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Restricted principal values of bounded analytic and harmonic functions

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1. Introduction.

Let f be a function defined on the open unit disc $D=\{z:|z|<1\}$. Let P be a point of the unit circle $C=\{z:|z|=1\}$. A cluster value α achieved by f at P (i.e., $\alpha=\lim f(z_n)$ for some sequence z_n in D with $z_n\rightarrow P$) is called a *principal value of* f *at* P if it is a cluster value of f along every curve in D terminating at P. A classical theorem of Gross [8] locates a large class of principal values of a bounded analytic function.

THEOREM 1.1 (Gross' Principal Value Theorem). *Let* f *be a bounded analytic function on* D. *Suppose* α *is a cluster value of* f *along the radius drawn to a point* P *on* C. *Suppose* α *is also an accumulation point of the set of complex numbers which* f *fails to achieve in* D *in some neighborhood of* P. *Then,* α *is a principal value of* f *at* P.

This theorem is considered to be quite deep and its most recent proof [6] is commensurately difficult. In contrast, the results in the present paper represent special cases and variations of the Gross theorem and depend on techniques which can be classified as maximum modulus type theorems. For example, one variation of the second condition on α in Theorem 1.1 which is thematic in this paper is that α be a boundary point of the collection of all cluster values of f at P. On the other hand, this stronger hypothesis often allows us to obtain more detailed information about the boundary behavior of f at P. For example, we show that whether or not it is a radial cluster value each such boundary point is a cluster value of f on every curve to P which is more tangential than some corresponding tangential curve. Our techniques are also adequate to completely characterize the one-sided tangential principal values of real bounded

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harmonic functions. These two examples typify another feature in this paper; namely, the investigation of what we call *restricted principal values*, those which are cluster values on each member of some pre-assigned family of curves. As can be seen from the examples we give, such restricted principal values occur in sufficient variation (without actually being principal values) to warrant their study.

2. Preliminaries.

The results of the succeeding sections are with few exceptions stated within the classical setting of boundary behavior theorems. Thus, one of the tasks of this section will be to introduce a number of cluster sets. On the other hand, our proofs practically always involve the study of the representations of bounded harmonic functions on the maximal ideal space, $\mathfrak{D}$, of the Banach algebra, H^∞, of bounded analytic functions on D. For most of the fundamental facts about this algebra and its connections with the algebra, L^∞, of bounded measurable functions on C, we refer the reader to Hoffman's book [**9**], and [**2**], [**3**], [**12**], [**10**]. We will, however, sketch the main outlines of this theory and supply precise statements of the theorems we need. In addition, we will provide where necessary the connections between the cluster sets just mentioned and the range of functions $f \in L^\infty$ on certain compact subsets of $\mathfrak{D}$.

The maximal ideal space, $\mathfrak{D}$, of H^∞ is a compactification of the disc D, [**4**]. Every homomorphism $h \in \mathfrak{D}$ is thus the w^* limit of a net in D along which every bounded function converges. The Gelfand representation of $f \in H^\infty$ is the unique continuous extension of f to $\mathfrak{D}$. Given $F \in L^\infty$, its Poisson extension f to D is a bounded (complex) harmonic function whose radial limits exist a.e. on C and recover F. This mapping is an isometry between the Banach space of bounded harmonic functions on D (with the sup-norm) and the Banach space L^∞ (with the essential sup-norm). Furthermore, this correspondence is a Banach algebraic isometry between H^∞ and a subalgebra of L^∞. The maximal ideal space of L^∞ can be identified with the Šilov boundary, $\check{S}$, of H^∞ and the Gelfand representation of L^∞ is the collection of all continuous complex functions on $\check{S}$. Each $h \in \mathfrak{D}$ has a unique representing measure, μ_h, supported on $\check{S}$. If $f \in H^\infty$, then μ_h is also a Jensen measure, i.e.,

$$\log|f(h)| \leqq \int_{\check{S}} \log|f|\, d\mu_h\,.$$

The representation on $\check{S}$ of each $f \in L^\infty$ can be extended continuously to all of $\mathfrak{D}$ by $f(h)=\int_{\check{S}} f d\mu_h$, and this extension agrees with the Poisson extension on D. Because of these facts we will usually simply consider a function f in L^∞ or

H^∞ to be a continuous function on $\mathscr{D}$, obviating the need for mentioning the various representations. Even when we are considering an L^∞ function on the circle we will take the fairly standard notational liberty of denoting it by $f(e^{i\theta})$.

Because our interest is in boundary behavior at a point $P\in C$ we will standardize $P=1$ and focus our attention on the fiber, $\mathscr{D}_1$, consisting of those $h\in\mathscr{D}$ which are the limits of nets in D whose limits in the topology of the plane are 1. Then, the restriction algebra $H^\infty|\mathscr{D}_1$ is uniformly closed, $\mathscr{D}_1$ is its maximal ideal space and its Šilov boundary is $\check{S}_1=\check{S}\cap\mathscr{D}_1$. The relative topology in $\mathscr{D}_1$ of $\check{S}_1$ has a basis of open-closed sets of the form

$$\tilde{M}=\{h\in\check{S}_1:\chi_M(h)=1\}$$

where M is some measurable subset of C and χ_M is its characteristic function. Furthermore, the range of $f\in L^\infty$ on $\tilde{M}$ is precisely the set of essential cluster values of $f(e^{i\theta})$ as $e^{i\theta}\to 1$ through M.

A net $\{z_\alpha=r_\alpha e^{i\theta_\alpha}\}$ in D tending to 1 and converging to $h\in\mathscr{D}_1$ is *non-tangential* if $\theta_\alpha/(1-r_\alpha)\to c\in(-\infty,\infty)$, *radial* if $c=0$, *upper tangential* if $\theta_\alpha/(1-r_\alpha)\to\infty$. The homomorphism h is correspondingly said to be a *non-tangential* or *Stolz homomorphism*, a *radial homomorphism*, or an *upper tangential homomorphism*. The collection of all Stolz homomorphisms is denoted by $\mathcal{S}$, all radial, by $\mathscr{R}$, and all upper tangential, by $\mathscr{T}^+$. At this point we make the following convention: Whenever a concept such as upper tangential homomorphism is defined we tacitly assume the dual definition for the "lower" concept. We will also, if necessary, use the obvious notation for such concepts, e.g., $\mathscr{T}^-$ is the set of all lower tangential homomorphisms. We point out that $\mathscr{D}_1$ is the disjoint union of $\mathscr{T}^-$, $\mathcal{S}$, and $\mathscr{T}^+$. The *Lindelöf homomorphisms*, $\mathscr{L}$, are the closure of $\mathcal{S}$, and the *upper barely tangential homomorphisms*, $\mathscr{B}^+$, are defined by $\mathscr{B}^+=\mathscr{T}^+\cap\mathscr{L}$.

