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C^*-algebras associated with shift dynamical systems

Dedicated to the memory of Professor Teishirô Saitô

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Introduction.

In this paper, we study the structure of C^*-algebras generated by shift operators on a separable Hilbert space $\mathfrak{H}$ associated with a fixed basis $\{e_n\}_{n\in Z}$ where Z is the set of all integers. A pair (Ω, σ) is said to be a topological dynamical system if Ω is a compact (Hausdorff) space and σ is a homeomorphism of Ω. According to O'Donovan [6], our C^*-algebras correspond to the class of topological dynamical systems which satisfy the condition: there exists a map ϕ of Z onto a dense subset of Ω such that $\sigma(\phi(n))=\phi(n+1)$ for each n in Z. We call each of these systems a shift dynamical system and denote by (Ω, σ, ϕ). Although O'Donovan studied mainly the C^*-algebras generated by a weighted shift, we examine ones generated by a family of shift operators.

Recently, by Rieffel [15] and Pimsner-Voiculescu [13], the irrational rotation C^*-algebras were completely classified by using the K_0-groups. Furthermore, Riedel [14] generalized their work to the C^*-algebras associated with minimal rotations on the dual groups of countable discrete subgroups of the one-dimensional torus $T=\{z\in C; |z|=1\}$ where C is the set of all complex numbers. Each of these C^*-algebras associated with minimal rotation can be considered as a C^*-algebra generated by shift operators in our sense.

In Section 1, we discuss some general properties and two kinds of conjugacies of shift dynamical systems, and consider the fundamental properties (e.g. simplicity, existence of tracial state) of C^*-algebras associated with those systems. In Section 2, we show that the structure of simple C^*-algebras corresponding to the discrete subgroups G (not necessarily countable subgroup) of T is completely determined by G. This generalizes Riedel's results. Furthermore we give a necessary and sufficient condition for a shift dynamical system to be one associated with a discrete subgroup of T. In Section 3, we discuss the case where ϕ is a homeomorphism of Z onto the subspace $\phi(Z)$ of Ω. This is equivalent

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to that the corresponding C^*-algebra contains the ideal of all compact operators on $\mathfrak{H}$. We here note that the results in this section are closely related to ones by Green [8].

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1. Shift dynamical systems.

We shall examine the C^*-algebras generated by shift operators on a Hilbert space $\mathfrak{H}$ and the relationship between these algebras and the shift dynamical systems. We fix a separable Hilbert space $\mathfrak{H}$ and an orthogonal basis $\{e_n\}_{n\in Z}$ for $\mathfrak{H}$. We define the shift operators on $\mathfrak{H}$ with respect to this basis.

DEFINITION 1.1. A bounded linear operator U on $\mathfrak{H}$ is called a shift operator if $Ue_n=z_{n+1}e_{n+1}$ for every n in $\boldsymbol{Z}$, where the absolute value $|z_n|$ of each complex number z_n is 1. In particular, throughout this paper, we denote by S the shift operator with $z_n=1$ for every n.

For a shift operator U with $Ue_n=z_{n+1}e_{n+1}$ $(n\in\boldsymbol{Z})$, we define an operator W, the diagonal part of U, by setting, $We_n=z_ne_n$ $(n\in\boldsymbol{Z})$, so that $U=WS$. Let $\mathcal{S}$ be a family of shift operators and $W(\mathcal{S})$ the diagonal part of $\mathcal{S}$: namely, $W(\mathcal{S})=\{W; U=WS, U\in\mathcal{S}\}$. Let $\boldsymbol{B}(\mathfrak{H})$ be the C^*-algebra of all bounded linear operators on $\mathfrak{H}$. For a subset $\mathcal{T}$ (resp. an element T) of $\boldsymbol{B}(\mathfrak{H})$, $C^*(\mathcal{T})$ (resp. $C^*(T)$) means the C^*-subalgebra of $\boldsymbol{B}(\mathfrak{H})$ generated by $\mathcal{T}$ (resp. T) and $C^*(\mathcal{T}, T)$ means the C^*-algebra $C^*(\mathcal{T}\cup\{T\})$. For any shift operator U, $C^*(U)$ is $*$-isomorphic to $C^*(S)$ because U is unitarily equivalent to S. Moreover, we easily find that, for any family $\mathcal{S}$ of shift operators, $C^*(\mathcal{S})$ is spatially isomorphic to $C^*(\mathcal{S}')$ for some $\mathcal{S}'$ which contains S. Hence, to examine the properties of C^*-algebras $C^*(\mathcal{S})$ generated by $\mathcal{S}$ in $\boldsymbol{B}(\mathfrak{H})$, it is enough to assume that $\mathcal{S}$ contains S. Under this assumption, $C^*(\mathcal{S})$ is generated by $W(\mathcal{S})$ and S. Furthermore we assume throughout this paper that $W(\mathcal{S})$ is a subgroup of unitary operators on $\mathfrak{H}$ such that $SW(\mathcal{S})S^*=W(\mathcal{S})$. Then $C^*(\mathcal{S})=C^*(W(\mathcal{S}), S)$ and also $SC^*(W(\mathcal{S}))S^*=C^*(W(\mathcal{S}))$.

According to O'Donovan [6], the C^*-algebra $C^*(\mathcal{S})$ corresponds to a shift dynamical system $\Sigma=(\Omega, \sigma, \phi)$ in the following manner. Let π be the natural representation of $l^\infty(\boldsymbol{Z})$ on $\mathfrak{H}$ defined by $\pi(\boldsymbol{a})e_n=a_ne_n$ for each $\boldsymbol{a}=(a_n)$ in $l^\infty(\boldsymbol{Z})$.

Let $\mathcal{A}$ be the C^*-algebra $C^*(W(\mathcal{S}))$ generated by the diagonal part of a family $\mathcal{S}$ of shift operators. Then the C^*-algebra $\mathcal{A}$ satisfies the following conditions:

$$(1)\quad \mathcal{A}\subset\pi(l^\infty(\boldsymbol{Z})) \qquad (2)\quad S\mathcal{A}S^*=\mathcal{A}.$$

On the other hand, if $\mathcal{A}$ is a C^*-subalgebra with unit of $\pi(l^\infty(\boldsymbol{Z}))$ satisfying $S\mathcal{A}S^*=\mathcal{A}$, then there exists a family $\mathcal{S}$ such that $C^*(W(\mathcal{S}))=\mathcal{A}$. Therefore, for our purpose, we study the structure of these C^*-algebras $C^*(\mathcal{A}, S)$. Since $\mathcal{A}$ is an abelian C^*-algebra, $\mathcal{A}$ is $*$-isomorphic to the algebra $C(\Omega)$ of all complex-valued continuous functions on a compact Hausdorff space Ω and we denote by $\hat{T}$ the Gelfand representation of T in $\mathcal{A}$. Define $\alpha(T)=STS^*$ for T in $\mathcal{A}$. Then α is a $*$-automorphism of $\mathcal{A}$. Hence, α induces a homeomorphism σ of Ω such that $\widehat{\alpha(T)}(\omega)=\hat{T}(\sigma^{-1}\omega)$ for every ω in Ω. Let ϕ be the mapping of $\boldsymbol{Z}$ into Ω defined by $\phi(n)(T)=a_n$ for n in $\boldsymbol{Z}$ and $T=\pi(\boldsymbol{a})$ in $\mathcal{A}$ where $\boldsymbol{a}=(a_n)\in l^\infty(\boldsymbol{Z})$. Then $T\boldsymbol{e}_m=\phi(m)\boldsymbol{e}_m$ for each m in $\boldsymbol{Z}$. Since $\hat{T}(\sigma(\phi(n)))=\widehat{S^*TS}(\phi(n))=a_{n+1}=\hat{T}(\phi(n+1))$ for every $T=\pi(\boldsymbol{a})$ in $\mathcal{A}$ where $\boldsymbol{a}=(a_n)\in l^\infty(\boldsymbol{Z})$, we have $\sigma(\phi(n))=\phi(n+1)$. Furthermore $\phi(\boldsymbol{Z})$ is dense in Ω because $\|\pi(\boldsymbol{a})\|=\sup\{|a_n|\,;\,n\in\boldsymbol{Z}\}=\sup\{|\widehat{\pi(\boldsymbol{a})}(\phi(n))|\,;\,n\in\boldsymbol{Z}\}$ for each $\pi(\boldsymbol{a})$ in $\mathcal{A}$.

