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Coincidence Lefschetz numbers for a pair of fibre preserving maps

(Dedicated to Professor T. Kudo on his 60th birhday)

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Introduction.

Dold studied in [**5, 6**] the fixed point index in connection with fibre preserving maps. In this paper we shall consider the coincidence Lefschetz numbers for a pair of fibre preserving maps, and prove various theorems which are variants of the Dold's results. Some of the results in [**5**] are obtained for generalized cohomology, but we shall be concerned with only the classical cohomology. Our method is different from that of Dold, and is the one employed by Becker and Gottlieb in their study [**1, 9**] of the transfer homomorphism.

Let $p: E\to B$ be a fibre bundle such that each fibre $M_b=p^{-1}(b)$ $(b\in B)$ is an oriented compact m-manifold, and such that the local system $\{H^m(M_b)\}_{b\in B}$ is trivial. For simplicity, such a fibre bundle will be called an *m-orientable fibre bundle*. In this paper we shall consider frequently a pair of fibre preserving maps $f, g: E\to E'$ from an m-orientable fibre bundle $p: E\to B$ to an m-orientable fibre bundle $p': E'\to B'$. Let $h, l: B\to B'$ denote the maps induced from f, g respectively. If $h=l$ we define an element $\wedge_{f,g}\in H^m(E)$, called the Lefschetz coincidence class for f and g (see § 2). This class is fundamental in our study.

For an m-orientable fibre bundle, the integration along the fibre can be defined, and also an orientation class is defined. We review briefly these facts in § 1. If b is a coincidence point of h and l, we have the coincidence Lefschetz number $\lambda(f_b, g_b)$, where $f_b, g_b: M_b\to M'_{b'}$ $(b'=h(b)=l(b))$ are induced by f, g respectively. We study in § 2 conditions under which $\lambda(f_b, g_b)$ is independent of the choice of b, and in § 3 relations among the coincidence Lefschetz numbers $\lambda(f, g)$, $\lambda(h, l)$ and $\lambda(f_b, g_b)$ in the case when B and B' are oriented compact n-manifolds. In § 4 we deal with $\lambda(f, g)$ for equivariant maps $f, g: M\to M$, where M is an oriented connected compact manifold on which a finite group acts. We show in § 5 that the coincidence transfer homomorphism $\tau_{f,g}$ can be

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defined in some cases, and in § 6 that the Lefschetz-Hopf trace formula holds for $\tau_{f,g}$ under some assumption. In § 7 we define the coincidence-coincidence index on the analogy of the coincidence-fixed-point index in Dold [**6**], and study its properties. The results obtained in § 2~§ 7 are generalized in § 8 to the case of fibre bundle having fibre manifold with boundary.

By a manifold we mean a topological manifold without boundary. The coefficient ring R of cohomology is to be a principal ideal domain.

1. Preliminaries.

If $p: E \to B$ is an m-orientable fibre bundle having fibre M, then we have a homomorphism

$$p_! : H^{q+m}(E\,;R) \to H^q(B\,;R),$$

called the *integration along the fibre*. In terms of the spectral sequence, this is defined to be the composite

$$H^{q+m}(E) = D^{0,q+m} = D^{q,m} \xrightarrow{\text{onto}} E_\infty^{q,m}$$

$$\subset E_2^{q,m} = H^q(B\,; H^m(M\,;R)) \xrightarrow{\kappa_*} H^q(B\,;R),$$

where κ_* is induced from $\kappa: H^m(M; R) \to R$ given by $\kappa(\alpha) = \langle \alpha, [M] \rangle$, the kronecker product of $\alpha \in H^m(M; R)$ and the fundamental homology class $[M]$.

The integration along the fibre satisfies the following properties ([**2**], [**9**]).

(1.1) If B is an oriented compact manifold, then $p_!$ agrees with the Gysin homomorphism induced by p.

(1.2) If $q: B \to D$ is an n-orientable fibre bundle, then $qp: E \to D$ is an $(m+n)$-orientable fibre bundle and we have

$$(qp)_! = q_! p_! .$$

(1.3) For $\beta \in H^*(B\,;R)$ and $\gamma \in H^*(E\,;R)$, we have

$$p_!(p^*\beta \smile \gamma) = \beta \smile p_! \gamma .$$

(1.4) Let $\tilde{p}: \tilde{E} \to Y$ be an m-orientable fibre bundle with fibre $\tilde{M}$, and let $\tilde{\varphi}: \tilde{E} \to E$ be a fibre preserving map such that $\tilde{\varphi}$ induces an isomorphism of $H^m(M; R)$ onto $H^m(\tilde{M}; R)$ preserving orientation. Then the following diagram commutes:

$$\begin{array}{ccc} H^*(E\,;R) & \xrightarrow{\tilde{\varphi}^*} & H^*(\tilde{E}\,;R) \\ \big\downarrow{\scriptstyle p_!} & & \big\downarrow{\scriptstyle \tilde{p}_!} \\ H^*(B\,;R) & \xrightarrow{\varphi^*} & H^*(Y\,;R), \end{array}$$

where $\varphi: Y \to B$ is the map induced by $\tilde{\varphi}$. In particular, this is true for a pull-back diagram

$$\begin{CD} \tilde{E} @>{\tilde{\varphi}}>> E \\ @V{\tilde{p}}VV @VV{p}V \\ Y @>{\varphi}>> B. \end{CD}$$

(1.5) Let $p_i: E_i \to B_i$ be an m_i-orientable fibre bundle ($i=1, 2$). Then $p_1 \times p_2: E_1 \times E_2 \to B_1 \times B_2$ is an (m_1+m_2)-orientable fibre bundle, and we have

$$(p_1 \times p_2)_!(\gamma_1 \times \gamma_2) = (-1)^{m_1(|\gamma_2|-m_2)} p_{1!}\gamma_1 \times p_{2!}\gamma_2$$

for $\gamma_i \in H^*(E_i; R)$.

If $p: E \to B$ is an m-orientable fibre bundle and $B_0 \subset B$, then we have

$$p_!: H^{q+m}(E, E_0; R) \to H^q(B, B_0; R)$$

$(E_0 = p^{-1}(B_0))$ defined similarly and satisfying (1.1)~(1.5) under appropriate conditions.

Next, we shall define the orientation class of an m-orientable fibre bundle $p: E \to B$. To begin with we shall recall the following fact ([**10**], [**11**]).

Let $p: (E, E') \to B$ be a fibre-bundle pair with fibre pair (M, M'). Assume that

$$H^q(M, M'; R) = \begin{cases} R & \text{if } q=m, \\ 0 & \text{if } q \neq m, \end{cases}$$

and the local system $\{H^m(M_b, M'_b; R)\}_{b \in B}$ is trivial. Then there exists the Thom class $u \in H^m(E, E'; R)$ which is uniquely characterized by the property that its restriction on every fibre pair is a preferred generator.

Given an m-orientable fibre bundle $p: E \to B$, we consider the fibre square $E \underset{B}{\times} E$ and its diagonal dE. Then the projection $p_1: E \times E \to E$ to the first factor yields a fibre-bundle pair $p_1: (E \underset{B}{\times} E, E \underset{B}{\times} E - dE) \to E$ with fibre pair $(M, M-pt)$. Since the pull-back $p_1: E \underset{B}{\times} E \to E$ is m-orientable and $H^m(M, M-pt; R) \cong H^m(M; R)$, it follows that the local system $\{H^m(M_{p(e)}, M_{p(e)}-e; R)\}_{e \in E}$ is trivial. Since $H^q(M, M-pt; R)=0$ if $q \neq m$, the fibre-bundle pair has the Thom class, which is denoted by $u(p) \in H^m(E \underset{B}{\times} E, E \underset{B}{\times} E - dE; R)$. We call $u(p)$ the *orientation class* or the *fundamental cohomology class* of the fibre bundle $p: E \to B$.

We put

$$\Delta(p) = u(p) | E \underset{B}{\times} E \in H^m(E \underset{B}{\times} E; R),$$

and call it the *diagonal cohomology class* of $p: E \to B$.

If B is a point, $u(p)$ agrees with the orientation class of the manifold M which will be denoted by $u(M) \in H^m(M \times M, M \times M - dM; R)$. Therefore, in this case, $\Delta(p)$ agrees with the diagonal cohomology class of M which will be denoted by $\Delta(M) \in H^m(M \times M; R)$.

The following properties (1.6) and (1.7) are easily verified.

(1.6) For the pull-back diagram in (1.4) we have

$$u(\tilde{p}) = (\tilde{\varphi} \times \tilde{\varphi})^* u(p), \quad \Delta(\tilde{p}) = (\tilde{\varphi} \times \tilde{\varphi})^* \Delta(p).$$

(1.7) With the notation in (1.5), we have

$$u(p_1 \times p_2) = T_{23}^*(u(p_1) \times u(p_2)),$$
$$\Delta(p_1 \times p_2) = T_{23}^*(\Delta(p_1) \times \Delta(p_2)),$$

where

$$T_{23}: (E_1 \underset{B_1}{\times} E_1,\ E_1 \underset{B_1}{\times} E_1 - dE_1) \times (E_2 \underset{B_2}{\times} E_2,\ E_2 \underset{B_2}{\times} E_2 - dE_2) \to (E \underset{B}{\times} E,\ E \underset{B}{\times} E - dE)$$

$(E = E_1 \times E_2,\ B = B_1 \times B_2)$ is the map interchanging the second and the third factors.

2. The independence of Lefschetz number from fibre.

Let M and M' be an oriented compact m-manifolds, and let $f, g: M \to M'$ be continuous maps. Then we denote by $\lambda(f, g)$ the *Lefschetz number* for f and g, i. e. the Lefschetz trace of the composite

$$H^*(M; Q) \xrightarrow{g_!} H^*(M'; Q) \xrightarrow{f^*} H^*(M; Q),$$

where $g_!$ is the Gysin homomorphism induced by g.

Consider the image of the diagonal cohomology class $\Delta(M')$ under the homomorphism $(f, g)^*: H^m(M' \times M') \to H^m(M)$. Then the following is known ([8], [**12**]):

$$\lambda(f, g) = \langle (f, g)^* \Delta(M'), [M] \rangle \in Z. \tag{2.1}$$

Let $p: E \to B$ and $p': E' \to B'$ be m-orientable fibre bundles, and $f, g: E \to E'$ be fibre preserving maps covering a map $h: B \to B'$. Then we have a homomorphism $(f, g)^*: H^*(E' \underset{B'}{\times} E'; R) \to H^*(E; R)$. For the diagonal cohomology class $\Delta(p')$, we put

$$\Lambda_{f, g} = (f, g)^* \Delta(p') \in H^m(E; R),$$

and call it *the Lefschetz coincidence class* for f and g.

