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On the deficiencies of algebroid functions

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1. Introduction. Let R be an n-sheeted covering surface of $|z|<\infty$ and $M(R)$ the field of meromorphic functions on R. Then, according to Selberg [2], the deficiency relation for $w\in M(R)$ (In all what follows, we assume that w is not constant.) is

$$\sum_{a}\delta(a, w)\leqq 2+\xi,$$

where $\xi=\limsup_{r\to\infty} N(r, R)/T(r, w)$. However, it is difficult for us to calculate ξ for each $w\in M(R)$. In this respect, when $w\in M(R)$ is proper on R (that is, R is the proper existence surface of w), Ullrich [4] showed by the branch point theorem that

$$\xi\leqq 2n-2.$$

Therefore, combining these two inequalities, we obtain an explicit upper bound for the deficiency sum in case that $w\in M(R)$ is proper on R;

$$(*)\qquad \sum_{a}\delta(a, w)\leqq 2n.$$

On the other hand, the value distribution of algebroid functions has been studied through the systems of their coefficients which are entire functions on $|z|<\infty$. Along this line, Cartan [1] and Toda [3] have given another form of the deficiency relation

$$\sum_{a}\delta(a, w)\leqq n+1+\lambda(n-\lambda),$$

where λ is the maximum number of $\boldsymbol{C}$-independent linear relations among these coefficients.

In this paper, we shall give some extentions of these studies. In **2**, we shall show the deficiency relation for $w\in M(R)$ which is not proper on R. In **3**, we shall extend the Nevanlinna theory of systems in the case of algebroid functions. In **4** and **5**, we shall consider the case where R is also an m-sheeted covering surface of another covering surface S which is k-sheeted over $|z|<\infty$, and show that, on such a surface R, the deficiency sum for $w\in M(R)$ being proper on R is estimated somewhat better. For instance, if $\limsup_{r\to\infty} N(r, S)/N(r, R)=0$ (that is, the ramification of S is negligible compared with that of R), then,

$$\sum_a \delta(a, w) \leqq m+1+\lambda(m-\lambda),$$

or

$$\sum_a \delta(a, w) \leqq 2m,$$

and if the sheet number k of S is prime and $\limsup_{r\to\infty} N(r, R_s)/N(r, R)=0$ (that is, the ramification of R over S is negligible compared with that of R), then,

$$\sum_a \delta(a, w) \leqq 2n-2m+2.$$

In **6**, we shall give an example which illustrates these two consequences.

We shall use the standard symbols of the Nevanlinna theory of algebroid functions except when the remark should be necessary (see Selberg [2]).

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2. First we summarize some theorems that are necessary for later use.

THEOREM A (Selberg [2]). *Let R and $M(R)$ be as in* **1**. *Then, for $w \in M(R)$, we have*

$$\| \ m(r, w'/w)=O(\log rT(r, w)),$$

where "$\|$" means that this relation holds for $r \to \infty$ except for the intervals of finite linear measure.

THEOREM B (Selberg [2]). *For $w \in M(R)$ and arbitrary $a_1, \cdots, a_q \in \boldsymbol{C}$, we have*

$$\| \ (q-2)\,T(r, w) \leqq \sum_{j=1}^{q} N(r, a_j, w)+N(r, R)+O(\log rT(r, w)).$$

THEOREM C (Ullrich [4]). *Let $w \in M(R)$ be proper on R. Then, for $N(r, R)$, we have*

$$N(r, R) \leqq (2n-2)\,T(r, w)+O(1).$$

THEOREM D (Valiron [5]). *As for the derivative w' of $w \in M(R)$, we have*

$$\| \ T(r, w') \leqq 2T(r, w)+N(r, R)+O(\log rT(r, w)).$$

Now, we give the following proposition.

PROPOSITION 1. *Let R and $M(R)$ be as above and $f \in M(R)$ be not proper on R. We denote by S the proper existence surface of f. Then, R is a covering surface of S and f is the lift of the meromorphic function on S. Here, if S is*

k-sheeted over $|z|<\infty$, then, k is a divisor of n and R is $m(=n/k)$-sheeted over S.

According to Proposition 1, Theorems B and C, we obtain the following theorem.