Conversely let $\Sigma=(\Omega, \sigma, \phi)$ be a shift dynamical system. For a function f in $C(\Omega)$, we denote by $\pi(f)$ the operator in $\boldsymbol{B}(\mathfrak{H})$ defined by $\pi(f)\boldsymbol{e}_n=f(\phi(n))\boldsymbol{e}_n$ $(n\in\boldsymbol{Z})$. We put $\mathcal{A}=\{\pi(f)\,;\,f\in C(\Omega)\}$. Then $\mathcal{A}$ is a C^*-subalgebra of $\pi(l^\infty(\boldsymbol{Z}))$ and is $*$-isomorphic to $C(\Omega)$ under the correspondence of $\pi(f)$ with f. Furthermore we have $S\mathcal{A}S^*=\mathcal{A}$ and $\widehat{(S\pi(f)S^*)}(\omega)=\widehat{\pi(f)}(\sigma^{-1}\omega)$. Namely $C^*(\mathcal{A}, S)$ is a C^*-algebra generated by shift operators corresponding to the given shift dynamical system $\Sigma=(\Omega, \sigma, \phi)$.

For a given shift dynamical system $\Sigma=(\Omega, \sigma, \phi)$, let $C^*(\Sigma)$ be the C^*-algebra generated by $\pi(C(\Omega))$ and the shift operator S, that is, $C^*(\Sigma)=C^*(\pi(C(\Omega)), S)$. We henceforth consider the relationship between the shift dynamical systems Σ and the C^*-algebras $C^*(\Sigma)$.

We here note a property of dynamical systems, and give a proof for the sake of completeness.

PROPOSITION 1.2. *Let $\Sigma=(\Omega, \sigma, \phi)$ be a shift dynamical system. Then the following statements are equivalent.*

(1) *Ω is an infinite set,*

(2) *ϕ is injective,*

(3) *$\phi(Z)$ is a proper subset of Ω.*

PROOF. The implication (2)$\Rightarrow$(1) is trivial. (1)$\Rightarrow$(3): Suppose that $\Omega=\bigcup_{n=1}^{\infty}\{\phi(n)\}$; then by Baire Category theorem, there is an n_0 such that $\overline{\{\phi(n_0)\}}=\{\phi(n_0)\}$ has a non-empty interior, so that each $\phi(n)$ is isolated. This contradicts the assumption that Ω is infinite and compact. (3)$\Rightarrow$(2): Suppose that ϕ is not injective. Then there exist distinct numbers n_0 and m_0 in $\boldsymbol{Z}$ such that $\phi(n_0)=\phi(m_0)$. Put $k=n_0-m_0$. Then we have

$$\phi(n)=\sigma^{n-m_0}(\phi(m_0))=\sigma^{n-m_0}(\phi(n_0))=\phi(n-m_0+n_0)=\phi(n+k)$$

for every n in $\boldsymbol{Z}$. Hence $\phi(\boldsymbol{Z})=\{\phi(0), \phi(1), \cdots, \phi(k-1)\}$. Since $\phi(\boldsymbol{Z})$ is dense in Ω, we have $\phi(\boldsymbol{Z})=\Omega$. q. e. d.

We first consider the case where Ω is finite. Suppose that Ω consists of n-points $\{\omega_0, \cdots, \omega_{n-1}\}$. By the property of ϕ, we can assume that $\sigma\omega_i=\omega_{i+1}$ $(0\leqq i\leqq n-2)$ and $\sigma\omega_{n-1}=\omega_0$. Then $\pi(C(\Omega))$ is the C^*-algebra generated by n projections $P_{n,i}=\sum_{k\in\boldsymbol{Z}} P_{nk+i}$ $(i=0, \cdots, n-1)$, where P_n is the projection of $\mathfrak{H}$ onto the one-dimensional subspace $[\boldsymbol{e}_n]$ generated by $\boldsymbol{e}_n$. Let V_n be the unitary operator of $\mathfrak{H}$ onto $\mathfrak{H}\otimes\mathfrak{H}_n$ defined by $V_n(\boldsymbol{e}_{nk+i})=\boldsymbol{e}_k\otimes\boldsymbol{e}_i$ $(i=0, \cdots, n-1;\ k\in\boldsymbol{Z})$, where $\mathfrak{H}_n$ is the n-dimensional Hilbert space with a basis $\{\boldsymbol{e}_0, \cdots, \boldsymbol{e}_{n-1}\}$. Then $C^*(\Sigma)$ is $*$-isomorphic to the C^*-tensor product $C^*(S)\otimes\boldsymbol{B}(\mathfrak{H}_n)$ on $\mathfrak{H}\otimes\mathfrak{H}_n$ by the spatial isomorphism $\Psi_n: T\to V_nTV_n^*$ $(T\in C^*(\Sigma))$ and thus it is $*$-isomorphic to $C(\boldsymbol{T})\otimes\boldsymbol{B}(\mathfrak{H}_n)$, where $C(\boldsymbol{T})$ is the C^*-algebra of all continuous functions on $\boldsymbol{T}$.

Since an α-invariant ideal in $\pi(C(\Omega))$ corresponds to a proper σ-invariant closed subset of Ω, it follows that $C^*(\Sigma)$ has no proper ideal if and only if σ has no proper invariant closed subset of Ω. In this case, σ is said to be minimal. The following proposition is known ([**6**]).

PROPOSITION 1.3. *Let $\Sigma=(\Omega, \sigma, \phi)$ be a shift dynamical system. Then we have the following.*

(1) *$C^*(\Sigma)$ is $*$-isomorphic to the C^*-crossed product $\pi(C(\Omega))\times_{\alpha}\boldsymbol{Z}$,*

(2) *If Ω is infinite, $C^*(\Sigma)$ is simple if and only if σ is minimal.*

We here remark that the mapping: $\mathcal{J}\to\mathcal{J}\cap\pi(C(\Omega))$ ($\mathcal{J}$ is an ideal in $C^*(\Sigma)$) in Theorem 2. 2. 3. in [**9**] is not always one-to-one. For a proper σ-invariant closed set E of Ω, we put $\mathcal{J}(E)=\{f\in C(\Omega);\ f$ vanishes on $E\}$ and denote by $\mathcal{J}(E)$ the ideal of $C^*(\Sigma)$ generated by $\mathcal{J}(E)$ and S. Conversely, according to the proof of Power [**12**], if σ admits an invariant ergodic positive measure with support Ω, each ideal $\mathcal{J}$ of $C^*(\Sigma)$ corresponds to a closed σ-invariant proper subset E of Ω, that is, $\mathcal{J}=\mathcal{J}(E)$. In the general case, the ideals of $C^*(\Sigma)$ do not necessarily correspond to the ideals of $C(\Omega)$ and such an example will be given in Section 3.

Next, in order to discuss the existence of tracial state of $C^*(\Sigma)$, we here consider the conditional expectation of $C^*(\Sigma)$ onto $\pi(C(\Omega))$. For T in $\boldsymbol{B}(\mathfrak{H})$, we put $E(T)=\sum_{n\in\boldsymbol{Z}} P_nTP_n$. Then E is the norm one projection of $\boldsymbol{B}(\mathfrak{H})$ onto $\pi(l^\infty(\boldsymbol{Z}))$. We denote by E_Σ the restriction of E to the C^*-algebra $C^*(\Sigma)$. Then for $\{f_k\}_{k=-n}^{n}$ in $C(\Omega)$, it follows that

$$E_\Sigma(\sum_{k=-n}^{n}\pi(f_k)S^k)=\pi(f_0) \qquad (\in\pi(C(\Omega)))\,.$$

Since E_Σ is a bounded linear map of $C^*(\Sigma)$ into $\pi(l^\infty(\boldsymbol{Z}))$, the range of E_Σ is the C^*-subalgebra $\pi(C(\Omega))$ of $C^*(\Sigma)$. Hence E_Σ is the norm one projection of $C^*(\Sigma)$ onto $\pi(C(\Omega))$, and $C^*(\Sigma)\cap\pi(l^\infty(\boldsymbol{Z}))=\pi(C(\Omega))$. For each tracial state Tr of $C^*(\Sigma)$, the restriction τ of Tr to $\pi(C(\Omega))$ induces a σ-invariant positive measure with total measure 1 and it follows that $\mathrm{Tr}=\tau\cdot E_\Sigma$. Conversely such a measure on Ω induces a tracial state of $C^*(\Sigma)$ in the above manner. Moreover we have that $\mathrm{Tr}=\tau\cdot E_\Sigma$ is faithful if and only if the support of corresponding measure to τ on Ω is the full space Ω. Since every minimal homeomorphism σ admits an invariant ergodic positive measure with support Ω (cf. [**3**, Chapter II, Exercise 9, (1) and (5)]), we get the following proposition. Though this is well known implicitly, we note the statement in the context of shift dynamical systems.

