\mid G=\mathcal{L}_v g$ *for some geodesic vector field* $v\}$.

In particular, this implies the following.

COROLLARY 2.14. *Let (M, g) be a compact Riemannian manifold. Then $\{G\in\mathcal{S}\mid G$ is harmonic w.r.t. g and $G\perp\ker\delta\}$ is finite-dimensional.*

Combining Proposition 2.13 and a result of Yano-Nagano [16] we also have the following corollary immediately.

COROLLARY 2.15. *Let (M, g) be a compact irreducible symmetric space. Then the nullity of the harmonic map $1_M:(M, g)\to(M, g)$ is greater than the dimension of the isometry group if and only if (M, g) is either a Hermitian symmetric space or the exceptional simple group manifold G_2.*

REMARK 2.2. Harvey-Lawson [7] discovered that a class of manifolds which is intimately related to the group G_2 has a remarkable analog of the Kaehlerian manifolds in connection with the minimal submanifolds, especially with their absolute minimum property.

We need the following.

LEMMA 2.16. *Let g and G be two Riemannian metrics on a surface M. Then $1_M:(M, g)\to(M, G)$ is conformal if and only if $1'_M\cdot 1_M$ is bi-harmonic for any conformal change of metric $1'_M:(M, G)\to(M, e^{2\rho}G)$, where $1'_M$ is also the identity map. By $1'_M\cdot 1_M$ being bi-harmonic, we mean that both $1'_M\cdot 1_M$ and $(1'_M\cdot 1_M)^{-1}$ are harmonic.*

PROOF. We put $\bar{G}=e^{2\rho}G$. Assume that

$$1'_M\cdot 1_M:(M, g)\xrightarrow{1_M}(M, G)\xrightarrow{1'_M}(M, \bar{G})$$

is bi-harmonic for any $\rho\in C^\infty(M)$. Then by taking $\rho=0$, we see that 1_M is bi-harmonic. Thus, we have

$$\begin{aligned}0&=g^{ba}(\nabla^{\bar{G}}_{X_b}X_a-\nabla_{X_b}X_a)=g^{ba}(\nabla^{\bar{G}}_{X_b}X_a-\nabla^{G}_{X_b}X_a)\\&=g^{ba}(\Phi(X_b)X_a+\Phi(X_a)X_b-G(X_b, X_a)U^G)\,,\end{aligned}$$

where $X_i=\partial/\partial x^i$, $\Phi=d\rho$, and U^G is the vector field associated with the 1-form Φ for G. From this we find

$$(\operatorname{tr} G)U^G=2U\,, \tag{2.13}$$

where U is the vector field associated with Φ for g. Because this is true for any function ρ on M, 1_M must be conformal on the whole surface M. The converse of this follows from a result of Eells and Sampson [5].

§ 3. Harmonic immersions.

Let $\phi:(M, g)\to(N, \tilde{g})$ be a map between two Riemannian manifolds. The ϕ is harmonic by definition if its tension field $\tau(\phi)=\operatorname{div}(\phi_*)$ vanishes or equivalently ϕ satisfies

$$g^{ji}(\nabla^{\tilde{g}}_{\phi_* X_j}\phi_* X_i-\phi_*\nabla^{g}_{X_j}X_i)=0 \tag{3.1}$$

where $X_j=\partial/\partial x^j$ and $\nabla^{\tilde{g}}$ and ∇^g denote the covariant derivatives of $(N, \tilde{g})$ and (M, g), respectively. Let $G=\phi^*\tilde{g}$ and ∇^G denote the covariant derivative of (M, G). Then (3.1) is equivalent to

$$g^{ji}(\nabla^{\tilde{g}}_{\phi_* X_j}\phi_* X_i-\phi_*\nabla^{G}_{X_j}X_i)=g^{ji}(\nabla^{G}_{X_j}X_i-\nabla^{g}_{X_j}X_i)=0\,. \tag{3.2}$$

Let h denote the second fundamental form of the isometric immersion $\phi:(M, G)\to(N, \tilde{g})$. Then $h(X_j, X_i)=\nabla^{\tilde{g}}_{\phi_* X_j}\phi_* X_i-\phi_*\nabla^{G}_{X_j}X_i$. And hence (3.2) gives the following.

PROPOSITION 3.1. *Let $\phi:(M, g)\to(N, \tilde{g})$ be an immersion between two Riemannian manifolds. Then ϕ is harmonic if and only if* (a) *$\phi^*\tilde{g}$ is a harmonic tensor with respect to g and* (b) *$\operatorname{tr}_g h=0$ identically.*

Note that $(1/n)\mathrm{tr}_g h$ is not the mean-curvature vector of $\phi:(M, \phi^*\tilde{g})\to(N, \tilde{g})$. From Proposition 3.1 we obtain easily the following [6].