THEOREM 1. *Let $f\in M(R)$ be not proper on R. Then, the deficiency relation for f becomes*

$$\sum_{a}\delta(a, f)\leqq 2k,$$

where k is the sheet number of S over $|z|<\infty$.

3. Let S be a k-sheeted covering surface of $|z|<\infty$. We call

$$f=(f_0, \cdots, f_m)$$

a system on S if $f_0, \cdots, f_m$ are holomorphic functions on S without common zeros. We define the characteristic function of f by

$$T(r, f)=\frac{1}{2\pi k}\int_{\Gamma(r)}\log u(y)\,d\theta,$$

where $y=re^{i\theta}\in S$, $u(y)=\max_{0\leqq j\leqq m}|f_j(y)|$ and $\Gamma(r)$ is the part of S over $|z|=r$. Let

$$X=\{F\,|\,F=a_0f_0+\cdots+a_mf_m,\ a_j\in \boldsymbol{C}\}$$

be the family of homogeneous linear combinations of $f_0, \cdots, f_m$ with constant coefficients whose arbitrary $m+1$ coefficient vectors $\{(a_0, \cdots, a_m)\}$ are linearly independent. We denote by λ the maximum number of $\boldsymbol{C}$-independent linear relations among $f_0, \cdots, f_m$ (see Cartan [**1**]).

We shall denote by $\tilde{m}(r, g)$, $\tilde{N}(r, g)$ and $\tilde{T}(r, g)$, respectively, the proximity, counting and characteristic functions of $g\in M(S)$.

As in the case of systems on $|z|<\infty$, we obtain the following propositions.

PROPOSITION 2. *For $F_1, F_2\in X$, we have*

$$\tilde{T}(r, F_2/F_1)\leqq T(r, f)+O(1).$$

In proving Proposition 2, we use Florack's theorem when F_1 and F_2 have common zeros.

PROPOSITION 3. *From $f_0, \cdots, f_m$, we can choose $m+1-\lambda$ functions $f_{a_1}, \cdots, f_{a_{m+1-\lambda}}$ which satisfy the following conditions:*

(1) *$f_{a_1}, \cdots, f_{a_{m+1-\lambda}}$ are linearly independent over $\boldsymbol{C}$.*

(2) *Each $F\in X$ can be represented as a $\boldsymbol{C}$-linear combination of $f_{a_1}, \cdots, f_{a_{m+1-\lambda}}$.*

We call $f_{a_1}, \cdots, f_{a_{m+1-\lambda}}$ a basis of $f_0, \cdots, f_m$.

PROPOSITION 4. *For* $F_1, \cdots, F_q$ $(q \geqq m+1)$ *and a fixed* $y \in S$, *we define* F_{b_j} *as* $|F_{b_1}(y)| \leqq \cdots \leqq |F_{b_q}(y)|$. *Then, there exists a constant* K *which is independent of* y *and satisfies*

$$u(y) \leqq K|F_{b_j}(y)|$$

for $j=m+1, \cdots, q$.

Propositions 3 and 4 are consequences of linear algebra.

PROPOSITION 5. *Setting* $\Delta = \|F_1, \cdots, F_p\| / F_1 \cdots F_p$ *for* $F_1, \cdots, F_p \in X$, *we have*

$$\| \ \tilde{m}(r, \Delta) = O(\log rT(r, f)),$$

where $\|F_1, \cdots, F_p\|$ *is the Wronskian of* $F_1, \cdots, F_p$.

PROOF. Let $\varphi_j = F_{j+1}/F_j$ $(j=1, \cdots, p-1)$. Then,

$$\tilde{m}(r, \Delta) \leqq \sum_{j=1}^{p-1} \sum_{h=1}^{p-1} K_h \tilde{m}(r, \varphi_j^{(h)}/\varphi_j^{(h-1)}) + O(1),$$

where K_h are constants. Here, by Theorem A, we have

$$\| \ \tilde{m}(r, \varphi_j^{(h)}/\varphi_j^{(h-1)}) = O(\log r\tilde{T}(r, \varphi_j^{(h-1)})),$$

and by Proposition 1, Theorems D and C,

$$\begin{aligned} \| \ \tilde{T}(r, \varphi_j') &\leqq 2\tilde{T}(r, \varphi_j) + N(r, S_j) + O(\log r\tilde{T}(r, \varphi_j)) \\ &= O(\tilde{T}(r, \varphi_j)), \end{aligned}$$

where S_j is the proper existence surface of φ_j. Further, by Proposition 2, we have

$$\tilde{T}(r, \varphi_j) \leqq T(r, f) + O(1).$$

Combining these inequalities, we obtain the proposition. Q. E. D.