Let $C_{f, g}$ denote the coincidence set of f and g:

$$C_{f, g} = \{e \in E \mid f(e) = g(e)\}.$$

PROPOSITION 2.1. *If $\wedge_{f,g}\neq 0$ then $C_{f,g}\neq\emptyset$.*

PROOF. We have a homomorphism

$$(f, g)^*: H^*(E'\times_{B'}E', E'\times_{B'}E'-dE'; R)\to H^*(E, E-C_{f,g}; R),$$

and it follows that $\wedge_{f,g}=(f, g)^*\Delta(p')$ is the restriction of $(f, g)^*u(p')\in H^*(E, E-C_{f,g}; R)$ on E. Therefore if $C_{f,g}=\emptyset$ then $\wedge_{f,g}=0$.

For $b\in B$, let $f_b, g_b: M_b\to M'_{h(b)}$ denote the maps induced from f, g respectively.

LEMMA 2.2. *For the inclusion $i_b: M_b\to E$, we have*

$$\langle i_b^*\wedge_{f,g}, [M_b]\rangle=\lambda(f_b, g_b)\in R.$$

PROOF. From a commutative diagram

$$\begin{tikzcd} M_b \arrow[r, "i_b"] \arrow[d, "(f_b, g_b)"] & E \arrow[d, "(f, g)"] \\ M'_{h(b)}\times M'_{h(b)} \arrow[r, "i'\times i'"] & E'\times_{B'}E' \end{tikzcd}$$

(i' : inclusion), we get

$$\begin{aligned} i_b^*\wedge_{f,g} &= i_b^*(f, g)^*\Delta(p') \\ &= (f_b, g_b)^*(i'\times i')^*\Delta(p')=(f_b, g_b)^*\Delta(M'_{h(b)}) \end{aligned}$$

which proves the result by (2.1).

THEOREM 2.3. *Let $p: E\to B$ and $p': E'\to B'$ be m-orientable fibre bundles, and let $f, g: E\to E'$ be fibre preserving maps covering a map $h: B\to B'$. Assume that B is pathwise connected. Then we have*

$$p_!\wedge_{f,g}=\lambda(f_b, g_b)1. \tag{2.2}$$

In particular, $\lambda(f_b, g_b)$ does not depend on $b\in B$.

PROOF. From a pull-back diagram

$$\begin{tikzcd} M_b \arrow[r, "i_b"] \arrow[d, "p|M_b"] & E \arrow[d, "p"] \\ \{b\} \arrow[r, "i_b"] & B \end{tikzcd}$$

and Lemma 2.2, it follows that

$$i_b^* p_! \wedge_{f,g} = (p|M_b)_! i_b^* \wedge_{f,g} = \langle i_b^* \wedge_{f,g}, [M_b] \rangle = \lambda(f_b, g_b).$$

Since B is pathwise connected, this shows (2.2).

A fibre bundle $p: E \to B$ is said to be *Q-orientable* if the local system $\{H^q(p^{-1}(b); Q)\}_{b \in B}$ is trivial for every q. A Q-orientable fibre bundle having fibre an oriented compact m-manifold is m-orientable.

We have the following theorem. (Compare Proposition (8.20) in [**5**] and Theorem 2 in [**3**].)

THEOREM 2.4. *Let $p: E \to B$ be an m-orientable fibre bundle over a pathwise connected space, and let $p': E' \to B'$ be a Q-orientable fibre bundle over a paracompact Hausdorff space such that each fibre is an oriented compact m-manifold. Let $f, g: E \to E'$ be fibre preserving maps covering $h, l: B \to B'$ respectively. Then $\lambda(f_b, g_b)$ does not depend on $b \in C_{h,l}$.*

PROOF. Take a lifting function $\rho: E' \underset{B'}{\times} B'^I \to E'^I$ for the fibre bundle $p': E' \to B'$ ([**11**]). For each path $\sigma \in B'^I$, define a continuous map $\varphi_\sigma: p'^{-1}(\sigma(0)) \to p'^{-1}(\sigma(1))$ by $\varphi_\sigma(e') = \rho(e', \sigma)(1)$.

Let $b_0, b_1 \in C_{h,l}$, and take a path $\omega \in B^I$ such that $\omega(0) = b_0$ and $\omega(1) = b_1$. Consider the pull-back $\tilde{p}: \tilde{E} \to I$ of $p: E \to B$ under $\omega: I \to B$. Define $f', g': \tilde{E} \to E'$ by

$$f'(t, e) = \varphi_{\sigma_t}(f(e)), \quad g'(t, e) = g(e),$$

where $t \in I$, $e \in E$, and $\sigma_t \in B'^I$ is defined by

$$\sigma_t(s) = \begin{cases} h\omega((1-2s)t) & \text{if } 0 \leqq s \leqq 1/2, \\ l\omega((2s-1)t) & \text{if } 1/2 \leqq s \leqq 1. \end{cases}$$

It follows that f' and g' are fibre preserving maps covering $l\omega: I \to B'$, and hence by Theorem 2.3 $\lambda(f'_t, g'_t)$ does not depend on $t \in I$, where $f'_t, g'_t: \tilde{p}^{-1}(t) \to p'^{-1}(l\omega(t))$ are induced from f', g' respectively. In particular, it holds that

$$(2.3) \qquad \lambda(f'_0, g'_0) = \lambda(f'_1, g'_1).$$

Regard $\tilde{p}^{-1}(0) = p^{-1}(b_0)$ and $\tilde{p}^{-1}(1) = p^{-1}(b_1)$ via the projection to the second factor. Then we have

$$f'_i = \varphi_{\sigma_i} f_{b_i}, \quad g'_i = g_{b_i}$$

for $i = 0, 1$. Since $p': E' \to B'$ is Q-orientable, $\varphi_{\sigma_i}^*: H^*(p'^{-1}(b'_i); Q) \to H^*(p'^{-1}(b'_i); Q)$ is the identity, where $b'_i = h(b_i) = l(b_i)$. Consequently it holds that

$$(2.4) \qquad \lambda(f'_i, g'_i) = \lambda(f_{b_i}, g_{b_i}).$$

From (2.3) and (2.4) we get $\lambda(f_{b_0}, g_{b_0}) = \lambda(f_{b_1}, g_{b_1})$. This completes the proof.

EXAMPLE. The following is an example of $b_0, b_1 \in C_{h,l}$ with $\lambda(f_{b_0}, g_{b_0}) \neq \lambda(f_{b_1}, g_{b_1})$. (cf. the final paragraph in [**5**]). Let E denote the quotient space obtained from the 3-dimensional torus $S^1\times S^1\times S^1$ by identifying (z_1, z_2, z_3) with $(-z_1, \bar{z}_2, \bar{z}_3)$, and define $p: E\to B=S^1$ by $p([z_1, z_2, z_3])=z_1^2$. Then we have a fibre bundle over S^1 having fibre $S^1\times S^1$. This fibre bundle is 2-orientable, but it is not Q-orientable. Define a fibre preserving map $f: E\to E$ by $f([z_1, z_2, z_3])=[\bar{z}_1, \bar{z}_2, \bar{z}_3]$, and put $g=id$. In this case, h is the reflexion on S^1 and $l=id$, and so $C_{h,l}$ is two points ± 1. We see that $\lambda(f_1, g_1)=4$ and $\lambda(f_{-1}, g_{-1})=0$.

3. Product formula for Lefschetz numbers.

We shall first prove

PROPOSITION 3.1. *Let $p: E\to B$ be an m-orientable fibre bundle over an oriented compact n-manifold. Then E is a compact $(m+n)$-manifold oriented canonically, and it holds that*

$$(p\times p)^*\Delta(B)\smile\Delta(p)=\Delta(E)\,|\,E\times_B E$$

for the diagonal cohomology classes $\Delta(B)\in H^n(B\times B; R)$, $\Delta(p)\in H^m(E\times_B E; R)$ and $\Delta(E)\in H^{m+n}(E\times E; R)$, where $(p\times p)^*: H^*(B\times B; R)\to H^*(E\times_B E; R)$.

PROOF. The first conclusion can be proved easily by making use of spectral sequence.

We say that a subset C of M is a small cell if C has an open neighborhood $O\subset M$ such that (O, C) is homeomorphic to $(R^m, \bar{V}^m)$, where $\bar{V}^m$ is the closed unit ball. Take a neighborhood T of dE in $E\times E$ such that $T_e=\{e'\in E\,|\,(e, e')\in T\}$ for each $e\in E$ satisfies the following condition: there exist a coordinate neighborhood (U, η) for $p: E\to B$ and a small cell C in M such that $\eta: U\times M\approx p^{-1}(U)$ maps $p(T_e)\times C$ onto T_e. Moreover, take an open neighborhood N of dB in $B\times B$ such that $N\subset(p\times p)T$ and $N_b=\{b'\in B\,|\,(b, b')\in N\}$ is a coordinate neighborhood around b which is homeomorphic to the open unit n-ball V^n. Put

$$W=(p\times p)^{-1}(N)\,.$$

The projection $p_1: E\times E\to E$ to the first factor yields a fibre-bundle pair $p_1: (E\times_B E, E\times_B E-T)\to E$ with fibre pair $(M, M-C)$, and also a fibre-bundle pair $p_1: (W, W-T)\to E$ with fibre pair $(V^n\times(M, M-C))$. Since $p: E\to B$ is m-orientable, so is the pull-back $p_1: E\times_B E\to E$. It follows that $p_1: E\times_B E\to E$ is a subbundle of $p_1: W\to E$ and the inclusion $E\times_B E\to W$ is a fibre homotopy equivalence. Therefore $p_1: W\to E$ is also m-orientable. From these facts and

$$H^q(M, M-C; R)=\begin{cases} H^m(M; R) & \text{if } q=m\,,\\ 0 & \text{if } q\neq m\,,\end{cases}$$

$$H^q(V^n\times(M,\ M-C)\,;\ R)=\begin{cases} H^m(V^n\times M;\ R) & \text{if } q=m\,, \\ 0 & \text{if } q\neq m\,, \end{cases}$$

we see that there exist the Thom classes

$$\begin{aligned} &u_0(p)\in H^m(E\underset{B}{\times}E,\ E\underset{B}{\times}E-T\,;\ R)\,, \\ &\bar{u}(p)\in H^m(W,\ W-T\,;\ R) \end{aligned} \tag{3.1}$$

such that

$$u(p)\,|\,(E\underset{B}{\times}E,\ E\underset{B}{\times}E-T)=u_0(p)\,,$$

$$\bar{u}(p)\,|\,(E\underset{B}{\times}E,\ E\underset{B}{\times}E-T)=u_0(p)\,,$$

and so

$$\bar{u}(p)\,|\,E\underset{B}{\times}E=\Delta(p)\,. \tag{3.2}$$

We have the following commutative diagrams:

$$\begin{tikzcd}[column sep=small] & H^n(B\times B,\ B\times B-dB) \arrow[r, "(p\times p)^*"] \arrow[dl] & H^n(W,\ W-E\underset{B}{\times}E) \arrow[d] \\ H^n(B,\ B-pt) \arrow[r] & H^n(V^n,\ V^n-pt) \arrow[r, "\times 1"] & H^n((V^n,\ V^n-pt)\times M)\,, \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] H^n(W,\ W-E\underset{B}{\times}E)\otimes H^m(W,\ W-T) \arrow[r, "\smile"] \arrow[d] & H^{m+n}(W,\ W-(E\underset{B}{\times}E\cap T)) \arrow[d] \\ H^n((V^n,\ V^n-pt)\times M)\otimes H^m(V^n\times(M,\ M-C)) \arrow[r, "\smile"] & H^{m+n}(V^n\times M,\ V^n\times M-pt\times C) \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] \longleftarrow H^{m+n}(W,\ W-dE) \arrow[d] & H^{m+n}(E\times E,\ E\times E-dE) \arrow[l] \arrow[d] \\ \longleftarrow H^{m+n}(V^n\times M,\ V^n\times M-pt) & H^{m+n}(E,\ E-pt)\,, \arrow[l] \end{tikzcd}$$

where unlabeled arrows are induced by inclusions, B and E being identified with $b\times B\subset B\times B$ and $e\times E\subset E\times E$. From these diagrams it follows that

$$(p\times p)^*u(B)\smile\bar{u}(p)=u(E)\,|\,(W,\ W-(E\underset{B}{\times}E\cap T))\,, \tag{3.3}$$

where $u(B)$ and $u(E)$ are the orientation classes of B and E. This and (3.2) imply the second conclusion, and the proof is complete.

The following theorem generalizes the product formula on Euler numbers. (Compare also (2) of p. 162 in [4].)

THEOREM 3.2. *Let $p: E\to B$ and $p': E'\to B'$ be m-orientable fibre bundles over oriented compact n-manifolds, and let $f,\ g: E\to E'$ be fibre preserving maps covering $h: B\to B'$. If B is connected, it holds that*

$$\lambda(f,\ g)=\lambda(h,\ h)\lambda(f_b,\ g_b)=(\deg h)\chi(B')\lambda(f_b,\ g_b) \tag{3.4}$$

for $b\in B$, *where* $\chi(B')$ *is the Euler number of* B'.

PROOF. The following diagram commutes:

$$\begin{tikzcd}
H^*(B'\times B') \arrow[r, "(p'\times p')^*"] \arrow[d, "(h,h)^*"] & H^*(E'\underset{B'}{\times}E') \arrow[d, "(f,g)^*"] & H^*(E'\times E') \arrow[l] \arrow[dl, "(f,g)^*"] \\
H^*(B) \arrow[r, "p^*"] & H^*(E)\,. &
\end{tikzcd}$$

Therefore it follows from Proposition 3.1 and Theorem 2.3 that

$$\begin{aligned}
&p_!(f,g)^*\Delta(E')=p_!(f,g)^*(\Delta(E')\,|\,E'\underset{B'}{\times}E')\\
&=p_!(f,g)^*((p'\times p')^*\Delta(B')\smile\Delta(p'))\\
&=p_!(p^*(h,h)^*\Delta(B')\smile(f,g)^*\Delta(p'))\\
&=(h,h)^*\Delta(B')\smile p_!\Lambda_{f,g}\\
&=\lambda(f_b,g_b)(h,h)^*\Delta(B')\,.
\end{aligned}$$

For the map $q: B\to pt$, we have

$$\lambda(f,g)=(qp)_!(f,g)^*\Delta(E')\,,$$

$$\lambda(h,h)=q_!(h,h)^*\Delta(B')\,.$$

Therefore it holds that

$$\begin{aligned}
&\lambda(f,g)=\lambda(f_b,g_b)q_!(h,h)^*\Delta(B')\\
&=\lambda(f_b,g_b)\lambda(h,h)\,.
\end{aligned}$$

Since it follows that

$$\begin{aligned}
&\lambda(h,h)=\langle(h,h)^*\Delta(B'),[B]\rangle\\
&=\langle h^*d^*\Delta(B'),[B]\rangle=\langle d^*\Delta(B'),h_*[B]\rangle\\
&=(\deg h)\langle d^*\Delta(B'),[B']\rangle=(\deg h)\chi(B')\,,
\end{aligned}$$

the proof is complete.

Theorem 3.2 can be generalized to the case f and g cover different maps. To begin with we shall recall some facts on the coincidence index.

Let M and M' be oriented compact m-manifolds, and let $f, g: M\to M'$ be continuous maps. Let V be an open set of M such that the coincidence set $C=C_{f|V,g|V}$ of $f|V$, $g|V: V\to M'$ is compact. Then we put

$$I_{f,g}^V=\langle(f|V,g|V)^*u(M'),o_C\rangle\in Z\,,$$

where $(f|V,g|V)^*: H^m(M'\times M', M'\times M'-dM')\to H^m(V, V-C)$ and o_C is the fundamental homology class around C. $I^V_{f,g}$ is called the *coincidence index* of $f|V$ and $g|V$. The following properties are easily verified (cf. [12]).

(i) If $C=\emptyset$ then $I^V_{f,g}=0$.

(ii) If K is a compact set such that $C\subset K\subset V$, then

$$I^V_{f,g}=\langle (f|V, g|V)^*u(M'), o_K\rangle .$$

(iii) If V' is an open set such that $C\subset V'\subset V$, then

$$I^V_{f,g}=I^{V'}_{f,g} .$$

(iv) If V is represented as a finite union of open sets V_i such that every $C_i=C_{f|V_i, g|V_i}$ is compact and $C_i\cap C_j=\emptyset$ $(i\neq j)$, then

$$I^V_{f,g}=\sum_i I^{V_i}_{f,g} .$$

(v) $I^M_{f,g}=\lambda(f, g)$.

Let C_μ be a pathwise connected component of $C_{f\,g}$. Take an open neighborhood V of C_μ such that $V\cap C_{f,g}=C_\mu$. Then, by (iii) above, $I^V_{f,g}$ does not depend on V. We write $I(f, g; C_\mu)=I^V_{f,g}$ and call it the *coincidence index of f and g around C_μ*.

The following theorem generalizes Theorem 3.2, and is a version of Theorem (8.18) in [5].

THEOREM 3.3. *Let $p: E\to B$ and $p': E'\to B'$ be m-orientable fibre bundles over oriented compact n-manifolds, and let $f, g: E\to E'$ be fibre preserving maps covering $h, l: B\to B'$ respectively. Let $\{C_\nu\}$ denote the set of pathwise connected components of $C_{h,l}$. Then it holds that*

$$\lambda(f, g)=\sum_\nu I(h, l; C_\nu)\lambda(f_{b_\nu}, g_{b_\nu}) \tag{3.5}$$

for $b_\nu\in C_\nu$.

In particular, if B is connected and $p': E'\to B'$ is Q-orientable, then we have

$$\lambda(f, g)=\lambda(h, l)\lambda(f_b, g_b) \tag{3.6}$$

for $b\in C_{h,l}$.

PROOF. First we notice that the following facts: (i) Since $C_{h,l}$ is compact, the set $\{C_\nu\}$ is finite and all C_ν are compact. (ii) $\lambda(f_{b_\nu}, g_{b_\nu})$ does not depend on $b_\nu\in C_\nu$ by Theorem 2.3, and $\lambda(f_b, g_b)$ in the particular case does not depend on $b\in C_{h,l}$ by Theorem 2.4.

It follows from the proof of Proposition 3.1 that there exist an open neighborhood N' of dB' in $B'\times B'$ and an element $\bar{\Delta}(p')\in H^m(W')$ satisfying

$$\bar{\Delta}(p')|E'\underset{B'}{\times}E'=\Delta(p') , \tag{3.2$'$}$$

$$(p'\times p')^*u(B')\smile\bar{\Delta}(p')=u(E')|(W', W'-E'\underset{B'}{\times}E') , \tag{3.3$'$}$$

where $W'=(p'\times p')^{-1}(N')$ and $(p'\times p')^*: H^n(B'\times B', B'\times B'-dB')\to H^*(W', W'-E'\underset{B'}{\times}E')$. In fact, if we put $\bar{\Delta}(p')=\bar{u}(p')|W'$ then (3.2) and (3.3) imply (3.2)′ and (3.3)′.

Take a family $\{U_\nu\}$ of disjoint pathwise connected open sets in B such that $U_\nu\cap C_{h,l}=C_\nu$ and $(h, l)U_\nu\subset N'$. Put $V_\nu=p^{-1}(U_\nu)$, $K_\nu=p^{-1}(C_\nu)$, $f_\nu=f|V_\nu$, $g_\nu=g|V_\nu$. Since

$$C_{f,g}\subset\bigcup_\nu V_\nu\,,\quad C_{f_\nu,g_\nu}\subset K_\nu\,,$$

it follows from the properties of coincidence index mentioned above that

$$\lambda(f, g)=I^E_{f,g}=\sum_\nu I^{V_\nu}_{f,g}$$

$$=\sum_\nu\langle(f_\nu, g_\nu)^*u(E'), o_{K_\nu}\rangle\,.$$

Therefore it suffices to prove

$$\langle(f_\nu, g_\nu)^*u(E'), o_{K_\nu}\rangle=I(h, l\,; C_\nu)\lambda(f_{b_\nu}, g_{b_\nu})\,. \tag{3.7}$$

Consider the following diagram:

$$\begin{array}{ccccc}
H^*(E) & \xleftarrow{\ j^*\ } & H^*(E, E-K_\nu) & \xrightarrow[\cong]{\ k^*\ } & H^*(V_\nu, V_\nu-K_\nu)\\
\downarrow{\scriptstyle p_!} & & \downarrow{\scriptstyle p_!} & & \downarrow{\scriptstyle p_{\nu!}}\\
H^*(B) & \xleftarrow{\ j^*\ } & H^*(B, B-C_\nu) & \xrightarrow[\cong]{\ k^*\ } & H^*(U_\nu, U_\nu-C_\nu)
\end{array}$$

where $p_\nu=p|V_\nu$ and j, k are inclusions. For $\gamma\in H^*(E, E-K_\nu)$ we have

$$\langle k^*\gamma, o_{K_\nu}\rangle=\langle\gamma, j_*[E]\rangle=\langle j^*\gamma, [E]\rangle$$

$$=\langle p_!j^*\gamma, [B]\rangle=\langle j^*p_!\gamma, [B]\rangle=\langle p_!\gamma, k_*o_{C_\nu}\rangle$$

$$=\langle k^*p_!\gamma, o_{C_\nu}\rangle=\langle p_{\nu!}k^*\gamma, o_{C_\nu}\rangle\,.$$