COROLLARY 3.2. *Let $\phi:(M, g)\to(N, \tilde{g})$ be a harmonic immersion from (M, g) into a complete, 1-connected Riemannian manifold $(N, \tilde{g})$ of nonpositive sectional curvature. Then M is not compact.*

PROOF. If M is compact and ϕ is an immersion from M into $(N, \tilde{g})$, then the second fundamental form h of the isometric immersion $\phi:(M, \phi^*\tilde{g})\to(N, \tilde{g})$ is positive-definite with respect to some normal direction at some point $p\in M$. Since g is positive-definite, $\mathrm{tr}_g h\neq 0$ at p. Thus Proposition 3.1 yields a contradiction. (Q.E.D.)

Now assume that $\phi:(M, g)\to(N, \tilde{g})$ is a harmonic immersion. Denote by h the second fundamental form of $\phi:(M, G)\to(N, \tilde{g})$, where $G=\phi^*\tilde{g}$. Let R^G and $R^{\tilde{g}}$ denote the curvature tensors of (M, G) and $(N, \tilde{g})$, respectively. Then the Gauss equation gives

$$R^G(X, Y; Z, W)=R^{\tilde{g}}(\phi_*X, \phi_*Y; \phi_*Z, \phi_*W) +\tilde{g}(h(X, W), h(Y, Z))-\tilde{g}(h(X, Z), h(Y, W)), \tag{3.3}$$

for X, Y, Z, W tangent to M. Assume that $\dim M=2$. Let E_1, E_2 be an orthonormal basis on M with respect to $G=\phi^*\tilde{g}$. Then by putting $X=W=E_1$ and $Y=Z=E_2$, we obtain from (3.3) the following.

$$2K_G(T_pM)=2K_{\tilde{g}}(\phi_*T_pM)+\tilde{g}(\mathrm{tr}_G h, \mathrm{tr}_G h)-\|h\|_{\tilde{g}}^2, \tag{3.4}$$

where K_G and $K_{\tilde{g}}$ denote the sectional curvatures of (M, G) and $(N, \tilde{g})$, respectively.

If $\mathrm{tr}_G h=0$ at a point $p\in M$, (3.4) gives the well-known inequality $K_G\leqq K_{\tilde{g}}$. Moreover, $K_G=K_{\tilde{g}}$ if and only if $h=0$.

If $\mathrm{tr}_G h\neq 0$ at p, we may choose an orthonormal basis $E_1, E_2, \xi_3, \cdots, \xi_m$ of T_pN w.r.t. $\tilde{g}$ such that E_1 and E_2 are tangent to M and they diagonalize the symmetric matrix (g^{ij}). Furthermore, we may assume that ξ_3 is parallel to $\mathrm{tr}_G h$. Since $\phi:(M, g)\to(N, \tilde{g})$ is harmonic, Proposition 3.1 then implies

$$g^{11}h(E_1, E_1)+g^{22}h(E_2, E_2)=0.$$

In particular, this gives $h_{22}^3=-(g^{11}/g^{22})h_{11}^3$, where $h_{ij}^r=\tilde{g}(h(E_i, E_j), \xi_r)$. Because (g^{ij}) is positive-definite, this implies $h_{11}^3h_{22}^3\leqq 0$. Moreover, since ξ_3 is assumed to be parallel to $\mathrm{tr}_G h$, we also have $h_{11}^r h_{22}^r\leqq 0$ for $r=4, \cdots, m$. Therefore, by using equation (3.3) of Gauss we conclude that $K_G(T_pM)\leqq K_{\tilde{g}}(\phi_*T_pM)$ and the equality holds if and only if $h=0$ at p. Consequently, Proposition 3.1 gives us the following decreasing property for harmonic immersions of a surface [12].

COROLLARY 3.3. *Let $\phi:(M, g)\to(N, \tilde{g})$ be a harmonic immersion between two*

Riemannian manifolds. If $\dim M=2$, *then* $\phi:(M, \phi^*\tilde{g})\to(N, \tilde{g})$ *decreases the sectional curvature unless this* ϕ *is totally geodesic. More precisely, the sectional curvature of* $(M, \phi^*\tilde{g})$ *and* $(N, \tilde{g})$ *satisfy* $K_{\phi^*\tilde{g}}(T_pM)\leqq K_{\tilde{g}}(\phi_*T_pM)$, *for* $p\in M$. *And if* $K_{\phi^*\tilde{g}}(T_pM)\equiv K_{\tilde{g}}(\phi_*T_pM)$, *then* $\phi:(M, \phi^*\tilde{g})\to(N, \tilde{g})$ *is totally geodesic.*

Proposition 3.1 also implies easily the following.

COROLLARY 3.4. *Let* $(N, \tilde{g})$ *be a Riemannian manifold of non-positive sectional curvature and* (M, g) *be a compact 2-dimensional Riemannian manifold. If* (M, g) *admits a harmonic immersion into* $(N, \tilde{g})$, *then the Euler characteristic of* M *satisfies* $\chi(M)\leqq 0$. *If* $\chi(M)=0$, *then any harmonic immersion from* (M, g) *into* $(N, \tilde{g})$ *is totally geodesic, i.e.,* $h\equiv 0$.

§4. Gauss maps.