PROPOSITION 1.4. *Let Σ be a minimal shift dynamical system. Then $C^*(\Sigma)$ has a faithful tracial state.*

We here give an example of $C^*(\Sigma)$, which is the largest one in the C^*-algebras associated with shift dynamical systems.

EXAMPLE 1.5. Let $\mathcal{S}$ be the set of all shift operators. For a completely regular topological space X, we denote by βX the Stone-Čech's compactification of X. Then $\Sigma=(\beta\boldsymbol{Z},\sigma,\phi)$ is the shift dynamical system associated with $C^*(\mathcal{S})$ where $\sigma(\{n_\alpha\})=\{n_\alpha+1\}$ for each net $\omega=\{n_\alpha\}$ of integers and ϕ is the natural embedding from $\boldsymbol{Z}$ into $\beta\boldsymbol{Z}$. Let E (resp. E_+, E_-) be the set of all cluster points of $\phi(\boldsymbol{Z})$ (resp. $\phi(\boldsymbol{Z}_+)$, $\phi(\boldsymbol{Z}_-)$) where $\boldsymbol{Z}_+$ (resp. $\boldsymbol{Z}_-$) means the set of all positive integers (resp. negative integers). Then E, E_+ and E_- are σ-invariant distinct closed proper subsets of Ω. Hence $C^*(\mathcal{S})$ $(=C^*(\Sigma))$ contains at least three ideals, thus $C^*(\mathcal{S})$ is a proper C^*-subalgebra of $\boldsymbol{B}(\mathfrak{H})$.

Now we consider a relationship between the $*$-isomorphic classes of $C^*(\Sigma)$ and the conjugate classes of the dynamical systems Σ. We recall that two dynamical systems (Ω_1,σ_1) and (Ω_2,σ_2) are said to be conjugate if there exists a homeomorphism h of Ω_1 onto Ω_2 such that $h\circ\sigma_1\circ h^{-1}=\sigma_2$. If shift dynamical systems $\Sigma_1=(\Omega_1,\sigma_1,\phi_1)$ and $\Sigma_2=(\Omega_2,\sigma_2,\phi_2)$ are conjugate, then there exists a $*$-isomorphism α of $C(\Omega_1)$ onto $C(\Omega_2)$ such that $\alpha\circ\sigma_1=\sigma_2\circ\alpha$. Since each C^*-algebra $C^*(\Sigma)$ is $*$-isomorphic to the crossed product $C(\Omega)\times_\sigma\boldsymbol{Z}$, $C^*(\Sigma_1)$ and $C^*(\Sigma_2)$ are $*$-isomorphic. Therefore the structure of C^*-algebras $C^*(\Sigma)$ is determined by the compact space Ω and the homeomorphism σ. But the following example shows that the converse does not hold.

EXAMPLE 1.6. Put $\Omega=\{\omega_1,\omega_2\}\cup\boldsymbol{Z}\cup\{\omega_3\}$, where ω_1 (resp. ω_2) is the limit point of $\{2n\,;\,n\in\boldsymbol{Z}_+\}$ (resp. $\{2n+1\,;\,n\in\boldsymbol{Z}_+\}$) and ω_3 is the limit point of $\boldsymbol{Z}_-$ and each n $(\in\boldsymbol{Z})$ is an isolated point. Let σ_1 and σ_2 be homeomorphisms of Ω

such that $\sigma_1\omega_1=\sigma_2\omega_1=\omega_2$, $\sigma_1\omega_2=\sigma_2\omega_2=\omega_1$, $\sigma_1\omega_3=\sigma_2\omega_3=\omega_3$ but $\sigma_1(n)=n+1$, $\sigma_2(n)=n-1$. Let ϕ_1 and ϕ_2 be maps of $\boldsymbol{Z}$ onto Ω such that $\phi_1(n)=n$, $\phi_2(n)=-n$. Then $\Sigma_1=(\Omega, \sigma_1, \phi_1)$ and $\Sigma_2=(\Omega, \sigma_2, \phi_2)$ are shift dynamical systems. Considering the unitary operator V defined by $V\boldsymbol{e}_n=\boldsymbol{e}_{-n}$, we find that $VC^*(\Sigma_1)V^*=C^*(\Sigma_2)$. However there exists no homeomorphism h of Ω such that $h\circ\sigma_1=\sigma_2\circ h$.

REMARK. Arveson and Josephson [2] proved that the isomorphic classes of the non-self-adjoint Banach algebras generated by shift operators and their diagonal parts correspond to the conjugate classes of the dynamical systems provided that σ admits an ergodic invariant measure with support Ω.

In the class of shift dynamical systems, we will define more strict conjugacy.

DEFINITION 1.7. Shift dynamical systems $\Sigma_1=(\Omega_1, \sigma_1, \phi_1)$ and $\Sigma_2=(\Omega_2, \sigma_2, \phi_2)$ are said to be strictly conjugate if there exists a homeomorphism h of Ω_1 onto Ω_2 such that $(h\circ\phi_1)(n)=\phi_2(n)$ for all n in $\boldsymbol{Z}$.

It is easy to see that strict conjugacy implies usual conjugacy and we have the following.

PROPOSITION 1.8. *Shift dynamical systems Σ_1 and Σ_2 are strictly conjugate if and only if $\pi(C(\Omega_1))=\pi(C(\Omega_2))$ on the Hilbert space $\mathfrak{H}$.*

PROOF. Suppose that there exists a homeomorphism h of Ω_1 onto Ω_2 such that $(h\circ\phi_1)(n)=\phi_2(n)$. For f in $C(\Omega_1)$, $g=f\circ h^{-1}$ belongs to $C(\Omega_2)$ and $\pi_2(g)\boldsymbol{e}_n=(f\circ h^{-1})(\phi_2(n))\boldsymbol{e}_n=f(\phi_1(n))\boldsymbol{e}_n=\pi_1(f)\boldsymbol{e}_n$ for every $\boldsymbol{e}_n$, thus $\pi_2(g)=\pi_1(f)$, where π_i means the representation π of $C(\Omega_i)$ on $\mathfrak{H}$ for each i $(i=1, 2)$. Obviously the map: $f\to f\circ h^{-1}$, is surjective. Conversely we suppose that $\pi_1(C(\Omega_1))=\pi_2(C(\Omega_2))$. Put $\alpha=\pi_2^{-1}\circ\pi_1$. Then α is a $*$-isomorphism of $C(\Omega_1)$ onto $C(\Omega_2)$. Thus α induces a homeomorphism h of Ω_1 onto Ω_2 such that $\alpha(f)(\omega)=f(h^{-1}\omega)$ for ω in Ω_2. For n in $\boldsymbol{Z}$ and f in $C(\Omega_1)$, we have

$$\begin{aligned} f(\phi_1(n))\boldsymbol{e}_n&=\pi_1(f)\boldsymbol{e}_n=\pi_2(\alpha(f))\boldsymbol{e}_n\\ &=(\alpha\circ f)(\phi_2(n))\boldsymbol{e}_n=f((h^{-1}\circ\phi_2)(n))\boldsymbol{e}_n\,. \end{aligned}$$

Hence $f(\phi_1(n))=f((h^{-1}\circ\phi_2)(n))$ for every f in $C(\Omega_1)$, thus $(h\circ\phi_1)(n)=\phi_2(n)$. q.e.d.

We will consider the relation between usual conjugacy and strict conjugacy for special shift dynamical systems. Let Γ be a monothetic compact abelian group, that is, there exists an injective group homomorphism ϕ of $\boldsymbol{Z}$ onto a dense set in Γ. Put $\sigma\omega=\omega+\phi(1)$ for ω in Γ where $+$ is the group operation in Γ. Then $\Sigma=(\Gamma, \sigma, \phi)$ is a shift dynamical system. These systems are fully studied in the next section.

PROPOSITION 1.9. *Let $\Sigma=(\Omega, \sigma, \phi)$ be a shift dynamical system satisfying either* (1) *or* (2) *of the following:*

(1) *Every $\phi(n)$ is an isolated point in Ω,*

(2) $\Sigma=(\Gamma, \sigma, \phi)$ *for some monothetic compact abelian group* Γ.

Then a shift dynamical system $\Sigma_0=(\Omega_0, \sigma_0, \phi_0)$ *is conjugate to* Σ *if and only if* Σ_0 *is strictly conjugate to* Σ.