By a curve, Γ, to 1 we mean the continuous image of $[0, 1]$ into $D\cup\{1\}$ such that the image of $[0, 1)$ lies in D while the image of 1 is 1. If $re^{i\theta}\in\Gamma$ and $c=\lim\theta/(1-r)$ as $re^{i\theta}\to 1$ along Γ exists in the extended sense, we say that Γ is a *non-tangential curve* if c is finite and an *upper tangential curve* if $c=\infty$. Note that unlike a net tending to a point in $\mathscr{D}_1$, a curve may be neither tangential nor non-tangential. A chord of the circle C passing through 1 is a *Stolz ray*. If this ray is the diameter to 1, then $c=0$; if this ray is above the radius, then $c>0$ and as $c\to\infty$ it approaches the tangent line to C at 1. If Γ is an upper tangential curve, we denote by $\Gamma^{\searrow}$ the collection of all $h\in\mathscr{T}^+$ which are the limits of nets in D which are eventually below Γ.

Let $f\in L^\infty$. The *(full) cluster set of f at* 1, $C(f, 1)$, is the set of all cluster values of f at 1. The *upper boundary cluster set of f at* 1, $C_B^+(f, 1)$, is given by

$$C_B^+(f, 1)=\bigcap_{n=1}^{\infty}\Big(\bigcup_{0<\theta<1/n} C(f, e^{i\theta})\Big)^-.$$

If Γ is a subset of D, the *cluster set along Γ of f at* 1, $C_\Gamma(f, 1)$, is the collection of all cluster values of f through Γ. If Γ_0 is an upper tangential curve, the *upper tangential cluster set below Γ_0 of f at* 1, $\Gamma_0^{\searrow}(f, 1)$, is defined as $\Gamma_0^{\searrow}(f, 1)=f(\Gamma_0^{\searrow})$. The *Stolz cluster set of f at* 1, $\mathcal{S}(f, 1)$, the *Lindelöf cluster set of f at* 1, $\mathcal{L}(f, 1)$, the *upper barely tangential cluster set of f at* 1, $\mathcal{B}^+(f, 1)$, and the *upper tangential cluster set of f at* 1, $\mathcal{T}^+(f, 1)$ are given by $\mathcal{S}(f, 1)=f(\mathcal{S})$, $\mathcal{L}(f, 1)=f(\mathcal{L})$, $\mathcal{B}^+(f, 1)=f(\mathcal{B}^+)$, and $\mathcal{T}^+(f, 1)=f(\mathcal{T}^+)$.

With the appropriate sympathy for the reader confronted by such an explosion of technical terms we admit that the use of many of these cluster sets is almost solely a notational convenience which keeps us (and traditionally others) from having to write too many long but descriptive sentences. On the other hand, most of the sets are geometrically appealing. One exception is the set $\mathcal{B}^+(f, 1)$ first introduced in [3]. This set, however, is as important for many one-sided results as is the classical radial cluster set ($f(\mathcal{R})$) for two-sided theorems. It has perhaps been overlooked in the past because it is difficult to describe.

We now state as a theorem a convenient list of these cluster sets together with various descriptions of them. Before doing this we momentarily require two additional cluster sets. Given constants $a<b$ let $\mathcal{S}_{a,b}(f, 1)$ denote the collection of all cluster values of f at 1 between the Stolz rays with constants a and b inclusive; and if Γ is an upper tangential curve, let $C_{a,\Gamma}(f, 1)$ denote the collection of all cluster values of f at 1 between the Stolz ray with constant a and the curve Γ inclusive.

THEOREM 2.1. *The following relations hold for each $f\in L^\infty$.*

1. $C(f, 1)=f(\mathcal{D}_1)$.
2. $C_\Gamma(f, 1)=f(\Gamma^- -\Gamma)$, *the closure being in $\mathcal{D}$.*
3. $\mathcal{T}^+(f, 1)$ *is the set of upper tangential cluster values of f at* 1.
4. $\mathcal{S}(f, 1)$ *is the set of non-tangential cluster values of f at* 1 (*and is the classical outer angular cluster set*).
5. $\mathcal{L}(f, 1)=\mathcal{S}(f, 1)^-$.
6. $\Gamma^{\searrow}(f, 1)$ *is the set of all cluster values of f approached on upper tangential nets eventually below or on Γ.*
7. $\mathcal{B}^+(f, 1)=\bigcap_{a>0}(\bigcap_{b>a}\mathcal{S}_{a,b}(f, 1))^-$

$$=\bigcap\{\Gamma^{\searrow}(f, 1): \Gamma \textit{ is an upper tangential curve}\}$$

$$=\bigcap_{a>0}\{C_{a,\Gamma}(f, 1): \Gamma \textit{ is an upper tangential curve}\}$$

$$=f(\mathcal{T}^+\cap\mathcal{L}).$$

That is, $\mathscr{B}^+(f, 1)$ *is the collection of all cluster values achieved on nets to* 1 *which are upper tangential but eventually above every Stolz ray and below every upper tangential curve.*

PROOF. The proof of 1. can be found in [**9**] and the proofs of 2., 3., 4., 5., and 6. follow almost by definition. The Stolz side characterization of $\mathscr{B}^+(f, 1)$ is found in [**3**] while the tangential side and Stolz-to-tangential characterizations of $\mathscr{B}^+(f, 1)$ are developed in [**14**].

With one exception we resist the temptation of using special terminology for each instance of the concept of restricted principal values which appears below and state our conclusions in straightforward descriptive terms. Some of these appear casually to be the same as others so we now list them in an effort to accentuate the differences:

α is a cluster value along

1. each curve to 1 (a principal value);
2. each Stolz ray to 1;
3. each curve to 1 which is either a Stolz ray or a tangential curve;
4. each upper tangential curve to 1 (in this case we call α an *upper tangential principal value*);
5. each curve to 1 above the radius (whether or not it has a definite order of contact);
6. each upper tangential curve to 1 which does not lie above a given such curve;
7. each curve to 1 which does not lie below a given curve;
8. each curve to 1 which lies between two given curves.

3. Some theorems on restricted principal values.

The special information which we will require in this section we now state as a theorem. These results are found in [**14**].

THEOREM 3.1. (a) *The maximal ideal space of the restriction algebra,* $(H^\infty | \mathscr{T}^+)^-$, *is* $\mathscr{T}^+$ *and its Šilov boundary is* $\check{S}_1^+ = \check{S}_1 \cap \mathscr{T}^+$.

(b) *Let* Γ *be any upper tangential curve to* 1. *Then, the Šilov boundary of* $L^\infty | \Gamma^{\searrow}$ *is contained in* $\Gamma^- - \Gamma$. *In particular,* $\|f\|_{\mathscr{B}^+} \leqq \|f\|_{\Gamma^- - \Gamma}$. *Also, if* Γ, Γ' *are both upper tangential and* Γ' *lies below* Γ, *then* $\|f\|_{\Gamma'^- - \Gamma'} \leqq \|f\|_{\Gamma^- - \Gamma}$ (*where* $\|f\|_K = \sup\{|f(h)| : h \in K\}$).

The result below shows that a radial cluster value which is not assumed in every neighborhood of 1 or which is in a sense a cluster value of largest modulus is already a cluster value along each Stolz ray. This is partly illustrated by any function which like $\exp((z+1)/(z-1))$ has a radial limit which is not a principal value.