Let $\phi: M\to(E^m, \tilde{g}_0)$ be an immersion from a surface M into the euclidean m-space E^m, where $\tilde{g}_0$ denotes the standard metric on E^m. *We denote by* g_0 *the induced metric on* M *via* ϕ. Let Q_{m-2} be the Grassmann manifold consisting of 2-dimensional oriented linear subspaces of E^m. It is well-known that Q_{m-2} admits a standard Riemannian metric $\tilde{G}_0$ which makes Q_{m-2} a symmetric space of rank two. Moreover, with respect to a natural complex structure, Q_{m-2} is holomorphically isometric to the complex $(m-2)$-dimensional complex quadric.

The *Gauss map* $\Gamma: M\to(Q_{m-2}, \tilde{G}_0)$ is the map which is obtained by parallel displacement of the tangent plane T_pM (more precisely, ϕ_*T_pM) of M at p in E^m. For simplicity, we always assume that the Gauss map is regular. So, *the Gauss map induces a metric, denoted by* G_0, *on* M. In this and the next sections *we will study relationship between these two canonical metrics* g_0 *and* G_0 *on* M.

Let H and h denote the mean-curvature vector and the second fundamental form of the isometric immersion $\phi:(M, g_0)\to(E^m, \tilde{g}_0)$, respectively. Then a result of Obata [**10**] gives

$$G_0(X, Y)=2\tilde{g}_0(H, h(X, Y))-Kg_0(X, Y) \tag{4.1}$$

where K denotes the Gaussian curvature of (M, g_0).

In Sections 4 and 5 we make the following

ASSUMPTION. $\phi: M\to(E^m, \tilde{g}_0)$ *is an immersion from a surface* M *into* E^m *such that its Gauss map is regular.*

In this section, we obtain the following relation between g_0 and G_0.

THEOREM 4.1. $\phi:(M, g_0)\to(E^m, \tilde{g}_0)$ *is harmonic if and only if* $\phi:(M, G_0)\to(E^m, \tilde{g}_0)$ *is harmonic.*

PROOF. Under the hypothesis, if $\phi:(M, g_0)\to(E^m, \tilde{g}_0)$ is harmonic, $\phi:(M, g_0)\to(E^m, \tilde{g}_0)$ is minimal. Thus equation (4.1) implies

$$(4.2)\qquad G_0(X, Y)=-Kg_0(X, Y).$$

Therefore, $\mathrm{tr}_{G_0}h=-(1/K)\mathrm{tr}_{g_0}h=0$. Hence, by Lemma 2.16 and (4.2) we conclude that $\phi:(M, G_0)\to(E^m, \tilde{g}_0)$ is also harmonic (Proposition 3.1).

Conversely, if $\phi:(M, G_0)\to(E^m, \tilde{g}_0)$ is harmonic, Proposition 3.1 implies

$$(4.3)\qquad \mathrm{tr}_{G_0}h=0, \qquad \text{and}$$

$$(4.4)\qquad \hat{G}_0^{ji}(\hat{\Gamma}_{ji}^k-\Gamma_{ji}^k)=0,$$

where $(\hat{G}_0^{ji})$ is the inverse matrix of $((G_0)_{ji})$ and $\hat{\Gamma}_{ji}^k$ and Γ_{ji}^k denote the Christoffel symbols of G_0 and g_0, respectively.

We put $M_1=\{p\in M \mid \mathrm{tr}_{g_0}h\neq 0 \text{ at } p\}$. Then M_1 is an open subset of M. On M_1, there is an orthonormal local frame $E_1, E_2, \xi_3, \cdots, \xi_m$ of E^m w.r.t. $\tilde{g}_0$ such that, restricted to M, E_1 and E_2 are tangent to M and ξ_3 is parallel to $H=(1/2)\mathrm{tr}_{g_0}h$. Furthermore, from (4.3) we see that $\dim(\mathrm{Im}\,h)\leqq 2$ at each point of M. Thus we may also assume that the second fundamental tensors $A_3, \cdots, A_m$, w.r.t. $E_1, E_2, \xi_3, \cdots, \xi_m$, take the following forms:

$$(4.5)\qquad A_3=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}, \qquad A_4=\begin{pmatrix}\alpha & \mu\\ \mu & -\alpha\end{pmatrix}, \qquad A_5=\cdots=A_m=0,$$

where $g_0(A_r(X), Y)=\tilde{g}_0(h(X, Y), \xi_r)$. From (4.1) we find

$$(4.6)\qquad (G_0)_{11}=\lambda_1^2+\alpha^2+\mu^2, \quad (G_0)_{12}=0, \quad (G_0)_{22}=\lambda_2^2+\alpha^2+\mu^2,$$

where $(G_0)_{ij}=G_0(E_i, E_j)$. Using (4.3), (4.5) and (4.6) we get

$$(4.7)\qquad (\lambda_1+\lambda_2)(\lambda_1\lambda_2+\alpha^2+\mu^2)=0, \qquad \alpha(\lambda_2^2-\lambda_1^2)=0.$$