Now, we shall show the following theorem. The proof will be given along the same line as in Cartan [1] and Toda [3].

THEOREM 2. *Let* f, X *and* λ *be as above. Then, for arbitrary* $F_1, \cdots, F_q \in X$, *we have*

$$\begin{aligned} \| \ &(q-m-1-\lambda(m-\lambda))\,T(r, f) \\ &\leqq \sum_{j=1}^{q} \tilde{N}(r, 0, F_j) + (\lambda+1)(m-\lambda)(m+1-\lambda)\,N(r, S) + O(\log rT(r, f)). \end{aligned}$$

PROOF. By Proposition 3, we choose a basis $f_{a_1}, \cdots, f_{a_{m+1-\lambda}}$ of $f_0, \cdots, f_m$ and set

$$H(y) = F_1 \cdots F_q / \|f_{a_1}, \cdots, f_{a_{m+1-\lambda}}\|^{\lambda+1}.$$

Then, according to the first main theorem of Selberg [2], we have

$$\frac{1}{2\pi k}\int_{\Gamma(r)}\log|H(y)|d\theta\leqq\tilde{N}(r,0,H)+O(1).$$

So, we may estimate these two terms. To estimate the first, according to Proposition 3, we rewrite $H(y)$ as

$$K\frac{F_{b_1}\cdots F_{b_q}}{\|F_{b_1},\cdots,F_{b_{m+1-\lambda}}\|\prod_{j=m+2-\lambda}^{m+1}\|F_{b_j},F_{b_2},\cdots,F_{b_{m+1-\lambda}}\|}$$

$$=K\frac{F_{b_{m+2}}\cdots F_{b_q}}{\dfrac{\|F_{b_1},\cdots,F_{b_{m+1-\lambda}}\|}{F_{b_1}\cdots F_{b_{m+1-\lambda}}}\prod_{j=m+2-\lambda}^{m+1}\dfrac{\|F_{b_j},F_{b_2},\cdots,F_{b_{m+1-\lambda}}\|}{F_{b_j}F_{b_2}\cdots F_{b_{m+1-\lambda}}}[F_{b_2}\cdots F_{b_{m+1-\lambda}}]^{\lambda}},$$

where K is a constant. Therefore, by using Propositions 4 and 5 and the definition of $T(r,f)$, we have

$$\|\ (q-m-1-\lambda(m-\lambda))\,T(r,f)$$
$$\leqq\frac{1}{2\pi k}\int_{\Gamma(r)}\log|H(y)|d\theta+O(\log rT(r,f)),$$

(see [1] and [3]). To estimate the second, we rewrite $H(y)$ as

$$K\frac{F_{b_{m+2}}\cdots F_{b_q}}{\dfrac{\|F_{b_1},\cdots,F_{b_{m+1-\lambda}}\|}{F_{b_1}\cdots F_{b_{m+1-\lambda}}}\prod_{j=m+2-\lambda}^{m+1}\dfrac{\|F_{b_j},F_{b_2},\cdots,F_{b_{m+1-\lambda}}\|}{F_{b_j}}},$$

and count the multiplicities of the poles of the denominator. Now, let $y\in S$ be an ordinary point, then, the estimate is as usual. Let $y\in S$ be a branch point of order $k'-1$, then, $F_j^{(h)}/F_j$ may have a pole of order $h\cdot k'$ at y, so that the multiplicity of the pole of the denominator is at most $(1/2)(\lambda+1)(m-\lambda)\cdot(m+1-\lambda)k'$. Therefore, taking $(1/2)k'\leqq k'-1$ into account, we have

$$\tilde{N}(r,0,H)\leqq\sum_{j=1}^{q}\tilde{N}(r,0,F_j)+(\lambda+1)(m-\lambda)(m+1-\lambda)N(r,S).$$

Combining these inequalities, we obtain the theorem. Q. E. D.