Therefore it holds that

$$\langle(f_\nu, g_\nu)^*u(E'), o_{K_\nu}\rangle=\langle p_{\nu!}(f_\nu, g_\nu)^*u(E'), o_{C_\nu}\rangle\,. \tag{3.8}$$

From a commutative diagram

$$\begin{tikzcd}
H^*(E'\times E', E'\times E'-dE') \arrow[d, "i^*"] \arrow[drr, "(f_\nu, g_\nu)^*"] & & \\
H^*(W', W'-E'\times_{B'} E') \arrow[rr, "(f_\nu, g_\nu)^*"] & & H^*(V_\nu, V_\nu-K_\nu) \\
H^*(B'\times B', B'\times B'-dB') \arrow[u, "(p'\times p')^*"] \arrow[rr, "(h_\nu, l_\nu)^*"] & & H^*(U_\nu, U_\nu-C_\nu) \arrow[u, "p_\nu^*"']
\end{tikzcd}$$

(i: inclusion) and (3.3)$'$, it follows that

$$\begin{aligned}(f_\nu, g_\nu)^*u(E') &= (f_\nu, g_\nu)^*((p'\times p')^*u(B')\smile\bar{\Delta}(p')) \\ &- p_\nu^*(h_\nu, l_\nu)^*u(B')\smile(f_\nu, g_\nu)^*\bar{\Delta}(p').\end{aligned}$$

From a commutative diagram

$$\begin{tikzcd}
H^*(W') \arrow[r, "(f_\nu, g_\nu)^*"] \arrow[d, "i^*"] & H^*(V_\nu) \arrow[r, "p_{\nu!}"] \arrow[d, "i^*"] & H^*(U_\nu) \arrow[d, "i^*"] \\
H^*(E'\times_{B'} E') \arrow[r, "(f_\nu, g_\nu)^*"] & H^*(K_\nu) \arrow[r, "(p\mid K_\nu)^*"] & H^*(C_\nu)
\end{tikzcd}$$

(i: inclusions), (3.2)$'$ and Theorem 2.3, it follows that

$$p_{\nu!}(f_\nu, g_\nu)^*\bar{\Delta}(p') = \lambda(f_{b_\nu}, g_{b_\nu})1\,,$$

because C_ν and K_ν are pathwise connected.

Consequently we have

$$\begin{aligned}&p_{\nu!}(f_\nu, g_\nu)^*u(E') \\ &= p_{\nu!}(p_\nu^*(h_\nu, g_\nu)^*u(B')\smile(f_\nu, g_\nu)^*\bar{\Delta}(p')) \\ &= (h_\nu, g_\nu)^*u(B')\smile p_{\nu!}(f_\nu, g_\nu)^*\bar{\Delta}(p') \\ &= \lambda(f_{b_\nu}, g_{b_\nu})(h_\nu, l_\nu)^*u(B').\end{aligned}$$

By this and (3.8), it follows that

$$\begin{aligned}&\langle (f_\nu, g_\nu)^*u(E'), o_{K_\nu}\rangle \\ &= \lambda(f_{b_\nu}, g_{b_\nu})\langle (h_\nu, l_\nu)^*u(B'), o_{C_\nu}\rangle \\ &= \lambda(f_{b_\nu}, g_{b_\nu})I(h, l\,; C_\nu).\end{aligned}$$

Thus (3.7) holds, and the proof of (3.5) is complete.

Since

$$\lambda(h, l)=\sum_{\nu} I(h, l; C_{\nu}),$$

(3.6) follows from Theorem 2.4 and (3.5). This completes the proof.

4. Lefschetz numbers of equivariant maps.

An application of Lemma 2.2 proves the following theorem. (Cf. Theorem 4 in [4].)

THEOREM 4.1. *Let M and M' be oriented compact m-manifolds on which a finite group Π acts, and let $f, g: M\to M'$ be equivariant maps. Assume that M is connected and the action on M is free. Then $\lambda(f, g)$ is divisible by the order $|\Pi|$ of Π.*

PROOF. (I) First we consider the case when Π acts by orientation preserving homeomorphisms.

Take the fibre bundles

$$p: E\Pi \underset{\Pi}{\times} M\to B\Pi, \quad p': E\Pi\underset{\Pi}{\times} M'\to B\Pi$$

associated to the universal bundle of Π. It follows that p and p' are m-orientable, and there are fibre preserving maps

$$\bar{f}=id\underset{\Pi}{\times}f, \ \bar{g}=id\underset{\Pi}{\times}g: E\Pi\underset{\Pi}{\times}M\to E\Pi\underset{\Pi}{\times}M'$$

covering $id: B\Pi\to B\Pi$. For the Lefschetz class $\wedge_{\bar{f},\bar{g}}\in H^{m}(E\Pi\underset{\Pi}{\times}M)$, it holds by Lemma 2.2 that

$$\langle i^{*}\wedge_{\bar{f},\bar{g}}, [M]\rangle=\lambda(f, g),$$

where $i: M\to E\Pi\underset{\Pi}{\times}M$ is the inclusion.

Since the action of Π on M is free, there is a natural isomorphism

$$H^{*}(E\Pi\underset{\Pi}{\times}M)\cong H^{*}(M/\Pi)$$

under which $i^{*}: H^{*}(E\Pi\underset{\Pi}{\times}M)\to H^{*}(M)$ corresponds to $\pi^{*}: H(M/\Pi)\to H^{*}(M)$ induced by the projection $\pi: M\to M/\Pi$. We know

$$\pi^{*}H^{m}(M/\Pi)\subset|\Pi|H^{m}(M).$$

Therefore $\lambda(f, g)$ is divisible by $|\Pi|$.

(II) Next we consider the general case.

For $N=M$ and M', consider the product manifold $S^{1}\times N$ of $S^{1}=\{z\in C\mid |z|=1\}$ and N, and define an action of Π on it by

$$\alpha\cdot(z, x)=\begin{cases}(z, \alpha\cdot x) & \text{if } \alpha \text{ preserves orientation,}\\ (\bar{z}, \alpha\cdot x) & \text{if } \alpha \text{ reverses orientation,}\end{cases}$$

where $\alpha\in\Pi$, $z\in S^1$, $x\in N$. It follows that Π acts on $S^1\times N$ by orientation preserving homeomorphisms and that the action on $S^1\times M$ is free. By identifying $x\in N$ with $(1, x)\in S^1\times N$, N may be regarded as an invariant subspace of $S^1\times N$. We have equivariant maps $i'fp_2, id\times g: S^1\times M\to S^1\times M'$, where $p_2: S^1\times M\to M$ is the projection to the second factor. Therefore, by the fact proved in Case I, $\lambda(i'fp_2, id\times g)$ is divisible by $|\Pi|$. Since

$$\lambda(i'fp_2, id\times g)=\lambda(f, g) \tag{4.1}$$

as is shown in the following, $\lambda(f, g)$ is divisible by $|\Pi|$.

$\lambda(i'fp_2, id\times g)$ is the Lefschetz trace of the composite

$$H^*(S^1\times M; Q)\xrightarrow{(id\times g)_!} H^*(S^1\times M'; Q)\xrightarrow{i'_*} H^*(M'; Q)$$
$$\xrightarrow{f^*} H^*(M; Q)\xrightarrow{p_2^*} H^*(S^1\times M; Q),$$

and a diagram

$$\begin{tikzcd} H^*(M) \arrow[r, "g_!"] \arrow[d, "p_2^*"] & H^*(M') \arrow[d, "p_2^*"] \\ H^*(S^1\times M) \arrow[r, "(id\times g)_!"] & H^*(S^1\times M') \end{tikzcd}$$

is commutative, as is easily checked. Therefore $\lambda(i'fp_2, id\times g)$ is the Lefschetz trace of $f^*i'^*p_2^*g_!=f^*g_!$. This completes the proof.

REMARK. Although no theorem of the type of Theorem 4.1 appears in [**5, 6**], the author was informed from Dold that Theorem (8.18) in [**5**] yields immediately the following result: *Let X be an ENR on which a finite group Π acts freely, and let $f: X\to X$ be an equivariant map such that the fixed point set of f is compact. Then the fixed point index of f is divisible by $|\Pi|$.*

5. Coincidence transfer homomorphism.

Let $p: E\to B$ and $p': E'\to B'$ be m-orientable fibre bundles, and let $f, g: E\to E'$ be fibre preserving maps covering a map $h: B\to B'$. Then, after the definition of the fixed point transfer homomorphism due to Becker-Gottlieb [**1**], we call the composite

$$(5.1)\qquad H^*(E\,;R)\xrightarrow{\smile\Lambda_{f,g}}H^*(E\,;R)\xrightarrow{p_!}H^*(B\,;R)$$

the *coincidence transfer homomorphism* of f and g, and denote it by $\tau_{f,g}$.

THEOREM 5.1. *The coincidence transfer homomorphism satisfies the following properties*:

(i) *If $H^*(E\,;R)$ is regarded as an $H^*(B\,;R)$-module via $p^*:H^*(B\,;R)\to H^*(E\,;R)$, then $\tau_{f,g}$ is an $H^*(B\,;R)$-homomorphism.*

(ii) *If B is pathwise connected, we have*

$$\tau_{f,g}(1)=\lambda(f_b,\,g_b)\,,\quad (b\in B)\,.$$

(iii) (*coincidence property*) *If $\tau_{f,g}\neq 0$ then $C_{f,g}\neq\emptyset$.*

(iv) (*pull-back property*) *Let*

$$\begin{array}{ccc}\tilde E & \xrightarrow{\tilde\varphi} & E\\ {\scriptstyle\tilde p}\downarrow & & \downarrow{\scriptstyle p}\\ Y & \xrightarrow{\varphi} & B\,,\end{array}\qquad\qquad \begin{array}{ccc}\tilde E' & \xrightarrow{\tilde\varphi'} & E'\\ {\scriptstyle\tilde p'}\downarrow & & \downarrow{\scriptstyle p'}\\ Y' & \xrightarrow{\varphi'} & B'\end{array}$$

be pull-back diagrams, and let $k:Y\to Y'$ be a continuous map such that $\varphi' k=h\varphi$. Define $\tilde f,\,\tilde g:\tilde E\to\tilde E'$ by

$$\tilde f(y,\,e)=(k(y),\,f(e)),\ \tilde g(y,\,e)=(k(y),\,g(e))$$

($y\in Y$, $e\in E$). Then the following diagram commutes:

$$(5.2)\qquad\begin{array}{ccc}H^*(E\,;R) & \xrightarrow{\tilde\varphi^*} & H^*(\tilde E\,;R)\\ {\scriptstyle\tau_{f,g}}\downarrow & & \downarrow{\scriptstyle\tau_{\tilde f,\tilde g}}\\ H^*(B\,;R) & \xrightarrow{\varphi^*} & H^*(Y\,;R)\,,\end{array}$$

(v) (*product property*) *For $i=1,2$, let $p_i:E_i\to B_i$ and $p'_i:E'_i\to B'_i$ be m_i-orientable fibre bundles, and let $f_i,\,g_i:E_i\to E'_i$ be fibre preserving maps covering $h_i:B_i\to B'_i$. Then the following diagram commutes:*

$$
(5.3)\qquad
\begin{tikzcd}
H^*(E_1;R)\otimes H^*(E_2;R) \arrow[r, "\times"] \arrow[d, "\tau_{f_1,g_1}\otimes\tau_{f_2,g_2}"] & H^*(E_1\times E_2;R) \arrow[d, "\tau_{f_1\times f_2,g_1\times g_2}"] \\
H^*(B_1;R)\otimes H^*(B_2;R) \arrow[r, "\times"] & H^*(B_1\times B_2;R)
\end{tikzcd}\ .
$$

PROOF. (i) follows from (1.3), and (ii) follows from Theorem 2.3. (iii) is obvious from Proposition 2.1. (iv) and (v) are easily shown by making use of (1.4)~(1.7).