On M_1, $\lambda_1+\lambda_2\neq 0$. Thus (4.7) gives $\alpha(\lambda_1-\lambda_2)=\lambda_1\lambda_2+\alpha^2+\mu^2=0$. If $\alpha\neq 0$, these imply $h=0$. Because the Gauss map is assumed to be regular, (4.1) yields a contradiction. Consequently, $\alpha=0$ identically on M_1. Hence, (4.6) and (4.7) reduce to

$$(4.8)\qquad (G_0)_{11}=\lambda_1^2+\mu^2, \quad (G_0)_{12}=0, \quad (G_0)_{22}=\lambda_2^2+\mu^2,$$

$$(4.9)\qquad \lambda_1\lambda_2+\mu^2=0, \qquad \lambda_1\lambda_2\neq 0, \qquad \mu\neq 0,$$

on M_1. For convenience we put

$$(4.10)\qquad \nabla_{E_j}^{G_0}E_i=\widetilde{\left\{\begin{matrix}a\\ j\,i\end{matrix}\right\}}E_a, \qquad \nabla_{E_j}^{g_0}E_i=\left\{\begin{matrix}a\\ j\,i\end{matrix}\right\}E_a.$$

From (4.8) and (4.10) we have (see, Kobayashi-Nomizu [**9**], vol. I, p. 160, for instance)

$$\widetilde{\left\{\begin{matrix}1\\ 11\end{matrix}\right\}}=\frac{1}{\lambda_1^2+\mu^2}\{\lambda_1(E_1\lambda_1)+\mu(E_1\mu)\},$$

$$(4.11)\qquad \widetilde{\begin{Bmatrix}1\\22\end{Bmatrix}}=\frac{-1}{\lambda_1^2+\mu^2}\{\lambda_2(E_1\lambda_2)+\mu(E_1\mu)-G_0([E_1,\ E_2],\ E_2)\},$$

$$\widetilde{\begin{Bmatrix}2\\11\end{Bmatrix}}=\frac{-1}{\lambda_2^2+\mu^2}\{\lambda_1(E_2\lambda_1)+\mu(E_2\mu)-G_0([E_2,\ E_1],\ E_1)\},$$

$$\widetilde{\begin{Bmatrix}2\\22\end{Bmatrix}}=\frac{1}{\lambda_2^2+\mu^2}\{\lambda_2(E_2\lambda_2)+\mu(E_2\mu)\}.$$

For simplicity, we locally define 1-forms ω_i^k by $\omega_i^k(E_j)=\begin{Bmatrix}k\\ij\end{Bmatrix}$. Then we obtain from (4.4), (4.8) and (4.10) the following

$$(4.12)\qquad \hat{G}_0^{11}\left(\widetilde{\begin{Bmatrix}k\\11\end{Bmatrix}}-\omega_1^k(E_1)\right)+\hat{G}_0^{22}\left(\widetilde{\begin{Bmatrix}k\\22\end{Bmatrix}}-\omega_2^k(E_2)\right)=0\,.$$

If $k=1$, (4.8), (4.11), and (4.12) give

$$(4.13)\qquad (\lambda_2^2+\mu^2)\lambda_1(E_1\lambda_1)-(\lambda_1^2+\mu^2)\lambda_2(E_1\lambda_2)$$
$$=\mu(\lambda_1^2-\lambda_2^2)(E_1\mu)+(\lambda_1^2+\mu^2)(\lambda_1^2-\lambda_2^2)\omega_2^1(E_2)\,.$$

On the other hand, (4.9) gives

$$(4.14)\qquad \lambda_2(E_1\lambda_1)=-\lambda_1(E_1\lambda_2)-2\mu(E_1\mu)\,.$$

Hence, by (4.9), (4.13) and (4.14) we obtain

$$(4.15)\qquad \mu(E_1\lambda_2)=\lambda_2(E_1\mu)-\frac{\lambda_2(\lambda_1^2+\mu^2)(\lambda_1+\lambda_2)}{\mu(\lambda_1-\lambda_2)}\omega_2^1(E_2)\,.$$

Similarly, we also have

$$(4.16)\qquad \mu(E_2\lambda_1)=\lambda_1(E_2\mu)-\frac{\lambda_1(\lambda_2^2+\mu^2)(\lambda_1+\lambda_2)}{\mu(\lambda_2-\lambda_1)}\omega_1^2(E_1)\,.$$

From the Codazzi equation of $\phi:(M,\ g_0)\to(E^m,\ \tilde{g}_0)$, we have $(\overline{\nabla}_{E_2}h)(E_1,\ E_1)=(\overline{\nabla}_{E_1}h)(E_2,\ E_1)$ (see [2] for instance). Thus

$$(4.17)\qquad \lambda_1(D_{E_2}\xi_3)+(E_2\lambda_1)\xi_3-2\omega_1^2(E_2)\mu\xi_4$$
$$=\mu(D_{E_1}\xi_4)+(E_1\mu)\xi_4-\omega_2^1(E_1)\lambda_1\xi_3-\omega_1^2(E_1)\lambda_2\xi_3\,,$$

where D denotes the normal connection of $\phi:(M,\ g_0)\to(E^m,\ \tilde{g}_0)$. If we put

$$(4.18)\qquad D\xi_r=\omega_r^s\xi_s\,,\qquad r,\ s=3,\ 4,\ \cdots,\ m\,,$$

then (4.17) gives

$$(4.19)\qquad E_2\lambda_1=\mu\omega_4^3(E_1)+(\lambda_1-\lambda_2)\omega_1^2(E_1)\,,$$