LEMMA 3.2. *Let $f \in H^{\infty}$ and suppose that α is a radial cluster value of f at 1. If either* (1) *$f(z) \neq \alpha$ near 1 in D; or* (2) *α is a boundary point of the Lindelöf cluster set, then α is a cluster value along each Stolz ray.*

PROOF. Let h be a radial homomorphism such that $f(h)=\alpha$. Suppose we have shown that either hypothesis (1) or (2) implies that f is constantly α on the part, $P(h)$, in which h lies. Hoffman [**10**] has shown that the part of a radial homomorphism is an analytic disc in $\mathcal{S}$ which intersects the closure of each Stolz ray. Thus, we immediately have that α is a cluster value along each Stolz ray.

Suppose (1) holds. Then, the Blaschke product in the canonical factorization of $f-\alpha$ is continuous at 1. Thus, we may assume without loss of generality that $f-\alpha$ does not vanish in D. Then, by a result of Royden in ([**10**], p. 78), f is constantly α on each part on which it assumes the value α.

Suppose (2) holds. Since $P(h)$ is an analytic disc, f is an open map on $P(h)$. Since the range of f on $P(h)$ is a subset of the Lindelöf cluster set and we are assuming α to be on its boundary, it follows again that f is constantly α on $P(h)$; and the theorem follows.

The next result holds in any function algebra. It will allow us not only to predict various restricted principal values but also to measure to what extent a value comes close to being a restricted principal value. In this theorem and the later related results we use d for the Euclidean distance between a point and a compact subset of the plane and the operator, ∂, to denote the topological boundary of a set.

THEOREM 3.3. *Let A be a function algebra on its maximal ideal space Σ. Let K be a non-empty compact subset of Σ with the property that K is the maximal ideal space of the closure of the restriction algebra $A|K$. Suppose B and B' are non-empty compact sets with $B \subset K$, $B' \subset \Sigma$ and such that for every $f \in A$*

$$\|f\|_{B} \leqq \|f\|_{B' \cap K}.$$

Then, for every $h \in B$ and every $f \in A$

$$d(f(h), f(B' \cap K)) \leqq 2d(f(h), \partial f(K)).$$

In particular $f(B) \cap \partial f(K) \subset f(B' \cap K)$.

PROOF. For $f \in A$, $h \in B$ let

$$\delta = d(f(h), f(B' \cap K)), \quad \eta = d(f(h), \partial f(K)).$$

If $\delta \leqq \eta$ there is nothing to prove, so we assume $\eta < \delta$. Given an arbitrary positive number $\varepsilon < \delta - \eta$ choose a complex number $z \notin f(K)$ such that $|f(h)-z| \leqq \eta + \varepsilon$. Since K is the maximal ideal space of $(A|K)^{-}$ the function $g=(f-z)^{-1} \in (A|K)^{-}$. Thus, since B, $B' \cap K$ are both subsets of K, and there is a sequence

$g_n \in A$ with $\|g_n - g\|_K \to 0$, we have $\|g\|_B \leqq \|g\|_{B' \cap K}$. We estimate for any $\varphi \in B' \cap K$,

$$\begin{aligned}\|f-z\|_{B'\cap K} &\geqq |f(\varphi)-z| \\ &\geqq |f(\varphi)-f(h)| - |f(h)-z| \\ &\geqq \delta-(\eta+\varepsilon).\end{aligned}$$

Thus,

$$\|g\|_{B'\cap K} \leqq \frac{1}{\delta-(\eta+\varepsilon)}.$$

Also,

$$|g(h)| \geqq \frac{1}{\eta+\varepsilon}.$$

Therefore,

$$\frac{1}{\eta+\varepsilon} \leqq \frac{1}{\delta-(\eta+\varepsilon)}$$

or

$$\delta-\eta-\varepsilon \leqq \eta+\varepsilon$$

and since ε is arbitrary

$$\delta \leqq 2\eta$$

as claimed.

We list three corollaries. The first one is typical of the one-sided results utilizing the upper barely tangential cluster set. As either Example 5.1 or Example 5.2 of § 5 shows, it is possible for α to be an upper tangential principal value without being a principal value. The second result is more technical and illustrates that restricted principal values might occur in a way which is indeed very restricted. The third will be mentioned again in connection with Theorem 3.7 from which it can also be derived.

COROLLARY 3.4. *Let $f \in H^\infty$. Suppose α is an upper barely tangential cluster value. Then, for each upper tangential curve Γ*

$$d(\alpha, C_\Gamma(f, 1)) \leqq 2d(\alpha, \partial\mathfrak{T}^+(f, 1)).$$

In particular, if α is a boundary point of the upper tangential cluster set of f, then α is a cluster value of f along each upper tangential curve.

PROOF. By Theorem 3.1, the maximal ideal space of $(H^\infty|\mathfrak{T}^+)^-$ is $\mathfrak{T}^+$ and the Šilov boundary of $H^\infty|\Gamma^\frown$ is contained in $\Gamma^- - \Gamma$ and for each $f \in H^\infty$, $\|f\|_{\mathfrak{B}^+} \leqq \|f\|_{\Gamma^- - \Gamma}$. Since $f(\Gamma^- - \Gamma) = C_\Gamma(f, 1)$, the result follows from Theorem 3.3.

COROLLARY 3.5. *Let $f \in H^\infty$ and suppose α is an upper barely tangential cluster value. Let Γ_0 be the upper semi-circle of a circle in D tangent to the unit circle at* 1 (*an oricycle*). *Then, for any upper tangential curve Γ lying below Γ_0*

$$d(\alpha, C_{\Gamma}(f, 1)) \leqq 2d(\alpha, \partial\Gamma_0^{\searrow}(f, 1)).$$

In particular, if α is on the boundary of the collection of all upper tangential cluster values approached below or on Γ_0, then α is a cluster value on each upper tangential curve Γ lying below Γ_0.

PROOF. It is a general function algebraic fact (see [7], p. 39) that the maximal ideal space of the uniform closure of the restriction of an algebra to a compact subset, K, consists of all those homomorphisms of the original algebra which can be represented by a probability measure on K. Since by Theorem 3.1, the Šilov boundary of $H^{\infty}|\Gamma_0^{\searrow}$ is contained in $\Gamma_0^{-}-\Gamma_0$ we see that $\Gamma_0^{\searrow}$ is a subset of the maximal ideal space of $(H^{\infty}|\Gamma_0)^{-}$. The function $\exp((1+z)/(1-z))$ has smaller modulus on $\Gamma_0^{\searrow}$ than at any homomorphism in $\mathfrak{T}^{+}-\Gamma_0^{\searrow}$ which rules out such homomorphisms from being in the maximal ideal space of $(H^{\infty}|\Gamma_0^{\searrow})^{-}$. Let u be the harmonic measure of the lower half of the unit circle and let $f=\exp(u+iv)$ where v is a harmonic conjugate of u. Then, $f \in H^{\infty}$ and if h is not an upper tangential homomorphism we have $|f(h)|>1$ while $|f|=1$ on $\Gamma_0^{\searrow}$. Consequently, the maximal ideal space of $(H^{\infty}|\Gamma_0^{\searrow})^{-}$ is exactly $\Gamma_0^{\searrow}$. By Theorem 3.1, if Γ is any upper tangential curve (lying below Γ_0) then $\|f\|_{\mathfrak{S}^{+}} \leqq \|f\|_{\Gamma^{-}-\Gamma}$. Thus, the hypotheses of Theorem 3.3 are satisfied and the theorem follows.