When we do not assume m-orientability of fibre bundle, we have the following theorem.

THEOREM 5.2. *Let $p: E\to B$ be a fibre bundle having fibre M a connected oriented compact manifold, and let $f, g: E\to E$ be fibre preserving maps covering $id: B\to B$. Then there exists a homomorphism $\tau_{f,g}: H^*(E;R)\to H^*(B;R)$ which agrees with $\tau_{f,g}$ in* (5.1) *if p is m-orientable, and which satisfies* (i), (ii), (iii) *in Theorem* 5.1. *Furthermore the following* (iv) *and* (v) *hold.*

(iv) *Let*

$$
\begin{tikzcd}
\tilde{E} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\tilde{p}"] & E \arrow[d, "p"] \\
Y \arrow[r, "\varphi"] & B
\end{tikzcd}
$$

be a pull-back diagram, and define $\tilde{f}, \tilde{g}: \tilde{E}\to\tilde{E}$ by

$$\tilde{f}(y, e)=(y, f(e)),\ \tilde{g}(y, e)=(y, g(e))$$

$(y\in Y,\ e\in E)$. *Then we have the commutative diagram* (5.2).

(v) *For $i=1, 2$, let $p_i: E_i\to B_i$ be a fibre bundle having fibre M_i a connected oriented compact manifold, and let $f_i, g_i: E_i\to E_i$ be fibre preserving maps covering $id: B_i\to B_i$. Then we have the commutative diagram* (5.3).

PROOF. We begin with constructing from $p: E\to B$ an $(m+1)$-orientable fibre bundle

$$\hat{p}: LE\to B$$

with fibre $S^1\times M$, where $m=\dim M$ and $S^1=\{z\in C\mid |z|=1\}$.

Take an atlas $\{(U_j, \eta_j)\}_{j\in J}$ for $p: E\to B$, and consider the topological sum

$$\bigcup_{j\in J} S^1\times M\times U_j .$$

Identify $(z, x, b)\in S^1\times M\times U_j$ with $(z', x', b')\in S^1\times M\times U'_j$ if $b'=b$, $\eta_{j',b}(x')=\eta_{j,b}(x)$, and $z'=z$ or $\bar{z}$ according as $\eta_{j',b}^{-1}\eta_{j,b}: M\to M$ preserves orientation or not, where $\eta_{j,b}: M\to p^{-1}(b)$ is the homeomorphism defined from $\eta_j: M\times U_j\to p^{-1}(U_j)$. Then LE is defined to be the quotient space, and $\hat{p}$ is defined by $\hat{p}[(z, x, b)]=b$, where $[(z, x, b)]\in LE$ is represented by $(z, x, b)\in S^1\times M\times U_j$. It is obvious that $\hat{p}$ is $(m+1)$-orientable fibre bundle having fibre $S^1\times M$.

By identifying $\eta_j(x, b)\in E$ with $[(1, x, b)]\in LE$, $p: E\to B$ may be regarded as a subbundle of $\hat{p}: LE\to B$. There is a fibre preserving retraction $r: LE\to E$ given by $r[(z, x, b)]=\eta_j(x, b)$ if $(z, x, b)\in S^1\times M\times U_j$.

We have a fibre preserving map

$$ifr: LE\to LE$$

covering $id: B\to B$, where $i: E\to LE$ is the inclusion. Define

$$Lg: LE\to LE$$

by $Lg([(z, x, b)])=[(z, y, b)]$, where (z, x, b), $(z, y, b)\in S^1\times M\times U_j$, and $y=\eta_{j,b}^{-1}g\eta_{j,b}(x)$. This is also a fibre preserving map covering $id: B\to B$.

We have now the coincidence transfer homomorphism $\tau_{ifr, Lg}: H^*(LE; R)\to H^*(B; R)$. Using this, we define $\tau_{f,g}$ to be the composite

$$\text{(5.4)}\qquad H^*(E; R)\xrightarrow{r^*} H^*(LE; R)\xrightarrow{\tau_{ifr, Lg}} H^*(B; R).$$

If $p: E\to B$ is m-orientable, then it follows that $\hat{p}=q\times p: S^1\times E\to pt\times B=B$, $ifr=\text{const}\times f$ and $Lg=id\times g$. Therefore, by (ii) and (v) of Theorem 5.1, we have

$$\begin{aligned}\tau_{ifr, Lg}r^*(\gamma)&=\tau_{\text{const}\times f, id\times g}(1\times\gamma)\\&=\tau_{\text{const}, id}(1)\times\tau_{f,g}(\gamma)=1\times\tau_{f,g}(\gamma)\\&=\tau_{f,g}(\gamma)\end{aligned}$$

for $\gamma\in H^*(E; R)$. This shows that the new $\tau_{f,g}$ agrees with the old $\tau_{f,g}$ if p is m-orientable.

Next, we shall show that $\tau_{f,g}$ satisfies the properties (i)$\sim$(v).

(i) For $\beta\in H^*(B; R)$ and $\gamma\in H^*(E; R)$, we have

$$\begin{aligned}\tau_{f,g}(p^*\beta\smile\gamma)&=\tau_{ifr, Lg}r^*(p^*\beta\smile\gamma)\\&=\tau_{ifr, Lg}(\hat{p}^*\beta\smile r^*\gamma)=\beta\smile\tau_{ifr, Lg}r^*\gamma\\&=\beta\smile\tau_{f,g}\gamma\end{aligned}$$

by (i) of Theorem 5.1.

(ii) The restriction of ifr, Lg on fibre are

$$S^1\times M_b \xrightarrow{p_2} M_b \xrightarrow{f_b} M_b \xrightarrow{i} S^1\times M_b\,,$$

$$S^1\times M_b \xrightarrow{id\times g_b} S^1\times M_b$$

respectively. Therefore, if B is pathwise connected, we have

$$\begin{aligned}\tau_{f,g}(1)&=\tau_{ifr,Lg}r^*(1)=\tau_{ifr,Lg}(1)\\&=\lambda(if_bp_2,\ id\times g_b)=\lambda(f_b,\ g_b)\end{aligned}$$

by (ii) of Theorem 5.1 and (4.1).

(iii) We have $C_{ifr,Lg}=C_{f,g}$. Therefore, if $C_{f,g}=0$ then $\tau_{ifr,Lg}=0$ by (iii) of Theorem 5.1, and so $\tau_{f,g}=0$.

(iv) Take an atlas $\{(U_j,\ \eta_j)\}_{j\in J}$ for $p: E\to B$, and consider an atlas $\{(V_j,\ \zeta_j)\}_{j\in J}$ for $\tilde{p}:\tilde{E}\to Y$ defined by $V_j=\varphi^{-1}(U_j)$, $\zeta_j(x,\ y)=(y,\ \eta_j(x,\ \varphi(y))$ with $y\in Y$, $x\in M$. Construct $\hat{p}: LE\to B$ and $(\tilde{p})^\wedge: L\tilde{E}\to Y$ by making use of these atlases. Define $\psi:(LE)^\sim\to L\tilde{E}$ by

$$\psi(y,\ [(z,\ x,\ b)])=[(z,\ x,\ y)]\,,$$

where $y\in Y$, $(z,\ x,\ b)\in S^1\times M\times U_j$ and $(z,\ x,\ y)\in S^1\times M\times V_j$. It follows that ψ is a bundle isomorphism of $(\hat{p})^\sim:(LE)^\sim\to Y$ to $(\tilde{p})^\wedge: L\tilde{E}\to Y$, and that $(ifr)^\sim$, $(Lg)^\sim:(LE)^\sim\to(LE)^\sim$ correspond respectively to $i\tilde{f}r$, $L\tilde{g}: L\tilde{E}\to L\tilde{E}$ under ψ. This and (iv) of Theorem 5.2 show

$$\tau_{\tilde{f},\tilde{g}}\tilde{\varphi}^*=\tau_{(ifr)^\sim,(Lg)^\sim}\tilde{\varphi}^*r^*=\varphi^*\tau_{ifr,Lg}r^*=\varphi^*\tau_{f,g}\,.$$

(v) Define $\psi: LE_1\times LE_2\to S^1\times L(E_1\times E_2)$ by

$$\begin{aligned}&\psi([(z_1,\ x_1,\ b_1)],\ [(z_2,\ x_2,\ b_2)])\\&=(z_1z_2,\ [(z_1,\ (x_1,\ x_2),\ (b_1,\ b_2))])\,,\end{aligned}$$

where $(z_i,\ x_i,\ b_i)\in S^1\times M_i\times U^i_j$, $\{(U^i_j,\ \eta^i_j)\}_{j\in J(i)}$ being an atlas for $p_i: E_i\to B_i$ $(i=1,\ 2)$. It follows that ψ is a bundle isomorphism of $\hat{p}_1\times\hat{p}_2: LE_1\times LE_2\to B_1\times B_2$ to $q\times(p_1\times p_2)^\wedge: S^1\times L(E_1\times E_2)\to pt\times B_1\times B_2=B_1\times B_2$, and that $(i_1f_1r_1)\times(i_2f_2r_2)$, $Lg_1\times Lg_2: LE_1\times LE_2\to LE_1\times LE_2$ correspond respectively to const$\times i(f_1\times f_2)r$, $id\times L(g_1\times g_2): S^1\times L(E_1\times E_2)\to S^1\times L(E_1\times E_2)$ under ψ. This and (v) of Theorem 5.1 show

$$\begin{aligned}&\tau_{f_1,g_1}(\gamma_1)\times\tau_{f_2,g_2}(\gamma_2)\\&=\tau_{i_1f_1r_1,Lg_1}r_1^*(\gamma_1)\times\tau_{i_2f_2r_2,Lg_2}r_2^*(\gamma_2)\\&=\tau_{i_1f_1r_1\times i_2f_2r_2,Lg_1\times Lg_2}(r_1\times r_2)^*(\gamma_1\times\gamma_2)\\&=\tau_{i(f_1\times f_2)r,L(g_1\times g_2)}r^*(\gamma_1\times\gamma_2)\\&=\tau_{f_1\times f_2,g_1\times g_2}(\gamma_1\times\gamma_2)\,.\end{aligned}$$

6. Lefschetz-Hopf theorem.

In this section we show that under some assumption a Lefschetz-Hopf trace formula holds for the coincidence transfer homomorphism $\tau_{f,g}$ in (5.1).