$$(4.20)\qquad E_1\mu=\lambda_1\omega_3^4(E_2)-2\mu\omega_1^2(E_2)\,.$$

Similarly, we also have

$$E_1\lambda_2=\mu\omega_4^3(E_2)+(\lambda_1-\lambda_2)\omega_1^2(E_2)\,, \tag{4.21}$$

$$E_2\mu=\lambda_2\omega_3^4(E_1)-2\mu\omega_2^1(E_1)\,. \tag{4.22}$$

Substituting (4.20) and (4.21) into (4.15) we find

$$\omega_1^2(E_2)=0\,. \tag{4.23}$$

Similarly, substituting (4.19) and (4.22) into (4.16), we find

$$\omega_1^2(E_1)=0\,. \tag{4.24}$$

From (4.23) and (4.24) we see that (M_1, g_0) is flat. Thus, the Gauss equation implies $\lambda_1\lambda_2-\mu^2=0$. Combining this with (4.9) we see that $M_1=\emptyset$. Thus $\phi:(M, g_0)\to(E^m, \tilde{g}_0)$ is minimal. Therefore, by Proposition 3.1, $\phi:(M, g_0)\to(E^m, \tilde{g}_0)$ is harmonic. (Q.E.D.)

REMARK 4.1. If $\dim M\geqq 3$, Theorem 4.1 is not true. In fact, there exists a hypersurface M in E^{n+1}, $n\geqq 3$, such that $\phi:(M, g_0)\to(E^{n+1}, \tilde{g}_0)$ is harmonic but $\phi:(M, G_0)\to(E^{n+1}, \tilde{g}_0)$ is not.

§ 5. Harmonic Gauss maps.

Let $\phi: M\to(E^m, \tilde{g}_0)$ be an immersion from a surface M into E^m such that its Gauss map is regular. Then, as we mentioned in Section 4, M admits two canonical metrics g_0 and G_0, one induced from ϕ, the other induced from its Gauss map Γ. In [11], Ruh and Vilms studied the problem "when is the Gauss map $\Gamma:(M, g_0)\to(Q_{m-2}, \tilde{G}_0)$ harmonic?" And they obtained a beautiful theorem which says that $\Gamma:(M, g_0)\to(Q_{m-2}, \tilde{G}_0)$ is harmonic if and only if the mean-curvature vector of $\phi:(M, g_0)\to(E^m, \tilde{g}_0)$ is parallel (in the normal bundle). In this section, we shall study the following

PROBLEM. *When is the Gauss map $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ harmonic?*

Since G_0 is the induced metric on M via Γ, this is equivalent to ask when is the Gauss map $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ minimal? For simplicity we denote by $\tilde{h}$ the second fundamental form of the isometric immersion $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$. Let X_1, X_2 be the local coordinate vector fields; $X_i=\partial/\partial x^i$. Then by a formula of [3], we know that the Gauss map $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ is harmonic if and only if

$$\hat{G}_0^{ji}\{(\overline{\nabla}_X h)(X_j, X_i)+h(\nabla^{g_0}_{X_j}X_i-\nabla^{G_0}_{X_j}X_i, X)\}=0\,, \tag{5.1}$$

where h denotes the second fundamental form of $\phi:(M, g_0)\to(E^m, \tilde{g}_0)$. It follows from definition that $\hat{G}_0^{ji}(\nabla^{g_0}_{X_j}X_i-\nabla^{G_0}_{X_j}X_i)=-\tau_0$, where τ_0 is the tension field of the identity map $1_M:(M, G_0)\to(M, g_0)$. Thus, (5.1) gives the following.

PROPOSITION 5.1. *The Gauss map $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ is harmonic if and only if, for any $X\in TM$, we have*

$$h(\tau_0, X)=\mathrm{tr}_{G_0}(\overline{\nabla}_X h). \tag{5.2}$$

Since it is a very large class of surfaces in E^m whose Gauss map $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ is harmonic (or minimal) (see [3] for surfaces whose Gauss images are totally geodesic), complete classification of such surfaces seems to be formidable. Thus we should study the problem under some additional assumptions. First we give the following.

PROPOSITION 5.2. *The Gauss map $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ is harmonic and $1_M:(M, G_0)\to(M, g_0)$ is conformal if and only if either* (a) *$\phi:(M, g_0)\to(E^m, \tilde{g}_0)$ is harmonic or* (b) *there is a hypersphere S^{m-1} (with the canonical metric) of E^m such that $\phi:(M, g_0)\to S^{m-1}$ is harmonic.*