COROLLARY 3.6. *Suppose $f \in H^{\infty}$ and α is a radial cluster value of f at 1. If α also lies on the boundary of the full cluster set, $C(f, 1)$, of f at 1, then α is a cluster value of f along each curve to 1 which is either a Stolz ray or a tangential curve.*

PROOF. The fact that $\alpha \in \partial C(f, 1)$ implies that α is on the boundary of any subset of $C(f, 1)$ which contains α. In particular, α is on the boundary of the Lindelöf cluster set. By Lemma 3.2, α is a cluster value of f on each Stolz ray. Consequently, α is in both the upper and lower barely tangential cluster sets and so, a fortiori, α is in the upper and lower tangential cluster sets. By the same reasoning as above, α is in turn a boundary point of each of these last two cluster sets. Applying Corollary 3.4 and its implied lower tangential version, we conclude that α is a cluster value on each tangential curve.

In case we are only interested in information about restricted principal values and not the more precise information involving the distance function, d, we may use Theorem 3.3 to prove Theorem 3.7 below. Recalling Gross' theorem, Theorem 1.1, one might wonder what role the two hypotheses play. Theorem 3.7 supplies a partial answer to this question. Each boundary point of the full cluster set, $C(f, 1)$, "desires" to be a principal value in that it is necessarily a restricted principal value either above some upper tangential curve or below some lower tangential curve. In fact, as soon as it is a cluster value along any tangential curve it is immediately a restricted principal value from that curve upwards (or downwards). It is not surprising then that if such a boundary point is a radial cluster value it will be (as the Gross theorem proves) a principal value. Although our techniques do not allow us to prove this special case of the Gross theorem, Corollary 3.6 above does illustrate the point we have been making. We have included in Theorem 3.7 also the more elementary fact that even the values in $C_B^+(f, 1)$ have some ambitions toward being principal values. Any such point is a cluster value on some upper tangential curve above any given one. Finally, we remark that the function $f(z)=\exp((z+1)/(z-1))$ illustrates that restricted principal values above tangential curves do occur. For, $\mathfrak{T}^+(f, 1)=D\cup C$ and each value $\alpha\in C$ is a cluster value on each curve more tangential than oricycles, but is not achieved at all on oricycles or less tangential curves.

THEOREM 3.7. *Let $f\in H^\infty$. Then,*

$$\partial\mathfrak{T}^+(f, 1)\subset\liminf C_\Gamma(f, 1)\subset\limsup C_\Gamma(f, 1)=C_B^+(f, 1),$$

where the limits are taken over all upper tangential curves directed by $\Gamma\leqq\Gamma'$ if Γ' lies above Γ, $\partial\mathfrak{T}^+(f, 1)$ is the boundary of the upper tangential cluster set of f, $C_\Gamma(f, 1)$ is the cluster set of f along Γ and $C_B^+(f, 1)$ is the upper boundary cluster set of f.

Furthermore, as soon as a value in $\partial\mathfrak{T}^+(f, 1)$ is achieved as a cluster value on some upper tangential curve Γ it is a restricted principal value from Γ on upwards.

PROOF. We begin by supposing that $\alpha\in C_B^+(f, 1)$. By the definition of $C_B^+(f, 1)$ there is a sequence of points $e^{i\theta_n}$ tending to 1 from above and cluster values, α_n, of f at $e^{i\theta_n}$ such that $\alpha_n\to\alpha$. There are many ways of choosing points z_n in the disc sufficiently close to $e^{i\theta_n}$ with, say, $f(z_n)$ within $1/n$ of α_n. In particular, we may choose these points so that the segmental curve joining them lies above any prescribed upper tangential curve. Thus,

$$\alpha\in\bigcap_{\substack{\Gamma_0 \text{ upper} \\ \text{tangential}}}\ \bigcup_{\Gamma\geqq\Gamma_0} C_\Gamma(f, 1)=\limsup C_\Gamma(f, 1),$$

and

$$C_B^+(f, 1)\subset\limsup C_\Gamma(f, 1).$$

On the other hand, suppose $\alpha\notin C_B^+(f, 1)$. Then, there is some interval, I, on the circle abutting 1 from above and some $\varepsilon>0$ such that $\{z\in D: |f(z)-\alpha|\leqq\varepsilon\}^-\cap C\cap I=\{1\}$. Thus, there is certainly an upper tangential curve to 1 which misses the set $\{z\in D: |f(z)-\alpha|\leqq\varepsilon\}$. Consequently, $\alpha\notin\limsup C_\Gamma(f, 1)$ and combining these facts with the conclusions of the previous paragraph we have

$$\limsup C_\Gamma(f, 1)=C_B^+(f, 1).$$

We assert that $\partial\mathfrak{T}^+(f, 1)\subset C_B^+(f, 1)$. This is actually a result of Doob ([**6**], Theorem G), but since we can also prove it easily we do so for completeness.

From Theorem 3.1 we have that $\mathfrak{T}^+$ is the maximal ideal space of $(H^\infty|\mathfrak{T}^+)^-$ and its Šilov boundary is $\check{S}_1^+$. By a general Banach algebraic result (see, for example, [11], p. 143),

$$\partial\mathfrak{T}^+(f, 1)=\partial f(\mathfrak{T}^+)\subset f(\check{S}_1^+)\,.$$

But it is clear from the techniques in [15] that $f(\check{S}_1^+)\subset C_B^+(f, 1)$ and the assertion follows. Therefore, by what we have proved before, if $\alpha\in\partial\mathfrak{T}^+(f, 1)$, then α is a cluster value on some upper tangential curve Γ. If Γ' is any upper tangential curve above Γ, then we have from Theorem 3.1 that $\|f\|_{\Gamma^- -\Gamma}\leqq\|f\|_{\Gamma'^- -\Gamma'}$. It is by now easy to see that the hypotheses of Theorem 3.3 are fulfilled with $K=\mathfrak{T}^+$, $B=\Gamma^- -\Gamma$ and $B'=\Gamma'^- -\Gamma'$. Thus, we may conclude that α is a cluster value along Γ'. Consequently,

$$\alpha\in\bigcup_{\substack{\Gamma_0 \text{ upper} \\ \text{tangential}}}\ \bigcap_{\Gamma\geq\Gamma_0} C_\Gamma(f, 1)\,.$$

Therefore, $\partial\mathfrak{T}^+(f, 1)\subset\liminf C_\Gamma(f, 1)$ completing the proof of the chain of containments. In the process of this last part of the proof we have also shown that a value $\alpha\in\partial\mathfrak{T}^+(f, 1)$ which is a cluster value along some upper tangential curve Γ is also a cluster value along any curve Γ' to 1 above Γ.

The final result of this section is the one promised in the introduction characterizing the upper tangential principal values of real bounded harmonic function.