PROOF. Suppose that there exists a homeomorphism k of Ω onto Ω_0 such that $k\circ\sigma=\sigma_0\circ k$. Then Σ_0 satisfies the same conditions as Σ does. We first consider the case (1). Since $\Omega_0=k(\phi(\boldsymbol{Z}))\cup k(\Omega-\phi(\boldsymbol{Z}))$ and $k(\phi(\boldsymbol{Z}))$ are open, $k(\Omega-\phi(\boldsymbol{Z}))$ is a σ_0-invariant closed set in Ω_0, so that $\phi_0(0)=k\circ\phi(m)$ for some m. We put $h=\sigma_0^m\circ k$. Then h is a homeomorphism of Ω onto Ω_0 such that $h(\phi(n))=\phi_0(n)$ for all n in $\boldsymbol{Z}$. In the case (2), we put $h(\omega)=k(\omega+k^{-1}(\phi_0(0)))$ for ω in Ω. Then we have that $h(\phi(n))=k(\phi(n)+k^{-1}(\phi_0(0)))=k(\sigma^n(k^{-1}(\phi_0(0))))=k(k^{-1}(\sigma_0^n(\phi_0(0))))=\phi_0(n)$. q. e. d.

Propositions 1.8, 1.9 and the fact $C^*(\Sigma)\cap\pi(l^\infty(\boldsymbol{Z}))=\pi(C(\Omega))$ imply the following equivalency.

COROLLARY 1.10. *Let* Σ *be as in Proposition* 1.9. *Then* Σ_0 *is conjugate to* Σ *if and only if* $C^*(\Sigma_0)=C^*(\Sigma)$.

2. C^*-algebras corresponding to discrete subgroups of $\boldsymbol{T}$.

We denote by $\boldsymbol{T}_d$ the one-dimensional torus with discrete topology and consider a unitary representation π of $\boldsymbol{T}_d$ on the Hilbert space $\mathfrak{H}$ with a basis $\{e_n\}_{n\in Z}$. For each element $e^{2\pi ix}$ in $\boldsymbol{T}_d$, let $\pi(e^{2\pi ix})$ be the unitary operator on $\mathfrak{H}$ defined by $\pi(e^{2\pi ix})e_n=e^{2\pi inx}e_n$ $(n\in\boldsymbol{Z})$. For a subgroup G of $\boldsymbol{T}_d$, we put $\pi(G)=\{\pi(g); g\in G\}$. Then $\pi(G)$ is a subgroup of all unitary operators in $\pi(l^\infty(\boldsymbol{Z}))$, so that $C^*(\pi(G), S)$ is a C^*-algebra generated by a family of shift operators. This section is devoted to the study of C^*-algebras associated with subgroups G of $\boldsymbol{T}_d$. We denote by Σ_G the shift dynamical system associated with the C^*-algebra $C^*(\pi(G), S)$. Then $C^*(\Sigma_G)$ coincides with the C^*-algebra $C^*(\pi(G), S)$. Riedel [**14**] has already studied the relations between C^*-algebras $C^*(\Sigma_G)$ and groups G in the case of countable groups, but we do not have such assumption for the groups G.

PROPOSITION 2.1. *Let* G *be an infinite subgroup of* $\boldsymbol{T}_d$. *Then* $\Sigma_G=(\Omega, \sigma, \phi)$ *has the following properties.*

(1) Ω *is homeomorphic to the dual group of* G, *which is a compact monothetic abelian group.*

(2) ϕ *is an injective homomorphism, that is,* $\phi(n+m)=\phi(n)+\phi(m)$ *for all* n, m *in* $\boldsymbol{Z}$.

(3) $\sigma\omega=\omega+\phi(1)$ *for* ω *in* Ω.

(4) σ *is minimal on* Ω.

In (2), (3) *and* (4), Ω *is identified with the dual group of* G.

Before going into the proof we remark that the dual group $\hat{G}$ of G is a compact monothetic abelian group with dense countable subset $\phi(\boldsymbol{Z})$, where ϕ is an injective homomorphism of $\boldsymbol{Z}$ into $\hat{G}$ such that $\langle e^{2\pi ix}, \phi(n)\rangle = e^{2\pi inx}$ for $e^{2\pi ix}$ in G and n in $\boldsymbol{Z}$ where $\langle e^{2\pi ix}, \gamma\rangle$ means the value of a character $\gamma\in\hat{G}$ at the point $e^{2\pi ix}$ in G (see [**16**, Theorem 2.3.3]).

PROOF OF PROPOSITION 2.1. (1) and (2): Let $\mathcal{A}$ be the linear subspace of $C^*(\pi(G))$ generated by $\pi(G)$. Then, since G is a subgroup of $\boldsymbol{T}_d$, $\mathcal{A}$ is a dense $*$-subalgebra of $C^*(\pi(G))$. Let η be the map of $\mathcal{A}$ into $C(\hat{G})$ defined by

$$\eta(\sum_{k=1}^{p}\alpha_k\pi(e^{2\pi ix_k}))(\gamma)=\sum_{k=1}^{p}\alpha_k\langle e^{2\pi ix_k}, \gamma\rangle$$

for $e^{2\pi ix_k}\in G$ $(k=1, 2, \cdots, p)$ and $\gamma\in\hat{G}$. Then, we find that η is an isometric map of $\mathcal{A}$ into $C(\hat{G})$ by the following equalities and so η is also well-defined;

$$\|\eta(\sum_{k=1}^{p}\alpha_k\pi(e^{2\pi ix_k}))\|=\sup\{|\sum_{k=1}^{p}\alpha_k\langle e^{2\pi ix_k}, \gamma\rangle|\,;\,\gamma\in\hat{G}\}$$

$$=\sup\{|\sum_{k=1}^{p}\alpha_k\langle e^{2\pi ix_k}, \phi(n)\rangle|\,;\,n\in\boldsymbol{Z}\}=\sup\{|\sum_{k=1}^{p}\alpha_k e^{2\pi inx_k}|\,;\,n\in\boldsymbol{Z}\}$$

$$=\|\sum_{k=1}^{p}\alpha_k\pi(e^{2\pi ix_k})\|\,.$$

Since $\pi(e^{2\pi ix})^*=\pi(e^{-2\pi ix})$, we can show that η is $*$-preserving. Hence, we shall show that η is multiplicative. For any $\sum_{k=1}^{p}\alpha_k\pi(e^{2\pi ix_k})$ and $\sum_{l=1}^{q}\beta_l\pi(e^{2\pi iy_l})$ in $\mathcal{A}$ and every $\gamma\in\hat{G}$, we have the following equalities;

$$\eta(\{\sum_{k=1}^{p}\alpha_k\pi(e^{2\pi ix_k})\}\{\sum_{l=1}^{q}\beta_l\pi(e^{2\pi iy_l})\})(\gamma)$$

$$=\eta(\sum_{k=1}^{p}\sum_{l=1}^{q}\alpha_k\beta_l\pi(e^{2\pi i(x_k+y_l)}))(\gamma)$$

$$=\sum_{k=1}^{p}\sum_{l=1}^{q}\alpha_k\beta_l\langle e^{2\pi i(x_k+y_l)}, \gamma\rangle=\sum_{k=1}^{p}\sum_{l=1}^{q}\alpha_k\beta_l\langle e^{2\pi ix_k}, \gamma\rangle\langle e^{2\pi iy_l}, \gamma\rangle$$

$$=\eta(\sum_{k=1}^{p}\alpha_k\pi(e^{2\pi ix_k}))(\gamma)\eta(\sum_{l=1}^{q}\beta_l\pi(e^{2\pi iy_l}))(\gamma)\,.$$

Thus, η is an isometric homomorphism of $\mathcal{A}$ into $C(\hat{G})$. Since $\mathcal{A}$ is dense in $C^*(\pi(G))$ and we can show that $\eta(\mathcal{A})$ is a dense $*$-subalgebra of $C(\hat{G})$, η can be extended to the isomorphism of $C^*(\pi(G))$ onto $C(\hat{G})$. Thus, Ω is homeomorphic to $\hat{G}$ and $\widehat{\pi(e^{2\pi ix})}(\phi(n))=\eta(\pi(e^{2\pi ix}))(\phi(n))$ for each n in $\boldsymbol{Z}$.

(3): For each n in $\boldsymbol{Z}$ and g in G, we have the equalities; $\widehat{\pi(g)}(\sigma(\phi(n)))=\widehat{\pi(g)}(\phi(n+1))=\widehat{\pi(g)}(\phi(n)+\phi(1))$. Hence $\widehat{\pi(g)}(\sigma(\omega))=\widehat{\pi(g)}(\omega+\phi(1))$ for every ω in Ω since $\phi(\boldsymbol{Z})$ is dense in Ω.