4. Let R be an n-sheeted covering surface of $|z|<\infty$. We consider the case where R is also an m-sheeted covering surface of another covering surface S which is k-sheeted over $|z|<\infty$. Now, let $w\in M(R)$ be proper on R. Then, w satisfies an irreducible equation of degree m whose coefficients are holomorphic functions on S without common zeros;

$$P(w,\ y)=f_0(y)\,w^m+\cdots+f_m(y)=0,\quad y\in S.$$

Let $f=(f_0,\ \cdots,\ f_m)$ be the system on S given by these coefficients.

Under these circumstances, we obtain the following proposition using the same method as in Valiron [5].

PROPOSITION 6. $T(r,\ w)=(1/m)\,T(r,\ f)+O(1)$.

According to this proposition combined with Theorem 2 and Theorem C, we obtain the following theorem.

THEOREM 3. *Let* R *and* S *be as above and* $\limsup_{r\to\infty} N(r,\ S)/N(r,\ R)=0$. *Then, the deficiency relation for* $w\in M(R)$ *being proper on* R *is*

$$\sum_a \delta(a,\ w)\leqq m+1+\lambda(m-\lambda).$$

5. Let R, S and w be as in **4**. We define the following counting function

$$N(r,\ R_s)=\frac{1}{n}\left[\int_0^r \frac{n(t,\ R_s)-n(0,\ R_s)}{t}dt+n(0,\ R_s)\log r\right],$$

where $n(r,\ R_s)$ is the sum of the orders of the branch points of R over the part of S over $|z|\leqq r$. Then, we obtain the following two propositions.

PROPOSITION 7. $N(r,\ R)=N(r,\ R_s)+N(r,\ S)$.

PROOF. Here, we look upon the ordinary point as the branch point of order 0. Now, let $y\in S$ be a branch point of order $k'-1$ and assume that there exist p branch points $x_1,\ \cdots,\ x_p$ over y of order $m_1-1,\ \cdots,\ m_p-1$ with respect to S $(m_1+\cdots+m_p=m)$. Then, we can easily see that

$$n(r,\ R)=\sum_y\Big(\sum_{j=1}^p (k'm_j-1)\Big)=\sum_y (k'm-p),$$

$$n(r,\ R_s)=\sum_y\Big(\sum_{j=1}^p (m_j-1)\Big)=\sum_y (m-p),$$

$$n(r,\ S)=\sum_y (k'-1),$$

where the sum $\sum_y$ extends over all $y\in S$ over $|z|\leqq r$. Thus, we obtain the equality

$$n(r,\ R)=n(r,\ R_s)+m\cdot n(r,\ S).$$

Dividing this equality by $n\cdot r$ and integrating it for r, we obtain the proposition. Q. E. D.

PROPOSITION 8. $N(r, R_s) \leqq (2m-2)T(r, w)+O(1)$.

Proposition 8 is proved by using the same method as in Ullrich [4].

Now, we shall show the following theorems.

THEOREM 4. *Let* $\limsup_{r\to\infty} N(r, S)/N(r, R)=u$. *If* $u<1$, *then the deficiency relation for* w *is*

$$\sum_a \delta(a, w) \leqq 2+(2m-2)/(1-u).$$

PROOF. By Proposition 7, we have

$$\liminf_{r\to\infty} N(r, R_s)/N(r, R)=1-u.$$

Therefore, combining this equality with Theorem B, we have for arbitrary small ε

$$\| \ (q-2)T(r, w)$$

$$\leqq \sum_{j=1}^{q} N(r, a_j, w)+N(r, R_s)/(1-u-\varepsilon)+O(\log rT(r, w)).$$

Here, by using Proposition 8, we obtain the theorem. Q. E. D.