THEOREM 3.8. *Let u be a real bounded harmonic function on D. Then, the collection of all upper tangential principal values of u coincides with its upper barely tangential cluster set, $\mathcal{B}^+(u, 1)$.*

PROOF. Suppose α is a cluster value of u on each upper tangential curve Γ, i.e., $\alpha\in C_\Gamma(u, 1)$. Then, for each such Γ, $\alpha\in\Gamma^\searrow(u, 1)$. By Theorem 2.1, part 7., $\alpha\in\mathcal{B}^+(u, 1)$. Thus, upper tangential principal values are necessarily in $\mathcal{B}^+(u, 1)$. (In fact, no special use was made of u being real here.)

For the converse we first recall that by Theorem 3.1, $\|f\|_{\mathcal{B}^+}\leqq\|f\|_{\Gamma^- -\Gamma}$ for any $f\in L^\infty$ and upper tangential curve Γ. In addition we see from [14] that both $\mathcal{B}^+$ and $\Gamma^- -\Gamma$ are compact connected sets. Therefore, if $u\in L^\infty$ and u is real, then $\mathcal{B}^+(u, 1)=u(\mathcal{B}^+)$ and $C_\Gamma(u, 1)=u(\Gamma^- -\Gamma)$ are compact real intervals. Since $\|u\|_{\mathcal{B}^+}\leqq\|u\|_{\Gamma^- -\Gamma}$, $\mathcal{B}^+(u, 1)\subset C_\Gamma(u, 1)$. Consequently, every upper barely tangential cluster value is also in $C_\Gamma(u, 1)$ for each upper tangential curve Γ and is therefore an upper tangential principal value.

Once again we may illustrate this theorem using the function $f(z)=\exp((z+1)/(z-1))$. Since the radial limit of f is zero, $\mathcal{B}^+(f, 1)=\{0\}$. This is by no means an upper tangential principal value. However, if $u=\mathrm{Re}\, f$, then u has cluster value zero on each (upper tangential) curve to 1 as predicted. One sees also that, in a sense, it is easier for a real L^∞ function to have restricted principal values than for a general L^∞ function.

4. Restricted principal values arising from even stronger hypotheses.

It turns out that there are some fixed approaches along curves to 1 for which we can predict in advance for a large class of L^∞ functions that their (restricted) principal values will in fact be cluster values along these approaches. Although this appears to us to be a very striking phenomenon it is probably just a testimony to the complexity of $\mathfrak{D}$ and ultimately to the axiom of choice. The type of hypothesis which we require to produce this behavior often differs from that of §3 in that we assume what corresponds to very explicit information about the behavior of an L^∞ function on the circle.

Recall that the Šilov boundary of $H^\infty|\mathfrak{D}_1$ is $\check{S}_1=\check{S}\cap\mathfrak{D}_1$. Thus, the unique representing measure, μ_h, of each $h\in\mathfrak{D}_1$ is supported on $\check{S}_1$. We will denote the minimum closed support of μ_h by $\mathfrak{S}_{\mu_h}$. Let M be a measurable subset of C. Then, the characteristic function $\chi_M\in L^\infty$ and its extension to $\mathfrak{D}$, which we will call u_M, agrees on D with the classical harmonic measure of M. Recalling the definition of $\tilde{M}$ we see that $u_M(h)=\mu_h(\tilde{M})$. This simple but fundamentally important fact is discussed in [12] and is the main tool used in [2] to prove Theorem 4.1 below. It is this theorem upon which the present section is based.

We require also the classical definitions of the one- and two-sided upper and lower densities at 1 of a measurable subset, M, of C. These are given by

$$D(M)=\overline{\lim_{\theta\to 0}}\frac{\lambda(M\cap I_\theta)}{2\theta},\quad d(M)=\underline{\lim_{\theta\to 0}}\frac{\lambda(M\cap I_\theta)}{2\theta},$$

$$D^+(M)=\overline{\lim_{\theta\to 0}}\frac{\lambda(M\cap J_\theta)}{\theta},\quad d^+(M)=\underline{\lim_{\theta\to 0}}\frac{\lambda(M\cap J_\theta)}{\theta},$$

where λ is Lebesgue measure, and $I_\theta=\{e^{it}: -\theta\leqq t\leqq\theta\}$, $J_\theta=\{e^{it}: 0\leqq t\leqq\theta\}$.

THEOREM 4.1. [13]. (a) *Let h be a radial homomorphism. For every curve Γ to 1 there exists a homomorphism $\varphi\in\Gamma^- -\Gamma$ such that $\mathfrak{S}_{\mu_\varphi}\subset\mathfrak{S}_{\mu_h}$. Furthermore, φ can be chosen to lie in the set $\{\psi\in\mathfrak{D}_1: u_M(\psi)=1$ for every $M\subset C$ with $d(M)=1\}$.*

(b) *Let h be an upper barely tangential homomorphism and let K be any closed subset of $\mathfrak{S}_{\mu_h}$ such that $\mu_h(K)=k>0$. Given any upper tangential curve Γ to 1 there exists a homomorphism $\varphi\in\Gamma^- -\Gamma$ such that $\mathfrak{S}_{\mu_\varphi}\subset\mathfrak{S}_{\mu_h}$ and $\mu_\varphi(K)\geqq k$.*

(c) *Let Γ_0 be any curve in D to 1 which lies above some Stolz ray. Let h be a homomorphism approached below Γ_0 which is not lower tangential. Suppose $K\subset\mathfrak{S}_{\mu_h}\cap\check{S}_1^+$, K is closed and $\mu_h(K)=k>0$. Given a curve Γ to 1 which lies above Γ_0 there exists a homomorphism $\varphi\in\Gamma^- -\Gamma$ such that $\mathfrak{S}_{\mu_\varphi}\subset\mathfrak{S}_{\mu_h}$ and $\mu_\varphi(K)\geqq k$.*

Using these facts we may prove a number of results concerning restricted principal values. The only immediate drawback of these theorems is that the hypotheses are somewhat abstract. We will, therefore, subsequently supply a number of corollaries with more concrete assumptions.

THEOREM 4.2. *Let $f \in L^\infty$. Suppose there is a radial homomorphism h such that f is constantly α on $\mathfrak{S}_{\mu_h}$. Then, α is a principal value of f at 1. Furthermore, for each curve Γ to 1 there is a universal net on Γ tending to 1 along which f tends to α and the choice of this net is independent of f.*

PROOF. Let Γ be any curve to 1. By Theorem 4.1, part (a), there is a $\varphi \in \Gamma^- - \Gamma$ such that $\mathfrak{S}_{\mu_\varphi} \subset \mathfrak{S}_{\mu_h}$. Choose a universal net in D tending along Γ to φ. Then, if $f \in L^\infty$ and $f \equiv \alpha$ on $\mathfrak{S}_{\mu_h}$, we also have $f \equiv \alpha$ on $\mathfrak{S}_{\mu_\varphi}$ so $f(\varphi) = \alpha$, i. e., f tends to α along the universal net chosen. Consequently, α is a principal value.

We remark that when $f \equiv \alpha$ on a support $\mathfrak{S}_{\mu_\varphi}$ we may conclude that $f(\varphi) = \alpha$ even for $f \in L^\infty$. Because μ_φ is a Jensen measure for H^∞, if $f \in H^\infty$, it is enough to have that $f \equiv \alpha$ on a set of μ_φ-positive measure in order to conclude that $f(\varphi) = \alpha$. The two theorems below are valid even for $f \in L^\infty$ in case the number k in their hypotheses is 1.