PROOF. If $1_M:(M, G_0)\to(M, g_0)$ is conformal, then 1_M is harmonic. Thus Proposition 5.1 shows that if the Gauss map $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ is harmonic, then $\mathrm{tr}_{G_0}(\overline{\nabla}_X h)=\lambda\,\mathrm{tr}_{g_0}(\overline{\nabla}_X h)=2D_X H=0$, where $G_0=(1/\lambda)g_0$. Thus, the mean-curvature vector is parallel. Moreover, from (4.1) we also have $\tilde{g}_0(H, h(X, Y))=\mu g_0(X, Y)$ for $\mu=(1/2)((1/\lambda)+K)$. This shows that $\phi:(M, g_0)\to(E^m, \tilde{g}_0)$ is pseudo-umbilical. Consequently, (M, g_0) is either minimal in $(E^m, \tilde{g}_0)$ or minimal in a hypersphere S^{m-1} of E^m via ϕ (see p. 69 of [2]). Since $\phi:(M, g_0)\to(E^m, \tilde{g}_0)$ is isometric, either $\phi:(M, g_0)\to(E^m, \tilde{g}_0)$ is harmonic or $\phi:(M, g_0)\to S^{m-1}$ is harmonic. The converse of this is easy to verify. (Q.E.D.)

In particular, if $1_M:(M, G_0)\to(M, g_0)$ is homothetic, we have the following.

COROLLARY 5.3. *The Gauss map $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ is harmonic and $1_M:(M, G_0)\to(M, g_0)$ is homothetic if and only if (M, g_0) is of constant curvature and there is a hypersphere S^{m-1} of E^m such that $\phi:(M, g_0)\to S^{m-1}$ is harmonic.*

PROOF. If $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ is harmonic and $1_M:(M, G_0)\to(M, g_0)$ is homothetic, then by (4.1) and Proposition 5.2 we may conclude that (M, g_0) has constant curvature. If case (a) of Proposition 5.2 occurs, (M, g_0) is a minimal surface of E^m via ϕ. Since (M, g_0) has constant curvature, this implies that $\phi:(M, g_0)\to(E^m, \tilde{g}_0)$ is totally geodesic by a result of Chen and Yau. Thus $\Gamma(M)$ is a point. This contradicts the regularity of Γ. Thus only case (b) of Proposition 5.2 may occur. The converse of this follows from (4.1) and Proposition 5.2. (Q.E.D.)

In order to classify surfaces in E^m such that $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ is harmonic and $1_M:(M, G_0)\to(M, g_0)$ is affine, we give the following lemmas.

LEMMA 5.4. *Let g and G be two Riemannian metrics on a surface M. If*

$1_M:(M, G)\to(M, g)$ *is affine, then either* (a) 1_M *is homothetic or* (b) *both* (M, g) *and* (M, G) *are flat.*

PROOF. Let H_0 denote the restricted linear holonomy group of M common to g and G. If H_0 is not trivial, H_0 is irreducible. Hence, by Schur's lemma, G and g are conformal, i.e., $G=e^{2\rho}g$ for some function ρ. Since 1_M is affine, ρ must be a constant. If H_0 is trivial, both (M, g) and (M, G) are flat.

Combining Corollary 5.3 and Lemma 5.4 we obtain

LEMMA 5.5. *If* $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ *is harmonic and* $1_M:(M, G_0)\to(M, g_0)$ *is affine, then either* (a) *both* (M, g_0) *and* (M, G_0) *are flat or* (b) (M, g_0) *is of constant curvature and* ϕ *maps* (M, g_0) *isometrically into a hypersphere* S^{m-1} *of* E^m *as a minimal surface.*

Corollary 5.3 shows that if case (b) of Lemma 5.5 occurs, then $1_M:(M, G_0)\to(M, g_0)$ is always affine and $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ is always harmonic. Therefore, we should only consider the case (a) in which both (M, g_0) and (M, G_0) are flat.

LEMMA 5.6. *If* (M, g_0) *is flat and* $1_M:(M, G_0)\to(M, g_0)$ *is affine, then* A_H *is parallel, where* A_H *is the second fundamental tensor given by* $g_0(A_HX, Y)=\tilde{g}_0(h(X, Y), H)$.

PROOF. Since (M, g_0) is flat, equation (4.1) gives $G_0(X, Y)=2g_0(A_HX, Y)$. For any fixed point $p\in M$, let (x^1, x^2) be an orthogonal coordinate system such that $X_i=\partial/\partial x^i$, $i=1, 2$, are parallel and they diagonalize the symmetric matrix (G_{ij}) at p. Then we have $\Gamma^k_{ji}\equiv 0$ and $G_{12}(p)=0$. Because $1_M:(M, G_0)\to(M, g_0)$ is affine, we also have ${}^{G_0}\Gamma^k_{ji}\equiv 0$. Hence G_{ij} are constants on the coordinate neighborhood. Because X_i are parallel vector fields, these imply that A_H is a parallel tensor. (Q.E.D.)

We put $M_1=\{p\in M \mid G_0(p)=\lambda g_0(p)$ for some $\lambda\}$ and $M_2=M-M_1$. Then M_2 is an open subset of M. The following result follows from Corollary 5.3.

LEMMA 5.7. *If* $1_M:(M, G_0)\to(M, g_0)$ *is affine, and* $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ *is harmonic, then each component of* $\mathrm{int}(M_1)$ *lies in a hypersphere of* E^m *as a minimal surface via* ϕ.

Now, we give the following main result of this section.