Let $p: E \to B$ be a fibre bundle having fibre M a compact manifold, and assume that p has a rational cohomology extension of the fibre, i.e. a linear map $\theta: H^*(M; Q) \to H^*(E; Q)$ such that the composite

$$H^*(M; Q) \xrightarrow{\theta} H^*(E; Q) \xrightarrow{i_b^*} H^*(M_b; Q)$$

is an isomorphism for each $b \in B$. Then, by the Leray-Hirsch theorem ([**11**]), an isomorphism

$$H^*(B; Q) \otimes H^*(M; Q) \cong H(E; Q)$$

of vector spaces is given by

$$\beta \otimes \alpha \longmapsto p^*\beta \smile \theta\alpha .$$

The following theorem is a version of Theorem (6.18) in [**5**].

THEOREM 6.1. *Let $p: E \to B$ and $p': E' \to B$ be m-orientable fibre bundles over an oriented compact manifold B, and assume that p' has a rational cohomology extension of the fibre. Let $f, g: E \to E'$ be fibre preserving maps covering $id: B \to B$. Then the image of $\gamma \in H^*(E; Q)$ under $\tau_{f,g}: H^*(E; Q) \to H^*(B; Q)$ equals the Lefschetz trace of the composite*

$$H^*(E'; Q) \xrightarrow{f^*} H^*(E; Q) \xrightarrow{\gamma\smile} H^*(E; Q) \xrightarrow{g_!} H^*(E'; Q)$$

regarded as an $H^*(B; Q)$-endomorphism, where $g_!$ is defined by means of the canonical orientation of E and E'.

We begin with

LEMMA 6.2. *If $p: E \to B$ is an m-orientable fibre bundle over a compact manifold B, and $p_i: E \times_B E \to E$ denote the projection to the i-th factor $(i=1, 2)$, then the composite*

$$H^*(E; R) \xrightarrow{p_2^*} H^*(E \times_B E; R) \xrightarrow{\smile\Delta(p)} H^*(E \times_B E; R) \xrightarrow{p_{1!}} H^*(E; R)$$

is the identity.

PROOF. If we consider the inclusion of each fibre

$$\begin{array}{ccc} M & \xrightarrow{\quad i \quad} & E \times_B E \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p_1} \\ pt & \xrightarrow{\quad i \quad} & E , \end{array}$$

it follows from the definition of $\Delta(p)$ that

$$i^* p_{1!}\Delta(p)=p_! i^*\Delta(p)=\langle i^*\Delta(p), [M]\rangle=1$$

and hence $p_{1!}\Delta(p)=1$. Therefore we have

$$p_{1!}(p_1^*\alpha\smile\Delta(p))=\alpha\smile p_{1!}\Delta(p)=\alpha$$

for $\alpha\in H^*(E; R)$. Thus it suffices to prove

$$p_1^*\alpha\smile\Delta(p)=p_2^*\alpha\smile\Delta(p).$$

It follows that there exist an open neighborhood W of dE in $E\times_B E$ and a homotopy $p_1|W\simeq p_2|W: W\to E$ (see p. 81 of [7]). Hence it holds that

$$(p_1 i)^*=(p_2 i)^*: H^*(E; R)\to H^*(W; R),$$

where $i: W\to E\times_B E$ is the inclusion. Therefore, for the excision

$$i^*: H^*(E\times_B E,\ E\times_B E-dE; R)\cong H^*(W,\ W-dE; R),$$

we have

$$i^*(p_1^*\alpha\smile u(p))=i^*p_1^*\alpha\smile i^*u(p)$$
$$=i^*p_2^*\alpha\smile i^*u(p)=i^*(p_2^*\alpha\smile u(p)).$$

Since this shows that

$$p_1^*\alpha\smile u(p)=p_2^*\alpha\smile u(p)$$

in $H^*(E\times_B E,\ E\times_B E-dE; R)$, we get the desired result by restricting to $H^*(E\times_B E)$.

PROOF OF THEOREM 6.1. Let $\theta: H^*(M'; Q)\to H^*(E'; Q)$ be a cohomology extension of the fibre for $p': E'\to B$. Let $p_i: E'\times_B E'\to E'$ denote the projection to the i-th factor ($i=1, 2$). Then $p_2^*\theta: H^*(M'; Q)\to H^*(E'\times_B E'; Q)$ is a cohomology extension of the fibre for $p_1: E'\times_B E'\to E'$. Therefore an isomorphism

$$H^*(B; Q)\otimes H^*(M'; Q)\otimes H^*(M'; Q)\cong H^*(E'\times_B E'; Q)$$

of vector spaces is given by

$$\beta\otimes\alpha_1\otimes\alpha_2\mapsto p_1^*p'^*\beta\smile p_1^*\theta\alpha_1\smile p_2^*\theta\alpha_2.$$

Take a basis $\{\alpha_i\}$ of $H^*(M'; Q)$, and put

$$\Delta(p')=\sum_{i,j} p_1^*p'^*\beta_{ij}\smile p_1^*\theta\alpha_i\smile p_2^*\theta\alpha_j$$

with $\beta_{ij}\in H^*(B; Q)$. From a pull-back diagram

$$\begin{tikzcd} E'\times_B E' \arrow[r, "p_2"] \arrow[d, "p_1"] & E' \arrow[d, "p'"] \\ E' \arrow[r, "p'"] & B, \end{tikzcd}$$

we have

$$p_{1!}p_2^* = p'^* p'_! .$$

By Lemma 6.2 we have

$$p_{1!}(\smile\Delta(p'))p_2^* = id .$$

Thus, putting

$$\sigma_i = \gamma\smile f^*\theta\alpha_i ,$$

it follows that

$$\begin{aligned}
&g_!(\gamma\smile f^*\theta\alpha_i)\\
&= p_{1!}(p_2^* g_!\sigma_i\smile\Delta(p'))\\
&= \sum_{k,j} p_{1!}(p_2^* g_!\sigma_i\smile p_1^* p'^*\beta_{kj}\smile p_1^*\theta\alpha_k\smile p_2^*\theta\alpha_j)\\
&= \sum_{k,j}(-1)^{|\alpha_j|\varepsilon_{kj}} p_{1!}(p_2^*(g_!\sigma_i\smile\theta\alpha_j)\smile p_1^*(p'^*\beta_{kj}\smile\theta\alpha_k))\\
&= \sum_{k,j}(-1)^{\varepsilon_{kj}} p_{1!}p_2^*(g_!\sigma_i\smile\theta\alpha_j)\smile p'^*\beta_{kj}\smile\theta\alpha_k\\
&= \sum_{k,j}(-1)^{\varepsilon_{kj}} p'^*(p'_!(g_!\sigma_i\smile\theta\alpha_j)\smile\beta_{kj})\smile\theta\alpha_k ,
\end{aligned}$$

where $\varepsilon_{kj}=|\beta_{kj}|+|\alpha_k|$. Since $\{\theta\alpha_i\}$ is an $H^*(B;Q)$-basis for $H^*(E';Q)$, the above expression shows that the Lefschetz trace of the $H^*(B;Q)$-endomorphism $g_!(\gamma\smile)f^*$ agrees with

$$\begin{aligned}
&\sum_{i,j}(-1)^{|\beta_{ij}|} p'_!(g_!\sigma_i\smile\theta\alpha_j)\smile\beta_{ij}\\
&= \sum_{i,j}(-1)^{|\beta_{ij}|} p'_! g_!(\sigma_i\smile g^*\theta\alpha_j)\smile\beta_{ij}\\
&= \sum_{i,j}(-1)^{|\beta_{ij}|} p_!(\sigma_i\smile g^*\theta\alpha_j)\smile\beta_{ij} .
\end{aligned}$$

On the other hand, it follows that

$$\begin{aligned}
&\tau_{f,g}(\gamma) = p_!(\gamma\smile(f, g)^*\Delta(p'))\\
&= \sum_{i,j} p_!(\gamma\smile(f, g)^*(p_1^* p'^*\beta_{ij}\smile p_1^*\theta\alpha_i\smile p_2^*\theta\alpha_j))\\
&= \sum_{i,j} p_!(\gamma\smile p^*\beta_{ij}\smile f^*\theta\alpha_i\smile g^*\theta\alpha_j)
\end{aligned}$$

$$=\sum_{i,j}(-1)^{|\beta_{ij}|}p_!(\sigma_i\smile g^*\theta\alpha_j)\smile\beta_{ij}.$$

This completes the proof.

COROLLARY 6.3. *Let $p: E\to B$ and $p': E'\to B'$ be m-orientable fibre bundles. Assume that B is an oriented compact manifold, and that p' has a rational cohomology extension of the fibre. Let $f, g: E\to E'$ be fibre preserving maps covering $h: B\to B'$. Then the image of $\gamma\in H^*(E;Q)$ under $\tau_{f,g}: H^*(E;Q)\to H^*(B;Q)$ equals the Lefschetz trace of the composite*

$$H^*(B\underset{B'}{\times}E';Q)\xrightarrow{\tilde{f}^*}H^*(E;Q)\xrightarrow{\smile\gamma}H^*(E;Q)\xrightarrow{\tilde{g}_!}H^*(B\underset{B'}{\times}E';Q)$$

regarded as an $H^*(B;Q)$-endomorphism, where $B\underset{B'}{\times}E'$ is the fibre product of $h: B\to B'$ and $p': E'\to B'$, and $\tilde{f}$, $\tilde{g}$ are given by $\tilde{f}(e)=(p(e), f(e))$, $\tilde{g}(e)=(p(e), g(e))$ $(e\in E)$.

PROOF. It follows from the pull-back property in Theorem 5.1 that

$$\tau_{\tilde{f},\tilde{g}}=\tau_{f,g}: H^*(E;Q)\to H^*(B;Q).$$

The pull-back $p_1: B\underset{B'}{\times}E'\to B$ of $p': E'\to B'$ under $h: B\to B'$ has a rational cohomology extension of the fibre, and $\tilde{f}, \tilde{g}: B\underset{B'}{\times}E'\to E$ cover $id: B\to B$. Therefore we get the result by Theorem 6.1.