THEOREM 4.3. *Let $f \in H^\infty$. Suppose there is an $h \in \mathscr{B}^+$ and a closed subset $K \subset \mathfrak{S}_{\mu_h}$ with $\mu_h(K) = k > 0$ and $f \equiv \alpha$ on K. Then α is an upper tangential principal value of f at 1. Furthermore, one may choose in advance a universal net on each upper tangential curve depending on h and K but not on f along which f tends to α.*

PROOF. This follows from Theorem 4.1, part (b), in much the same way that Theorem 4.2 did from part (a). The only real difference is explained in the remarks preceding the statement of the present theorem.

Similarly we prove

THEOREM. 4.4. *Let $f \in H^\infty$. Let Γ_0 be any curve in D to 1 which lies above some Stolz ray. Let h be a homomorphism approached below Γ_0 which is not lower tangential. Suppose $K \subset \mathfrak{S}^+_{\mu_h}$ $(= \mathfrak{S}_{\mu_h} \cap \check{S}^+_1)$ with $\mu(K) = k > 0$, K closed. If $f \equiv \alpha$ on K, then α is a cluster value of f along every curve to 1 lying above Γ_0. Furthermore, one may choose in advance a universal net on each such curve depending on h and K but not on f along which f tends to α.*

We shall now investigate the hypothesis that a function in L^∞ be identically α on a subset of positive measure of a support. We begin with the most apparent method for producing this phenomenon. If $h \in \mathscr{D}_1$ and $|f(h)| = \|f\|_{\mathscr{D}_1}$, then since $f(h)$ is the integral of f with respect to a probability measure we have f constantly $f(h)$ on $\mathfrak{S}_{\mu_h}$. From this observation and the previous three theorems we easily prove the following results. (Part (a) for $f \in H^\infty$ is a result of Doob [5] and is proved in [2] for $f \in L^\infty$.)

COROLLARY 4.5. *Let $f \in L^\infty$ and let α be a cluster value of f of maximum modulus. Then,*

(a) *If α is a radial cluster value, it is a principal value.*

(b) *If α is an upper barely tangential cluster value, it is an upper tangential principal value.*

(c) *As soon as α is a cluster value along some curve, Γ_0, lying above a Stolz ray, it is a cluster value on every curve above Γ_0.*

One may obtain the same conclusions by using the concept of "peak point" of a compact subset, K, of the plane for the algebra, $R(K)$, of continuous functions on K which can be uniformly approximated by rational functions with poles off K. (See [7], Chapter 8 for characterizations of peak points.) For such a point, z_0, there is a $g \in R(K)$ such that $g(z_0)=1$ while for $z \in K$, $z \neq z_0$ we have $|g(z)|<1$. There is clearly an operational calculus using the functions in $R(K)$. That is, if A is a function algebra with maximal ideal space Σ, and if $f \in A$ and $f(\Sigma) \subset K$, then $g \circ f \in A$ for every $g \in R(K)$. Then, if $h \in \mathfrak{D}_1$, $f \in H^\infty$ and $f(h)$ is a peak point of $f(\mathfrak{D}_1)$, there is a $g \in R(f(\mathfrak{D}_1))$ with $g \circ f \in H^\infty$, $(g \circ f)(h) =1$, and $\|g \circ f\|_{\mathfrak{D}_1}=1$. This forces $g \circ f$ to be constantly 1 on $\mathfrak{S}_{\mu_h}$ and, in turn, f to be constantly $f(h)$ on $\mathfrak{S}_{\mu_h}$. We state the implied counterpart of Corollary 4.5, part (a), below since it represents a special case of the Gross principal value theorem. To see the connection with Corollary 3.6, we notice that a peak point of $f(\mathfrak{D}_1)$ is necessarily a boundary point of $C(f, 1)$, in fact, "most" boundary points are peak points.

COROLLARY 4.6. *Let $f \in H^\infty$ and let α be a radial cluster value of f. If α is a peak point for $R(C(f, 1))$, then α is a principal value of f at 1. Furthermore, for each curve Γ to 1, α is a cluster value along a universal net on Γ with the property that for each measurable subset $M \subset C$ with $d(M)=1$ the harmonic measure u_M tends to 1.*

An important special case of this last result is by now not difficult.

COROLLARY 4.7. *Let f be a one-to-one bounded analytic function on D. Then, every radial cluster value of f (at 1) is a principal value of f (at 1).*

PROOF. If f is one-to-one, then it is clear that $C(f, 1)$ has no interior and is disjoint from $f(D)$. Let α be a radial cluster value of f at 1. Then, the image under f of the radius to 1 is a curve in the complement of $C(f, 1)$ which clusters on α. Thus, given any disc D_r of radius r centered at α it is clear that we may find a connected piece of this curve whose diameter is as close to r as we please. But then (see [7], Chapter 8) the analytic capacity of this curve is almost r so that the analytic capacity, $\gamma(D_r - C(f, 1))$, of the complement of $C(f, 1)$ in D_r is also almost r. Consequently,

$$\overline{\lim_{r \to 0}} \frac{\gamma(D_r - C(f, 1))}{r} > 0$$

which verifies a condition due to Curtis sufficient to imply that α is a peak point for $R(C(f, 1))$ (see [7], Chapter 8). Corollary 4.6 applies and the result follows.

Given $f \in L^\infty$, a complex number α, and $\varepsilon > 0$, we let $S_\varepsilon = S_\varepsilon(f, \alpha) = \{e^{i\theta} : |f(e^{i\theta}) - \alpha| < \varepsilon\}$. We will also have need of $S_\varepsilon^+ = S_\varepsilon \cap \{e^{i\theta} : 0 < \theta < \pi\}$. Following Doob [5] we define f to be *quasi-approximately continuous with value* α provided $D(S_\varepsilon(f, \alpha)) = 1$ for every $\varepsilon > 0$. Doob proved in [5] (see also [2]) that a quasi-approximate continuity value of $f \in H^\infty$ of maximum modulus is also a principal value. From what we have done so far we can show that the hypothesis that the value be of maximum modulus is entirely superfluous even for $f \in L^\infty$.

COROLLARY 4.8. *Let $f \in L^\infty$. Every quasi-approximate continuity value of f is a principal value of f.*

PROOF. It was already known in [2] that for a function $f \in L^\infty$, α is a quasi-approximate continuity value if and only if $f \equiv \alpha$ on the support of some radial homomorphism. The result is thus immediate from Theorem 4.2.

A similar one-sided result was proved for H^∞ functions in [2] and we state this result here.

THEOREM 4.9. *Let $f \in H^\infty$. Suppose there exists a $k > 0$ such that $D^+(S_\varepsilon^+(f, \alpha)) \geqq k$ for some α and all $\varepsilon > 0$; or more generally,*

$$\lim_{\varepsilon \to 0} D^+(S_\varepsilon^+(f, \alpha)) \log \varepsilon = -\infty .$$

Then, f has the cluster value α on each curve to 1 which lies above the radius.