THEOREM 5.8. *The Gauss map* $\Gamma:(M, G_0)\to(Q_{m-2}, \tilde{G}_0)$ *is harmonic and* $1_M:(M, G_0)\to(M, g_0)$ *is affine if and only if* (M, g_0) *is of constant curvature and either*

(a) (M, g_0) *is immersed in a hypersphere of* E^m *as a minimal surface via* ϕ, *or*

(b) (M, g_0) *is immersed as an open portion of the product surface of two planar circles via* ϕ.

PROOF. Under the hypothesis, if 1_M is affine, then either 1_M is homothetic or both (M, g_0) and (M, G_0) are flat (Lemma 5.4). If 1_M is homothetic, (M, g_0) is of constant curvature and ϕ immerses (M, g_0) into a hypersphere of E^m as a minimal surface (Corollary 5.3) provided that $\Gamma: (M, G_0) \to (Q_{m-2}, \tilde{G}_0)$ is harmonic.

If both (M, g_0) and (M, G_0) are flat and $\Gamma: (M, G_0) \to (Q_{m-2}, \tilde{G}_0)$ is harmonic, then A_H is parallel (Lemma 5.6). In this case, we may choose local coordinates (x^1, x^2) such that

$$\nabla X_i = 0, \qquad (G_0)_{ij} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \tag{5.3}$$

where a_i are positive constants and ∇ stands for ∇^{g_0}. From (4.1) we have

$$\mathrm{tr}_{g_0} G_0 = 4\tilde{g}_0(H, H) = a_1 + a_2 .$$

In particular, $\|H\|_{\tilde{g}_0}$ is constant. Since 1_M is affine, (5.3) gives

$${}^{G_0}\Gamma_{ji}^k = {}^{g_0}\Gamma_{ji}^k = 0 .$$

Because $\Gamma: (M, G_0) \to (Q_{m-2}, \tilde{G}_0)$ is harmonic, Proposition 5.1 implies

$$\frac{1}{a_1}(\overline{\nabla}_X h)(X_1, X_1) + \frac{1}{a_2}(\overline{\nabla}_X h)(X_2, X_2) = 0 . \tag{5.4}$$

Combining this with (5.3) we obtain

$$D_X h(X_1, X_1) = \frac{2a_1}{a_1 - a_2} D_X H , \tag{5.5}$$

$$D_X h(X_2, X_2) = \frac{2a_2}{a_2 - a_1} D_X H , \tag{5.6}$$

on $M_2 = M - M_1$, where $M_1 = \{p \in M \mid a_1 = a_2 \text{ at } p\}$. Since 1_M is affine, 1_M is homothetic on $\mathrm{int}(M_1)$.

Let N be a component of M_2. Because $\|H\|_{\tilde{g}_0}$ is constant, (5.5) and (5.6) imply

$$\tilde{g}_0(D_X h(X_1, X_1), H) = \tilde{g}_0(D_X h(X_2, X_2), H) = 0 .$$

Combining this with Lemma 5.6 and the Codazzi equation, we may find

$$\tilde{g}_0((\nabla_X h)(Y, Z), H) = \tilde{g}_0(h(X, Y), D_Z H) = 0 \tag{5.7}$$

for any X, Y, Z tangent to N. From equations (5.5) and (5.6) we also have

$$\tilde{g}_0(D_{X_i} D_{X_j} h(X_k, X_k), H) = -\tilde{g}_0(D_{X_j} h(X_k, X_k), D_{X_i} H) .$$

Let R^D denote the curvature tensor associated with the normal connection D of $\phi: (M, g_0) \to (E^m, \tilde{g}_0)$. Then we have

$$\begin{aligned} 2\tilde{g}_0(R^D(X_1, X_2) h(X_1, X_1), H) = {} & \tilde{g}_0(D_{X_2} h(X_2, X_2), D_{X_1} h(X_1, X_1)) \\ & - \tilde{g}_0(D_{X_1} h(X_2, X_2), D_{X_2} h(X_1, X_1)) . \end{aligned}$$

Combining this with (5.5), (5.6) and the Ricci equation, we may find

$$[A_H, A_{h(X_1,X_1)}]=[A_H, A_{h(X_2,X_2)}]=0, \qquad \text{on } N. \tag{5.8}$$

Since $G_0(X, Y)=2\tilde{g}_0(H, h(X, Y))=2g_0(A_H X, Y)$, (5.3) gives

$$A_H=\begin{pmatrix} a_1/2 & 0 \\ 0 & a_2/2 \end{pmatrix}. \tag{5.9}$$

Because $a_1 \neq a_2$ on N, (5.8) and (5.9) show that $A_{h(X_1,X_1)}$ takes the following form;

$$A_{h(X_1,X_1)}=\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}. \tag{5.10}$$