7. Coincidence-coincidence index.

Let $p: E\to B$ and $p': E'\to B'$ be m-orientable fibre bundles over oriented compact n-manifolds, and let $f, g: E\to E'$ be fibre preserving maps covering a map $h: B\to B'$. Let $\varphi: E\to B$ be a continuous map. With these data we shall define an integer $J(f, g;\varphi)$, called *the coincidence-coincidence index* of $(f, g;\varphi)$, as follows. This is a variation of the coincidence-fixed-point index in [**6**].

Consider the diagonal cohomology class $\Delta(B)\in H^n(B\times B)$, then we have

$$(p, \varphi)^*\Delta(B)\in H^n(E)$$

and hence

$$(p, \varphi)^*\Delta(B)\smile\Lambda_{f,g}\in H^{m+n}(E).$$

Giving E the canonical orientation, we define

$$J(f, g;\varphi)=\langle(p, \varphi)^*\Delta(B)\smile\Lambda_{f,g}, [E]\rangle.$$

PROPOSITION 7.1. *If $J(f, g;\varphi)\neq 0$ then $C_{p,\varphi}\cap C_{f,g}\neq\emptyset$.*

PROOF. Consider the orientation classes $u(B)\in H^n(B\times B, B\times B-dB)$ and $u(p')\in H^m(E'\underset{B'}{\times}E', E'\underset{B'}{\times}E'-dE')$. Then we have elements $(p, \varphi)^*u(B)\in$

$H^n(E, E-C_{p,\varphi})$ and $(f, g)^*u(p')\in H^m(E, E-C_{f,g})$. It follows that $(p, \varphi)^*\Delta(B)\smile(f, g)^*\Delta(p')$ is the image of $(p, \varphi)^*u(B)\smile(f, g)^*u(p')\in H^{m+n}(E, E-(C_{p,\varphi}\cap C_{f,g}))$ in E. Therefore if $C_{p,\varphi}\cap C_{f,g}=\emptyset$ then $J(f, g;\varphi)=0$. This proves the result.

The following is a version of Theorem (2.1) in [6].

THEOREM 7.2. *$J(f, g;\varphi)$ equals the Lefschetz trace of the composite*

$$H^*(B;Q)\xrightarrow{\varphi^*}H^*(E;Q)\xrightarrow{\tau_{f,g}}H^*(B;Q).$$

PROOF. It is known that

$$(\beta\frown[B])\backslash\Delta(B)=(-1)^q\beta$$

for $\beta\in H^q(B;R)$ ([10]). Therefore we have a commutative diagram

$$\begin{tikzcd}
H^q(E) \arrow[r, "\smile\Lambda_{f,g}"] \arrow[d, "\frown[E]"'] & H^{q+m}(E) \arrow[r, "p_!"] \arrow[d, "\frown[E]"'] & H^q(B) \\
H_{m+n-q}(E) \arrow[r, "(-1)^{qm}\Lambda_{f,g}\frown"] & H_{n-q}(E) \arrow[r, "(-1)^q p_*"] & H_{n-q}(B). \arrow[u, "\backslash\Delta(B)"']
\end{tikzcd}$$

Take a basis $\{\beta_j\}$ of $H^*(B;Q)$ and put

$$\Delta(B)=\sum_{i,j}c_{ij}\beta_i\times\beta_j\in H^n(B\times B;Q)$$

with $c_{ij}\in Q$. Then, in virtue of the above diagram, it follows that

$$\begin{aligned}
\tau_{f,g}\varphi^*(\beta_j)&=p_!(\varphi^*\beta_j\smile\Lambda_{f,g})\\
&=(-1)^{|\beta_j|(m+1)}p_*(\Lambda_{f,g}\frown(\varphi^*\beta_j\frown[E]))\backslash\Delta(B)\\
&=(-1)^{|\beta_j|(m+1)}\sum_{i,k}c_{ik}\langle\beta_i, p_*(\Lambda_{f,g}\frown(\varphi^*\beta_j\frown[E]))\rangle\beta_k\\
&=(-1)^{|\beta_j|}\sum_{i,k}c_{ik}\langle p^*\beta_i\smile\varphi^*\beta_j\smile\Lambda_{f,g}, [E]\rangle\beta_k.
\end{aligned}$$

Therefore the Lefschetz trace of $\tau_{f,g}\varphi^*$ agrees with

$$\begin{aligned}
&\sum_{i,j}c_{ij}\langle p^*\beta_i\smile\varphi^*\beta_j\smile\Lambda_{f,g}, [E]\rangle\\
&=\langle(p, \varphi)^*\Delta(B)\smile\Lambda_{f,g}, [E]\rangle\\
&=J(f, g;\varphi).
\end{aligned}$$

This completes the proof.

COROLLARY 7.3. *If $\varphi\simeq\varphi'\circ p: E\to B$ with $\varphi': B\to B$ and B is connected, then*

$$J(f, g; \varphi)=\lambda(\varphi')\lambda(f_b, g_b),$$

where $\lambda(\varphi')=\lambda(\varphi', id)$ *is the Lefschetz number of* φ'.

PROOF. By Theorem 7.2, $J(f, g; \varphi)$ equals the Lefschetz trace of the composite

$$H^*(B; Q) \xrightarrow{\varphi'^*} H^*(B; Q) \xrightarrow{p^*} H^*(E; Q) \xrightarrow{\tau_{f,g}} H^*(B; Q),$$

which send β to $\lambda(f_b, g_b)\varphi'^*(\beta)$ by Theorem 5.1. Therefore we have the result.

COROLLARY 7.4. *If* $B=B'$, $h=id$ *and* B *is connected, then*

$$J(f, g; \varphi)=\lambda(f, g)$$

holds for any $\varphi: E\to B$ *homotopic to* $p: E\to B$.

PROOF. We have $J(f, g; \varphi)=\chi(B)\lambda(f_b, g_b)$ by Corollary 7.3, and we have $\lambda(f, g)=\chi(B)\lambda(f_b, g_b)$ by Theorem 3.2. Therefore $J(f, g; \varphi)=\lambda(f, g)$.

Let $p: E\to B$ be a fibre bundle having $\theta: H^*(M; Q)\to H^*(E; Q)$, a rational cohomology extension of the fibre. Then, for a homomorphism $\rho: H^*(B; Q)\to H^*(E; Q)$ and an $H^*(B; Q)$-endomorphism $\nu: H^*(E; Q)\to H^*(E; Q)$, we define an endomorphism $\{\rho, \nu\}$ of $H^*(B; Q)\otimes H^*(M; Q)=H^*(E; Q)$ by

$$\{\rho, \nu\}(\beta\otimes\alpha)=\rho\beta\smile\nu\theta\alpha.$$

For $\gamma\in H^*(E; Q)$, let $\sigma_\nu(\gamma)$ denote the Lefschetz trace of the composite

$$H^*(E; Q) \xrightarrow{\nu} H^*(E; Q) \xrightarrow{\gamma\smile} H^*(E; Q)$$

regarded as an $H^*(B; Q)$-endomorphism. Then we have a homomorphism $\sigma_\nu: H^*(E; Q)\to H^*(B; Q)$.

The following is proved in (3.6) in [**6**].

LEMMA 7.5. *The Lefschetz trace of* $\{\rho, \nu\}$ *equals that of* $\sigma_\nu\rho$.

The following is a version of Proposition (3.5) in [**6**].

PROPOSITION 7.6. *Assume that* $B=B'$, $h=id$, *and both* p *and* p' *have rational cohomology extensions of the fibre. Then* $J(f, g; \varphi)$ *equals the Lefschetz trace of the endomorphism* $\{\varphi^*, f^*g_!\}$.

PROOF. By the definition, $\sigma_{f^*g_!}\varphi^*\beta$ ($\beta\in H^*(B; Q)$) is equal to the Lefschetz trace of the composite

$$H^*(E; Q) \xrightarrow{g_!} H^*(E'; Q) \xrightarrow{f^*} H^*(E; Q) \xrightarrow{\varphi^*\beta\smile} H^*(E; Q)$$

regarded as an $H^*(B; Q)$-endomorphism. Therefore, by Theorem 6.1 we have

$$\sigma_{f^*g_!}\varphi^*=\tau_{f,g}\varphi^*: H^*(B; Q)\to H^*(B; Q).$$

Since

$$\mathrm{Tr}(\sigma_{f^*g_!}\varphi^*)=\mathrm{Tr}\{\varphi^*, f^*g_!\}$$

$$\mathrm{Tr}(\tau_{f,g}\varphi^*)=J(f, g; \varphi)$$

by Lemma 7.5 and Theorem 7.2, we have the result.

8. Generalizations to manifolds with boundary.

In this section, the theorems proved in the preceding sections will be generalized to the case of fibre bundles having fibres manifolds with boundary.

Let M and M' be oriented compact m-manifolds with boundary, and let $f, g: M \to M'$ be continuous maps. If g maps the boundary $\dot{M}$ in the boundary $\dot{M}'$, then the *Lefschetz number* $\lambda(f, g)$ is defined, similarly to the case of manifold without boundary, to be the Lefschetz trace of the composite

$$H^*(M; Q) \xrightarrow{g_!} H^*(M'; Q) \xrightarrow{f^*} H^*(M; Q).$$

For a compact manifold M with boundary, let DM denote the double of M, i.e. the quotient space obtained from the topological sum $M \cup M_-$ of M and its copy M_- by identifying $x \in \dot{M}$ with its copy $x_- \in M_-$. DM is a manifold, and if M is oriented then DM is canonically oriented. There is a retraction $r: DM \to M$ defined by $r(x)=r(x_-)=x$ $(x \in M)$. For a continuous map $g: (M, \dot{M}) \to (M', \dot{M}')$, the double $Dg: DM \to DM'$ of g is defined by $(Dg)(x)=g(x)$, $(Dg)(x_-)=g(x)_-$ $(x \in M)$.

LEMMA 8.1. *Let M and M' be oriented compact manifolds with boundary, and let $f: M \to M'$ and $g: (M, \dot{M}) \to (M', \dot{M}')$ be continuous maps. Then, for $i'fr$, $Dg: DM \to DM'$, we have*

$$\lambda(f, g)=\lambda(i'fr, Dg),$$

where $i': M' \to DM'$ is the inclusion.