In § 5 we will compare the second part of this theorem with Theorem 4.3. For now we wish to illuminate the first part of the theorem by proving a slightly more general theorem using the present techniques. The key is the following lemma.

LEMMA 4.10. *Let $f \in L^\infty$. There exists a radial homomorphism, h, such that $f \equiv \alpha$ on a subset of $\mathfrak{S}_{\mu_h} \cap \check{S}_1^+$ of positive μ_h-measure if and only if there is a $k > 0$ such that $D^+(S_\varepsilon^+(f, \alpha)) \geqq k$ for every $\varepsilon > 0$.*

PROOF. Let $S_\varepsilon^+ = S_\varepsilon^+(f, \alpha)$, $\mathfrak{S}_{\mu_h}^+ = \mathfrak{S}_{\mu_h} \cap \check{S}_1^+$. Suppose there is a $k > 0$ such that $D^+(S^+) \geqq k$ for every $\varepsilon > 0$. From [13] we see that there is a radial homomorphism h_ε such that $u_{S_\varepsilon^+}(h_\varepsilon) = \mu_h(\tilde{S}_\varepsilon^+) \geqq \delta_k$ where $\delta_k = \pi^{-1} \tan^{-1}(k/2(1-k)^{1/2})$. Thus, for each $\varepsilon > 0$ the set $E_\varepsilon = \{h \in \mathfrak{R} : u_{S_\varepsilon^+}(h) \geqq \delta_k\}$ is non-empty and compact. Since the collection of sets S_ε^+ form a filter base on the upper semi-circle it is clear that the collection of sets E_ε form a filter base on $\mathfrak{R}$, and $\cap \{E_\varepsilon : \varepsilon > 0\} \neq \emptyset$. Let h be in this intersection and let $K = \cap \{\tilde{S}_\varepsilon^+ : \varepsilon > 0\} \cap \mathfrak{S}_{\mu_h}$. Since the range of f on $\tilde{S}_\varepsilon^+$ is the set of essential cluster values of f at 1 through S_ε^+ we have $|f - \alpha| \leqq \varepsilon$ on $\tilde{S}_\varepsilon^+$. Thus, $f \equiv \alpha$ on K. Since $\mu_h(\tilde{S}_\varepsilon^+) \geqq \delta_k$ for all $\varepsilon > 0$, $\mu_h(K) > 0$ and we have verified the "only if" portion of the theorem.

Conversely, suppose $K\subset\mathfrak{S}^{+}_{\mu_h}$, $h\in\mathfrak{R}$, $\mu_h(K)=k'>0$ and $f\equiv\alpha$ on K. Clearly, for each $\varepsilon>0$, $\tilde{S}^{+}_{\varepsilon}\supset K$ so that $\mu_h(\tilde{S}^{+}_{\varepsilon})\geqq k'$. From [**13**] we have the estimate that $D^{+}(S^{+}_{\varepsilon})\geqq 2k'=k>0$ and the proof is completed.

We mention now one additional well-known fact which follows from a result of Bishop [**1**] and the analysis of the Stolz homomorphisms in [**10**]. Given an $h\in\mathfrak{R}$ and any Stolz ray there is a φ in the closure of that ray such that $c^{-1}\mu_h\leqq\mu_\varphi\leqq c\mu_h$ for some c. In particular, $\mathfrak{S}_{\mu_h}=\mathfrak{S}_{\mu_\varphi}$ and if K is a subset with positive measure with respect to one of the measures, it is also with respect to the other. Using this in combination with Lemma 4.10 and Theorem 4.4 we have

COROLLARY 4.11. *Let $f\in H^{\infty}$. Suppose there exists a $k>0$ such that $D^{+}(S^{+}(f,\alpha))\geqq k$ for some α and all $\varepsilon>0$. Then, f has the cluster value α on each curve to* 1 *which lies above some Stolz ray.*

5. Some examples.

In this section we first list three examples, discuss their relevance, and then verify the examples.

EXAMPLE 5.1. Let $a>1$, and let $B(z)$ denote the Blaschke product with zeros $\{1-a^{-n}\}$, i.e.,

$$B(z)=\prod_{n=1}^{\infty}\frac{z-(1-a^{-n})}{1-(1-a^{-n})z}.$$

Then, every Stolz cluster value of B has modulus less than 1 while every tangential cluster value of B has modulus 1.

EXAMPLE 5.2 (See [**10**], p. 107). Let $f(z)$ be the principal branch of the relation $[(z+1)/(z-1)]^{i}$. Then, f maps the disc D onto the annulus $\{z: e^{-\pi/2}<|z|<e^{\pi/2}\}$, $f(S)$ is a subset of this annulus while each number of modulus $e^{-\pi/2}$ $[e^{\pi/2}]$ is a cluster value on each upper [lower] tangential curve to 1.

EXAMPLE 5.3. Let $h\in\mathfrak{B}^{+}$. Then there exists a function $f\in H^{\infty}$ such that $f\equiv 0$ on $\mathfrak{S}_{\mu_h}$ while $\varliminf_{\varepsilon\to 0^{+}} D^{+}(S^{+}_{\varepsilon}(f,0))\log\varepsilon>-\infty$.

Using Corollary 3.4 we see from Example 5.1 that each value of maximum modulus $(=1)$ of B is a tangential principal value without being a principal value.

We see from Example 5.2 that the set of lower tangential and upper tangential principal values can be disjoint. Also, though the numbers of modulus $e^{-\pi/2}$ are not maximum they are on the boundary of $C(f,1)$ and, in fact, are clearly peak points.

Example 5.3 shows that Theorem 4.9 which predicts upper principal values is not comparable to Theorem 4.3 which gives upper tangential principal values. Even when the subtle behavior of the second condition of Theorem 4.9 is absent (in effect the zero set of f is "very" disjoint from the set of all upper radial supports) it is still possible to have zero as an upper tangential principal value.

To verify Example 5.1 we note that $|B(z)|=\prod \chi(z, 1-a^{-n})$ where $\chi(a, b)=|(a-b)/(1-\bar{a}b)|$. To estimate this product we find it convenient to study

$$\alpha_n(z)=1-\chi^2(z, 1-a^{-n}).$$

It is clear by symmetry that we need only prove the result for approaches on the radius or above; so we assume that $0\leqq\theta<\pi/2$. Let

$$\gamma=\gamma(z)=\frac{r\sin\theta}{(1-2r\cos\theta+r^2)}, \quad \delta=\delta(z)=r\frac{\cos\theta-r}{1-2r\cos\theta+r^2}, \quad \beta=\beta(z)=\delta^2+\gamma^2.$$