(4): For any γ in $\hat{G}$, the set $O(\gamma)=\gamma+\psi(\boldsymbol{Z})$ is homeomorphic to a dense subset $\psi(\boldsymbol{Z})$ of $\hat{G}$. q. e. d.

Since every dual group $\hat{G}$ of G has a unique σ-invariant measure, $C^*(\Sigma_G)$ is $*$-isomorphic to the C^*-algebra $\mathcal{A}_\sigma$, of which the ideals were studied in [**12**]. In particular, when $G=\{e^{2\pi in\theta}; n\in\boldsymbol{Z}\}$ where θ $(0<\theta<1/2)$ is an irrational number, $\mathcal{A}_\sigma$ is just an irrational rotation C^*-algebra $\mathcal{A}_\theta$, of which Rieffel [**15**] and Pimsner-Voiculescu [**13**] examined the properties. For $\mathcal{A}_\theta$, the maps σ_θ and ϕ_θ in our consideration are as follows; $\sigma_\theta(e^{2\pi ix})=e^{2\pi i(x+\theta)}$ for $e^{2\pi ix}$ in $\boldsymbol{T}=\hat{G}_\theta$ and $\phi_\theta(n)=e^{2\pi in\theta}$. We easily find that $\mathcal{A}_\theta$ is simple and has the unique tracial state Tr. Furthermore Rieffel and Pimsner-Voiculescu showed that $\mathrm{Tr}((\mathcal{A}_\theta)_p)=(\boldsymbol{Z}+\boldsymbol{Z}\theta)\cap[0, 1]=\{t\in[0, 1]; e^{2\pi it}\in G_\theta\}$ where $(\mathcal{A}_\theta)_p$ is the set of all projection in $\mathcal{A}_\theta$. In general, a tracial state τ of a C^*-algebra $\mathcal{A}$ induces a natural homomorphism $\tilde{\tau}$ of the K_0-group $K_0(\mathcal{A})$ into $\boldsymbol{R}$ where $\boldsymbol{R}$ is the set of all real numbers. We denote by $R_\tau(\mathcal{A})$ the image of $K_0(\mathcal{A})$ by $\tilde{\tau}$. In the above arguments, we will in particular consider the last assertion for an arbitrary infinite subgroup G of $\boldsymbol{T}_d$. To prove it, we will use many arguments in [**14**]. But Riedel discussed the above problem for only countable groups.

Let G be a (not necessarily countable) infinite subgroup of $\boldsymbol{T}_d$. Then there exists a monotone increasing net $\{G_\lambda\}_{\lambda\in\Lambda}$ of finitely generated subgroups of G such that $\bigcup_{\lambda\in\Lambda} G_\lambda=G$. By the definition of $C^*(\Sigma)$, we see that $C^*(\Sigma_{G_1})$ is a C^*-subalgebra of $C^*(\Sigma_{G_2})$ if G_1 is a subgroup of G_2. For λ in Λ, let η_λ be the identity map of $C^*(\Sigma_{G_\lambda})$ into $C^*(\Sigma_G)$. Then $C^*(\Sigma_G)$ is the inductive limit of the system $\{C^*(\Sigma_{G_\lambda}), \eta_\lambda\}_{\lambda\in\Lambda}$ in the sense of Takeda [**17**]. For the unique tracial state Tr of $C^*(\Sigma_G)$, let Tr_λ be the restriction of Tr to $C^*(\Sigma_{G_\lambda})$. Using the theory of inductive limit of C^*-algebras introduced by Takeda [**17**], we can prove that

$$R_{\mathrm{Tr}}(C^*(\Sigma_G))=\bigcup_{\lambda\in\Lambda} R_{\mathrm{Tr}_\lambda}(C^*(\Sigma_{G_\lambda})).$$

This can be shown by a similar way as Riedel did in [**14**, Lemma 3.1]. If G_λ is an infinite subgroup of $\boldsymbol{T}_d$, by [**14**, Proposition 3.5], it follows that $R_{\mathrm{Tr}}(C^*(\Sigma_{G_\lambda}))=\{t\in\boldsymbol{R}; e^{2\pi it}\in G_\lambda\}$. When G_λ is finite, say $G_\lambda=\{e^{2\pi ik/n}; k=0, 1, \cdots, n-1\}$, we have already seen that $C^*(\Sigma_{G_\lambda})$ is $*$-isomorphic to $C(\boldsymbol{T})\otimes\boldsymbol{B}(\mathfrak{H}_n)$, thus the above equality holds too. Therefore we get $R_{\mathrm{Tr}}(C^*(\Sigma_G))=\{t\in\boldsymbol{R}; e^{2\pi it}\in G\}$.

THEOREM 2.2. *Let G be an arbitrary infinite subgroup of $\boldsymbol{T}_d$. Then we have the following:*

(1) *$C^*(\Sigma_G)$ is a simple C^*-algebra with the unique tracial state* Tr,

(2) $R_{\mathrm{Tr}}(C^*(\Sigma_G))=\{t\in\boldsymbol{R}; e^{2\pi it}\in G\}$.

REMARK. When G is a finite subgroup of $\boldsymbol{T}_d$, the corresponding C^*-algebra $\mathcal{A}_G$ in Riedel's paper is equal to $\boldsymbol{B}(\mathfrak{H}_n)$. Thus, if G is an infinite torsion subgroup of $\boldsymbol{T}_d$, the C^*-algebra $\mathcal{A}_G$ in his paper $(=C^*(\Sigma_G))$ cannot be an inductive limit of $\{\mathcal{A}_{G_n}\}$ for any increasing sequence $\{G_n\}$ of finitely generated subgroups such that $\bigcup_{n=1}^{\infty} G_n=G$. Hence the proof by Riedel [**14**, Theorem 3.6] was divided into two cases; G is not a torsion subgroup and G is a torsion subgroup, and the latter was reduced to the former.

Theorem 2.2 implies that $C^*(\Sigma_{G_1})$ is $*$-isomorphic to $C^*(\Sigma_{G_2})$ if and only if $G_1=G_2$ because the group $R_{\mathrm{Tr}}(C^*(\Sigma_G))$ is an isomorphism invariant. Now we give some examples of shift dynamical systems $\Sigma_G=(\Omega, \sigma, \phi)$ and C^*-algebras $C^*(\Sigma_G)$ associated with subgroup G of $\boldsymbol{T}_d$.

EXAMPLE 2.3. Let G be the subgroup generated by $e^{\pi i}$ and $e^{2\pi i\theta}$ where θ $(0<\theta<1)$ is an irrational number. Then $\Omega=\hat{G}=\boldsymbol{Z}/2\boldsymbol{Z}\times\boldsymbol{T}$, $\sigma(k, e^{2\pi ix})=((k+1)(\mathrm{mod.}\ 2), e^{2\pi i(x+\theta)})$ where $k=0$ or 1 and $\phi(n)=(n(\mathrm{mod.}\ 2), e^{\pi in\theta})$. Furthermore we have $R_{\mathrm{Tr}}(C^*(\Sigma_G))=\boldsymbol{Z}\cdot(1/2)+\boldsymbol{Z}\cdot\theta$.

EXAMPLE 2.4. Let $\mathcal{A}_k$ be the set of periodic sequences in $l^\infty(\boldsymbol{Z})$ with period k. Then $\pi(\mathcal{A}_k)=C^*(\pi(G_k))$ where G_k is the finite group generated by $e^{2\pi i/k}$. Put $\mathcal{B}_\infty=\bigcup_{k=1}^{\infty}\mathcal{A}_k$ (resp. $\mathcal{B}_p=\bigcup_{n=1}^{\infty}\mathcal{A}_{p^n}$). Then $C^*(\pi(\mathcal{B}_\infty))=C^*(\pi(G_{\boldsymbol{Q}}))$ (resp. $C^*(\pi(\mathcal{B}_p))=C^*(\pi(G_p))$) where $G_{\boldsymbol{Q}}=\{e^{2\pi ix}\,;\,x\in\boldsymbol{Q}\cap[0, 1]\}$ (resp. $G_p=\{e^{2\pi ik/p^n}\,;\,k=0, 1, \cdots, p^n, n\in\boldsymbol{N}\}$) and $\boldsymbol{Q}$ means the set of all rational numbers. Hence the character space Ω of $C^*(\pi(\Sigma_{G_{\boldsymbol{Q}}}))$ (resp. $C^*(\pi(\Sigma_{G_p}))$) is a profinite group $\hat{\boldsymbol{Z}}$ (resp. $\boldsymbol{Z}_p$ the group of p-adic integers)) (cf. [**6**, Chapter V.1.5]) and $\hat{\boldsymbol{Z}}$ (resp. $\boldsymbol{Z}_p$) contains naturally all integers $\boldsymbol{Z}$ as a dense subgroup, so that $\sigma\omega=\omega+1$ in $\hat{\boldsymbol{Z}}$ (resp. $\boldsymbol{Z}_p$) and $\phi(n)$ is the integer n in $\hat{\boldsymbol{Z}}$ (resp. $\boldsymbol{Z}_p$). Furthermore we have $R_{\mathrm{Tr}}(C^*(\Sigma_G))=\boldsymbol{Q}$ (resp. $=\{k/p^n\,;\,k\in\boldsymbol{Z}$ and $n\in\boldsymbol{N}\}$).