PROOF. Consider a diagram

$$\begin{tikzcd}
H^*(M) \arrow[r, "r^*"] \arrow[dr, "id"'] & H^*(DM) \arrow[r, "(Dg)_!"] \arrow[d, "i^*"] & D^*(DM') \arrow[d, "i'^*"] \\
& H^*(M) \arrow[r, "g_!"] & H^*(M')
\end{tikzcd}$$

of cohomology with coefficients in Q. Since $i'^*(Dg)_!=g_!i^*$ as is shown below, the diagram commutes. Therefore we have

$$\lambda(i'fr, Dg)=\mathrm{Tr}(r^*f^*i'^*(Dg)_!)$$

$$=\mathrm{Tr}(f^*i'^*(Dg)_!r^*)$$

$$=\mathrm{Tr}(f^*g_!)=\lambda(f, g).$$

The following diagram commutes:

$$\begin{tikzcd} H^*(DM) \arrow[rr, "i^*"] \arrow[d, "\frown[DM]"] & & H^*(M) \arrow[d, "\frown[M]"] \\ H_*(DM) \arrow[r, "j_*"] & H_*(DM, M_-) & H_*(M, \dot{M}) \arrow[l, "k_*"', "\cong"] \end{tikzcd}$$

where i, j and k are the inclusions. Consider the similar diagram for M', and consider the homomorphisms between the corresponding groups induced by g and Dg. Then, for $\alpha \in H^*(DM; R)$, we have

$$k'_*(i'^*(Dg)_!\alpha \frown [M']) = j'_*((Dg)_!\alpha \frown [DM'])$$

$$= j'_*(Dg)_*(\alpha \frown [DM]) = (Dg)_* j_*(\alpha \frown [DM])$$

$$= (Dg)_* k_*(i^*\alpha \frown [M]) = k'_* g_*(i^*\alpha \frown [M])$$

$$= k'_*(g_! i^*\alpha \frown [M']).$$

This shows $i'^*(Dg)_! = g_! i^*$, and completes the proof.

Let $p: E \to B$ be a fibre bundle having fibre M a compact manifold with boundary. We write

$$\dot{E} = \bigcup_{b \in B} \dot{M}_b,$$

and call it the *boundary* of E. Then the double DE of E is defined similarly to the double of a manifold with boundary. If B is a compact manifold, then E is a compact manifold with boundary, and $\dot{E}$, DE agree with the boundary, the double of the manifold E respectively.

Define $\bar{p}: DE \to B$ by $\bar{p}(x) = \bar{p}(x_-) = p(x)$ $(x \in E)$. It follows that $\bar{p}$ is a fibre bundle having fibre DM. We call $\bar{p}: DE \to B$ the *double* of the fibre bundle $p: E \to B$.

There exists a fibre preserving retraction $r: DE \to E$ given by $r(x) = r(x_-) = x$. Therefore we have a split exact sequence

$$(8.1) \qquad 0 \longrightarrow H^*(E, \dot{E}; R) \xrightarrow{j^*} H^*(DE; R) \underset{r^*}{\overset{i^*}{\rightleftarrows}} H^*(E; R) \longrightarrow 0.$$

Let $p: E' \to B'$ be also a fibre bundle having fibre M' a compact manifold with boundary, and let $g: (E, \dot{E}) \to (E', \dot{E}')$ be a fibre preserving map. Consider the doubles $\bar{p}: DE \to B$ and $\bar{p}': DE' \to B'$. Then we have a fibre preserving map $Dg: DE \to DE'$ defined by $Dg(x)=g(x)$, $Dg(x_-)=g(x)_-$, $(x \in E)$.

The notion of m-orientability is generalized as follows. Let $p: E \to B$ be a fibre bundle such that each fibre $M_b=p^{-1}(b)$ is an oriented compact m-manifold with boundary. If the local system $\{H^m(M_b, \dot{M}_b)\}_{b \in B}$ is trivial, we say that $p: E \to B$ is *m-orientable.*

It follows that if $p: E \to B$ is m-orientable, so is $\bar{p}: DE \to B$.

For an m-orientable fibre bundle $p: E \to B$, we have the integration along the fibre

$$p_!: H^{q+m}(E, \dot{E}; R) \to H^q(B; R).$$

Let $p: E \to B$ and $p': E' \to B'$ be m-orientable fibre bundles, and let $f, g: E \to E'$ be fibre preserving maps covering $h: B \to B'$. Assume

$$g(\dot{E}) \subset \dot{E}'.$$

Then the *Lefschetz class* $\wedge_{f,g} \in H^m(E, \dot{E}; R)$ is defined by

$$j^* \wedge_{f,g} = (id - r^* i^*) \wedge_{i'fr, Dg}$$

in terms of (8.1), where $\wedge_{i'fr, Dg} \in H^m(DE; R)$ is the Lefschetz coincidence class for the fibre preserving maps $i'fr$, $Dg: DE \to DE'$. It follows that $j^* i_b^* \wedge_{f,g} = i_b^* \wedge_{i'fr, Dg} \in H^m(DM_b; R)$ for the homomorphism $j^*: H^m(M_b, \dot{M}_b; R) \to H^m(DM_b; R)$.

Since $C_{f,g}=C_{i'fr,Dg}$, Proposition 2.1 remains valid. Applying Lemma 2.2 to $i'fr$, Dg, it follows from Theorem 8.1 that the lemma remains valid. Therefore we see from the proofs that Theorems 2.3 and 2.4 remain valid. Similarly it follows that Theorems 3.2 and 3.4 remain valid, and that Theorem 4.1 remains valid if $g(\dot{M}) \subset \dot{M}$.

The *coincidence transfer homomorphism* $\tau_{f,g}$ is defined to be the composite

$$H^*(E; R) \xrightarrow{\smile \wedge_{f,g}} H^*(E, \dot{E}; R) \xrightarrow{p_!} H^*(B; R)$$

similarly to (5.1), or to be the composite

$$H^*(E; R) \xrightarrow{r^*} H^*(DE; R) \xrightarrow{\tau_{i'fr,Dg}} H^*(B; R)$$

by making use of the coincidence transfer homomorphism of $i'fr$, $Dg: DE \to DE'$.

Theorem 5.1 remains valid. In fact, the properties (i)~(iv) are easily verified, and (v) is proved as follows.

Proof of the product property of $\tau_{f,g}$. Define $q: DE_1 \times DE_2 \to D(E_1 \times E_2)$ by

$$q(x_1, x_2)=q(x_{1-}, x_{2-})=(x_1, x_2),$$

$$q(x_{1-},\ x_2)=q(x_1,\ x_{2-})=(x_1,\ x_2)_-$$

$(x_1 \in E_1,\ x_2 \in E_2)$. Then we have

$$rq=r_1\times r_2\,,$$

where $r: D(E_1\times E_2)\to E_1\times E_2$, $r_i: DE_i\to E_i$ $(i=1,\ 2)$ are the retractions. Define $q': DE'_1\times DE'_2\to D(E'_1\times E'_2)$ similarly. Then we have

$$i'(f_1\times f_2)rq=q'(i'_1f_1r_1\times i'_2f_2r_2)\,,$$

$$D(g_1\times g_2)q=q'(Dg_1\times Dg_2)\,.$$

Consider the homomorphism $(q'\times q')^*: H^*(D'\underset{B'}{\times} D',\ D'\underset{B'}{\times} D'-dD'\,;\ R)\to H^*((D'_1\times D'_2)\underset{B'}{\times}(D'_1\times D'_2)-d(D'_1\times D'_2)\,;\ R)$, where $D'=D(E'_1\times E'_2)$, $D'_i=D(E'_i)$ and $B'=B'_1\times B'_2$. It is easily checked that

$$(q'\times q')^*u(\overline{p'_1\times p'_2})=u(\bar{p}'_1\times \bar{p}'_2)\,.$$

Therefore we have

$$(q'\times q')^*\Delta(\overline{p'_1\times p'_2})=\Delta(\bar{p}'_1\times \bar{p}'_2)\,.$$

It follows from (1.4) that a diagram

$$\begin{tikzcd}
H^*(D(E_1\times E_2)\,;\ R) \arrow[rr, "q^*"] \arrow[dr, "\overline{(p_1\times p_2)}_!"'] & & H^*(DE_1\times DE_2\,;\ R) \arrow[dl, "(\bar{p}_1\times \bar{p}_2)_!"] \\
& H^*(B_1\times B_2\,;\ R) &
\end{tikzcd}$$

commutes.

Thus, using the product property in Theorem 5.1, the following equalities hold for $\gamma_i\in H^*(E_i\,;\ R)$ $(i=1,\ 2)$, $\gamma=\gamma_1\times\gamma_2$, $p=p_1\times p_2$, $p'=p'_1\times p'_2$, $f=f_1\times f_2$ and $g=g_1\times g_2$.

$$\begin{aligned}
&\tau_{f_1,\, g_1}\gamma_1\times\tau_{f_2,\, g_2}\gamma_2\\
&=\tau_{i'_1f_1r_1,\, Dg_1}r_1^*\gamma_1\times\tau_{i'_2f_2r_2,\, Dg_2}r_2^*\gamma_2\\
&=\tau_{i'_1f_1r_1\times i'_2f_2r_2,\, Dg_1\times Dg_2}(r_1\times r_2)^*\gamma\\
&=\tau_{i'_1f_1r_1\times i'_2f_2r_2,\, Dg_1\times Dg_2}q^*r^*\gamma\\
&=(\bar{p}_1\times\bar{p}_2)_!(q^*r^*\gamma\smile(i'_1f_1r_1\times i'_2f_2r_2,\ Dg_1\times Dg_2)^*\Delta(\bar{p}'_1\times\bar{p}'_2))\\
&=(\bar{p}_1\times\bar{p}_2)_!(q^*r^*\gamma\smile(i'_1f_1r_1\times i'_2f_2r_2,\ Dg_1\times Dg_2)^*(q'\times q')^*\Delta(\bar{p}'))\\
&=(\bar{p}_1\times\bar{p}_2)_!q^*(r^*\gamma\smile(ifr,\ Dg)^*\Delta(\bar{p}'))
\end{aligned}$$

$$=\bar{p}_!(r^*\gamma\smile(ifr,\ Dg)^*\Delta(\bar{p}'))$$

$$=\tau_{ifr,Dg}r^*\gamma$$

$$=\tau_{f,g}\gamma\,.$$

This proves the product property.

For a fibre bundle $p: E\rightarrow B$ having fibre a connected oriented compact manifold with boundary and for fibre preserving maps $f,\ g: E\rightarrow E$ covering $id: B\rightarrow B$, the conclusion of Theorem 5.2 is still valid if $g(\dot{E})\subset\dot{E}$. The proof is completely analogous to that of Theorem 5.2.

We see that Theorem 6.1 remains valid if we assume the existence of rational cohomology extension of the fibre for the fibre-bundle pair $p': (E',\ \dot{E}')\rightarrow B$ as well as for $p': E'\rightarrow B$.

The coincidence-coincidence index can be also generalized, and the theorems in Section 7 remain valid.

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