THEOREM 5. *Let the sheet number* k *of* S *be prime and* $\limsup_{r\to\infty} N(r, R_s)/N(r, R)=v$. *If* $v<1$, *then, the deficiency relation for* w *is*

$$\sum_a \delta(a, w) \leqq 2+(2n-2m)/(1-v).$$

PROOF. As in the proof of Theorem 4, by Proposition 7 and Theorem B, we have for arbitrary small ε

$$\| \ (q-2)T(r, w)$$

$$\leqq \sum_{j=1}^{q} N(r, a_j, w)+N(r, S)/(1-v-\varepsilon)+O(\log rT(r, w)).$$

Let

$$P(w, y)=f_0(y)w^m+\cdots+f_m(y)=0, \quad y\in S$$

be an irreducible defining equation of w. Then, by Proposition 1, since k is prime, there exists at least one ratio f_j/f_0 that is proper on S. According to Theorem C, we have

$$N(r, S) \leqq (2k-2)T(r, f_j/f_0)+O(1),$$

and by Propositions 2 and 6,

$$T(r, f_j/f_0) \leqq m \cdot T(r, w) + O(1).$$

Combining these inequalities, we obtain the theorem. Q. E. D.

We remark that in Theorems 4 and 5, these estimates are better than the usual ones when $u < n(k-1)/k(n-1)$ and $v < (m-1)/(n-1)$. The case $u = n(k-1)/k(n-1)$ and $v = (m-1)/(n-1)$ can occur, for example, when R is regularly branched.

6. Example. Let S_1 be a 2-sheeted covering surface of $|z| < \infty$ whose branch points are above $z = n \cdot i$ $(n = 0, \pm 1, \cdots)$ and let R be a 2-sheeted covering surface of S_1 whose branch points are above the points of S_1 above $z = \sqrt{n}$ $(n = 1, 2, \cdots)$. Then, $N(r, S_1) \sim r$ and $N(r, R) \sim r^2/4$, so that the condition of Theorem 3 "$\limsup_{r\to\infty} N(r, S_1)/N(r, R) = 0$" is satisfied. Now, let $w \in M(R)$ be proper on R, then, according to Theorem 3,

$$\sum_a \delta(a, w) \leqq 2 + 1 + \lambda(2-\lambda).$$

Here, since $\lambda = 0$ or 1, the deficiency relation for w is

$$\sum_a \delta(a, w) \leqq 4.$$

This result can also be obtained by using Theorem 4.

On the other hand, from another point of view, R is a 2-sheeted covering surface of S_2 which is a 2-sheeted covering surface of $|z| < \infty$ whose branch points are above $z = \sqrt{n}$ $(n = 1, 2, \cdots)$. Then, $N(r, R_{s_2}) \sim r$, and the conditions of Theorem 5 "k is prime and $\limsup_{r\to\infty} N(r, R_{s_2})/N(r, R) = 0$" are satisfied. So, let $w \in M(R)$ be proper on R, then, according to Theorem 5, the deficiency relation for w is

$$\sum_a \delta(a, w) \leqq 6.$$

In this case, the former estimate is better.

Now, let $f \in M(R)$ be not proper on R. Then, the proper existence surface of f is reduced to a 2-sheeted covering surface of $|z| < \infty$ or $|z| < \infty$ itself. According to Theorem 1, the deficiency relation for f becomes

$$\sum_a \delta(a, w) \leqq 4 \text{ or } 2.$$

Thus, for all $w \in M(R)$, the deficiency sum for w is at most equal to 4. This result is better than the usual one obtained by (*).

References

[1] H. Cartan, Sur les zéros des combinaisons linéaires de p fonctions holomorphes données, Mathematica, **7** (1933), 5-31.
[2] H.L. Selberg, Algebroide Funktionen und Umkehrfunktionen Abelscher Integrale, Avh. Norske Vid. Akad. Oslo, **8** (1934), 1-72.
[3] N. Toda, Sur les combinaisons exceptionnelles de fonctions holomorphes; applications aux fonctions algébroïdes, Tôhoku Math. J., **22** (1970), 290-319.
[4] F.E. Ullrich, Uber den Einfluss der Verzweigtheit einer Algebroide auf ihre Wertverteilung, J. Reine Angew. Math., **167** (1932), 198-220.
[5] G. Valiron, Sur la dérivée des fonctions algébroïdes, Bull. Soc. Math. France, **59** (1931), 17-39.

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