We have seen that $C^*(\pi(\mathcal{A}_k))$ is spatially isomorphic to $C^*(S)\otimes\boldsymbol{B}(\mathfrak{H}_n)$ by Ψ_n in Section 1. Considering $C^*(S)$ as the C^*-algebra $C(\boldsymbol{T})$ of all continuous functions on $\boldsymbol{T}$, the inductive limit of $\{C(\boldsymbol{T})\otimes\boldsymbol{B}(\mathfrak{H}_n)\}_{n=1,2,\dots}$ (resp. $\{C(\boldsymbol{T})\otimes\boldsymbol{B}(\mathfrak{H}_{p^n})\}_{n=1,2,\dots}$) with respect to embeddings $\{\psi_n\}_{n=1,2,\dots}$ (resp. $\{\psi_{p^n}\}_{n=1,2,\dots}$) becomes the crossed product $C(\hat{\boldsymbol{Z}})\times_\sigma\boldsymbol{Z}$ (resp. $C(\boldsymbol{Z}_p)\times_\sigma\boldsymbol{Z}$).

EXAMPLE 2.5. Let $G=\boldsymbol{T}_d$. Then $\Omega=\mathrm{b}\boldsymbol{Z}$ (the Bohr compactification of $\boldsymbol{Z}$), $\phi(n)$ is the integer n in $\mathrm{b}\boldsymbol{Z}$, $\sigma\omega=\omega+1$ and $R_{\mathrm{Tr}}(C^*(\Sigma_G))=\boldsymbol{R}$.

For a given shift dynamical system $\Sigma=(\Omega, \sigma, \phi)$, we will consider the condition for Σ under which Σ is conjugate to Σ_G for some subgroup G of $\boldsymbol{T}_d$. To see this, let $M(\Omega)$ denote the semi-group of all maps of Ω into itself with the pointwise convergence topology and $E(\Sigma)$ denote the closure of the group $\{\sigma^n\}_{n\in\boldsymbol{Z}}$ in $M(\Omega)$. The semi-group $E(\Sigma)$ is called an Ellis semi-group [**3**, Chapter II.6]. Though $\{\sigma^n\}_{n\in\boldsymbol{Z}}$ is contained in the semi-group of continuous map on Ω, a map in $E(\Sigma)$ is not always continuous. However if $\{\sigma^n\}_{n\in\boldsymbol{Z}}$ is equicontinuous on Ω and Σ is minimal, where Ω can be regarded as a uniform space since Ω is compact, then every map in $E(\Sigma)$ is a homeomorphism of Ω and $E(\Sigma)$ becomes a group ([**3**, Proposition 2.9]). In this case $E(\Sigma)$ is a compact abelian group with a dense subgroup $\{\sigma^n\}_{n\in Z}$, that is, $E(\Sigma)$ is a monothetic group. The map $\tau\to\sigma\circ\tau$ $(\tau\in E(\Sigma))$ is a homeomorphism of $E(\Sigma)$ and it is denoted by T_σ. Therefore if $\{\sigma^n\}_{n\in Z}$ is equicontinuous, then $\Sigma'=(E(\Sigma), T_\sigma, \phi)$ is a shift dynamical system where $\phi(n)=\sigma^n$ and $E(\Sigma)$ is a monothetic compact abelian group. By [**16**, Theorem 2.3.3], Σ' is conjugate to $\Sigma_G=(\hat{G}, \sigma_{\hat{G}}, \phi_{\hat{G}})$ for some subgroup G of $\boldsymbol{T}_d$ by a homeomorphism h of $E(\Sigma)$ onto $\hat{G}$ such that $h(\sigma^n)=\phi_{\hat{G}}(n)$. Namely Σ' is strictly conjugate to Σ_G. From this fact, we can see that Σ is strictly conjugate to Σ_G. In fact, the map $\beta:\delta\to\delta(\phi(0))$ $(\delta\in E(\Sigma))$ is a homeomorphism of $E(\Sigma)$ onto Ω such that $\beta(\sigma^n)=\sigma^n(\phi(0))=\phi(n)$ (cf. [**3**, Exercise 21]). By what we mentioned above, Proposition 1.9 and Corollary 1.10, we get the following.

THEOREM 2.6. *Let $\Sigma=(\Omega, \sigma, \phi)$ be a shift dynamical system. Then the following statements are equivalent.*

(1) *$\{\sigma^n\}_{n\in Z}$ is equicontinuous on Ω and Σ is minimal.*

(2) *Σ is conjugate to Σ_G for some subgroup G of $\boldsymbol{T}_d$.*

(3) *Σ is strictly conjugate to Σ_G for some subgroup G of $\boldsymbol{T}_d$.*

(4) *$C^*(\Sigma)=C^*(\Sigma_G)$.*

REMARK. Let $\Sigma=(\Omega, \sigma, \phi)$ be a shift dynamical system. We do not know whether or not $\{\sigma^n\}_{n\in Z}$ is always equicontinuous if $C^*(\Sigma)$ is $*$-isomorphic to $C^*(\Sigma_G)$ for some subgroup G of $\boldsymbol{T}_d$.

By Furstenberg's example [**10**, p. 585], we get an example of a simple C^*-algebra which is not $*$-isomorphic to $C^*(\Sigma_G)$ for any subgroup G of $\boldsymbol{T}_d$.

EXAMPLE 2.7. Let $\Omega=\boldsymbol{T}^2=[0, 1)\times[0, 1)$ and $\sigma(x, y)=(T_\theta x, f(x)+y)$ where T_θ is the rotation on $\boldsymbol{T}$ associated with an irrational number θ $(0<\theta<1)$ and f is a continuous map of $\boldsymbol{T}$ into itself. By Furstenberg [**10**], σ becomes a minimal non-uniquely (that is, there exist at least two σ-invariant measures on Ω) ergodic homeomorphism for a suitable θ and f. Since every ergodic homeomorphism is minimal, $\Sigma=(\Omega, \sigma, \phi)$ is a minimal shift dynamical system where $\phi(n)=\sigma^n(0, 0)$. Thus $C^*(\Sigma)$ is simple and has at least two tracial states. By Theorem 2.2 (1), $C^*(\Sigma)$ is not $*$-isomorphic to C^*-algebra $C^*(\Sigma_G)$ for any subgroup G of $\boldsymbol{T}_d$. In the case where $f(x)=x$, Anzai [**1**] showed that σ is a minimal uniquely ergodic homeomorphism on Ω. In this case, $C^*(\Sigma)$ is a simple C^*-algebra which has a unique tracial state. It is unknown whether this C^*-algebra is $*$-isomorphic to $C^*(\Sigma_G)$ for a subgroup G of $\boldsymbol{T}_d$.

REMARK. Let Σ be a shift dynamical system (Ω, σ, ϕ) such that Ω is homeomorphic to the one-dimensional torus $\boldsymbol{T}$. According to the classical theorem of Poincaré, every homeomorphism β of $\boldsymbol{T}$ such that $\bar{O}(\omega)=\boldsymbol{T}$ for some ω in $\boldsymbol{T}$

is of the form $\beta=h\circ T_\theta\circ h^{-1}$ where T_θ is as in Example 2.7 and h is a homeomorphism of $\boldsymbol{T}$ onto itself. Therefore Σ is conjugate to $\Sigma_\theta=(\boldsymbol{T}, \sigma_\theta, \phi_\theta)$ and thus $C^*(\Sigma)$ is $*$-isomorphic to $\mathcal{A}_\theta$ $(=C^*(\Sigma_\theta))$ for some irrational number θ.

3. C^*-algebras $C^*(\Sigma)$ and compact operators.

In this section, we will consider the shift dynamical systems $\Sigma=(\Omega, \sigma, \phi)$ such that $\phi(\boldsymbol{Z})$ is open in Ω.