Then, it is easy to compute that

$$\alpha_n(z)=\frac{\beta(1-r^2)}{r^2}\,\frac{2a^n-1}{a^n}f(n),$$

where

$$f(n)=\frac{a^n}{(a^n+\delta)^2+\gamma^2}.$$

For fixed z (and $\operatorname{Re} z>1/2$ so $\beta>1$), one sees that $f(n)$ is increasing for $n\leqq\log_a\beta^{1/2}\equiv x_0$, and decreasing for larger n. Let n_0 be the greatest integer in x_0. Then $f(n_0)\geqq f(x_0-1)=\dfrac{a}{(1+a^2)\sqrt{\beta}+2a\alpha}$ so that

$$\alpha_{n_0}(z)\geqq\frac{\beta(1-r)}{r^2}\,\frac{a}{(1+a^2)\sqrt{\beta}+2a\alpha}$$

$$=\frac{a}{r(1+a^2)\left(1+2r\dfrac{1-\cos\theta}{(1-r)^2}\right)^{1/2}+2ar\left(1-\dfrac{1-\cos\theta}{1-r}\right)}.$$

Suppose now that $re^{i\theta}$ tends to 1 non-tangentially, say $\lim\dfrac{1-\cos\theta}{(1-r)^2}=\tau<\infty$. Then, $\lim\dfrac{1-\cos\theta}{1-r}=0$ and

$$\underline{\lim}\ \alpha_{n_0}(z)\geqq\frac{a}{(1+a^2)(1+2\tau)^{1/2}+2a}>0.$$

Thus,

$$\overline{\lim}\,|B(z)|\leqq\overline{\lim}\,\chi(z, 1-a^{-n_0})$$

$$=\overline{\lim}\,(1-\alpha_{n_0}(z))^{1/2}<1$$

as claimed.

To estimate $|B(z)|$ for tangential approach we make use of the simple inequality, $\Pi(1-b_n)\geqq 1-\sum b_n$ for $0\leqq b_n<1$. We choose $b_n=1-\chi_n(z, 1-a^{-n})$ and notice that

$$|B(z)|=\Pi(1-b_n)\geqq 1-\sum b_n\geqq 1-\sum\alpha_n(z).$$

Thus, to prove that $|B(z)|$ tends to 1 it is enough to show that $\sum\alpha_n(z)$ tends to zero. Recalling the definitions of n_0 and x_0 we have

$$\begin{aligned}\sum_{n=1}^{\infty} f(n) &\leqq \int_1^{\infty} f(x)dx+\max(f(n_0), f(n_0+1)) \\ &\leqq \int_1^{\infty} f(x)dx+f(x_0) \\ &=\frac{1}{\gamma \log a}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{a+\delta}{\gamma}\right)\right]+\frac{1}{2(\sqrt{\beta}+\delta)}.\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty}\alpha_n(z)\leqq\frac{4(1-r)\beta}{r^2\gamma\log a}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{a+\delta}{\gamma}\right)\right]+\frac{2(1-r)\beta}{r^2(\sqrt{\beta}+\delta)}=I_1+I_2.$$

We now suppose that z tends to 1 tangentially so that, e.g., $\sin\theta/(1-r)\to\infty$ and $(1-\cos\theta)/(1-r)^2\to\infty$. Then, it is not hard to show that $\gamma\to\infty$, $\delta/\gamma\to 0$, $\beta(1-r)/\gamma\to 0$, $\sqrt{\beta}/\gamma\to 1$. This makes it clear that $I_1\to 0$. Writing

$$I_2=\frac{2}{r^2}\,\frac{\beta(1-r)}{\gamma}\left[\frac{\sqrt{\beta}}{\gamma}-\frac{\delta}{\gamma}\right]$$

we see that $I_2\to 0$ and the result is clear.

The verification of Example 5.2 is straightforward and we leave these computations to the reader.

For Example 5.3 we require a lemma which is interesting in its own right.

LEMMA 5.4. *Let $\{M_n\}$ be a sequence of measurable subsets of C such that the sets $\{\tilde{M}_n\}$ from a strictly decreasing sequence in $\check{S}_1$. Then, there exists a function $f\in H^{\infty}$ such that*

$$\tilde{M}_n=\left\{\varphi\in\check{S}_1 : |f(\varphi)|\leqq\frac{1}{2^n}\right\}.$$

PROOF. We can assume that the sets $\{M_n\}$ also form a strictly decreasing sequence so that for each $n=1, 2, 3, \cdots$, $\emptyset\neq(M_n-M_{n+1})^{\sim}=\tilde{M}_n-\tilde{M}_{n+1}$. Define $M_0=C$ and for each $n=0, 1, 2, \cdots$ define

$$f_n(z)=\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\chi_{M_n-M_{n+1}}(e^{it})dt\right\}.$$

Then, $f_n \in H^\infty$, $|f_n|=1$ on $\tilde{M}_n - \tilde{M}_{n+1}$ and $f_n = 0$ on $\check{S}_1 - (\tilde{M}_n - \tilde{M}_{n+1})$. Define

$$f = \sum_{n=0}^{\infty} \frac{f_n}{2^n}.$$

The series converges uniformly to f in D so $f \in H^\infty$ and for any $\varphi \in \mathscr{D}_1$, $f(\varphi) = \sum 2^{-n} f_n(\varphi)$. If $\varphi \in \tilde{M}_n - \tilde{M}_{n+1}$, then $f(\varphi) = 2^{-n} f_n(\varphi)$ and $|f(\varphi)| = 2^{-n}$, since the "rings", $\tilde{M}_n - \tilde{M}_{n+1}$, are all disjoint. It is now clear that for all $\varphi \in \check{S}_1$, $|f(\varphi)| \leqq 2^{-n}$ if and only if $\varphi \in \tilde{M}_n$.

We use this now to verify Example 5.3. We need a subtle result from [**13**] which states that if $h \in \mathscr{T}^+$ then

$$\inf \{D^+(M) : \tilde{M} \supset \mathfrak{S}_{\mu_h}\} = 0 .$$

Let $h \in \mathscr{B}^+$. Then, for each integer n, there is a set $M_n \subset C$ such that $\tilde{M}_n \supset \mathfrak{S}_{\mu_h}$ while $D^+(M_n) < (\log 2^n)^{-1}$. We can clearly assume that the sets $\{\tilde{M}_n\}$ are strictly decreasing. By Lemma 5.4 there is $f \in H^\infty$ such that $\tilde{M}_n = \{\varphi \in \check{S}_1 : |f(\varphi)| \leqq 2^{-n}\} = \check{S}_{2^{-n}}(f, 0)$. Let $0 < \varepsilon < 1$ be given. There is an n such that $2^{-(n+1)} \leqq \varepsilon < 2^{-n}$. Hence $S_{2^{-(n+1)}} \subset S_\varepsilon \subset S_{2^{-n}}$ (almost everywhere) so that

$$D^+(S_{2^{-(n+1)}}) \leqq D^+(S_\varepsilon) \leqq D^+(S_{2^{-n}}) .$$

But we have $D^+(S_{2^{-n}}) < (n \log 2)^{-1}$. So since $2^{-(n+1)} \leqq \varepsilon$ we have

$$\begin{aligned} D^+(S_\varepsilon) &\leqq D^+(S_{2^{-n}}) \\ &\leqq \frac{1}{n \log 2} \\ &\leqq \frac{n+1}{n} \frac{1}{\log \varepsilon^{-1}} \\ &\leqq 2 \frac{1}{\log \varepsilon^{-1}} . \end{aligned}$$

Thus, $D^+(S_\varepsilon) \log \varepsilon \geqq -2$. Since ε can be chosen arbitrarily small the result follows.

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