Hence, $g_0(A_{h(X_1,X_2)}X_1, X_1)=g_0(A_{h(X_1,X_1)}X_1, X_2)=0$. Similar argument yields $g_0(A_{h(X_1,X_2)}X_2, X_2)=0$. Therefore,

$$A_{h(X_1,X_2)}=\begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix} \tag{5.11}$$

for some μ on N. Now, we shall claim that the first normal space $\operatorname{Im} h$ is of dimension $\leqq 2$ on N. If it is not, then H, $h(X_1, X_1)$ and $h(X_1, X_2)$ are linearly independent. Because a_i are constants, (5.5) and (5.6) give

$$R^D(X_1, X_2)h(X_1, X_1)=\frac{2a_1}{a_1-a_2}R^D(X_1, X_2)H.$$

Therefore, the Ricci equation implies

$$[A_{h(X_1,X_1)}, A_{h(X_1,X_2)}]=\frac{2a_1}{a_1-a_2}[A_H, A_{h(X_1,X_2)}]. \tag{5.12}$$

From (5.9), (5.10), (5.11) and (5.12) we get $a_1=b_1-b_2$. Similarly, we may also get $a_2=b_2-b_1$. Therefore, we have $a_1+a_2=0$. Thus, N is minimal in E^m via ϕ. In particular, we have $\dim \operatorname{Im} h \leqq 2$ on N. This yields a contradiction. Consequently, we always have $\dim(\operatorname{Im} h) \leqq 2$ on M_2.

Now, if $H \wedge h(X_1, X_1) \neq 0$ almost everywhere, then $\operatorname{Im} h=\operatorname{Span}\{H, h(X_1, X_1)\}$. By (5.9) and (5.10) we may assume that A_r take the following forms;

$$A_3=\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \quad A_4=\begin{pmatrix} e & 0 \\ 0 & -e \end{pmatrix}, \quad A_5=\cdots=A_m=0, \tag{5.13}$$

with respect to a suitable basis. Since $\|H\|_{\tilde{g}_0}$ and a_i are constant, c_1, c_2 and e are constant by virtue of (4.1). From (5.13) we have

$$h(X_1, X_1)=c_1\xi_3+e\xi_4, \quad h(X_1, X_2)=0, \quad h(X_2, X_2)=c_2\xi_3-e\xi_4. \tag{5.14}$$

Combining (5.3) and (5.14) we may find

$$D_{X_1}h(X_1, X_2)=D_{X_2}h(X_1, X_1)=0.$$

Thus we have

$$c_1 D_{X_2}\xi_3 + e D_{X_2}\xi_4 = 0\,. \tag{5.15}$$

On the other hand, (5.3), (5.4), and (5.14) also give

$$0 = \frac{1}{a_1}\{c_1 D_X \xi_3 + e D_X \xi_4\} + \frac{1}{a_2}\{c_2 D_X \xi_3 - e D_X \xi_4\}\,.$$

Combining this with (5.15) we obtain $D_{X_2}\xi_3 = D_{X_2}\xi_4 = 0$. Similarly, we may also obtain $D_{X_1}\xi_3 = D_{X_1}\xi_4 = 0$. Thus ξ_3 and ξ_4 are parallel in the normal bundle. Since $\dim(\mathrm{Im}\,h)=2$ and $\mathrm{Im}\,h = \mathrm{Span}\{\xi_3, \xi_4\}$, a reduction theorem of Erbacher shows that each component N of $\mathrm{int}(M_2)$ lies in a linear 4-space of E^m. Because ξ_4 is a parallel nondegenerate minimal section (p. 124 of [2]), N is an open portion of the product surface of two planar circles (Proposition 5.4 of [2, p. 128]). By continuity we may prove that N is the whole surface M.

If $h(X_1, X_1)\wedge H = 0$ on some connected open subset U of M_2, then $h(X_2, X_2)\wedge H = 0$ on U. In this case, we may choose an orthonormal normal frame $\xi_3, \cdots, \xi_m$ such that

$$A_3 = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}, \qquad A_5 = \cdots = A_m = 0\,.$$

Since (M, g_0) is flat, $c_1 c_2 = e^2$. Thus $\dim(\mathrm{Im}\,h) \equiv 2$ on U. Moreover, we have

$$h(X_1, X_1) = c_1\xi_3, \quad h(X_1, X_2) = e\xi_4, \quad h(X_2, X_2) = c_2\xi_3\,, \tag{5.16}$$

$$(G_0)_{11} = (c_1 + c_2)c_1, \quad (G_0)_{12} = 0, \quad (G_0)_{22} = (c_1 + c_2)c_2\,. \tag{5.17}$$

By using these and a similar argument as we have given in the previous case, we may also conclude that ξ_3 and ξ_4 are parallel in the normal bundle. Thus by continuity and similar argument as before, we may also conclude that $M = U$ and M is immersed as an open portion of the product surface of two planar circles via ϕ.

Conversely, if ϕ is an immersion of case (a), then Corollary 5.3 shows that 1_M is affine and $\Gamma\colon (M, G_0) \to (Q_{m-2}, \tilde{G}_0)$ is harmonic.

If ϕ is an immersion of case (b), then the Gauss map $\Gamma\colon (M, G_0) \to (Q_{m-2}, \tilde{G}_0)$ is totally geodesic (see Main Theorem of [3]) and by direct computation we may easily prove that 1_M is affine. (Q.E.D.)

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Bang-yen CHEN
Department of Mathematics
Michigan State University
East Lansing, MI 48824
U. S. A.

Tadashi NAGANO
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556
U. S. A.