LEMMA 3.1. *The set $\phi(\boldsymbol{Z})$ is open in Ω if and only if $\phi(n)$ is an isolated point for every n in $\boldsymbol{Z}$.*

PROOF. Suppose that $\phi(\boldsymbol{Z})$ is open. Then there exists a neighbourhood U of $\phi(0)$ whose closure $\bar{U}$ is contained in the countable set $\phi(\boldsymbol{Z})$. By Baire category theorem, there exists $\phi(n_0)$ in $\bar{U}$ such that $\phi(n_0)$ is isolated, so that each $\phi(n)$ is isolated. q. e. d.

If $\phi(\boldsymbol{Z})$ consists of finite points, then $C^*(\Sigma)$ does not contain compact operator by the remark before Proposition 1.3. But, if $\phi(\boldsymbol{Z})$ consists of infinite points, we can give a necessary and sufficient condition for that the C^*-subalgebra $C^*(\Sigma)$ of $\boldsymbol{B}(\mathfrak{H})$ contains all compact operators.

PROPOSITION 3.2. *We assume that $\phi(\boldsymbol{Z})$ is an infinite set. Then the set $\phi(\boldsymbol{Z})$ is open if and only if the C^*-subalgebra $C^*(\Sigma)$ of $\boldsymbol{B}(\mathfrak{H})$ contains all compact operators.*

PROOF. The necessity is shown by [8, Lemmas 1 and 3]. So we show the sufficiency. Suppose that $C^*(\Sigma)$ contains the one-dimensional projections P_n of $\mathfrak{H}$ onto $[\boldsymbol{e}_n]$. Let $\boldsymbol{a}=(\delta_{m,n})_{m=-\infty}^{\infty}$ then $P_n=\pi(\boldsymbol{a})$ and so $P_n\in\pi(l^\infty(\boldsymbol{Z}))$. Thus P_n belongs to $\pi(C(\Omega))$ by the fact $C^*(\Sigma)\cap\pi(l^\infty(\boldsymbol{Z}))=\pi(C(\Omega))$ denoted in the remark after Proposition 1.3. Hence $P_n=\pi(f)$ for some continuous function f on Ω. Since $P_n\boldsymbol{e}_m=\pi(f)\boldsymbol{e}_m=f(\phi(m))\boldsymbol{e}_m=\delta_{m,n}\boldsymbol{e}_m$ for each $m\in\boldsymbol{Z}$, $f(\phi(n))=1$ and $f(\phi(m))=0$ if $m\neq n$. Since $\phi(\boldsymbol{Z})$ is dense in Ω, f is just the characteristic function $\chi_{\phi(n)}$. Therefore $\phi(n)$ is an isolated point in Ω. q. e. d.

COROLLARY 3.3. *Let $\Sigma_1=(\Omega_1, \sigma_1, \phi_1)$ and $\Sigma_2=(\Omega_2, \sigma_2, \phi_2)$ be shift dynamical systems such that $C^*(\Sigma_1)$ and $C^*(\Sigma_2)$ are $*$-isomorphic. Then $\phi_1(\boldsymbol{Z})$ is open in Ω_1 if and only if $\phi_2(\boldsymbol{Z})$ is open in Ω_2.*

PROOF. If $\phi_1(\boldsymbol{Z})$ is an infinite set, then $C^*(\Sigma_1)$ has an irreducible representation on an infinite dimensional separable Hilbert space. Hence, if one of $\phi_1(\boldsymbol{Z})$ or $\phi_2(\boldsymbol{Z})$ is a finite set, the other is also finite because $C^*(\Sigma_1)$ and $C^*(\Sigma_2)$ are $*$-isomorphic. Thus, we may assume that both of $\phi_1(\boldsymbol{Z})$ and $\phi_2(\boldsymbol{Z})$ are infinite sets. Suppose $\phi_1(\boldsymbol{Z})$ is open in Ω_1. By Proposition 3.2, $C^*(\Sigma_1)$ contains all compact operators. Hence $C^*(\Sigma_2)$ contains the ideal generated by minimal projections in itself. Since $C^*(\Sigma_2)$ acts irreducibly on $\mathfrak{H}$, every minimal projection in $C^*(\Sigma_2)$

is one-dimensional. Thus $C^*(\Sigma_2)$ contains all compact operators. Therefore $\phi_2(\boldsymbol{Z})$ is open in Ω_2. q. e. d.

Let $\boldsymbol{C}(\mathfrak{H})$ denote the ideal of all compact operators on $\mathfrak{H}$ and Φ the canonical map of $\boldsymbol{B}(\mathfrak{H})$ onto the quotient C^*-algebra $\boldsymbol{A}(\mathfrak{H})=\boldsymbol{B}(\mathfrak{H})/\boldsymbol{C}(\mathfrak{H})$, which is called the Calkin algebra on $\mathfrak{H}$. Let $\Sigma=(\Omega, \sigma, \phi)$ be a shift dynamical system such that $\phi(\boldsymbol{Z})$ is open in Ω. If the complement $\Omega_c=\Omega-\phi(\boldsymbol{Z})$ contains a point ω_0 such that $O(\omega_0)=\{\sigma^n(\omega_0); n\in\boldsymbol{Z}\}$ is dense in Ω_c, then $\Sigma_c=(\Omega_c, \sigma_c, \phi_c)$ is also a shift dynamical system where σ_c is the restriction of σ to Ω_c and $\phi_c(n)=\sigma^n(\omega_0)$ for each n in $\boldsymbol{Z}$. We note that the C^*-algebra $C^*(\Sigma_c)$ does not depend on the choice of ω_0 by Proposition 1.3. By [8, Lemma 1 (iii) and Lemma 3], it follows that the C^*-algebra $C^*(\Sigma_c)$ is $*$-isomorphic to the quotient C^*-algebra $\Phi(C^*(\Sigma))=C^*(\Sigma)/\boldsymbol{C}(\mathfrak{H})$. Hence we have the following.

PROPOSITION 3.4. *Let $\Sigma=(\Omega, \sigma, \phi)$ be a shift dynamical system such that $\phi(\boldsymbol{Z})$ is not open in Ω. Then $C^*(\Sigma)$ is $*$-isomorphic to the quotient C^*-algebra $\Phi(C^*(\Sigma_{\boldsymbol{Z}}))$ for some shift dynamical system $\Sigma_{\boldsymbol{Z}}$.*

PROOF. Let $\Omega_{\boldsymbol{Z}}$ be the disjoint union of Ω and $\boldsymbol{Z}$. We define a topological structure on $\Omega_{\boldsymbol{Z}}$ as follows. The family of open sets in $\Omega_{\boldsymbol{Z}}$ consists of subsets U of $\Omega_{\boldsymbol{Z}}$ such that, for each ω in $U\cap\Omega$, there exists a neighbourhood V of ω in Ω such that $U\cup F\supset V\cup \boldsymbol{Z}_V$, where F is a finite set in $\boldsymbol{Z}$ and $\boldsymbol{Z}_V=\{n; \phi(n)\in V\}$. Then every n in $\boldsymbol{Z}=\Omega_{\boldsymbol{Z}}-\Omega$ is an isolated point in $\Omega_{\boldsymbol{Z}}$ and $\boldsymbol{Z}$ is a dense subset of $\Omega_{\boldsymbol{Z}}$. We put $\sigma_{\boldsymbol{Z}}(\omega)=\sigma(\omega)$ for ω in Ω, $\sigma_{\boldsymbol{Z}}(n)=n+1$ and $\phi_{\boldsymbol{Z}}(n)=n$. Then $\Sigma_{\boldsymbol{Z}}=(\Omega_{\boldsymbol{Z}}, \sigma_{\boldsymbol{Z}}, \phi_{\boldsymbol{Z}})$ is a shift dynamical system such that $(\Sigma_{\boldsymbol{Z}})_c=\Sigma$. Therefore, by the fact mentioned before this proposition, $C^*(\Sigma)$ is $*$-isomorphic to the quotient $\boldsymbol{C}^*$-algebra $C^*(\Sigma_{\boldsymbol{Z}})/\boldsymbol{C}(\mathfrak{H})$. q. e. d.

COROLLARY 3.5. *Let Σ be a minimal shift dynamical system. Then $C^*(\Sigma)$ is $*$-isomorphic to $\Phi(C^*(\Sigma_{\boldsymbol{Z}}))$.*

COROLLARY 3.6. *If $C^*(\Sigma)$ is simple, then it is $*$-isomorphic to $\Phi(C^*(\Sigma_{\boldsymbol{Z}}))$.*

For positive integers p and q, let $\mathcal{A}_{p,q}$ be the C^*-subalgebra of $l^\infty(\boldsymbol{Z})$ consisting of sequences $\boldsymbol{a}=(a_n)$ of complex numbers such that the sequences $\{a_{-np+i}\}_{n=1}^{\infty}$ and $\{a_{nq+j}\}_{n=1}^{\infty}$ converge for each $i=0, 1, \cdots, p-1$; $j=0, 1, \cdots, q-1$ and $\mathcal{A}_r$ be the set of sequences in $\mathcal{A}_{r,r}$ such that $\lim_{n\to\infty} a_{-nr+i}=\lim_{n\to\infty} a_{nr+i}$ for each $i=0, 1, \cdots, r-1$. Let $\Sigma_{p,q}$ (resp. Σ_r) $=(\Omega, \sigma, \phi)$ be the shift dynamical system associated with the C^*-algebra $C^*(\pi(\mathcal{A}_{p,q}), S)$ (resp. $C^*(\pi(\mathcal{A}_r), S)$). Then Ω is the compact set $\{x_0, \cdots, x_{p-1}\}\cup\boldsymbol{Z}\cup\{y_0, \cdots, y_{q-1}\}$ where x_i and y_j are the limit points of $\{-np+i\}_{n\in\boldsymbol{Z}_+}$ and $\{nq+j\}_{n\in\boldsymbol{Z}_+}$ respectively for each i, j and $\sigma(n)=n+1$, $\sigma(x_i)=x_{i+1}$ for $i=0, 1, \cdots, p-2$; $\sigma(x_{p-1})=x_0$, $\sigma(y_j)=y_{j+1}$ for $j=0, 1, \cdots, q-2$; $\sigma(y_{q-1})=y_0$. Clearly $\phi(n)=n$. When $p=q$ and $x_i=y_i$ for each i, the above system becomes the shift dynamical system associated with $\mathcal{A}_r$. We can prove that, by using an elementary calculation, the C^*-algebra $C^*(\Sigma_{p,q})$ (resp. $C^*(\Sigma_r)$) contains the ideal $\boldsymbol{C}(\mathfrak{H})$ of all compact operators and its quotient C^*-algebra by $\boldsymbol{C}(\mathfrak{H})$ is $*$-isomorphic to the direct sum of $C(\boldsymbol{T})\otimes\boldsymbol{B}(\mathfrak{H}_p)$ and $C(\boldsymbol{T})\otimes\boldsymbol{B}(\mathfrak{H}_q)$ (resp. $C(\boldsymbol{T})\otimes\boldsymbol{B}(\mathfrak{H}_r)$). In the case of Σ_r, this result is an immediate corollary to the fact mentioned before Proposition 3.4 that $C^*(\Sigma_c)$ and $C^*(\Sigma)/\boldsymbol{C}(\mathfrak{H})$ are $*$-isomorphic and $C^*(\Sigma_r)$ is an example of a C^*-algebra which contains ideals corresponding to no proper σ-invariant closed subset of Ω. In fact, for the ideal $\mathcal{J}=\{f\in C(\boldsymbol{T});\ f(1)=0\}$ of $C(\boldsymbol{T})$, the set $\{T\in C^*(\Sigma_r);\ \Phi(T)\in\mathcal{J}\otimes\boldsymbol{B}(\mathfrak{H}_r)\}$ is such an ideal.

In the following, we shall show that there exists no shift dynamical system except the systems $\Sigma_{p,q}$ and Σ_r in the case where $\Omega-\phi(\boldsymbol{Z})$ is a finite set. We here remark that $\Sigma_{1,1}$ and Σ_1 are the only cases which appear in [8, Section 1].

PROPOSITION 3.7. *Suppose that $\Omega-\phi(\boldsymbol{Z})$ is a finite set. Then Σ is strictly conjugate to $\Sigma_{p,q}$ or Σ_r for some positive numbers p, q or r.*

PROOF. We first suppose that every point of $\Omega-\phi(\boldsymbol{Z})$ is a cluster point of $\boldsymbol{Z}_+$ and put $\Omega-\phi(\boldsymbol{Z})=\{x_0, \cdots, x_{r-1}\}$. Then there exist mutually disjoint neighbourhoods $U(x_i)$ of x_i $(i=0, 1, \cdots, r-1)$. Let s be the number such that $\sigma^s(x_0)=x_0$ and $\sigma^j(x_0)\neq x_0$ for all j $(1\leqq j\leqq s-1)$. We can assume that $V=U(x_0)\cap\{\phi(ns);\ n\in\boldsymbol{Z}_+\}$ is an infinite set (by replacing x_0 by $\sigma^{-j}(x_0)$ if V is finite and $U(x_0)\cap\{\phi(ns+j);\ n\in\boldsymbol{Z}_+\}$ is infinite for $j\neq 0$) because $U(x_0)$ intersects countably many points in $\phi(\boldsymbol{Z})$. Put $\boldsymbol{N}_1=\{n\in\boldsymbol{N};\ \phi(ns)\in V\}$ and $\boldsymbol{N}_2=\{n\in\boldsymbol{N};\ \phi(ns)\notin V\}$. Then $\boldsymbol{N}$ is the disjoint union of $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$. If $\boldsymbol{N}_2$ is an infinite set, so is the intersection $(\boldsymbol{N}_1+1)\cap\boldsymbol{N}_2$. Though the infinite set $\{\phi((n+1)s);\ n\in\boldsymbol{N}_1\}\cap\{\phi(ns);\ n\in\boldsymbol{N}_2\}$ is contained in $\sigma^s V$, which has the only one limit point $\sigma^s(x_0)$ $(=x_0)$, a point x_0 is not a cluster point of $\{\phi(ns);\ n\in\boldsymbol{N}_2\}$. This contradiction implies that $\{\phi(ns);\ n>N\}$ is contained in $U(x_0)$ for a large positive number N. Thus $\phi(ns+j)$ converges to $\sigma^j(x_0)$ for each j $(0\leqq j\leqq s-1)$. Since each x_i is a cluster point of $\phi(\boldsymbol{Z})$, we have $r=s$. Namely Σ is strictly conjugate to Σ_r. For the remaining case, as in the above, we can show that Σ is strictly conjugate to $\Sigma_{p,q}$ for some positive numbers p and q. q. e. d.

Let $\{n_k\}$ be a strictly increasing sequence of integers with n_k dividing n_{k+1} for all k and $\mathcal{A}_{\{n_k\}}$ the norm closure of $\bigcup_{k=1}^{\infty}\mathcal{A}_{n_k}$ in $l^\infty(\boldsymbol{Z})$. Then $\mathcal{A}_{\{n_k\}}$ is a C^*-subalgebra of $l^\infty(\boldsymbol{Z})$ such that $S\pi(\mathcal{A}_{\{n_k\}})S^*=\pi(\mathcal{A}_{\{n_k\}})$ and there exists a canonical map ϕ of $\boldsymbol{Z}$ onto the dense open subset of the character space of $\pi(\mathcal{A}_{\{n_k\}})$. By Proposition 3.2 and the fact of the $*$-isomorph of $C^*(\Sigma_c)$ and $C^*(\Sigma)/\boldsymbol{C}(\mathfrak{H})$, $C^*(\pi(\mathcal{A}_{\{n_k\}}), S)$ contains the ideal $\boldsymbol{C}(\mathfrak{H})$ of all compact operators and its quotient C^*-algebra by $\boldsymbol{C}(\mathfrak{H})$ is $*$-isomorphic to $C^*(\Sigma_G)$ where G is the subgroup of $\boldsymbol{T}_d$ generated by $\{e^{2\pi i m/n_k};\ m=0, 1, \cdots, n_k-1$ and $k=1, 2, \cdots\}$. Furthermore we find that $C^*(\Sigma_G)$ is $*$-isomorphic to the quotient C^*-algebra $\mathfrak{A}(n_k)/\boldsymbol{C}(\mathfrak{K})$ where $\mathfrak{A}(n_k)$ is the C^*-algebra generated by all periodic weighted unilateral shift operators on a Hilbert space $\mathfrak{K}$ with respect to a basis $\{\boldsymbol{f}_n\}_{n\in\boldsymbol{N}}$ of $\mathfrak{K}$ of period n_k for some k. These C^*-algebras $\mathfrak{A}(n_k)/C(\mathfrak{K})$ were partially studied by Bunce-Deddens [4], and furthermore Green [8] and Ghatage-Phillips [7] have already stated that the inductive limit of a sequence $\{\mathfrak{A}(n_k)/C(\mathfrak{K})\}$ satisfying some conditions was determined by the corresponding torsion subgroup of $\boldsymbol{T}_d$. Furthermore Ghatage-Phillips conjectured in the introduction of [7] that $C^*(\Sigma_G)$ were determined by subgroups G of $\boldsymbol{T}_d$.

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